Lecture - 19

Monday, October 19, 2020 1:58 PM

Discrete Fourier Transform (DFT)

- A Fourier transform for the computers / machines
- Efficient implementations available (FFT)
- recall chart from before:

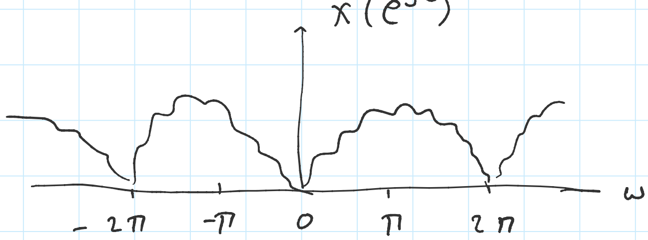
	Continuous-Time	Discrete-Time
Periodic (time)	FS	DFT
Aperiodic (time)	CTFT	DTFT

So far: either time or freq. is continuous or infinite series (if discrete)
these are hard to implement / represent in a computer.

recall sampling theorem: impulse-train sampling in time-domain
$\Rightarrow$ copies of spectrum in freq. domain.
allows $x(t)$ to be represented using $x[n]$ for band-limited signals

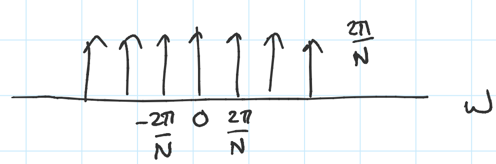
Sample in freq domain the DTFT:

$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$



impulse-train sampling using

$$S(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta\left(\omega - \frac{2\pi}{N}k\right)$$

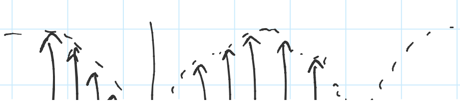


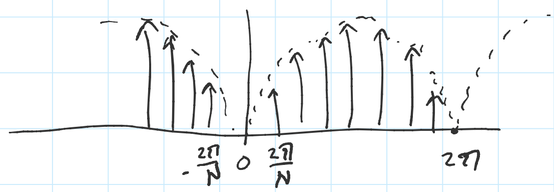
periodic (2π)

$X_p(e^{j\omega}) = X(e^{j\omega})\, S(e^{j\omega}) \xleftrightarrow{DTFT} x[n] * s[n]$

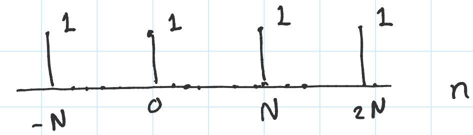
Show that (HW)

$$s[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN]$$





$l=-\infty$



$x_p[n] = x[n] * s[n]$

$$= x[n] * \left(\sum_{-\infty}^{\infty} \delta[n-lN]\right) = \sum_{-\infty}^{\infty} x[n-lN]$$ ... copies of $x[n]$ (shifted & added)

If $x[n]$ is time-limited (between 0 to $M-1$) & $M \leq N$

$\Rightarrow x[n] = 0$ for $n < 0$ & $n > M$

→ $x_p[n] = x[n] * s[n]$ will have all information about $x[n]$

↪ periodic with period N.

→ $\Rightarrow X_p(e^{j\omega}) = X(e^{j\omega})\, S(e^{j\omega})$ has all info. about $x[n]$.

$$X_p(e^{j\omega}) = X(e^{j\omega}) \cdot \frac{2\pi}{N} \sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{N}k\right)$$

$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

→ * $$X_p(e^{j\omega}) = \frac{2\pi}{N} \sum_{-\infty}^{\infty} X\left(e^{j\frac{2\pi}{N}k}\right) \delta\left(\omega - \frac{2\pi}{N}k\right)$$

In an interval of 2π we have N samples.

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} = X\left(e^{j\frac{2\pi}{N}k}\right)$$

From samples $X[k]$ we can obtain $x[n]$.

$k = 0, 1, 2, \cdots N-1$ $\qquad$ $n = 0, 1, 2, \cdots N-1$

$\Rightarrow$ a $M \leq N$ length signal $x[n]$, N samples are sufficient

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}\Big|_{\omega = \frac{2\pi}{N}k}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$ N-point DFT

DFT - discrete Fourier transform
↳ because freq. is discretized.

we know

$$x[n] = \begin{cases} x_p[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x_p[n] = \frac{1}{2\pi} \int_0^{2\pi} X_p(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X\left(e^{j\frac{2\pi}{N}k}\right) e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

$$\Rightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$ Inverse DFT (IDFT)

$n = 0, 1, \dots N-1$

$$\underset{n=0,1,\dots N-1}{x[n]} \xleftrightarrow{N\text{-DFT}} \underset{k=0,1,\dots N-1}{X[k]}$$

$$x_N = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \quad \& \quad X_N = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

DFT as a linear Transformation :

$$\text{DFT}: \quad X[k] = \sum_{0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = \sum_{0}^{N-1} x[n]\, W_N^{nk}$$

set $W_N = e^{-j\frac{2\pi}{N}}$

$$X_N = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{N-1} \\ \vdots & & & \\ W_N^0 & W_N^{N-1} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\begin{bmatrix} x[N-1] \end{bmatrix} \quad \begin{bmatrix} \vdots & & & \\ W_N^0 & W_N^{N-1} & \cdots & W_N^{(N-1)} \end{bmatrix} \begin{bmatrix} x[N-1] \end{bmatrix}$$

DFT matrix

$$X_N = F_N \, x_N$$

Similarly using IDFT : $x_N = F_N^{-1} X_N$

Show that : $F_N^{-1} = \frac{1}{N} F_N^*$

$$\Rightarrow F_N F_N^{-1} = I_N = \frac{1}{N} F_N F_N^*$$

$$\Rightarrow F_N F_N^* = N \cdot I_N.$$

orthogonal matrix

write down : F_2 & F_4 explicitly.

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \& \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

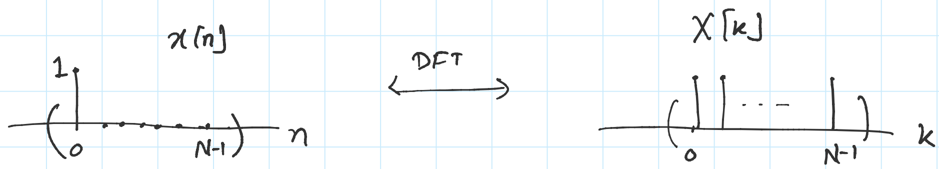
$$W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_2 = e^{-j\pi} = -1$$

$$\Rightarrow W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$F_N(i,j) = W_N^{(i-1)(j-1)}$$

Ex. $x[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$ Find N-point DFT.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = 1 \quad , \; k = 0, 1, 2, \cdots N-1$$



Ex.

Ex.

$$x[n] = \begin{cases} 1, & n = 0, 1, \ldots L-1 \\ 0, & \text{otherwise} \end{cases}, \quad \text{find } N \geq L \text{ point DFT.}$$

$$X[k] = \sum_{n=0}^{L-1} 1 \cdot e^{-j\frac{2\pi}{N}kn} = \frac{\left[e^{-j\frac{2\pi}{N}k}\right]^L - 1}{e^{-j\frac{2\pi}{N}k} - 1}$$

$$X[k] = \frac{\sin(\pi k L/N)}{\sin(\pi k/N)} e^{-j\frac{\pi}{N}k(L-1)} \qquad k = 0, 1, 2, \ldots N-1.$$

samples

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \frac{\sin\left(\frac{\omega L}{2}\right)}{\sin(\omega/2)} e^{-j\frac{\omega}{2}(L-1)}$$

as N is increased we get finer samples of $X(e^{j\omega})$