Lecture – 12

Monday, September 21, 2020 1:54 PM

Last class:

- Discrete-time signals $x[n]$ – common examples
- periodic signals in DT.
- differences between $\begin{cases} e^{j\omega t} & \text{– unique for } \omega \in (-\infty, \infty) \\ e^{j\omega n} & \text{– unique only for } \omega \in [-\pi, \pi) \\ & \text{– periodic only when } \omega = 2\pi \frac{k}{N}. \end{cases}$
- Discrete-time systems, focus on LTI systems
- LTI systems :

$h[n]$ – impulse response

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$ --- convolution sum (discrete-time)

Impulse response $h[n]$ fully characterizes an LTI system.

Today's class :

- Properties of convolution : (Proof – HW)
  - (a) commutative : $x[n] * h[n] = h[n] * x[n]$
  - (b) distributive : $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
  - (c) associative : $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

- Properties of unit impulse : $\delta[n]$ – Kronecker delta function $= \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$
  - (a) $x[n] \cdot \delta[n-m] = x[m] \cdot \delta[n-m]$
  - (b) $x[n] * \delta[n] = \sum_{m=-\infty}^{\infty} x[m]\, \delta[n-m] = x[n]$
  - (c) $x[n] * \delta[n-k] = x[n-k]$

- Discrete-time Fourier transform (DTFT) : for general aperiodic signals. $x[n]$

CTFT | DTFT

CTFT

$x(t) \overset{FT}{\longleftrightarrow} X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$e^{-j\omega t}$ is unique for $\omega \in (-\infty, \infty)$

IFT: $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

DTFT

$x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega})$ $g(\omega)$

DTFT: $$g(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$e^{-j\omega n}$ is unique for $\omega \in [-\pi, \pi)$

Inv. DTFT: $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = g(\omega) = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$g(\omega + 2\pi) = \sum_{-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n} = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n} \cdot \underbrace{e^{-j2\pi n}}_{=1} = g(\omega)$$

DTFT is periodic with period 2π.

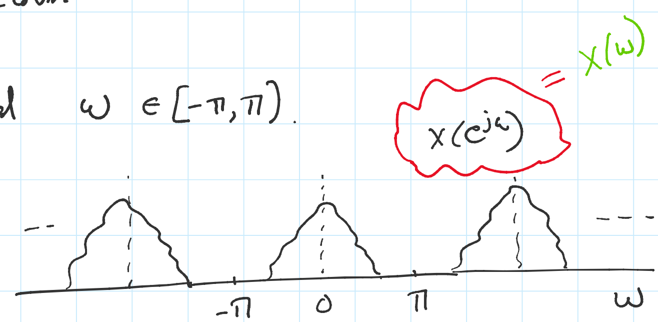
we will use $X(e^{j\omega})$ to denote DTFT of $x[n]$.

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega} \cdot e^{j2\pi}) = X(e^{j\omega})$$

This notation captures periodicity of DTFT.

$x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega})$ spectrum.

$X(e^{j\omega})$ we will specify in interval $\omega \in [-\pi, \pi)$.



* highest frequency : $\omega = \pm\pi$
* lowest frequency : $\omega = 0$

$$e^{j\pi n} = (-1)^n = \cos(\pi n) + j\sin(\pi n)$$

Inverse DTFT : consider $$I = \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

use definition of DTFT.

$$\int_{-\pi}^{\pi} \Big(\sum^{\infty} \quad e^{-j\omega k} \Big)\, e^{j\omega n}$$

use definition of DTFT.

$$I = \int_{-\pi}^{\pi} \left(\sum_{-\infty}^{\infty} x[k] e^{-j\omega k} \right) e^{j\omega n} \, d\omega$$

$$I = \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} e^{j\omega(n-k)} \, d\omega$$

$$\int_{-\pi}^{\pi} e^{j\omega m} \, d\omega = \begin{cases} 2\pi, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

$$\Rightarrow \quad I = x[n] \cdot 2\pi \quad \Rightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

Inverse DTFT

i.e. $x[n]$ is linear combination of various $e^{j\omega n}$

n – discrete time

ω – continuous frequency.

	Continuous-Time	Discrete-Time
Periodic (time)	FS	? (DFT)
Aperiodic (time)	FT	DTFT

Fourier Analysis

$$\text{FT} \rightarrow \text{Laplace Transform} \qquad \text{DTFT} \rightarrow \text{Z transform}$$

generalizations of Fourier Analysis

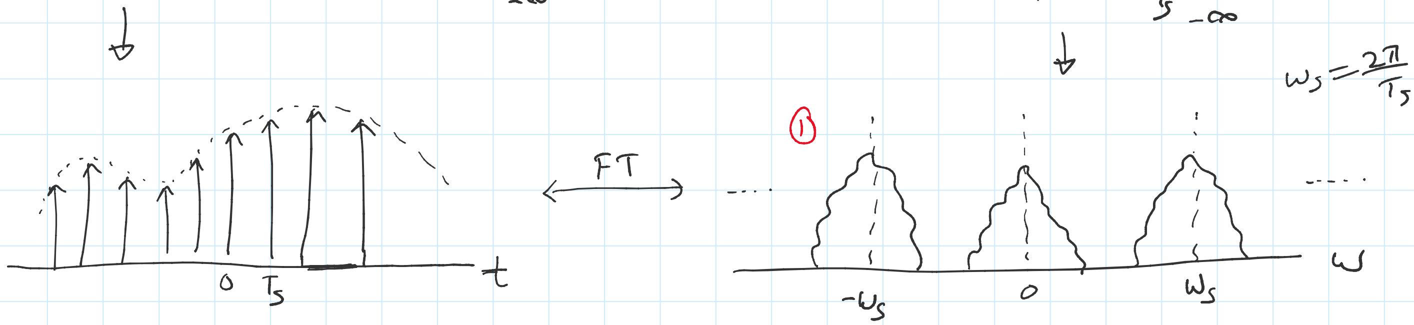
Remarks :

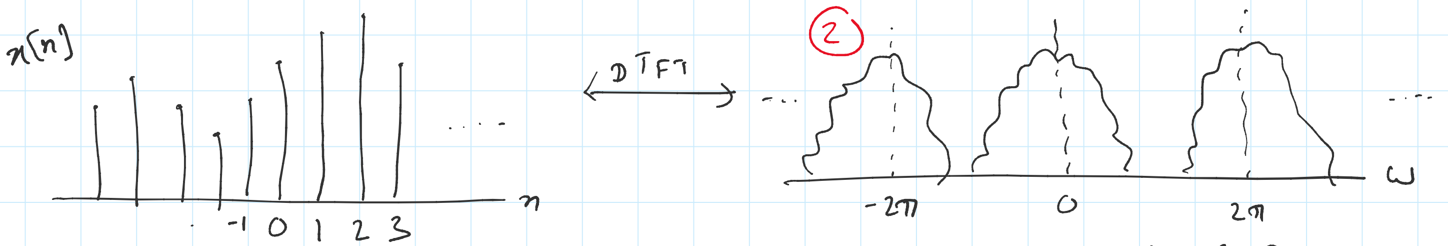
(A) Compare DTFT with CTFT of an impulse-train sampling of a band-limited signal

$$x_p(t) = x(t)p(t) = \sum_{-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$$

$$X_p(\omega) = \frac{1}{T_s} \sum_{-\infty}^{\infty} X(\omega - k\omega_s)$$

$\downarrow$ $\downarrow$ 2π





always periodic (2π)

③ Compare DTFT with FS analysis

periodic $x(t) \xleftrightarrow{FS} a_k$ (FS coefficients)

time t – continuous & $x(t)$ periodic

freq. k – discrete

DTFT: $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

time n – discrete

freq. ω – continuous & $X(e^{j\omega})$ is periodic

FS $\quad \omega_0 = \frac{2\pi}{T}$	DTFT
$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt$	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$
$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$

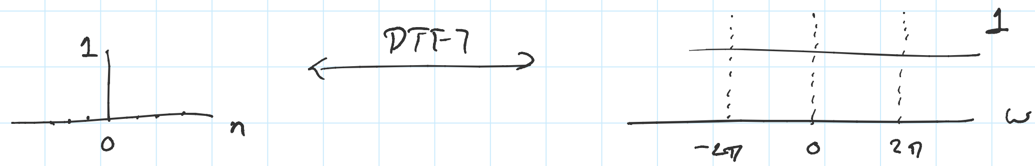
* Role of time & frequency is exchanged.

* mathematics is very similar, our interpretation of the signals & their domain is different.

Examples:

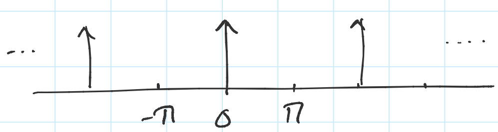
① Impulse in time: $x[n] = \delta[n]$, find $X(e^{j\omega})$.

$\rightarrow$ $$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n} = 1$$



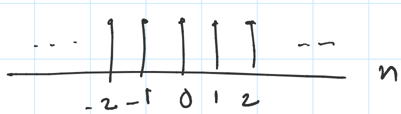
② Impulse in frequency:

$$X(e^{j\omega}) = \delta(\omega) \quad ; \quad \omega \in [-\pi, \pi) \text{ \& periodic } (2\pi)$$



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega$$



$$x[n] = \frac{1}{2\pi}$$

③ $X(e^{j\omega}) = \delta(\omega - \omega_0)$; $\omega \in [-\pi, \pi)$ & periodic (2π) & $\omega_0 \in [-\pi, \pi)$

$$x[n] = \frac{1}{2\pi} e^{j\omega_0 n}$$ ... single freq.