Lecture -11
Thursday, September 17, 2020 2:10 PM

Last class: Some remarks on Sampling Theorem
Quantization

Todays class: Discrete-time (DT) signals and systems:

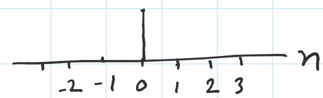
notation : $x[n]$, $n = 0, \pm 1, \pm 2, \cdots$ (n is always integer, $n \in \mathbb{Z}$)

$\mathbb{Z}$
$\mathbb{R}$
$\mathbb{C}$

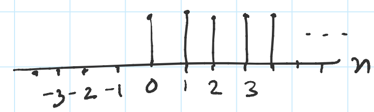
Standard examples :

Sketch

(a) unit impulse : $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$

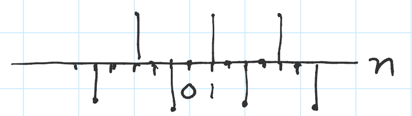


(b) unit step : $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

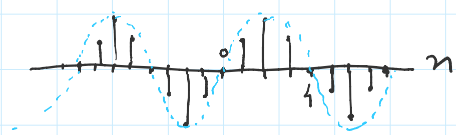


(c) Sinusoids : $x[n] = \sin(\omega_0 n)$

$\omega_0 = \frac{\pi}{2} \rightarrow \sin\left(\frac{\pi n}{2}\right)$



$\omega_0 = \frac{\pi}{4} \rightarrow \sin\left(\frac{\pi n}{4}\right)$

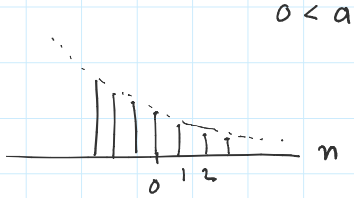


Similarly sketch for $\cos(\omega_0 n)$

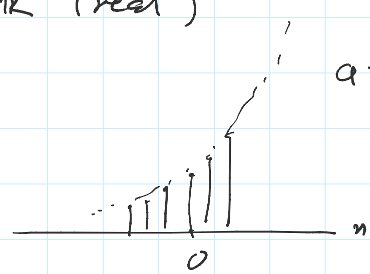
(d) Exponential : $x[n] = a^n$ $\qquad a \in \mathbb{R}$ (real)

$a > 0$

$0 < a < 1$

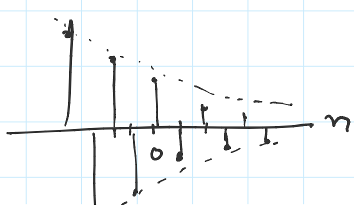


$a > 1$

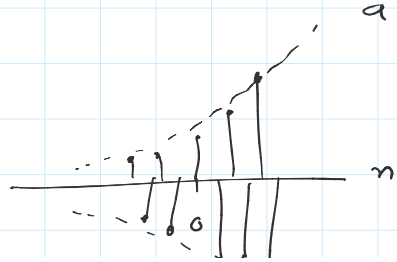


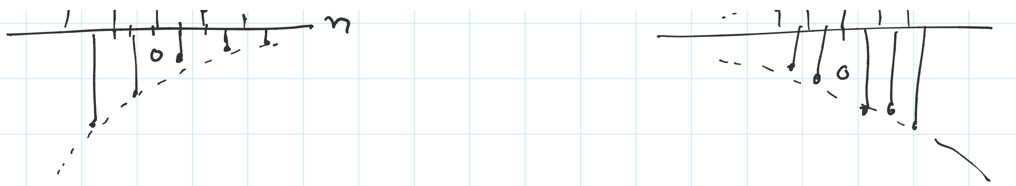
$a < 0$

$-1 < a < 0$



$a < -1$





$a \in \mathbb{C}$, $x[n] = a^n$ (complex exponential)

$a = re^{j\theta} \rightarrow r \in [0, \infty)$ & $\theta \in [0, 2\pi)$

plot: real , imag , magnitude , phase / angle

when $a = e^{j\omega_0} \Rightarrow x[n] = e^{j\omega_0 n}$ (complex sinusoid)

$= \cos(\omega_0 n) + j \sin(\omega_0 n)$

Some definitions:

energy : $E = \sum_{-\infty}^{\infty} |x[n]|^2$, if $E < \infty$, energy signal.

power : $P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$, if $P < \infty$, power signal

Even signals : $x[-n] = x[n]$, ex. $\cos(\omega_0 n)$

Odd signals : $x[-n] = -x[n]$, ex. $\sin(\omega_0 n)$

Periodic signals: $x[n]$ is periodic if

$x[n+N] = x[n]$ for some N & $\forall n$

periodic with period N , N is the Integer

Smallest such N $\rightarrow$ fundamental period.

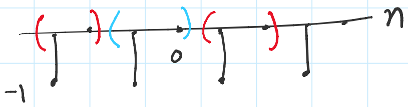
If N is period $\rightarrow$ any kN is also a period , $k = 1, 2, 3, \ldots$

Ex. Find fundamental period of $x[n] = \sin(5\pi n)$

$x[n+N] = x[n] \Rightarrow \sin(5\pi n + 5\pi N) = \sin(5\pi n)$

$\Rightarrow \cancel{5\pi n} + 5\pi N = \cancel{5\pi n} + 2\pi k \qquad k \in \mathbb{Z}$

$\Rightarrow N = \dfrac{2k}{5}, \quad k \in \mathbb{Z}$



$\Rightarrow N = 2$

find fundamental period:

ⓐ $\sin\left(\frac{\pi n}{5}\right)$ $\qquad N = 10$

ⓑ $\cos\left(\frac{2\pi n}{3} + \phi\right)$ $\qquad N = 3$

ⓒ $e^{j\frac{\pi}{4}n}$ $\qquad N = 8$ $\qquad \left\} \begin{array}{l} e^{j\theta_1} = e^{j\theta_2} \\ \Rightarrow \theta_1 = \theta_2 + 2\pi k. \end{array} \right.$

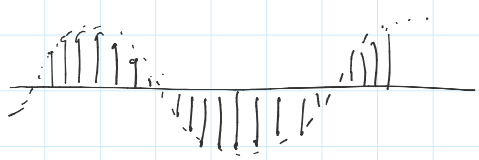
ⓓ $e^{j\left(\frac{\pi n}{3} + \phi\right)}$ $\qquad N = 6$

ⓔ $\sin(5n)$ $\qquad$ Not Periodic

$x[n+N] = x[n] \Rightarrow \sin(5n + 5N) = \sin(5n)$

$\Rightarrow \cancel{5n} + 5N = \cancel{5n} + 2\pi k, \quad k \in \mathbb{Z}$

$\Rightarrow \boxed{N = 2\pi \frac{k}{5}}$ no k which gives N as integer.



Continuous-Time

$\sin(\omega t)$ or $\cos(\omega t)$ or $e^{j\omega t}$

Discrete-time

$\sin(\omega n)$ or $\cos(\omega n)$ or $e^{j\omega n}$

$\sin(\omega t)$ or $\cos(\omega t)$ or $e^{j\omega t}$

(a) frequency $= \omega = 2\pi f$

period, $T = \frac{2\pi}{\omega} = \frac{1}{f}$

(b) each ω gives a unique signal.

$\sin(\omega n)$ or $\cos(\omega n)$ or $e^{j\omega n}$

(a) For periodicity : $\sin(\omega n + \omega N) = \sin(\omega n)$

$\Rightarrow \omega N = 2\pi k \quad , k \in \mathbb{Z}$

$\omega = 2\pi \frac{k}{N} \quad , \quad k \in \mathbb{Z}$

$\omega = 2\pi \left(\frac{k}{N}\right) = 2\pi \times (\text{rational num.})$

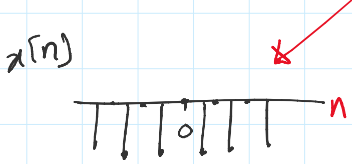
(b) if $\omega \to \omega + 2\pi$??

$x[n] = \sin((\omega + 2\pi) n) = \sin(\omega n)$

$x[n]$ is unique for $\omega \in [0, 2\pi)$

or $\omega \in [-\pi, \pi)$

Ex. $\sin(5\pi n) \begin{cases} = \sin(3\pi n) = \sin(\pi n) = \sin(-\pi n) = \cdots \\ = \sin(7\pi n) = \sin(9\pi n) = \cdots \end{cases}$



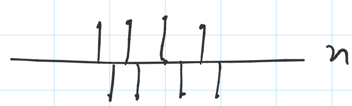
$\sin(\omega_0 n)$ – we will call ω_0 as frequency parameter

Q. what freq. is highest ? for DT $e^{j\omega n}$ / $\sin(\omega n)$ / $\cos(\omega n)$.

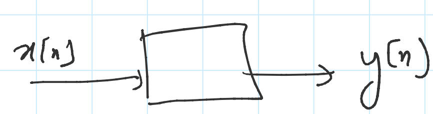
ω from 0 to 2π.

$e^{j\omega n}$: max freq. is $\omega = \pi$

$\omega = \pi \qquad e^{j\pi n}$:



Discrete-Time Systems



$x[n] \to y[n]$

note: In practice we will have digital signals & systems
but theory will use only DT —"— —

We focus on LTI systems:

Linearity: $x_1[n] \rightarrow y_1[n]$ & $x_2[n] \rightarrow y_2[n]$

$\hookrightarrow \alpha x_1[n] + \beta x_2[n] \rightarrow \alpha y_1[n] + \beta y_2[n]$; $\alpha, \beta \in \mathbb{C}$.

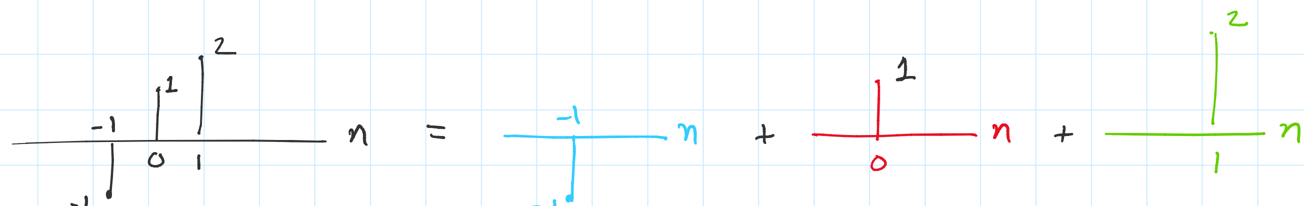
Time-invariance: $x[n] \rightarrow y[n]$

$\hookrightarrow \Rightarrow x[n-k] \rightarrow y[n-k]$; $k \in \mathbb{Z}$

Examples:	Linearity	Time-Invariant
(a) $y[n] = x[n-n_0]$	Yes	Yes
(b) $y[n] = x^2[n]$	No	Yes
(c) $y[n] = x[n] + x[n-1]$	Yes	Yes
(d) $y[n] = n \cdot x[n]$	Yes	No

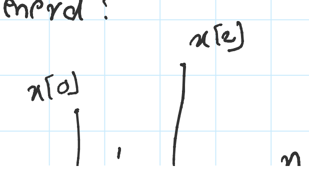
Representation of DT Signals $x[n]$ using impulses:

Ex.

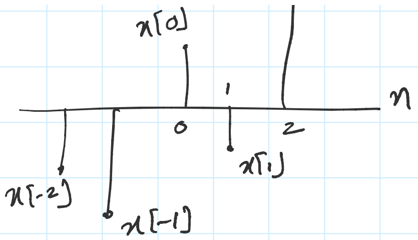


$= (-1)\,\delta[n+1] + 1 \cdot \delta[n] + 2\,\delta[n-1]$

In general:



$\Rightarrow x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \cdots$



$$\Rightarrow x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \cdots$$
$$+ x[-1]\,\delta[n+1] + x[-2]\,\delta[n+2] + \cdots$$

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$$

Any $x[n]$ is linear comb. of Scaled & Shifted Impulse

Compare with: $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

LTI System: Input $x[n] \longrightarrow y[n]$ Output

let $y[n] = f(x[n])$ → operates on whole signal $x[n]$.

$$y[n] = f\left(\sum_{-\infty}^{\infty} x[m]\,\delta[n-m]\right)$$

$$= \sum_{-\infty}^{\infty} x[m]\, f\left(\delta[n-m]\right) \quad \text{.... using linearity}$$

let $\delta[n] \longrightarrow h[n]$ ... impulse response of LTI system.

i.e. $f(\delta[n]) = h[n]$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \quad \text{.... using time-invariance}$$

⟶ Convolution Sum.

$y[n] = x[n] * h[n]$ ... discrete-time convolution.

Compare with: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$

$h[n]$ - Fully represents the DT System.