Lecture - 13

Thursday, September 24, 2020 2:04 PM

Last class: * Discrete-time Fourier transform (DTFT)

$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ *

$x[n]$ - aperiodic in general

DTFT: $$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse DTFT: $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

Spectrum $X(e^{j\omega})$ - periodic with period 2π. *

* Comparison of DTFT with CTFT (of sampled Band-limited signal)
* Comparison of DTFT with FS analysis
* Some examples

Today's class: more examples:

① $x[n] = a^n u[n]$; $|a| < 1$. Find $X(e^{j\omega})$

$$\rightarrow \quad X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{0}^{\infty} a^n e^{-j\omega n} = \sum_{0}^{\infty} (a e^{-j\omega})^n$$

$|ae^{-j\omega}| = |a| < 1$

$b + br + br^2 + \dots \rightarrow \frac{b}{1-r}$ if $|r| < 1$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$ ... complex valued (even if a is real)

Plot $|X(e^{j\omega})|$ for $a \in (-1, 1)$

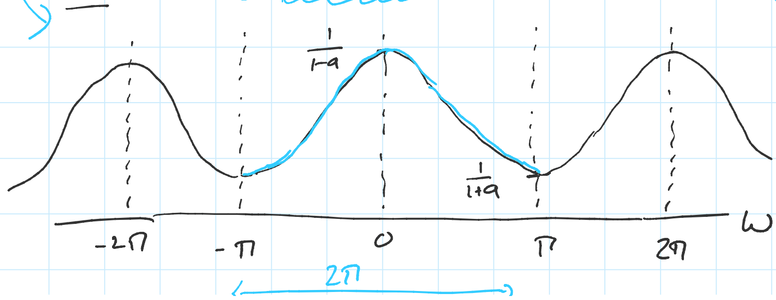
$$|X(e^{j\omega})| = \left|\frac{1}{1 - a(\cos\omega - j\sin\omega)}\right| = \left|\frac{1}{(1 - a\cos\omega) + j\, a\sin\omega}\right|$$

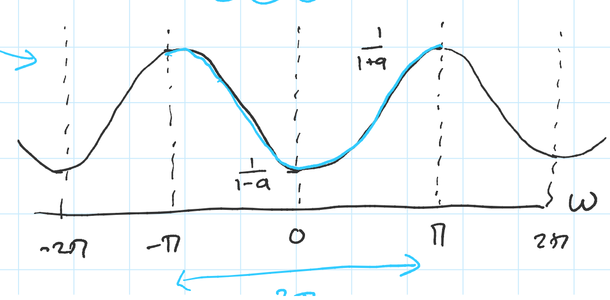
$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + (a\sin\omega)^2}} = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

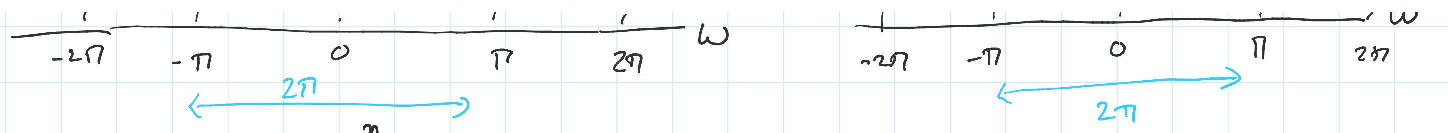
$\omega = 0 \rightarrow \frac{1}{1-a}$

$\omega = \pi \rightarrow \frac{1}{1+a}$

Plot: ① $a \in (0, 1)$ ② $a \in (-1, 0)$



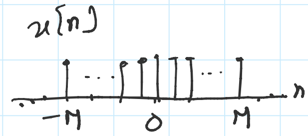




$x[n] = a^n u[n]$

HW: $x[n] = a^{|n|}$, $|a|<1$, Find $X(e^{j\omega})$ & sketch $|X(e^{j\omega})|$ for $a \in \mathbb{R}$

(2) Rectangular pulse in time: $x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$



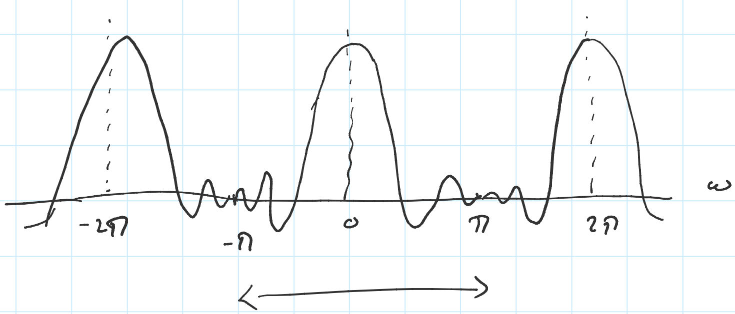
$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{-M}^{M} 1 \cdot e^{-j\omega n}$$

finite geometric sum
$b + br + \cdots br^n = \frac{b(1-r^{n+1})}{(1-r)}$

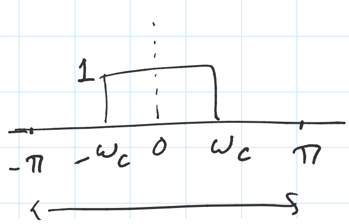
$$X(e^{j\omega}) = \frac{e^{j\omega M}\left(1-\left(e^{-j\omega}\right)^{2M+1}\right)}{1-e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{e^{j\omega M} \cdot e^{-j\omega\left(\frac{2M+1}{2}\right)}\left[e^{j\omega\left(\frac{2M+1}{2}\right)} - e^{-j\omega\left(\frac{2M+1}{2}\right)}\right]}{e^{-j\omega/2} \cdot \left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

$$X(e^{j\omega}) = \frac{\sin\left(\frac{2M+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \quad \cdots \text{periodic } (2\pi)$$

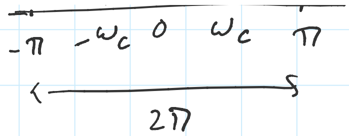


(3) Rectangular wave in frequency: $X(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$



& periodic (2π)

Find $x[n]$.



Find $x[n]$.

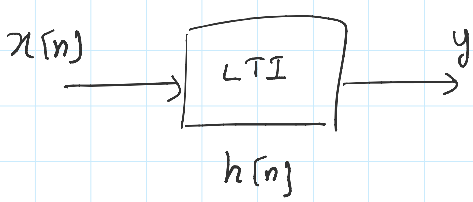
Inverse DTFT :

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$x[n] = \frac{1}{2\pi} \left. \frac{e^{j\omega n}}{jn} \right|_{-\omega_c}^{\omega_c} = \frac{\sin(\omega_c n)}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\, d\omega = \frac{\omega_c}{\pi}.$$

Discrete-time LTI systems & DTFT :



$y[n] = x[n] * h[n]$

(A) If $x[n] = e^{j\omega_0 n}$, output $y[n] = ?$

$$y[n] = \sum_{-\infty}^{\infty} h[m]\, x[n-m] = \sum_{-\infty}^{\infty} h[m]\, e^{j\omega_0 (n-m)} = \left(\sum_{-\infty}^{\infty} h[m]\, e^{-j\omega_0 m}\right) e^{j\omega_0 n}$$

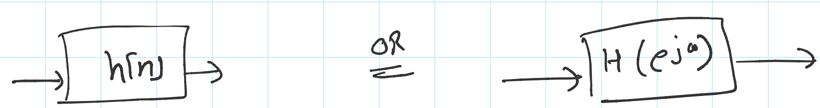
$$y[n] = H(e^{j\omega_0})\, e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \xrightarrow{LTI} H(e^{j\omega_0})\, e^{j\omega_0 n}$$

eigenfunctions

$$h[n] \xleftrightarrow{DTFT} H(e^{j\omega})$$

Impulse-response Frequency response



(B) Convolution property of DTFT :

$x[n] \longleftrightarrow X(e^{j\omega})$ & $h[n] \longleftrightarrow H(e^{j\omega})$

$y[n] = x[n] * h[n]$, $Y(e^{j\omega}) = \ ?$ $= X(e^{j\omega}) H(e^{j\omega})$

Proof: $$Y(e^{j\omega}) = \sum_{-\infty}^{\infty} y[n] e^{-j\omega n}$$

$$= \sum_n \left(\sum_m h[m] x[n-m] \right) e^{-j\omega n}$$

$$= \sum_m h[m] \left(\sum_n x[n-m] e^{-j\omega n} \right)$$ ... put $n-m=k$, $n = m+k$

$$= \sum_m h[m] \left(\sum_k x[k] e^{-j\omega k} \right) e^{-j\omega m}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$x[n] * h[n] \xleftrightarrow{DTFT} X(e^{j\omega}) H(e^{j\omega})$$

$\Rightarrow$ An LTI system acts a frequency filter

* No new frequencies can be added in the output.

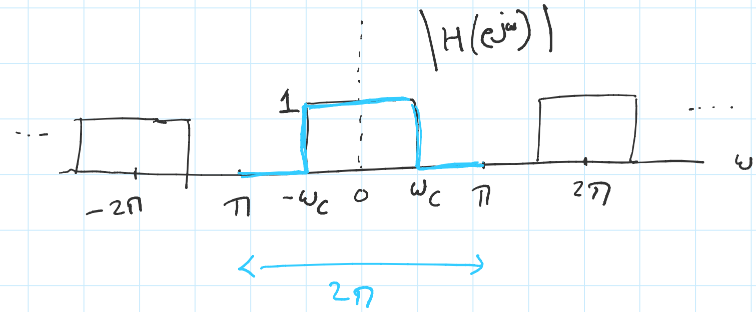
$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{j \angle H(e^{j\omega})}$$

freq. response — magnitude response — phase response

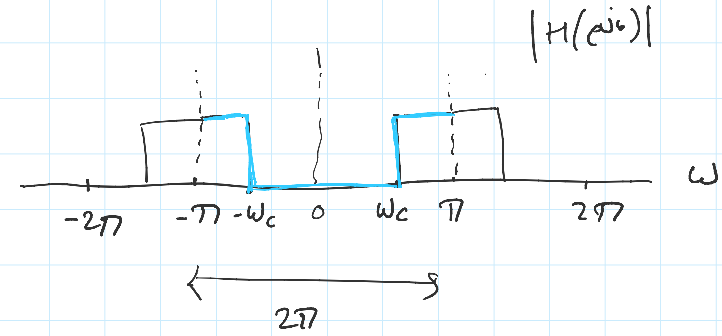
(c) "Ideal" filters in discrete-time

Ideal low pass filter (LPF)

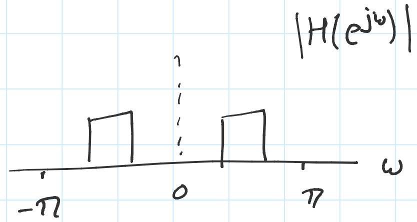
cutoff freq: ω_c



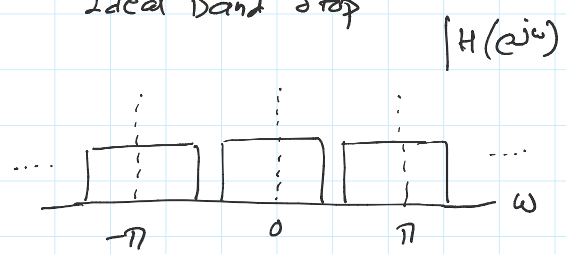
Ideal HPF (ω_c)



Ideal BPF

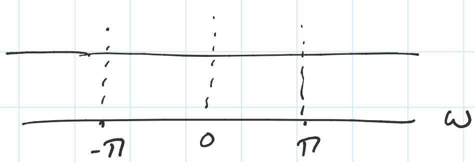


Ideal Band Stop



All pass filters: $|H(e^{j\omega})| = 1$ magnitude response is constant

phase response may not be 0



Ex. delay system: $h[n] = \delta[n-n_0]$, find $H(e^{j\omega})$

$$H(e^{j\omega}) = \sum_{-\infty}^{\infty} \delta[n-n_0]\, e^{-j\omega n} = e^{-j\omega n_0}$$

$$\left.\begin{aligned} |H(e^{j\omega})| &= 1 \quad \text{(constant)} \\ \measuredangle H(e^{j\omega}) &= -\omega n_0 \quad \text{(linear phase)} \end{aligned}\right\} \text{All pass filter}$$

D non-ideal filters

Ex. $h[n] = a^n u[n]$; $a \in (-1, 1)$

what in the nature of the filter / LTI system

→ $a \in (0,1)$ it behaves as a LPF

$a \in (-1, 0)$ it behaves as a HPF

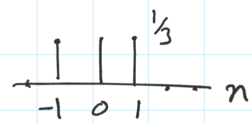
$a \in (-1, 0)$ it behaves as a HPF

HW : **Ex.** An LTI system is given by the relation:

$$y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right) = x[n] * h[n]$$

(a) Find: $h[n]$ & $H(e^{j\omega})$ & plot magnitude & phase response

$$h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right) \equiv \left\{\frac{1}{3}, \underset{\uparrow}{\frac{1}{3}}, \frac{1}{3}\right\}$$



(a) What is the nature of this filter / system

HW : Ex. Repeat above for : $y[n] = \frac{1}{4}\left(2x[n] - x[n+1] - x[n-1]\right)$

* Properties of DTFT :

(1) Convolution property

(2) Symmetry properties: $x[n] \longleftrightarrow X(e^{j\omega})$

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

(a) $x[n]$ is real $X(e^{j\omega}) = X^*(e^{-j\omega})$

(b) $x[n]$ is real & even $\Rightarrow$ $X(e^{j\omega})$ real & even

(c) $x[n]$ is real & odd $\rightarrow$ $X(e^{j\omega})$ imaginary & odd

(3) Linearity : $x[n] \longleftrightarrow X(e^{j\omega})$ & $y[n] \longleftrightarrow Y(e^{j\omega})$

$$\alpha x[n] + \beta y[n] \longleftrightarrow \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

④ Shift in time: $x[n] \longleftrightarrow X(e^{j\omega})$

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

⑤ Shift if frequency: $x[n] \longleftrightarrow X(e^{j\omega})$

$$e^{j\omega_0 n} x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

⑥ Parseval's Relation:

$$\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(e^{j\omega})|^2 \, d\omega$$

⑦ Multiplication (in time):

$$y[n] = x_1[n] \, x_2[n]$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X_1(e^{j\theta}) \, X_2\left(e^{j(\omega-\theta)}\right) d\theta$$

--- (periodic convolution)

Ex. using properties find DTFT of ⓐ $\sin(\omega_0 n)$ ⓑ $\cos(\omega_0 n)$