Linear Algebra Final Exam Review

Reduced Row Echelon Form

This is a reduced matrix that will give you the ability to solve systems of equations. There are a few rules that need to be followed.

- The first non-zero entry must be a 1
- All numbers in the same column as a leading 1 must be a 0
- Leading 1 on a next row must be shifted one position to the right

If a reduced matrix follows all of these rules, then you have found a solution to the system of equations. You first start with an augmented matrix, this is where you take all of the coefficients and put them in a matrix.

Rules for finding solutions to a system of equations

Here we will use a term called the rank of a matrix in its rref form. The rank of this type of a matrix is the number of leading 1s the matrix has. For an $n \times m$ matrix A, where n is the number of rows and m is the number of columns in the matrix. The following conditions hold true if met:

- if $m < n$ or $m > n$, there are infinitely many solutions or no solutions because you have either too many variables(columns) or too few variables(columns).
- if $n = m$ then there is a unique solution if $rank(A) = n$.

Linear Transformations

A transformation is considered to be linear if the following two conditions are met.

- T(+)=T()+T()
- T(k)=kT()

Where $k \epsilon \mathbb{R}$ and $\vec{w}, \vec{v} \epsilon \mathbb{R}^n$

Matrices

Now we will talk about matrices and their properties. For our discussion, let's assume all matrices are $n \times n$, unless otherwise stated.

Properties of Matrices

- AB may not equal BA
- $(AB)C = A(BC)$ only if the order is maintained
- $k(AB) = (kA)B = A(kB)$
- If A is an $n \times m$ matrix and we let I_n and I_m be $n \times n$ and $\times m$ matrices respectively, then $AI_m = A = I_nA$

Where $A, B, C \epsilon \mathbb{R}^{n \times n}$ and $k \epsilon \mathbb{R}$

Multiplying Matrices with Vectors

You basically multiply the vector's components by the corresponding components in the row of a matrix then add them together and place it in the corresponding position of the vector.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

Multiplying Matrices

This is basically like multiplying a matrix by a bunch of vectors, the columns in the other row

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_3 & ax_2 + bx_4 \\ cx_1 + dx_3 & cx_2 + dx_4 \end{bmatrix}$$

Notice that the columns of one must math the rows of the other and that a $n \times m$ multiplied by a $m \times n$ gives an $n \times n$ matrix

The Kernel

The kernel represents a set of all vectors that when transformed by a matrix, go to 0. You can find the kernel by finding the rref of the matrix and setting up equations, from each row. Then with these equations, you will need to set variables equal to an arbitrary constant. The number of variables you need to do this for is equal to the number of columns that have non-zero numbers in them or are not following the rules for rref. With these you should be able to solve for all the other variables. Then construct your vectors to form your basis. You should have a separate vector for every variable you have set to be an arbitrary constant. Here are some facts about the kernel:

- $\vec{0}$ is always in the kernel
- if $\vec{u}$ and $\vec{v}$ are in the kernel, then so is $\vec{v} + \vec{u}$
- if $\vec{u}$ is in the kernel, then so is $k\vec{u}$

Image

The image is all of the vectors that a matrix outputs after a transformation. This is found by simply multiplying the matrix by an arbitrary vector. But, make sure they are linearly independent vectors! You can check this by taking the rref of the matrix, and looking at the columns that have leading 1s. The corresponding columns in the original matrix will be in the image of the matrix, the other columns, not having leading 1s, will be in the kernel of the matrix.g

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} a \\ c \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \end{bmatrix}$$

Here are some facts about the image

- $\vec{0}$ is always in the image
- if $\vec{u}$ and $\vec{v}$ are in the image, then so is $\vec{v} + \vec{u}$
- if $\vec{u}$ is in the image, then so is $k\vec{u}$

Types of Transformations

Here are a list of possible transformations:

- Orthogonal Projection $\begin{bmatrix} u^2 & uv \\ uv & v^2 \end{bmatrix}$ Where $\begin{bmatrix} u \\ v \end{bmatrix}$ is unit vector
- horizontal shear $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ Where k is a constant
- Vertical Shear $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
- Scaling $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ Where $k \epsilon \mathbb{R}$
- Rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Inverse of a Matrix

To find an inverse of a matrix, augment the matrix with the identity matrix of the same size. Then find the rref, the side where you put the identity matrix will give you the inverse of the matrix you started with.

Invertibility Rules

If A is an $n \times n$ matrix, then the following holds true:

- A is invertible
- $rank(A) = n$
- $rref(A) = I_n$
- There exists a unique solution for $A\vec{x} = \vec{y}$ for $\vec{y} \epsilon \mathbb{R}^n$
- Kernel of A is the set of the zero vector
- Image of A is $\mathbb{R}^n$
- The determinant of A does not equal 0
- if A and B are invertible, then so is AB

Linear Sub-spaces

A set of vectors forms a linear subspace,W, if the following conditions are met:

- The zero vector is in W
- $\vec{u}$ and $\vec{v}$ are in W, then so is $\vec{v} + \vec{u}$
- if $\vec{u}$ is in W, then so is $k\vec{u}$, where $k \epsilon \mathbb{R}$

Basis of a subset

The basis of a subset is a set of the fewest number of vectors that can span the entire subspace through just linear combinations(Sum of the scalar multiples of the vector).These vectors have to linearly independent.

$$r_1\vec{v_1} + r_2\vec{v_2} + ... + r_p\vec{v_p} = 0$$

No, you can just make all of the constant equal 0, that would be too easy, thus trivial. So there can't be a trivial result in order for the vectors to be linearly independent.

Rank-Nullity Theorem

For an $n \times m$ matrix A,

$$dim(ker(A)) + dim(Img(A)) = m$$

This basically says that all of the variables either need to transferred correctly to after the transformation or that if they are not they need to go into the kernel.

Vector Spaces

It turns out that vectors can be defined to be anything, just as long as they follow these rules:

- Commutative property holds under addition
- The associative property holds true for addition
- There is a zero value
- You need to have some vector that when added to another vector gives you zero. Basically, you need a negative number LOL!
- Scalars need to distribute across addition
- Vectors need to distribute across addition
- Associative property holds true for multiplication
- You need to have a vector that when multiplied to another vector, gives the other vector back. Basically, you need a 1.

Here are some facts about Vector Spaces

- A vector $v \epsilon V$ is a linear combination in $v_1, v_2, ..., v_n \epsilon V$ if there exists scalars $c_1, c_2, .., c_n$ such that $v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$
- Dimension of a Subspace is the number of vectors in any basis for V

Change of Coordinates

The figure below shows the change of basis diagram. You can basically generate formulas based on this diagram and be able to get from one place to another.

$$
\begin{array}{ccc}
 & A & \\
\mathbb{R}^n & \longrightarrow & T(\mathbb{R}^n) \\
S^{-1} \Big\downarrow \vdots & & \vdots \Big\downarrow S^{-1} \\
\mathbb{R_B}^n & \longrightarrow & T(\mathbb{R_B}^n) \\
 & G &
\end{array}
$$

Here S is the matrix transformation that will take you from one basis to another. A is some transformation matrix. G is given by transforming the basis vectors of $\mathbb{B}$, with A and converting vectors you get after the transformation to the new basis by multiplying by S^{-1}. Doing this sort of transformation comes in handy when doing lengthy calculations, such as taking a very high power of a matrix. Here are some important formulas to keep in mind.

- $AS = SG$
- $G = S^{-1}AS$
- $A = SGS^{-1}$

Also, remember that two matrices are similar if the $G = S^{-1}AS$ holds true given that S is invertible.

Matrix of Transformation

This is where you take abstract vector spaces and apply the change of coordinates diagram. This is done by the following steps:

1. Create an isomorphism, an invertible matrix that can transform abstract vectors to actual vectors
2. Transform the Basis of the Abstract vectors space through the given Transformation
3. Then convert the transformed abstract vector basis to the actual vector basis using the transformation in step 1. Then you will have your matrix of transformation once you concatenate the vectors to form a matrix.

Orthonormal Basis

This is where you make a basis that is both orthogonal and normal. That is each vector is perpendicular to every other vector and each vector has a magnitude of one. A matrix that consists of orthonormal vectors are called orthogonal matrices. Here are some facts about Orthogonal Matrices:

- There, can't be more than n orthonormal vectors in a set that is supposed to span $\mathbb{R}^n$
- Any set of n orthonormal vectors form a basis of $\mathbb{R}^n$
- Any subspace V of $\mathbb{R}^n$ has an orthonormal basis
- Orthogonal Transformations preserve length

Graham Scmitt Process

This is how you actually find an orthonormal basis given a regular(doesn't mean anything mathematical in this instance) basis. The steps are listed below for a given set of vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$

1. Replace $\vec{v_1}$ with $\frac{\vec{v_1}}{||\vec{v_1}||} = \vec{u}$
2. $\vec{V_2}^{\perp}$ then replace $\vec{v_2}$ with $\vec{u_2} = \frac{\vec{v_2}}{||\vec{v_2}||}$ In order to do so, use these formulas

$$\vec{x}^{||} = proj_v(\vec{x}) = (\vec{u_1} \cdot \vec{x})\vec{u_1} + ... + (\vec{u_n} \cdot \vec{x})\vec{u_n}$$

   This is to then find the perpendicular:

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{||}$$

3. Find $\vec{v_3}$ using $\vec{u_1}$ and $\vec{u_2}$ then replace $\vec{v_3}$ with $\vec{u_3} = \frac{\vec{v_3}}{||\vec{v_3}||}$

QR factorization

The QR factorization is a way to write an orthogonal matrix as a product of two matrices, like shown below.

$$M = QR$$

Where M is the orthogonal matrix that you get after doing graham-scmitt, Q is the original matrix you start with, and R is the weird one described below.

- The diagonals are the magnitudes of $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$
- The part below the diagonal is all zero!
- The the rest of the positions are the dot products you took to convert each $\vec{v_n}$ to $\vec{u_n}$ in step 2 and 3.

Transpose

The transpose is an operation you can do on the matrix that changes the layout. This operation is given by the following equation:

$$A_{ij} = A_{ji}^T$$

Basically, you reflect the elements across the diagonal.

Properties of the Transpose

Here are the properties of the transpose:

- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$
- $rank(A^T) = rank(A)$
- $det(A) = det(A^T)$
- $(A^T)^{-1} = (A^{-1})^T$
- $A^T A = I_n$

Determinants

We know how to take determinants for 2×2 matrices , but you take the determinant of larger matrices using the following steps.

1. Pick a column or row
2. Look at the first element and remove the column/row that the element is in.
3. Find the determinant of the rest of the matrix not being excluded.
4. Then multiply the determinant from step 2 with the element being covered.
5. Then multiply by $(-1)^{j+i}$ where i is the row position and j is the column position
6. Repeat steps 1 through 5 for all elements in the row or column picked in step 1.

Here are some properties of the determinant:

- The determinant of a diagonal matrix is the product of the diagonal elements
- The determinant of a upper and lower is the product of the diagonal elements
- Multiply the determinant by -1 for every row switch you make in the matrix
- If an column is duplicated, the determinant becomes zero
- If you multiply a row by a scalar k, then you will get the determinant multiplied by k. If you multiply multiple rows by k, then simply multiply the determinant by k^n where n is the number of rows you scale by k
- $det(A^T) = det(A)$
- If a column or row is a multiple of another row or column respectively, then the determinant is zero

- $det(S) = \frac{1}{det(S^{-1}}$

- If A and B are similar then, $det(A) = det(B)$

- $det(AB) = det(A)det(B)$

- Adding a multiple of one row to another, will not change the determinant

Adjoint

The adjoint is found by taking the co factor and placing the co factor in each position of the new matrix, in the same position that was picked when finding the co factor. With this new matrix, if you take the transpose of the matrix, you will have the adjoint. This is given by this formula

$$C_{ij} = (-1)^{i+1}det(A_{ij})$$
$$C = adj(A)$$

Where A_{ij} is taking the determinant of a sub matrix where the i, j row and column respectively are excluded. With this we get the following:

$$A^{-1} = \frac{1}{\det(A)}adj(A)$$

EigenStuff and Diagonalization

In this section we will deal with eigenvalues, eigenvectors, and diagonalization. A diagonal matrix is one where there are only entries in the diagonals of the matrix. We know if a matrix is diagonizable if it is similar to a diagonal matrix, that is it satisfies the similar property of $A = SDS^{-1}$. It is also diagonizable only when there is a basis $\vec{v_1}, ..., \vec{v_n}$ so that $A\vec{v_1} = \lambda_i \vec{v_i}$

Finding Eigenvalue and Eigenvectors

To find the Eigenvalues associated with a matrix, solve this equation for lambda

$$det(A - \lambda I_n) = 0$$

This will give you all of the eigenvalues the matrix has. The number of times an eigenvalue is repeated is the **algebraic multiplicity or** $almu(\lambda)$.With this you can find the eigenvectors by solving the equation for the vector $\vec{v}$.

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

What you are solving for here is the eigenspace of the eigenvalue. The eigenspace consists of the kernel of $a - \lambda I_n$. The dimension of the eigenspace for a specific value is known as the geometric multiplicity of the eigenvalue or $gemu(\lambda)$. You can find out if a matrix is diagonizable if the matrix has n eigenvectors in the aggregate eigenbasis. This also means the aggregate $gemu(A)$ and $almu(A)$ are equal. The aggregate $gemu$ simply counting all of the eigenvectors in all of the eigenvectors for all eigenvalues. The aggregate for the $almu$ is the sum total of all the $almu$ for all the eigenvalues.

Once you find all of this, you can construct the diagonal matrix by simply putting all of the eigenvalues in the diagonals of the matrix. The matrix has be the same dimensions as the original matrix.You can also find S from the equation: $A = SDS^{-1}$, by plugging in the eigenvectors corresponding to the eigenvalue you placed in the diagonalized matrix

Some interesting properties about similar matrices A and B:

- A and B have the same characteristic polynomial:$f_A(\lambda) = f_b(\lambda)$ The characteristic polynomial is the polynomial given when solving for the eigenvalues.This is given by the equation below:

$$\lambda^2 - trace(A)(-\lambda)^{k-1} + ... + det(A)$$

Where n is the number of rows or columns in the square matrix and k is the geometric multiplicity of the eigenvalue.

- A and B have the same eigenvalues, with the same algebraic and geometric multiplicities, but they might not have the same eigenvectors

- $rank(A) = rank(B)$ and thus the nullities are the same as well!

- A and B have the same determinant and trace.