# Lecture - 08

Thursday, September 3, 2020 1:57 PM

**Last class:** ★ periodic sampling : $x[n] = x_c(nT_s)$

★ Band-limited signals : $X_c(\omega) = 0$ for $|\omega| > \omega_B$

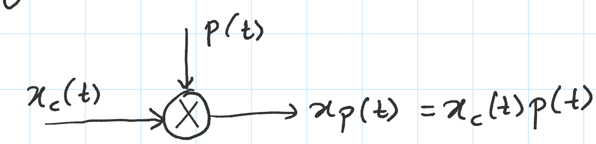
Nyquist rate $= 2\omega_B$ ☆

★ Sampling Theorem : perfect reconstruction of band-limited signals is possible if $\omega_s > 2\omega_B$.

## Today's Class :

Sampling Theorem
- → non-trivial result
- → hard to prove in time-domain
- → easy analysis using Fourier transform

To go from continuous-time signal $x_c(t)$ to is samples $x[n]$, we will use Impulse-train sampling




Let $x_c(t)$ be band-limited signal. ($\omega_M$)

let $p(t)$ be an impulse - train

$$p(t) = \sum_{-\infty}^{\infty} \delta(t - nT_s)$$

$$P(\omega) = \frac{2\pi}{T_s} \sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T_s}k\right) \qquad \omega_s = \frac{2\pi}{T_s}$$

$x_p(t) = x_c(t)\,p(t)$ ∴ using multiplication property of F.T.

$$X_p(\omega) = \frac{1}{2\pi}\left[X_c(\omega) * P(\omega)\right]$$

$$= \frac{1}{2\pi} X_c(\omega) * \frac{2\pi}{T_s}\left[\sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T_s}k\right)\right]$$

$$= \frac{1}{T_s} \sum_{-\infty}^{\infty} X_c(\omega) * \delta\left(\omega - \frac{2\pi}{T_s}k\right)$$

$$X_p(\omega) = \frac{1}{T_s} \sum^{\infty} X_c\left(\omega - \frac{2\pi}{T_s}k\right)$$ .... shifted & scaled copies of $X_c(\omega)$

$$X_p(\omega) = \frac{1}{T_s}\sum_{-\infty}^{\infty} X_c\left(\omega - \frac{2\pi}{T_s}k\right)$$ .... shifted & scaled copies of $X_c(\omega)$

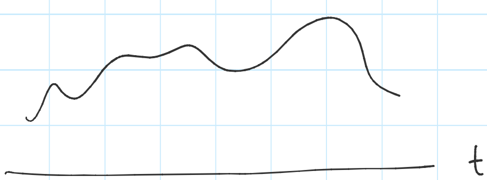
$$x_p(t) = x_c(t)\,p(t) = x_c(t)\left[\sum_{-\infty}^{\infty}\delta(t-nT_s)\right]$$

$$x_p(t) = \sum_{-\infty}^{\infty} x_c(nT_s)\,\delta(t-nT_s)$$ .... $\{ f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$

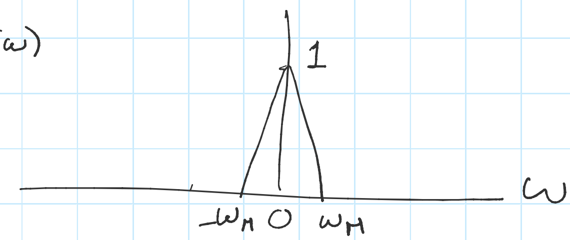
## Time-domain

## Frequency-domain

$x_c(t)$




$X_c(\omega)$




$\otimes$

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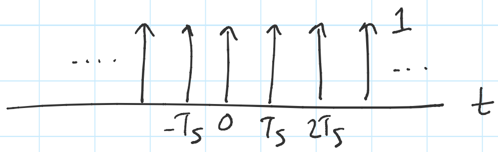
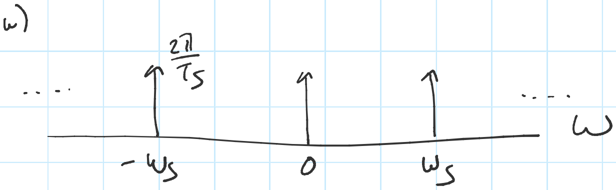
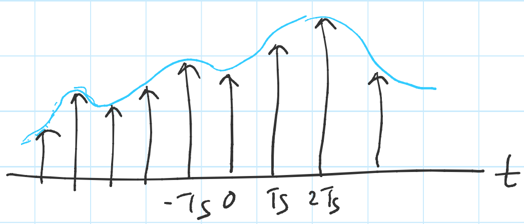
$p(t)$



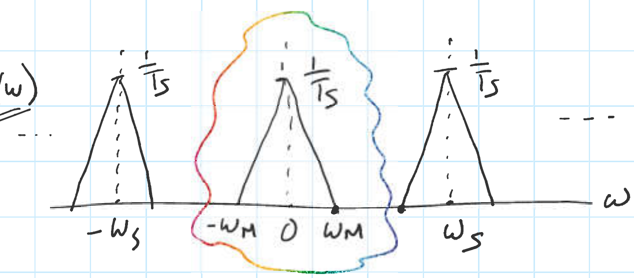

$P(\omega)$




$x_p(t)$

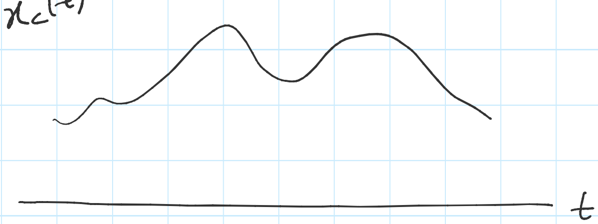



$X_p(\omega)$

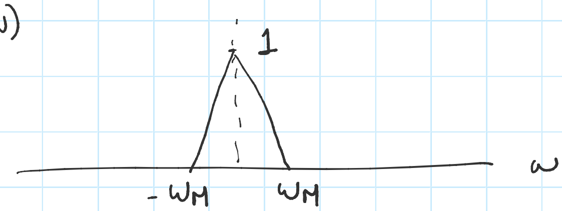



Gain = $T_s$ ↓ LPF ($\omega_c$)

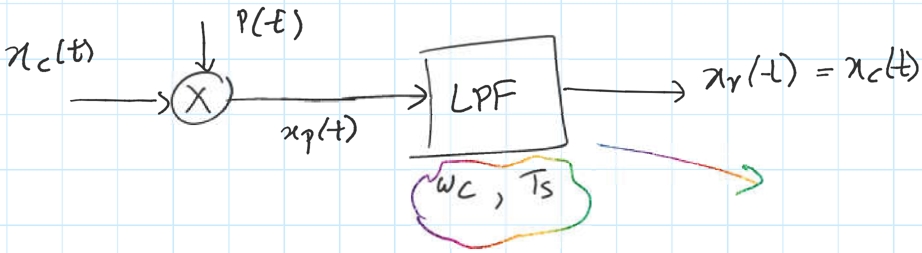
$x_r(t) = x_c(t)$

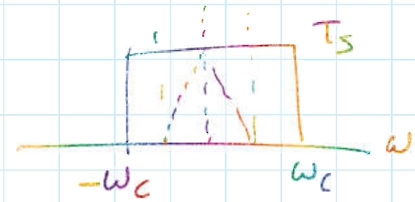



$X_r(\omega)$




$x_c(t)$ ↓ $p(t)$ $x_r(t) = x_c(t)$




Q: When is $x_r(t)$ same as $x_c(t)$?

(A) copies of $X_c(\omega)$ in $X_p(\omega)$ should not overlap

$$\Rightarrow \quad \omega_M < \omega_s - \omega_M \quad \Rightarrow \quad \omega_s > 2\omega_M$$

Sampling frequency > Nyquist rate of $x_c(t)$

(B) Ideal LPF: with gain $= T_s$

& cut-off : $\omega_M < \omega_c < \omega_c - \omega_M$

**Sampling Theorem:** Let $x_c(t)$ be band-limited signal with $X_c(\omega) = 0$ for $|\omega| > \omega_M$. Then $x_c(t)$ is uniquely determined by its samples $x_c(nT_s)$, $n = 0, \pm 1, \pm 2, \dots$; if $\omega_s > 2\omega_M$, $\omega_s = 2\pi / T_s$

**Exact reconstruction:** given samples $x[n] = x_c(nT_s)$, construct

$$x_p(t) = \sum_{-\infty}^{\infty} x_c(nT_s)\,\delta(t - nT_s)$$

& pass it through an ideal LPF with gain $T_s$ & cut-off: $\omega_M < \omega_c \;\; \omega_s - \omega_M$. The o/p of this filter will be $x_r(t) = x_c(t)$ .... (assuming $\omega_s > 2\omega_M$).

**Interpolation:** Freq. domain:

$$X_r(\omega) = X_p(\omega)\, H_{LPF}(\omega)$$

$$H_{LPF}(\omega) = \begin{cases} T_s ; & |\omega| \le \omega_c \\ 0 ; & |\omega| > \omega_c \end{cases}$$

$$H_{LPF}(\omega) = \begin{cases} T_s ; & |\omega| \le \omega_c \\ 0 ; & |\omega| > \omega_c \end{cases}$$

Time-domain: By convolution property of FT

$$x_r(t) = x_p(t) * h_{LPF}(t)$$

$$x_p(t) = \sum_{-\infty}^{\infty} x_c(nT_s)\, \delta(t - nT_s)$$

$$h_{LPF}(t) = T_s \frac{\sin(\omega_c t)}{\pi t}$$

$$x_r(t) = \left[ \sum_{-\infty}^{\infty} x_c(nT_s)\, \delta(t - nT_s) \right] * T_s \cdot \frac{\sin(\omega_c t)}{\pi t}$$

$$x_r(t) = T_s \sum_{-\infty}^{\infty} \underbrace{x_c(nT_s)}_{x[n]} \frac{\sin[\omega_c(t - nT_s)]}{\pi(t - nT_s)}$$

... linear combination of shifted & scaled sinc shapes.

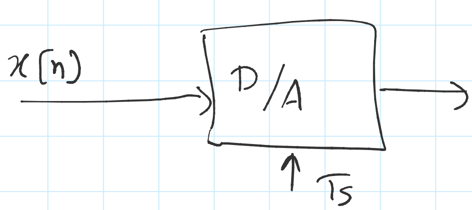
Ideal reconstruction – Impractical.

$x_r(t) = x_c(t)$ if (A) $\omega_s > 2\omega_M$

(B) $\omega_M < \omega_c < \omega_s - \omega_M$ & gain $T_s$

If (A) is satisfied, easy to choose $\omega_c = \frac{\omega_s}{2}$.

In practice we typically use less accurate but simpler methods for interpolation.




Digital to Analog convertors (DAC) (implemented using ICs).