Monday, August 17, 2020   2:02 PM

Last class : - Fourier Series examples
- from Fourier series to Fourier Transform
(Periodic) $\longrightarrow$ (aperiodic)

Notation:

$X(\omega)$ or $X(j\omega)$

frequency: ω or f

$\omega = 2\pi f$

ω: rad/s, f: s^{-1} or Hz

Fourier Transform: (Analysis)

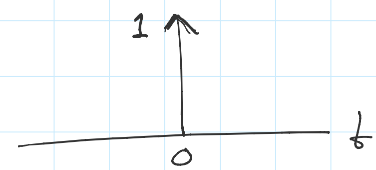
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Inverse Fourier Transform: (synthesis)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Ex. $x(t) = \delta(t)$ Impulse - signal

definition of $\delta(t)$: $\begin{cases} \delta(t) = 0 , & t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \end{cases}$



find $X(\omega)$: $X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$

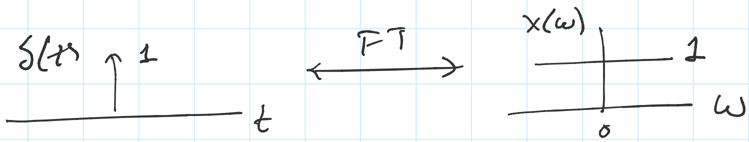
$\Rightarrow X(\omega) = 1 \quad \forall \omega$

$\delta(t) \xleftrightarrow{FT} 1$

Properties of $\delta(t)$

1) $\delta(t-t_0) f(t) = \delta(t-t_0) f(t_0)$

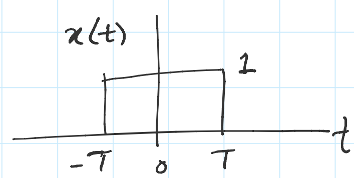
2) $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$



Time domain   Freq. domain

FT: allows us to express any $x(t)$ as linear combination of complex exponentials (ie $e^{j\omega t}$)

Ex. Rectangular pulse : $x(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases}$



find $X(\omega)$.

$\rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} 1 \cdot e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T}^{T}$

$X(\omega) = \dfrac{e^{-j\omega T} - e^{j\omega T}}{-j\omega} = \dfrac{2j\sin(\omega T)}{j\omega} = \dfrac{2\sin(\omega T)}{\omega}$ ... Sinc shape

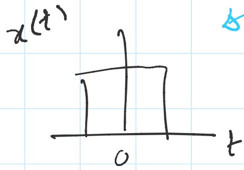
$$X(\omega) = \frac{e^{\dots} - e^{\dots}}{-j\omega} = \frac{2j\sin(\omega T)}{j\omega} = \frac{2\sin(\omega T)}{\omega} \quad \text{sinc shape}$$

$\sin(\omega T) = 0$

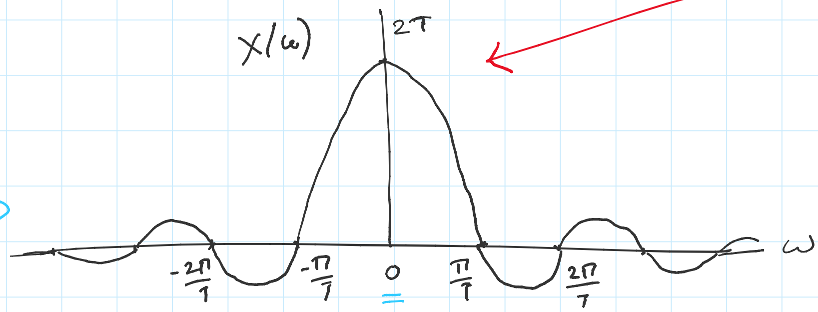
$\Rightarrow \omega T = k \cdot \pi$

$\Rightarrow \omega = \frac{\pi}{T} \cdot k$

Sketch $X(\omega)$:

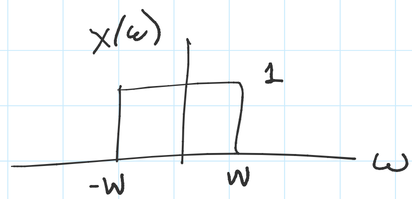


FT



Ex. $X(\omega)$ – pulse in freq. domain

$$X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



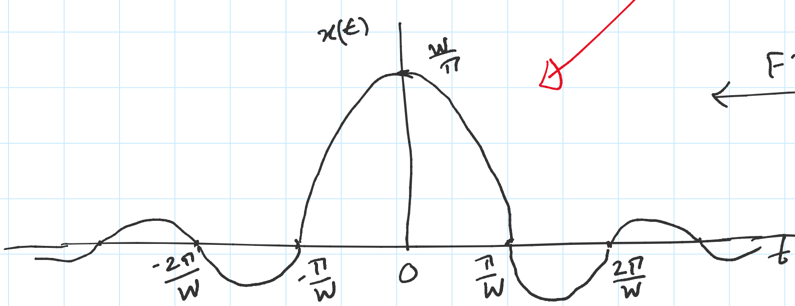
what is $x(t)$?

$$\rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{j\omega t}\, d\omega$$

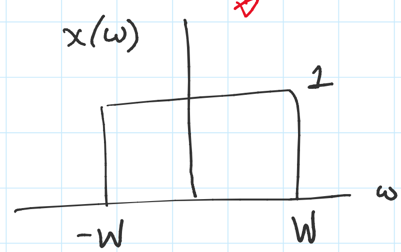
$$x(t) = \frac{1}{2\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{-W}^{W} = \frac{1}{2\pi} \frac{e^{jWt} - e^{-jWt}}{jt} = \frac{1}{2\pi \cdot jt} \cdot 2j\sin(Wt)$$

$$x(t) = \frac{\sin(Wt)}{\pi t} \quad \dots \text{sinc shape}$$

Pulse in freq.



FT



Time		Freq.
Pulse	$\longleftrightarrow$	sinc
sinc	$\longleftrightarrow$	pulse

impulse $\longleftrightarrow$ constant

constant ? $\longleftrightarrow$ impulse

Ex $\quad X(\omega) = \delta(\omega) \Rightarrow x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega = \dfrac{1}{2\pi} = \text{constant}$

$$\frac{1}{2\pi} \xleftrightarrow{FT} \delta(\omega)$$

$$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$

Ex. $\quad x(t) = \delta(t - t_0) \Rightarrow X(\omega) = e^{-j\omega t_0}$

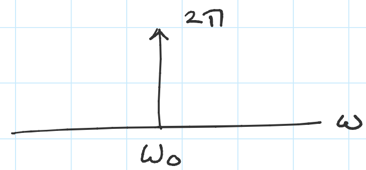
Ex. $\quad X(\omega) = 2\pi\,\delta(\omega - \omega_0), \quad x(t) = ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega$$

$$x(t) = e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega - \omega_0)$$

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

$$e^{j\omega_0 t} \xleftrightarrow{FT}$$



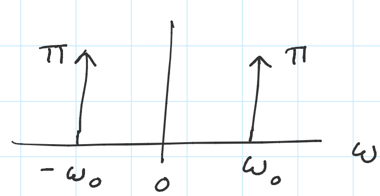
negative frequency

$\sin(\omega_0 t)$

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

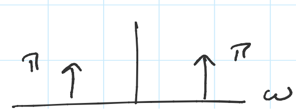
Ex. $\quad X(\omega) = \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$

$x(t) = ? \qquad \cos(\omega_0 t)$



Ex. $\quad X(\omega) = \dfrac{\pi}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$

$x(t) = ? \qquad \sin(\omega_0 t)$

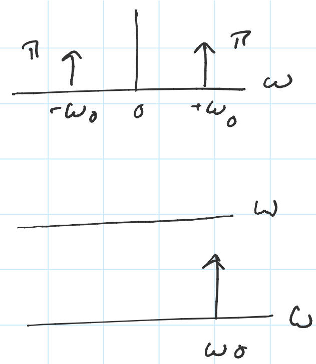


FT

$$\cos(\omega_0 t) \xleftrightarrow{FT} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$\sin(\omega_0 t) \xleftrightarrow{FT} \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$

$$\underbrace{e^{j\omega_0 t}}_{} \xleftrightarrow{FT} 2\pi\,\delta(\omega-\omega_0)$$



These are periodic

$\Rightarrow$ we have F.S. representation.

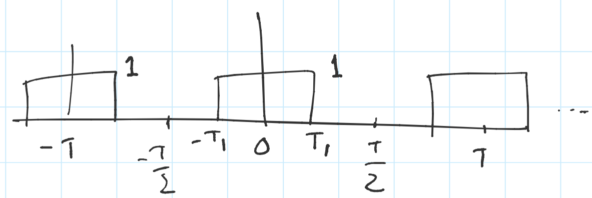
Ex. (A) $X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \rightarrow$ find $x(t)$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ ... Synthesis Eq. of F.S.

$$2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \xleftrightarrow{FT} \underset{\text{Periodic}}{x(t)} \xleftrightarrow{FS} \{a_k\}$$

periodic $x(t) \rightarrow$ find $a_k \longrightarrow$ write down $X(\omega)$ from (A)

Ex. HW: Find FT. of rectangular wave



Ex. Impulse - Train : (1) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. Find $X(\omega)$

Sketch : $x(t)$ & $X(\omega)$

$x(t)$ is periodic

find a_k.

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt \qquad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} \quad \forall\ k$$

$$\Rightarrow \quad x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{T}\sum_{-\infty}^{\infty} e^{jk\omega_0 t}$$

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$$X(\omega) = \frac{2\pi}{T}\sum_{-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$\omega_0 = \frac{2\pi}{T}$$