Lecture -16

Monday, October 5, 2020 2:05 PM

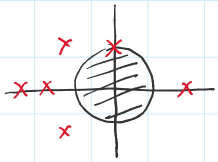
Last week: z - Transform $$x[n] \xleftrightarrow{z} X(z)$$

- ★ Various examples
- ★ Region of Convergence (ROC)
- ★ z-plane : poles & zeros
- ★ Rational form : $X(z) = \frac{N(z)}{D(z)}$, pole-zero plot in z-plane

★ Finite duration x[n] : ROC entire z-plane except possibly 0 and/or ∞.

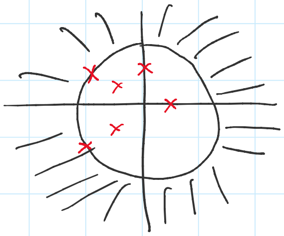
★ Infinite duration x[n] :

x[n] left-sided

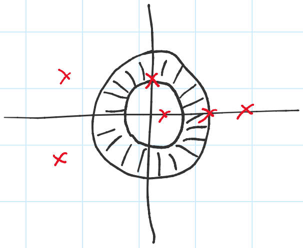


▨ - ROC

x[n] right-sided



x[n] two-sided



Todays class :

Ex. $x[n] = 2^n \cos(n) u[n]$, find $X(z)$ & ROC.

$$x[n] = \frac{2^n}{2} (e^{jn} + e^{-jn}) u[n]$$

$$= \frac{1}{2} \left[(2e^{j})^n + (2e^{-j})^n \right] u[n]$$

$$x[n] = \underbrace{\frac{1}{2} (2e^{j})^n u[n]}_{\downarrow} + \underbrace{\frac{1}{2} (2e^{-j})^n u[n]}_{\downarrow}$$

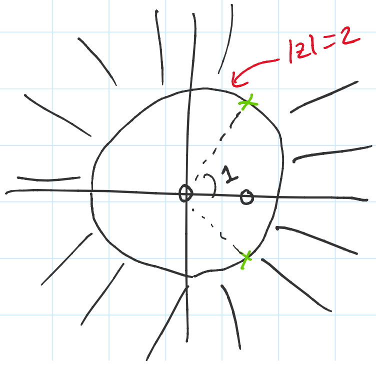
↓

$$X(z) = \frac{1}{2} \frac{z}{z - 2e^{j}} + \frac{1}{2} \frac{z}{z - 2e^{-j}} \quad \& \text{ ROC } \left\{ \begin{matrix} |z| > |2e^{j}| \ \& \\ |z| > |2e^{-j}| \end{matrix} \right\}$$

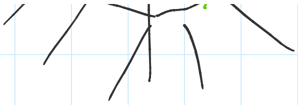
↓

$$X(z) = \frac{1}{2} \times \frac{2z^2 - 2z(e^{j} + e^{-j})}{(z - 2e^{j})(z - 2e^{-j})} \quad \& \text{ ROC : } |z| > 2$$

$$X(z) = \frac{z(z - 2\cos(1))}{(z - 2e^{j})(z - 2e^{-j})} \quad \& \text{ ROC : } |z| > 2.$$



$(z-2e^{j})(z-2e^{-j})$



zeros @ $z=0$ & $z=2\cos(1)$ $re^{j\theta}$

poles @ $z=2e^{j}$ & $z=2e^{-j}$

$\downarrow$ $2(\cos(1)+j\sin(1))$ $\rightarrow$ $2(\cos(1)-j\sin(1))$

Here, poles appear in complex conjugate pair.

Remark: In general, if the signal $x[n]$ is real valued, the poles & zeros will appear in complex conjugate pairs.

Proof:

$$x[n] \xleftrightarrow{z} X(z)$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*)$$

If $x[n]$ is real : $x^*[n] = x[n]$

$$\Rightarrow X(z) = X^*(z^*)$$

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

$$X^*(z) = \sum_{-\infty}^{\infty} x^*[n] (z^*)^{-n}$$

$$X^*(z^*) = \sum_{-\infty}^{\infty} x^*[n] z^{-n}$$

zeros : $X(z)=0 \Rightarrow X^*(z^*)=0 \Rightarrow X(z^*)=0$

poles : $X(z)=\infty \Rightarrow X^*(z^*)=\infty \Rightarrow X(z^*)=\infty$

complex zeros/poles always appear in complex-conjugate pairs.

For real signal $x[n]$, if $X(z) = \frac{N(z)}{D(z)}$ i.e. ratio of polynomials then both $N(z)$ & $D(z)$ will have real coefficients.

Inverse Z-Tranform :

① General : using Cauchy integral theorem :

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Complex integration over some closed contour in the ROC.

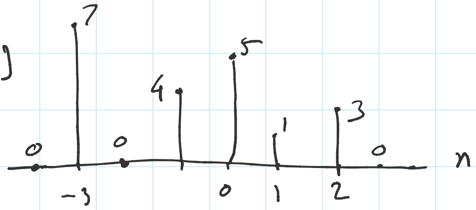
We will never use this directly in this course.

② Special cases: finite polynomial terms.

Ex. $X(z) = 3z^{-2} + z^{-1} + 5 + 4z + 7z^{3}$ & $0 < |z| < \infty$

find $x[n]$.

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$



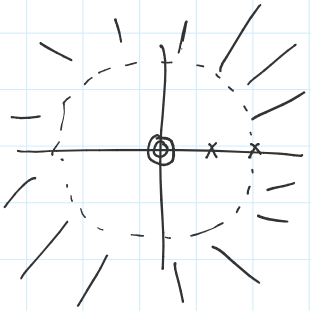
$$\Rightarrow x[n] = \{7, 0, 4, \underset{\uparrow}{5}, 1, 3\}$$

$$x[n] = 7\delta[n+3] + 4\delta[n+1] + 5\delta[n] + \delta[n-1] + 3\delta[n-3]$$

③ Special case: ratio of polynomials – partial fractions

Ex. $X(z) = \dfrac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$ & ROC: $|z| > 1$, find $x[n]$.

$$\rightarrow X(z) = \frac{z^2}{z^2 - \frac{3}{2}z + \frac{1}{2}} = \frac{z^2}{(z-\frac{1}{2})(z-1)}$$



use partial fraction

may be incorrect

$$X(z) = \frac{A}{z - 1/2} + \frac{B}{z-1}$$

, find A & B $\xrightarrow{\text{Solving}}$ A = , B =

① ✱ $X(z) = \dfrac{2z}{z-1} - \dfrac{z}{z-1/2}$

$$\begin{array}{r|l} & 1 \\ \hline z^2 - \frac{3}{2}z + \frac{1}{2} & z^2 \\ \ominus & z^2 - \frac{3}{2}z + \frac{1}{2} \\ \hline & \frac{3}{2}z - \frac{1}{2} \end{array}$$

$$\Rightarrow X(z) = 1 + \frac{\frac{3}{2}z - 1/2}{z^2 - \frac{3}{2}z + 1/2}$$

→ Apply partial fractions

$$X(z) = 1 + \frac{A}{z - 1/2} + \frac{B}{z-1} \longrightarrow A = -1/2, \quad B = 2$$

② ★ $X(z) = 1 + \frac{2}{z-1} - \frac{1}{2z-1}$ → can find $x[n]$.

Use ① to find $x[n]$.

★ $X(z) = 2\frac{z}{z-1} - \frac{z}{z-1/2}$ & $|z|>1$

$u[n]$ or $-u[-n-1]$

$|z|>1$ or $|z|<1$

$(\frac{1}{2})^n u[n]$ or $-(\frac{1}{2})^n u[-n-1]$

$|z|>\frac{1}{2}$ or $|z|<1/2$

$$x[n] = 2u[n] - (\tfrac{1}{2})^n u[n]$$

If: ROC: $|z| < 1/2 \Rightarrow x[n] = -2u[-n-1] + (\frac{1}{2})^n u[-n-1]$

If ROC: $\frac{1}{2} < |z| < 1 \Rightarrow x[n] = -2u[-n-1] - (\frac{1}{2})^n u[n]$

Ex. $X(z) = \frac{7-13z^{-1}}{1-2z^{-1}-3z^{-2}}$ & ROC : $|z|>1$ (HW)

Ex. $x[n] = 2^n\left(u[n]-u[n-M]\right)$, $M>0$

$$X(z) = \sum_0^M 2^n z^{-n} = \sum_0^M (2z^{-1})^n = \frac{1(1-(2z^{-1})^{M+1})}{1-2z^{-1}}$$

$X(z) = \frac{z^{M+1} - 2^{M+1}}{(z-2)\,z^M}$ .... $(z-2)$ present in numerator as well.

Note: poles & zeros at infinity

$X(z) = \frac{N(z)}{D(z)}$ let p & q be order of N(z) & D(z)

If $p = q$, we have p zeros & q poles

If $p > q$, we say $(p-q)$ poles at ∞

If $p < q$, we say $(q-p)$ zeros at ∞