Lecture - 23
Monday, November 2, 2020 2:02 PM

* Last class :

* Some remarks on FFT
* Processing Analog signals using digital filters

Correction :

sampling : $x_c(t) \longrightarrow \boxed{\text{Sampler}} \longrightarrow x_d[n]$

$x_c(t) \overset{FT}{\longleftrightarrow} X_c(\omega)$, $\quad T_s = \frac{2\pi}{\omega_s}$, $\quad x_d[n] \overset{DTFT}{\longleftrightarrow} X_d(e^{j\omega})$

relation : $$X_d(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} X_c\left(\frac{\omega - 2\pi k}{T_s}\right)$$ *

* Digital filter design : *

 * Ideal filters not practical < infinite length impulse response * ; non causal *
 * practical filter response < ripples in passband / stopband * ; transition regions *

 * Focus on filters with system function :

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{l=1}^{N} a_l z^{-l}}$$

→ $\{a_i = 0\} \Rightarrow$ FIR filters
→ In general → IIR filters

This rational form of $H(z)$ can be easily implemented (finite computations)

Today's class :

* Some remarks on filter design :
 1. we design frequency selective filters
 2. Design process - many softwares available.
 3. FIR vs IIR filters
  a. FIR are used when linear-phase is required.

③ FIR vs IIR Filters

ⓐ FIR are used when linear-phase is required.
(IIR filters do not have linear-phase)

ⓑ In general, for similar freq. response, IIR filter needs fewer coefficients (than FIR)

★ FIR filter design:

★ Importance of linear-phase

ⓐ For example:

$h[n]$		$H(e^{j\omega})$	
$\delta[n]$	$\longleftrightarrow$	1	(all pass filter)
$\delta[n-n_0]$	$\longleftrightarrow$	$e^{-j\omega n_0}$	(all pass filter)

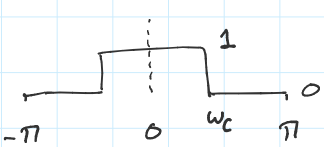
↳ shifts the signal by n_0

↳ $|H(e^{j\omega})| = 1$

$\measuredangle H(e^{j\omega}) = -\omega n_0$ ← linear phase

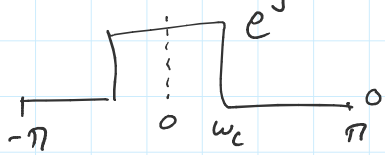
ⓑ Ideal LPF:

$H(e^{j\omega})$ $\overset{DTFT}{\longleftrightarrow}$ $h[n]$



$$\frac{\sin(\omega_c n)}{\pi n}$$

$$H = \begin{cases} e^{-j\omega n_0}, & |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| < \pi \end{cases} =$$



$\overset{DTFT}{\longleftrightarrow}$ $$\frac{\sin(\omega_c(n-n_0))}{\pi(n-n_0)}$$

↳ In the passband, we have linear-phase.

(c) In practice, delay i.e. linear-phase is acceptable / desirable

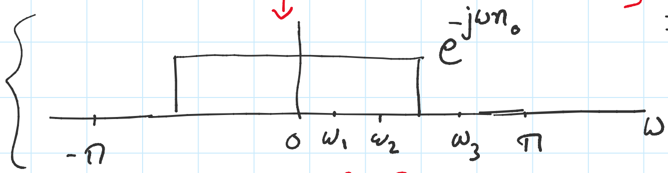
In general, we design linear-phase FIR filters.

linear-phase : shifts all frequencies equally (in the output)

non-linear phase : shift is frequency dependent

Ex. (1) $x[n] = e^{j\omega_1 n} + e^{j\omega_2 n} + e^{j\omega_3 n}$ $\qquad \omega_1 < \omega_2 < \omega_3$

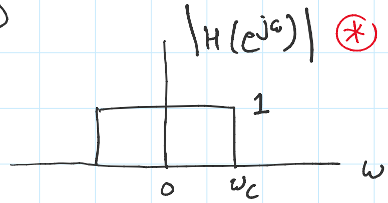
$x[n] \rightarrow \boxed{H(e^{j\omega})} \rightarrow y[n] = H(e^{j\omega_1})e^{j\omega_1 n} + H(e^{j\omega_2})e^{j\omega_2 n} + H(e^{j\omega_3})e^{j\omega_3 n}$



$= e^{-j\omega_1 n_0} e^{j\omega_1 n} + e^{-j\omega_2 n_0} e^{j\omega_2 n} + 0$

$= e^{j\omega_1 (n-n_0)} + e^{j\omega_2 (n-n_0)}$

$\measuredangle H(e^{j\omega}) = -\omega n_0$ (*)



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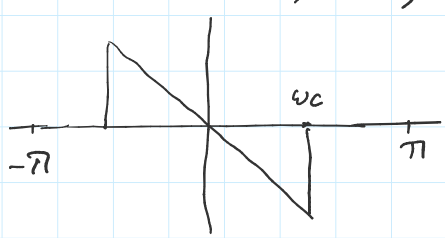
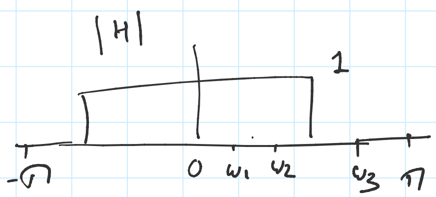


(2) Non-linear phase (still LTI system)

$x[n]$ same as above.

$H(e^{j\omega}) \equiv$



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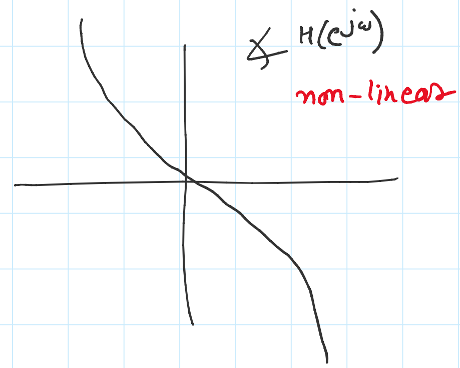


As above,

$\hookrightarrow y[n] = H(e^{j\omega_1})e^{j\omega_1 n} + H(e^{j\omega_2})e^{j\omega_2 n} + 0$

$= e^{j\omega_1(n-n_1)} + e^{j\omega_2(n-n_2)}$ $\qquad n_1 \neq n_2$

$\hookrightarrow$ signal is distorted.

When linear-phase is a strict requirement – use FIR filters

★ Linear-phase & FIR filters:

In general, FIR filters may or may not be linear phase

Requirement of linear phase → imposes conditions on $h[n]$

Specifically – symmetry & anti-symmetry conditions

★ An FIR filter (causal) has linear-phase if its impulse response $h[n]$ satisfies

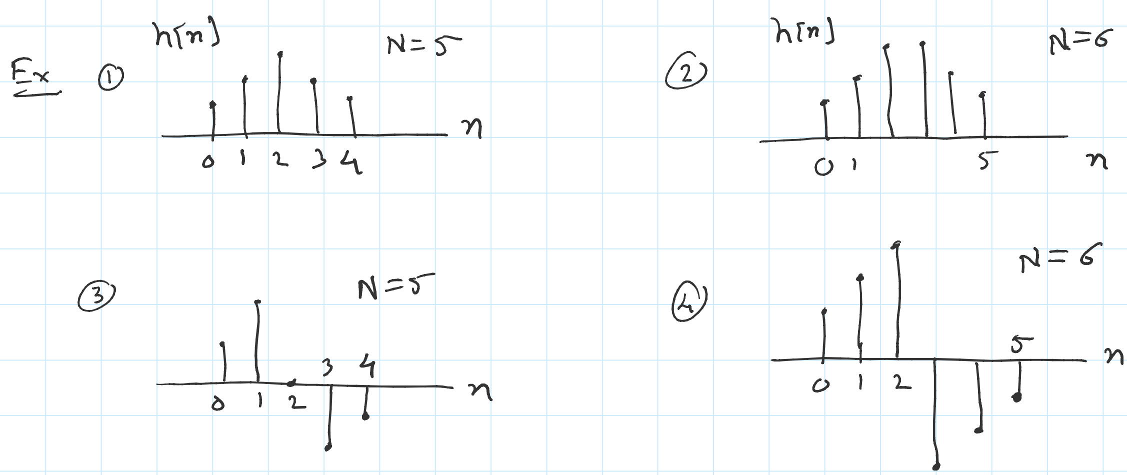
N-length filter

$$h[n] = \pm h[(N-1)-n] \quad , \quad n = 0, 1, 2, \cdots N-1$$

(+) symmetric FIR

(−) anti-symmetric FIR

Ex



① & ② – symmetric FIR

③ & ④ – anti-symmetric FIR

★ Linear-phase FIR filters using windows:

Steps: – Start with desired $H_d(e^{j\omega})$

– find impulse response $h_d[n] \xleftrightarrow{DTFT} H_d(e^{j\omega})$

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– find impulse response $h_d[n] \xleftrightarrow{DTFT} H_d(e^{j\omega})$

– apply window ; $h[n] = h_d[n] \cdot w[n]$

↳ gives FIR

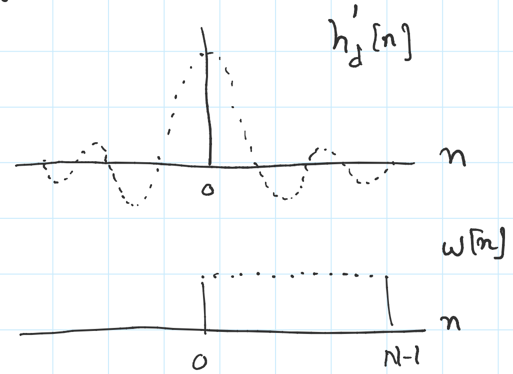
multiplication property

make sure $h[n]$ has symmetry / anti-symmetry.

$$H(e^{j\omega}) = \frac{1}{2\pi}\left[H_d(e^{j\omega}) \circledast W(e^{j\omega})\right]$$

many choices for selecting $w[n]$.

Ex. consider ideal LPF & rectangular window



$$w[n] = \begin{cases} 1, & n = 0, 1 \cdots N-1 \\ 0, & \text{otherwise} \end{cases}$$

take N - odd & $M = \frac{N-1}{2}$

select : $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega M}, & |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| < \pi \end{cases}$ ← Ideal LPF

$\updownarrow$

$$h_d[n] = \frac{\sin(\omega(n-M))}{\pi(n-M)} \quad \text{... shifted sinc}$$

$$h[n] = h_d[n] \cdot w[n] = \begin{cases} \frac{\sin(\omega(n-M))}{\pi(n-M)}, & n = 0, 1, \cdots N-1 \\ 0, & \text{otherwise.} \end{cases}$$

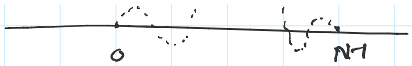
by our design is symmetric & hence has linear-phase.

$$h[0] = \frac{\sin(\omega M)}{\pi M} \quad \& \quad h[N-1] = \frac{\sin(\omega M)}{\pi M}$$

& similarly for other entries

⇒ symmetric shape

$W(e^{j\omega}) = \quad \ldots \quad \frac{\sin(\omega N/2)}{\ldots} \; e^{-j\omega M}$

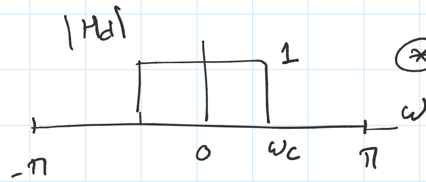


$\Rightarrow$ symmetric shape

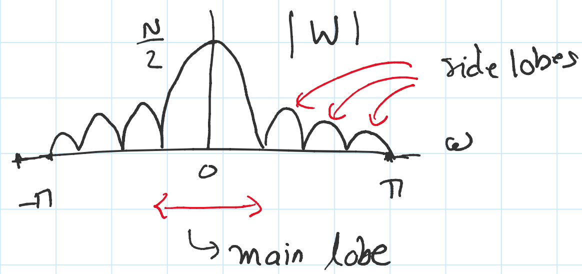
$$H(e^{j\omega}) = \frac{1}{2\pi}\left[H_d(e^{j\omega}) \circledast W(e^{j\omega})\right]$$

$$W(e^{j\omega}) \rightarrow \frac{\sin(\omega N/2)}{\sin(\omega/2)}\, e^{-j\omega M}$$

$$H(e^{j\omega}) \rightarrow \sum_{n=0}^{N-1} h[n]\, e^{-j\omega n}$$



$\circledast$



- side-lobes — affect ripples
- main-lobe — affects transition region

non-ideal LPF

- shows oscillations & Gibbs phenomenon

same plots with normalized frequency i.e.

$$f = \frac{\omega}{2\pi}$$

$$\omega : [0, \pi] \rightarrow f\,[0, 1/2]$$

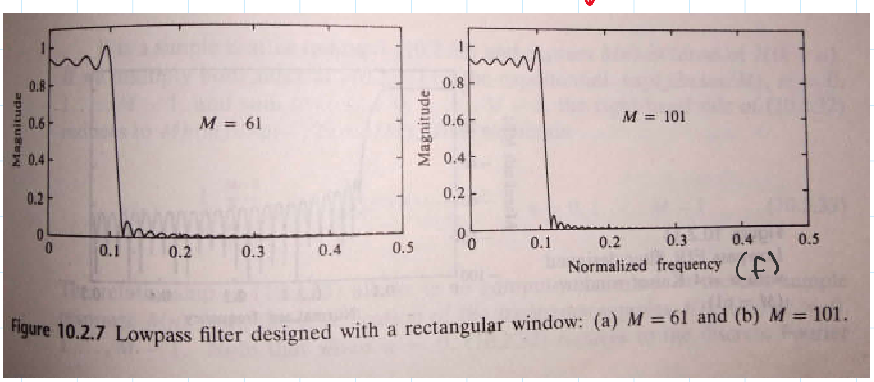


Figure 10.2.7 Lowpass filter designed with a rectangular window: (a) $M = 61$ and (b) $M = 101$.

- Various kinds of windows $w[n]$

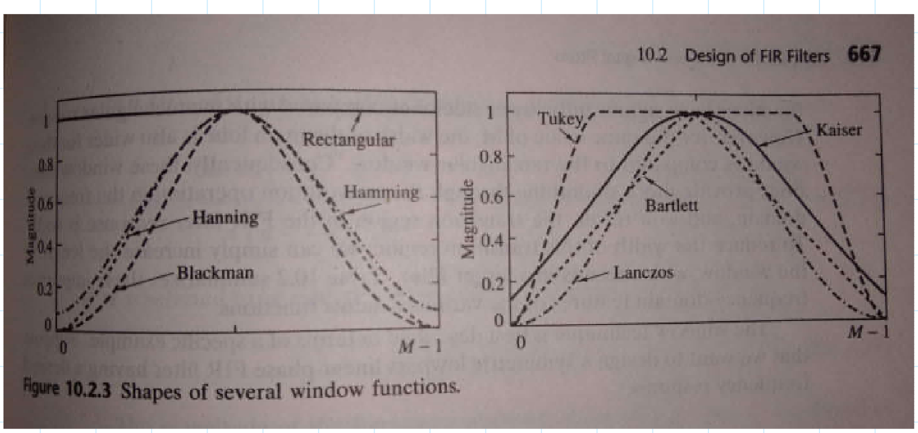


Figure 10.2.3 Shapes of several window functions.

★ Window freq. response i.e. $W(e^{j\omega})$

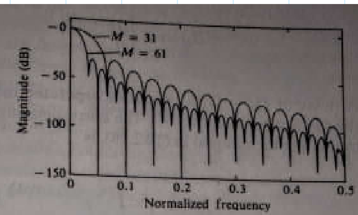


Figure 10.2.4 Frequency responses of Hanning window for (a) $M = 31$ and (b) $M = 61$.

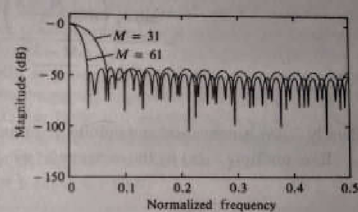


Figure 10.2.5 Frequency responses for Hamming window for (a) $M = 31$ and (b) $M = 61$.

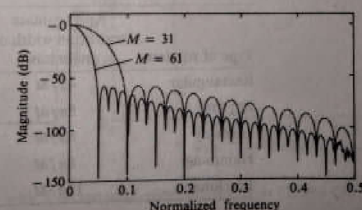


Figure 10.2.6 Frequency responses for Blackman window for (a) $M = 31$ and (b) $M = 61$.

★ Window formula

TABLE 10.1 Window Functions for FIR Filter Design

Name of window	Time-domain sequence, $h(n),\ 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \frac{M-1}{2}\right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$

Chapter 10 Design of Digital Filters

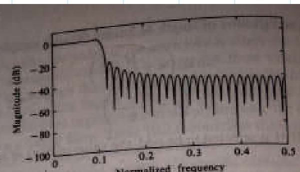


Figure 10.2.8 Lowpass FIR filter designed with rectangular window ($M = 61$).

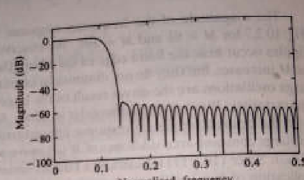


Figure 10.2.9 Lowpass FIR filter designed with Hamming window ($M = 61$).

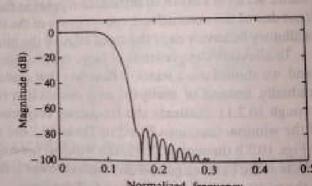


Figure 10.2.10 Lowpass FIR filter designed with Blackman window ($M = 61$).

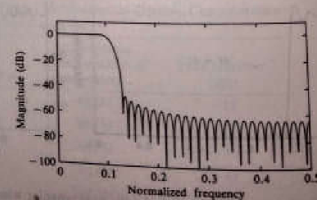


Figure 10.2.11 Lowpass FIR filter designed with $\alpha = 4$ Kaiser window ($M = 61$).

★ The obtained freq. response of the LPF designed using various windows. ★

Note : y-axis is in decibels i.e $20\log_{10}|H(e^{j\omega})|$

General trade-off: [for $W(e^{j\omega})$]

↳ narrow main lobe — higher side lobes.

wider —"— — lower —"—