Lecture -17

Thursday, October 8, 2020 2:13 PM

Last class:
- ★ Z-Transforms and ROC
- ★ Inverse Z-Transform methods ★
- ★ Various Examples

Todays class:

★ Z-Transform properties ★

① **Linearity:** $x_1[n] \xleftrightarrow{z} X_1(z)$, and ROC R_1

$x_2[n] \xleftrightarrow{z} X_2(z)$, R_2

$\Rightarrow a x_1[n] + b x_2[n] \xleftrightarrow{z} a X_1(z) + b X_2(z)$, ROC contains $R_1 \cap R_2$

Ex. $x[n] = 2^n u[n] + (-3)^n u[n]$ Ex. $x[n] = 2^n u[n] - 2^n u[n-1]$

find $X(z)$ & ROC.

② **Time-shift:** $x[n] \xleftrightarrow{z} X(z)$, ROC is R

$x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z)$, ROC is $R \pm \{0\} \pm \{\infty\}$

Ex. find inverse z-transform of $\frac{1}{z-a}$ with ROC $|z| > |a|$.

$\rightarrow$ recall $a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}$, $|z| > |a|$

using time-shift

$a^{n-1} u[n-1] \xleftrightarrow{z} \frac{1}{z-a}$, $|z| > |a|$

$$\left\{ \begin{aligned} X(z) &= \frac{z}{z-a} \\ z^{-1} X(z) &= \frac{1}{z-a} \end{aligned} \right.$$

Ex. $X(z) = \frac{1+2z^{-1}}{1+z^{-1}}$, $|z| > 1$ find $x[n]$.

$\rightarrow X(z) = 1 + \frac{z^{-1}}{1+z^{-1}} \Rightarrow X(z) = 1 + \frac{1}{z+1} \Rightarrow x[n] = \delta[n] + (-1)^{n-1} u[n-1]$

Ex. $X(z) = \frac{1}{z(z+1/2)}$, $|z| > 1/2$, find $x[n]$

$\rightarrow X(z) = \frac{z^{-1}}{z+1/2} = z^{-2} \frac{z}{z+1/2} \Rightarrow x[n] = (-\frac{1}{2})^{n-2} u[n-2]$

$z(z+1/2)$

$$\rightarrow \quad X(z) = \frac{z^{-1}}{z+1/2} = \underset{\substack{\uparrow \\ \text{Shift}}}{z^{-2}} \frac{z}{z+1/2} \Rightarrow x[n] = \left(-\frac{1}{2}\right)^{n-2} u[n-2]$$

③ Time reversal:

$$x[n] \xleftrightarrow{z} X(z), \quad \text{ROC is } R \qquad \Big| \qquad X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \quad \text{ROC is } \frac{1}{R}$$

If $z = z_0$ is in R, $\frac{1}{z_0}$ is in $\frac{1}{R}$

④ Initial Value Theorem: For a causal signal $x[n]$ i.e. $x[n] = 0$ for $n < 0$

$$x[0] = \lim_{z \to \infty} X(z)$$

⑤ Convolution:

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad R_2$$

$$\Rightarrow \quad x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z) X_2(z), \quad \text{ROC contains } (R_1 \cap R_2)$$

Convolution in time $\longleftrightarrow$ product in Z-domain.

★ LTI Systems:

$$x[n] \longrightarrow \underset{\text{LTI}}{\boxed{h[n]}} \longrightarrow y[n] = x[n] * h[n]$$

$$h[n] \xleftrightarrow{z} H(z)$$

impulse response — Transfer function / System function

we know: ① $z^n \longrightarrow H(z) z^n$

② $Y(z) = X(z) H(z)$ ....from convolution

Frequency response: $H(e^{j\omega}) = H(z) \Big|_{z = e^{j\omega}}$

★ multiple representations for a LTI system:

$H(e^{j\omega})$

★ multiple representations for a LTI system:

$h[n] \leftrightarrow H(e^{j\omega})$

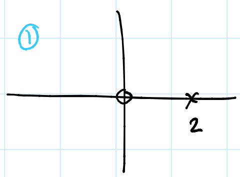
$h[n] \leftrightarrow H(z)$ (+ ROC) } or pole-zero plot (+ ROC)

★ ALL of these are equivalent representations of an LTI system.

★ Using pole-zero plot of $H(z)$ to infer properties of LTI system

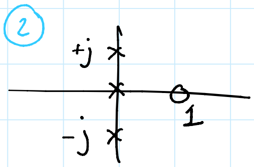
(A) $H(z)$ from pole-zero plot [for rational $H(z)$ i.e. $H(z) = \frac{N(z)}{D(z)}$]

Ex. (1)



$$H(z) = \frac{A\,z}{z-2}$$, A is the gain / constant.

Ex (2)



and $H(-1) = 1/2$. find $H(z)$.

$$\Rightarrow H(z) = \frac{A\,(z-1)}{z(z-j)(z+j)} = \frac{(z-1)\,A}{z(z^2+1)}$$

$H(-1) = 1/2 \Rightarrow A = 1/2$.

To find $h[n]$, we additionally need ROC information.

For (1) — two possible $h[n]$ depending on $|z|>2$ or $|z|<2$

For (2) — two possible $h[n]$ —"— $0<|z|<1$

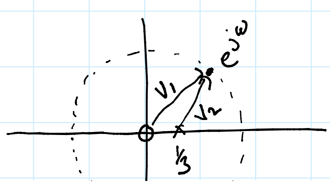
$|z|>1$

(B) Plot of frequency (magnitude) response from pole-zero plot [rough sketch]

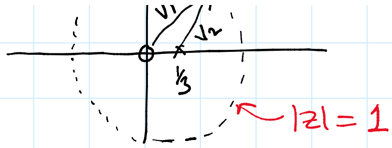
★ Geometric evaluation of the Fourier Transform

Sketch $|H(e^{j\omega})|$, $-\pi \le \omega \le \pi$

Ex. (1)



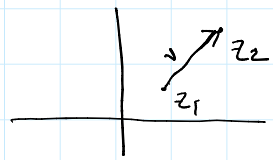
find $H(z)$ & evaluate on unit circle.



find $H(z)$ & evaluate on unit circle.

$$H(z) = \frac{z}{z - 1/3} \quad \text{\& put } z = e^{j\omega}.$$

$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - 1/3}$$

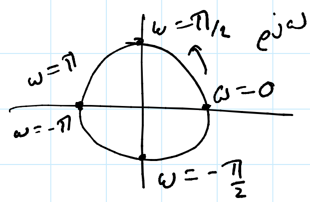


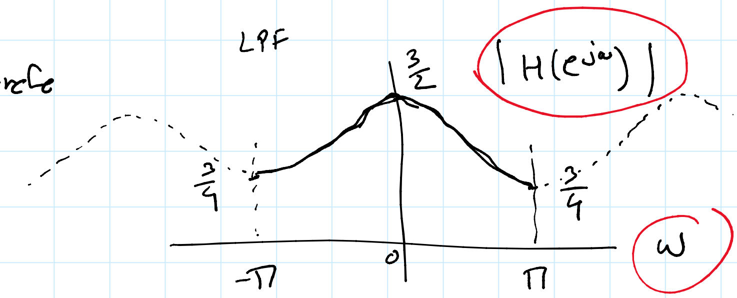
$v = z_2 - z_1$

$|v| = |z_2 - z_1|$

★ $$|H(e^{j\omega})| = \frac{|e^{j\omega}|}{|e^{j\omega} - 1/3|} = \frac{|e^{j\omega} - 0|}{|e^{j\omega} - 1/3|} = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}$$

$e^{j\omega}$ is a point on the unit circle

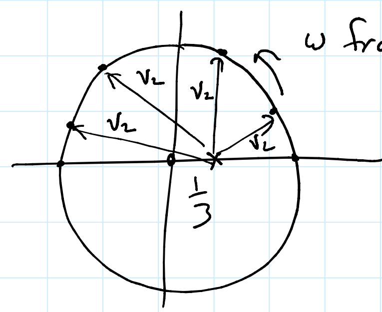




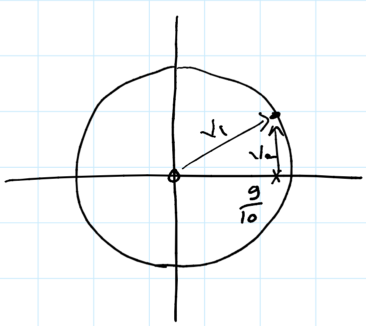
* ★ $e^{j\omega}$ point on unit circle
* ★ ω from 0 to π is upper half circle
* ★ ω from 0 to $-\pi$ is lower half circle.

★ To find: $|H(e^{j\omega})|$

★ $$|H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}$$



Ex. ②

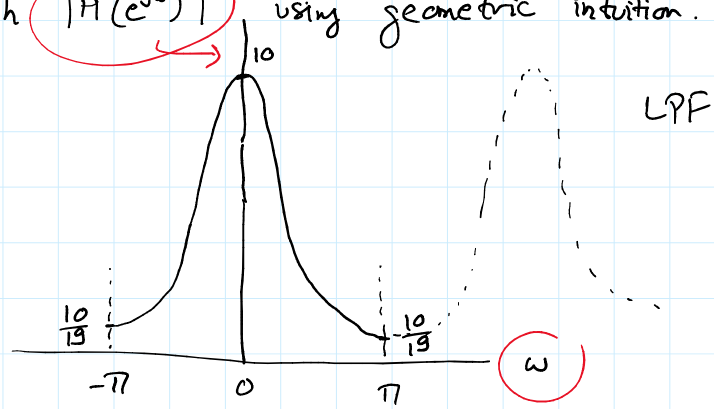


zero at $z = 0$

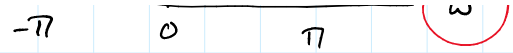
pole at $z = \frac{9}{10}$

Sketch $|H(e^{j\omega})|$ using geometric intuition.

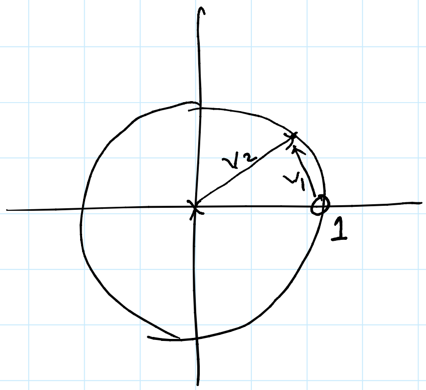
$$|H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}$$



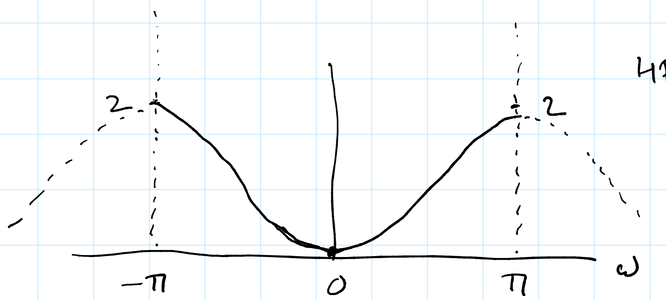
Ex. ③



Ex. ③



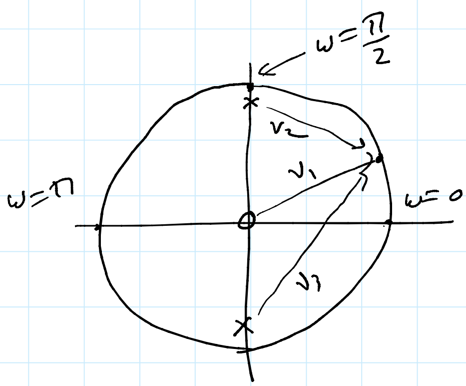
Sketch $|H(e^{j\omega})|$



HPF

$$|H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{|v_1(\omega)|}{1}$$

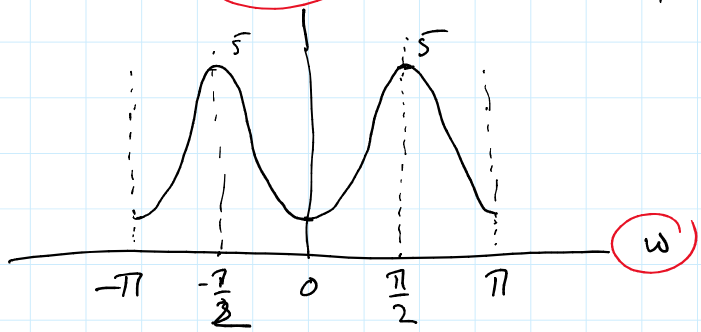
Ex. ④



zero at $z = 0$

poles at $z = \frac{9}{10}j$ & $z = -\frac{9}{10}j$

Sketch $|H(e^{j\omega})|$ Band-pass filter



$$|H(e^{j\omega})| = \frac{|v_1|}{|v_2||v_3|} = \frac{1}{|v_2||v_3|}$$

$$\omega = \frac{\pi}{2}, \quad |H(e^{j\omega})| = \frac{10}{19/10} = \frac{100}{19}$$

☆ Visualize 2-D surface $|H(z)| = 1$

- ☆ zeros anchor surface to z-plane
- ☆ poles make surface go to ∞.

Ex. ⑤