Lecture - 25
Monday, November 9, 2020 2:20 PM

Last Class

* FIR filters and linear-phase - requirements of symmetry & anti-symmetry
* FIR filter design by method of frequency sampling

Today's class

IIR Filter Design *

General remarks on IIR filter design:

(1) We focus on filters of the form $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{l=1}^{N} a_l z^{-l}}$ system function

here $a_l \neq 0$ for some of the l

then $h[n]$ would be IIR

(2) We cannot have linear phase (for causal IIR)
(Since symmetry or anti-symmetry is not possible)

Hence, we focus on satisfying the magnitude response requirement.

(3) For given filter requirements (i.e. transition region, passband ripples, stopband ripples, etc)
IIR filter has in general fewer parameters than FIR.

(4) Focus on designing IIR digital filters from IIR Analog filters

- start with Analog filter & convert it to digital filter
- many well-known existing methods to design Analog filters.

Ex. Butterworth filters
Chebyshev filters
Elliptic filters. (equiripple filter)

* Characterization of Analog filters :

h(t) - impulse response

✱ Characterization of Analog filters :

$h_a(t)$ - impulse response.

$$x(t) \longrightarrow \boxed{h_a(t)} \longrightarrow y(t)$$

$$h_a(t) \xleftrightarrow{\text{Laplace}} H_a(s)$$

System function :

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{1 + \sum_{l=1}^{N} \alpha_l s^l} \quad \text{... rational form}$$

Typical Analog filters designed with $H_a(s)$ of above form.

① Impulse response : $h_a(t)$

$$h_a(t) \xleftrightarrow{\text{Laplace}} H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{-st} dt \quad \text{... Laplace Transform.}$$

② Constant coefficient differential equations:

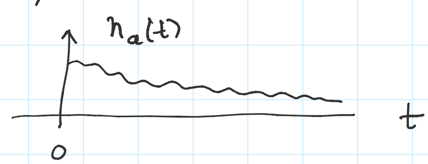
$$y(t) + \sum_{l=1}^{N} \alpha_l \frac{d^l y}{dt^l} = \sum_{k=0}^{M} \beta_k \frac{d^k x}{dt^k} \quad \text{.... (because of rational system function)}$$

$$y(t) \longleftrightarrow Y(s) \quad \Rightarrow \quad \frac{dy}{dt} \longleftrightarrow sY(s)$$

✱ IIR filter design by method of Impulse Invariance.

Idea: sample the analog impulse response.

$$h[n] = h_a(nT_s) \quad \text{--- sampling.}$$



ⓐ relation in Fourier domain.

$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H_a\left(\frac{\omega - 2\pi k}{T_s}\right)$$

T_s = sampling interval

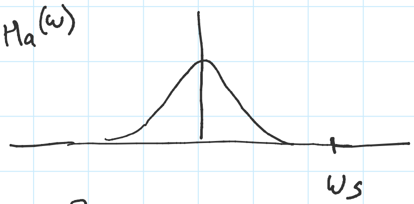
$$T_s = \frac{2\pi}{\omega_s}$$

sampling in time ⟶ copies in frequency domain.

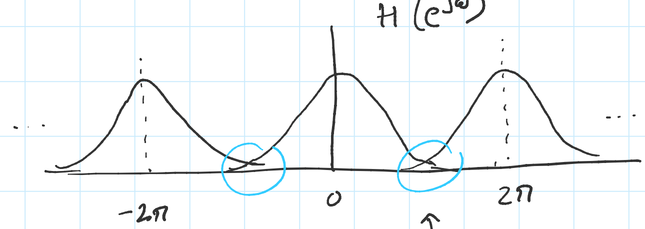
$T_s = \frac{2\pi}{\omega_s}$

$h_a(t) \overset{FT}{\longleftrightarrow} H_a(\omega)$

sampling in time $\rightarrow$ copies in frequency domain.



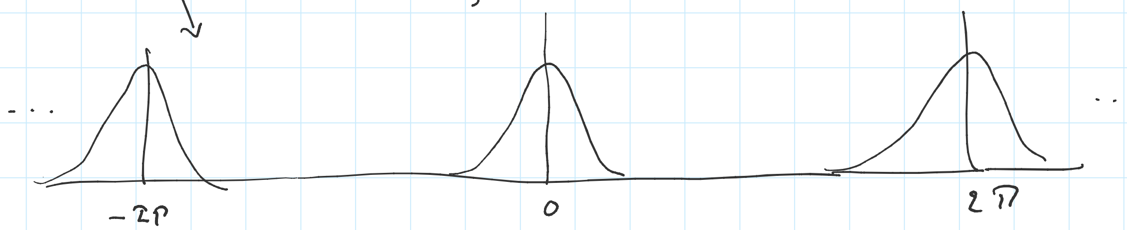
low-pass filter



overlap in general

low-pass filter

large ω_s



remark: If we use $H_a(\omega)$ to be high pass filter, we will have lot of aliasing in the digital filter. Hence this method is not used to design HPF.

(b) relation between $H_a(s)$ & $H(z)$

Laplace transform — z-transform

$H_a(s)$ which is rational, can be expressed as partial fractions.

$$\Rightarrow H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \quad \text{(A)}$$

.... (assumed all poles p_k are distinct)

here p_k are poles of $H_a(s)$.

we know:

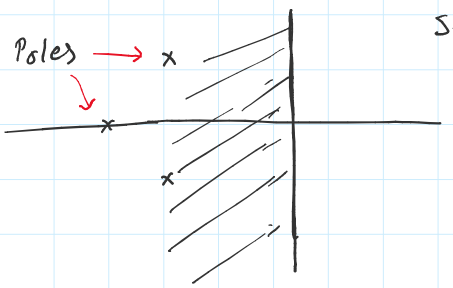
$$\frac{1}{s-a} \longleftrightarrow e^{at} u(t) \quad \text{--- continuous time}$$

$$\frac{z}{z-a} \longleftrightarrow a^n u[n] \quad \text{--- discrete time.}$$

$\Rightarrow$ from (A) $\quad h_a(t) = \sum_{k=1}^{N} C_k \cdot e^{p_k t} \, u(t)$ .... Causal.

also, for stability, $\quad \text{real}(p_k) < 0$

(For casal & stable system function of analog systems all poles must be in the left half plane)



$\text{real}(p_k) < 0$ for stability

After sampling: $\quad h[n] = h_a(nT_s)$

$$h[n] = \sum_{k=1}^{N} C_k \, e^{p_k n T_s} \, u(nT_s)$$

$$\updownarrow z$$

$$H(z) = \sum_{n=0}^{\infty} h[n] \, z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k} C_k \, e^{n p_k T_s} \right) z^{-n}$$

$$= \sum_{k=1}^{N} C_k \left(\sum_{n=0}^{\infty} e^{n p_k T_s} \, z^{-n} \right) = \sum_{k=1}^{N} C_k \left(\frac{1}{1 - e^{p_k T_s} z^{-1}} \right)$$

.... assuming convergence.

i.e. $\quad \left| e^{p_k T_s} z^{-1} \right| < 1$

$\Rightarrow \quad |z| > \left| e^{p_k T_s} \right|. \quad \forall k.$

$$\frac{N(z)}{D(z)} \equiv H(z) = \sum_{k=1}^{N} C_k \, \frac{z}{z - e^{p_k T_s}}$$

.... rational system function.

poles @ $\quad z = e^{p_k T_s}$

since $\quad \text{real}(p_k) < 0 \quad \Rightarrow \quad \left| e^{p_k T_s} \right| < 1 \qquad \forall k$

Since real $(p_k) < 0$ $\Rightarrow$ $|e^{p_k T_s}| < 1$ $\forall k$

$\Rightarrow$ poles of $H(z)$ inside unit circle.

$\Rightarrow$ $H(z)$ gives a stable (& causal) system.