Lecture - 22

Thursday, October 29, 2020 1:48 PM

Last Class:

- Fast Fourier Transform (FFT)
  - direct DFT – $O(N^2)$ multiplications & additions
  - radix-2 FFT – $O(N \log_2 N)$ mult. & additions, $N = 2^m$

    → efficient implementation based on computing shorter DFTs recursively & using properties of phase factors W_N.
  - we saw decimation-in-time approach

Todays class:

Some remarks on FFT:

(a) Inverse DFT has almost identical efficient implementation in $O(N \log_2 N)$.
– verify by looking at the IDFT formula.

(b) We have radix-4 algorithm as well (& other variations)

(c) In general when N is composite we can use divide & conquer approach i.e. Cooley-Tukey algorithm (& other variations)

(d) Decimation-in-time requires "bit-reversed" input ordering (of time)

(e) We also have decimation-in-frequency approach (radix-2)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad \text{say } N = 2^m.$$

k-even : $$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{2rn} \quad , k = 2r$$

k-odd : $$X[2r+1] = \sum_{n=0}^{N-1} x[n] W_N^{(2r+1)n} \quad , k = 2r$$

also, $W_{N/2}^{r(n+N/2)} = W_{N/2}^{rn}$

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_{N/2}^{rn} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_{N/2}^{rn} + \sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}] W_{N/2}^{rn}$$

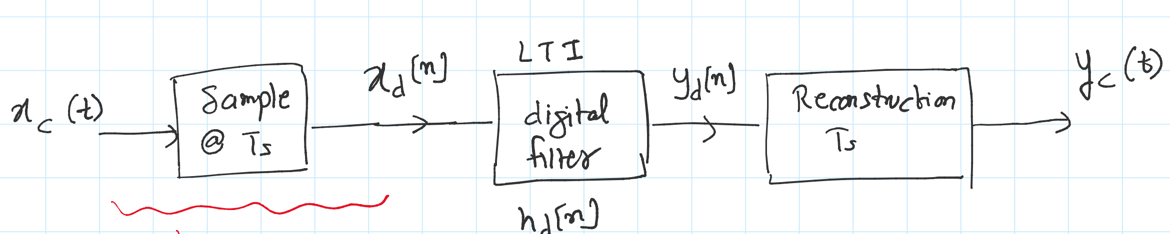
$$X[\ \ ] = \sum^{N/2-1} \left(x[n] + x[n+\tfrac{N}{2}]\right) W_{N/2}^{rn} \qquad r = 0, 1, \cdots \tfrac{N}{2}-1$$

$$X[2r] = \sum_{n=0}^{N/2-1} \left(x[n] + x[n+\tfrac{N}{2}]\right) W_{N/2}^{rn} \qquad r = 0, 1, \cdots \tfrac{N}{2}-1$$

$\frac{N}{2}$-point DFT of $y[n] = x[n] + x[n+N/2]$, $n = 0, 1, \cdots \frac{N}{2}-1$.

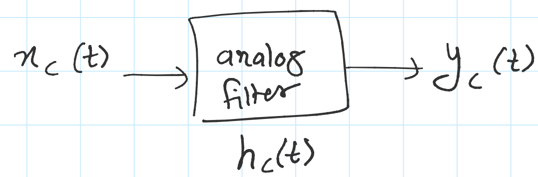
(f) performing convolution using FFT. - can be done efficiently in $O(N \log_2 N)$ where as direct convolution sum would be $O(N^2)$.

Processing analog signals using digital filters.



we implement using this

we want to do this

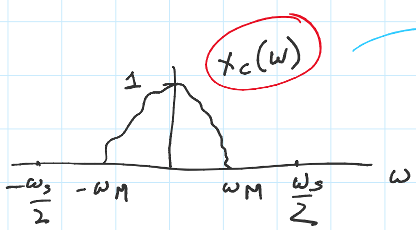


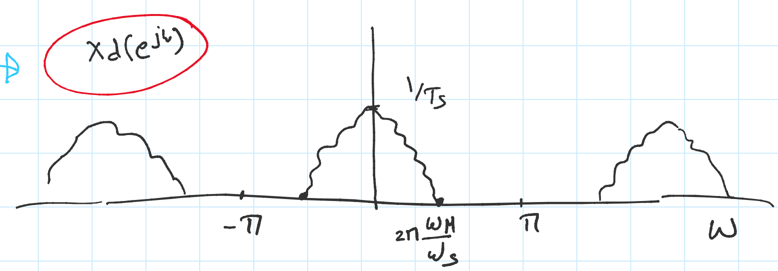
$$\omega_s = \frac{2\pi}{T_s}$$

$$x_c(t) \xleftrightarrow{FT} X_c(\omega)$$

$$x_d[n] \xleftrightarrow{DTFT} X_d(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left((\omega - k\omega_s)T_s\right)$$

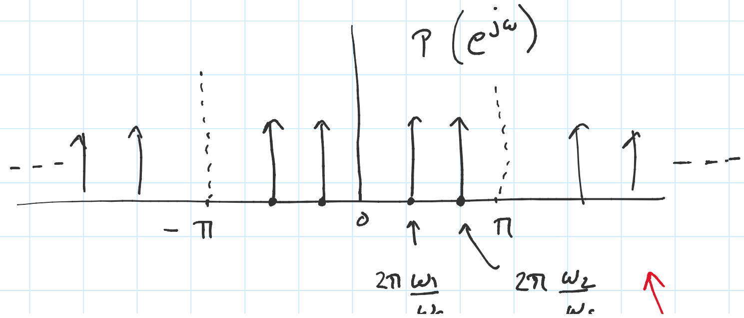
$x_d[n] = x_c(nT_s)$





LAB : 8.1

$\cos(\omega_1 t) + \cos(\omega_2 t)$



$-\omega_2 \quad -\omega_1 \quad 0 \quad \omega_1 \quad \omega_2$

$\cos(\omega_1 t) + \cos(\omega_2 t)$

$-\pi \quad 0 \quad \pi$

$2\pi \frac{\omega_1}{\omega_s} \quad 2\pi \frac{\omega_2}{\omega_s}$

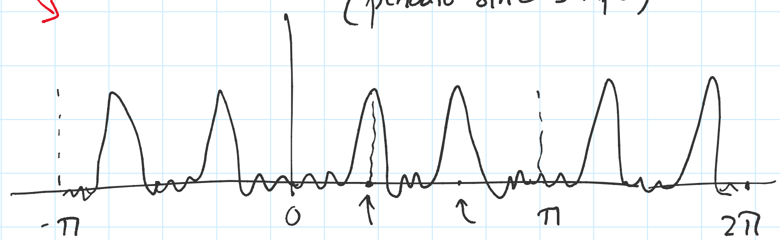
$$p[n] = \cos(\omega_1 n T_s) + \cos(\omega_2 n T_s)$$

$$= \cos\left(2\pi \frac{\omega_1}{\omega_s} n\right) + \cos\left(2\pi \frac{\omega_2}{\omega_s} n\right)$$

$$x[n] = p[n] \cdot w[n]$$

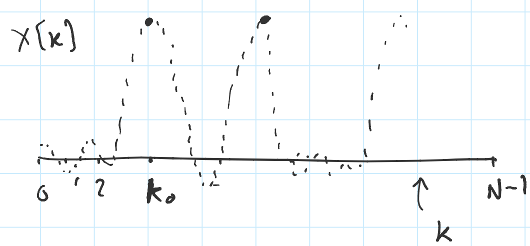
$$X(e^{j\omega}) = \frac{1}{2\pi}\left[P(e^{j\omega}) * W(e^{j\omega})\right]$$

for rectangular (periodic sinc shape)



k_0 corresponds to freq. $2\pi \frac{k_0}{N}$.

DFT : $X[k]$ i.e. samples of $X(e^{j\omega})$



1st peak at k_0

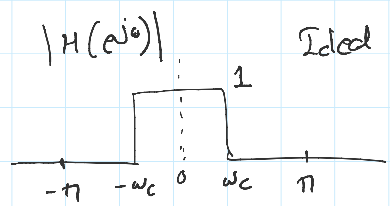
$$\Rightarrow 2\pi \frac{\hat{\omega}_1}{\omega_s} = 2\pi \frac{k_0}{N}$$

$$\Rightarrow \hat{\omega}_1 = \frac{k_0 \omega_s}{N}. \quad \text{estimate}$$

True $\omega_1 = \frac{\omega_s}{8}$

✱ Digital Filter Design ✱

Ideal filters :



Ideal LPF $\longleftrightarrow$ $h[n]$

$$h[n] = \frac{\sin(\omega_c n)}{\pi n}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, \underbrace{H(e^{j\omega})}_{\text{filter response}}$$

not practical because

- infinite length
- not causal.

* We focus on digital filters of the form:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{N(z)}{D(z)}$$

difference equation relation between $x[n]$ & $y[n]$.

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{l=1}^{N} a_l y[n-l]$$

} can be implemented with finite computation.

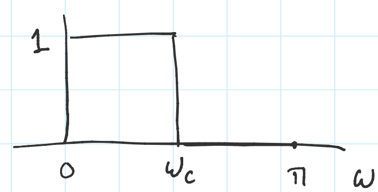
Cases (a) $a_i = 0 \quad \forall\, i = 1, 2, \cdots N.$

$\Rightarrow$ Finite Impulse Response (FIR) filters

(b) otherwise: Infinite impulse response (IIR) filters

* Filter design: given $H_D(e^{j\omega})$ i.e. desired filter find $h[n]$ so that its DTFT is close to $H_D(e^{j\omega})$.

Ideal



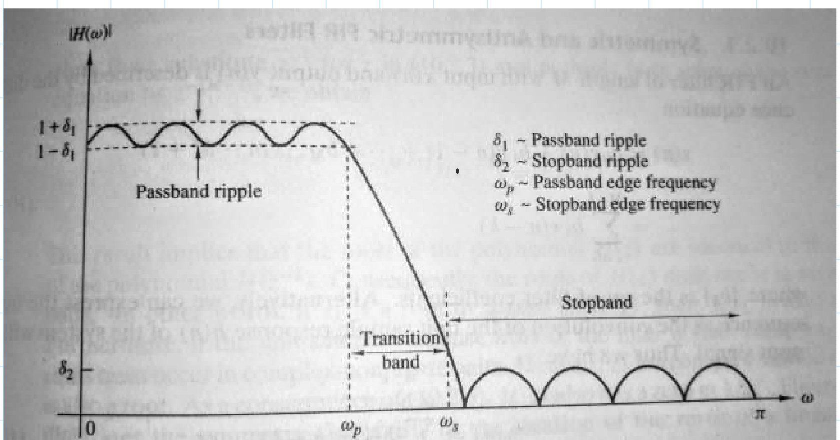


Figure 10.1.2 Magnitude characteristics of physically realizable filters.

In practice.

← From proakis