Lecture - 14

Monday, September 28, 2020 2:19 PM

Last Class:

- DTFT examples
- DTFT & LTI systems
  - Convolution property & filters
  - Some ideal filter response
- Properties of DTFT

Today's Class: Z-Transform

Continuous-Time	Discrete-Time
CTFT: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$	DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n}$
Laplace Transform: $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$	Z-Transform: $X(z) = \sum_{-\infty}^{\infty} x[n]\, z^{-n}$
$s = \sigma + j\omega$ ... (complex)	$z = re^{j\theta}$
$X(\omega)$ — ω real; $X(s)$ — s complex	$X(e^{j\omega})$ — ω real; $X(z)$ — z complex

DTFT: useful for signal analysis & filters.

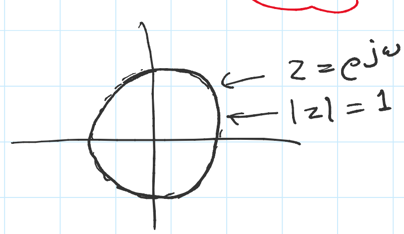
Z-Transform: useful for system analysis (stability, causality, etc.)

DTFT $X(e^{j\omega})$ is $X(z)$ evaluated $z = e^{j\omega}$

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega})$$

$z = e^{j\omega}$

z-plane



$g(\omega) = X(e^{j\omega})$ – complex valued function of a real variable (ω)

$X(z)$ – complex valued function of a complex variable (z) ; $z = re^{j\omega}$

- LTI systems & Z-Transform:

LTI systems & Z-Transform:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

LTI

$$e^{j\omega_0 n} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

(HW to show) $\quad z^n \longrightarrow H(z)\, z^n \quad , \quad \boxed{H(z) = \sum_{-\infty}^{\infty} h[n] z^{-n}}$

System function }
Transfer function }

Z-Transform:

$$x[n] \xleftrightarrow{Z} X(z)$$

The (infinite) sum : $\quad X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} \quad$ may not always converge (depends on z)

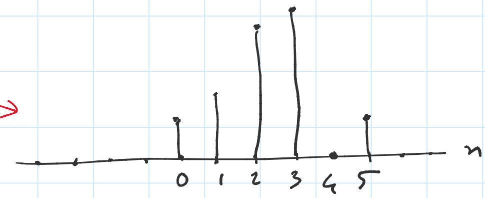
Region of Convergence (ROC): Set of all values of z for which $X(z)$ attains a finite value.

Z-Transform:

1. expression $X(z)$
2. region of convergence

Ex. ① $x[n] = \{ \underset{\uparrow}{1}, 2, 5, 7, 0, 1 \}$

$\uparrow$ indicates location of $x[0]$



$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{0}^{5} x[n] z^{-n}$$

$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

ROC: all of z-plane except $z = 0$.

$x[n]$		$X(z)$	ROC
② $\{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$	$\xleftrightarrow{Z}$	$z^2 + 2z + 5 + 7z^{-1} + z^{-3}$	all plane except 0 & ∞

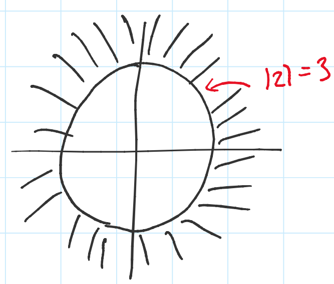
② $\{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$ $\longleftrightarrow$	$z^2 + 2z + 5 + 7z^{-1} + z^{-3}$	all plane except 0 & ∞
③ $\{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$	$z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$	all plane except 0
④ $\{1, 2, \underset{\uparrow}{5}\}$	$z^2 + 2z + 5$	all plane except ∞
⑤ $\delta[n]$	1	all plane
⑥ $\delta[n-n_0]$, $n_0 > 0$	z^{-n_0}	all plane except 0
⑦ $\delta[n+n_0]$, $n_0 > 0$	z^{+n_0}	all plane except ∞

* For finite duration signals, ROC is entire z-plane except possible 0 & ∞.

⑧ $x[n] = 3^n u[n]$, find $X(z)$. .... (DTFT does not exist)

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} = \sum_{0}^{\infty} 3^n z^{-n} = \sum_{0}^{\infty} (3z^{-1})^n$$

$$X(z) = \underbrace{\frac{1}{1-3z^{-1}}}_{\text{expression}} \quad \& \quad |3z^{-1}| < 1 \quad \text{i.e.} \quad \underbrace{|z| > 3}_{\text{ROC}}$$



* ROC always has circular symmetry

$z = re^{j\omega}$, $\quad X(z) = \sum_{-\infty}^{\infty} x[n] (re^{j\omega})^{-n}$

Within ROC, $|X(z)| < \infty$ i.e. finite

$$|X(z)| = \left| \sum_{-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} \right|$$

$$\leq \sum_{-\infty}^{\infty} \left| x[n] r^{-n} e^{-j\omega n} \right|$$

$$\leq \sum_{-\infty}^{\infty} \left| x[n] r^{-n} \right|$$

$$\le \sum_{-\infty} |x[n]\, r^{-n}|$$

$$|X(z)| \le \sum_{-\infty}^{-1} |x[n]\, r^{-n}| + |x[0]| + \sum_{1}^{\infty} |x[n]\, r^{-n}|$$

$$|X(z)| \le \underbrace{\sum_{1}^{\infty} |x[-n]\, r^{n}|}_{A} + |x[0]| + \underbrace{\sum_{1}^{\infty} \left|\frac{x[n]}{r^{n}}\right|}_{B}$$

For convergence of A, in general $\longrightarrow$ $r < r_1 < \infty$

for convergence of B, in general $\longrightarrow$ $r > r_2$

In general, for $|X(z)|$ to be finite

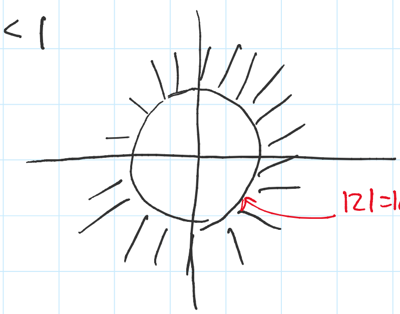
ROC should satisfy : $r < r_1$ & $r > r_2$

If $r_1 > r_2 \Rightarrow r_2 < r < r_1$ } annular ROC.

$r_1 < r_2 \Rightarrow$ No convergence & no z-transform.

Ex. $x[n] = a^n u[n]$ --- (right sided or causal signal)

$$X(z) = \sum_{0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} \quad \text{if } |az^{-1}| < 1$$

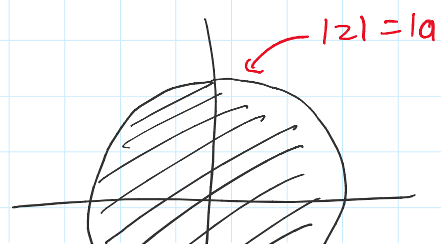


$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}} \quad \& \quad ROC: |z| > |a|$$

Ex. $x[n] = -a^n u[-n-1]$ --- (left sided or anti causal signal)

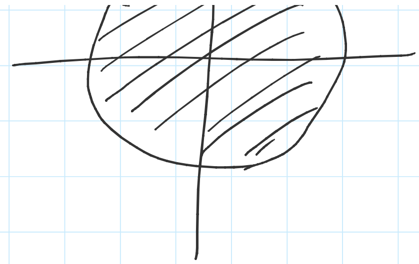
$$X(z) = \sum_{-\infty}^{\infty} x[n]\, z^{-n} = \sum_{-\infty}^{-1} (-a^n)\, z^{-n} = \sum_{1}^{\infty} -(a^{-1}z)^n$$

$$X(z) = -\frac{a^{-1}z}{1-a^{-1}z} \quad \& \quad |a^{-1}z| < 1$$



$$X(z) = \frac{1}{\ldots} \quad \& \quad ROC: |z| < |a|$$

$1 - az$



$$X(z) = \frac{1}{1 - az^{-1}} \quad \& \quad \text{ROC} : |z| < |a|.$$

plot $x[n]$ for $a = 3$.

Two different signals → can have same $X(z)$ expression
→ but ROC will be different

<u>Ex.</u> $x[n] = a^n u[n] + b^n u[-n-1]$