Lecture - 26
Thursday, November 12, 2020 2:10 PM

Last class

- Some remarks on IIR filter design
- Focus on - rational system function
  - conversion from analog IIR filters.
- Method of Impulse invariance
  - samples in time domain : $h_d[n] = h_a(nT_s)$
  - relation in frequency domain : aliasing (hence cannot use for HPF)
  - relation between $H_a(s)$ and $H_d(z)$:
    pole at p_k ⇒ pole at $e^{p_k T_s}$
  - stable & causal $h_a(t)$ ⇒ stable & causal $h_d[n]$.

equivalent

Today's class / Last class

- Characterization of Analog filter / system :

① Impulse response : $h_a(t) \longrightarrow$ impulse invariance.

② Differential Equation :

$$y(t) + \sum_{\ell=1}^{N} \alpha_\ell \frac{d^\ell y}{dt^\ell} = \sum_{k=0}^{M} \beta_k \frac{d^k x}{dt^k} \equiv H_a(s) = \frac{\sum_{k=0}^{M} \beta_k s^k}{1 + \sum_{l=0}^{N} \alpha_l s^l}$$

rational system function

Laplace

$Y(s) + \alpha_1 s Y(s) + \alpha_2 s^2 Y(s) + \dots$ Take Laplace Transform & $H_a(s) = \dfrac{Y(s)}{X(s)}$

- IIR filter design by approximation of derivatives :

① $$\left.\frac{dy}{dt}\right|_{t=nT} \approx \frac{y(nT) - y(nT-T)}{T}$$ ... backward difference approximation

② $$\left.\frac{d^2y}{dt^2}\right|_{t=nT} \approx \frac{y(nT) - 2y(nT-T) + y(nT-2T)}{T^2}$$

....

Put $y(nT) \equiv y[n]$

difference Eq^n

The differential equation can be approximated as: difference Eq^n

$$y[n] + \alpha_1\left(\frac{y[n]-y[n-1]}{T}\right) + \alpha_2\left(\frac{y[n]-2y[n-1]+y[n-2]}{T^2}\right) + \cdots = \beta_0 x[n] + \beta_1\left(\frac{x[n]-x[n-1]}{T}\right) + \cdots$$

* Take z-Transform on both sides:

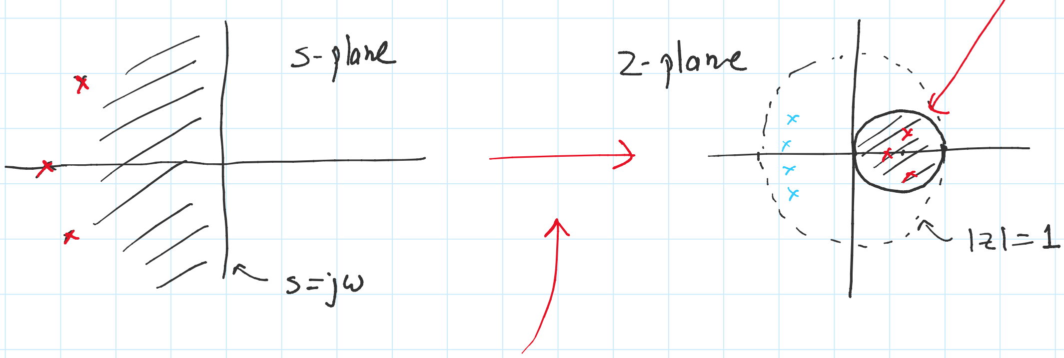
$$Y(z) + \alpha_1\left(\frac{1-z^{-1}}{T}\right)Y(z) + \alpha_2\left(\frac{1-z^{-1}}{T}\right)^2 Y(z) + \cdots = \beta_0 X(z) + \beta_1\left(\frac{1-z^{-1}}{T}\right)X(z) + \cdots$$

$$\Rightarrow H_d(z) = \frac{Y(z)}{X(z)} = H_a(s)\Big|_{s=\left(\frac{1-z^{-1}}{T}\right)}$$

Mapping: From s-plane to z-plane given by relation

$$s = \frac{1-z^{-1}}{T} \Rightarrow z = \frac{1}{1-sT}$$

If $s = j\omega \Rightarrow z = \dfrac{1}{1-j\omega T}$ -- we can show that this maps to circle $\left|z-\frac{1}{2}\right| = \frac{1}{2}$



* In fact the left half of s-plane maps to $\left|z-\frac{1}{2}\right| \le \frac{1}{2}$.

* $s = \sigma + j\omega$ LHP $\Rightarrow \sigma < 0$

* This method is useful for designing low pass filters & some band pass filters.

band pass filters.

* Cannot be used for HPF design.

Ex. $H_a(s) = \dfrac{1}{(s+0.1)^2+9}$. Find the digital filter by appr. of derivative.

☆ IIR filter design by Bilinear Transformation ☆

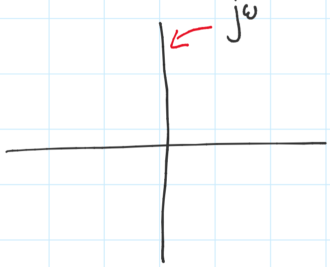
$$s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T}\,\frac{z-1}{z+1} \Rightarrow z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$$

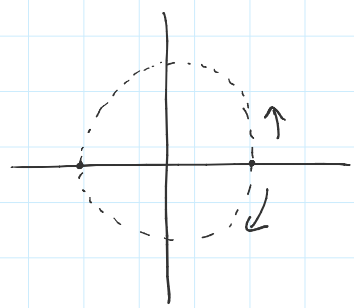
Mapping

(a) This mapping, maps $j\omega$-axis in s-plane to the unit circle in the z-plane only once.

$$s = j\omega \Rightarrow z = \frac{2/T + j\omega}{2/T - j\omega} \Rightarrow |z| = 1.$$

$$z = \frac{(2/T + j\omega)^2}{\left(\frac{2}{T}\right)^2+\omega^2} = \frac{\left(\frac{2}{T}\right)^2-\omega^2}{\left(\frac{2}{T}\right)^2+\omega^2} + j\,\frac{\frac{4\omega}{T}}{\left(\frac{2}{T}\right)^2+\omega^2}$$

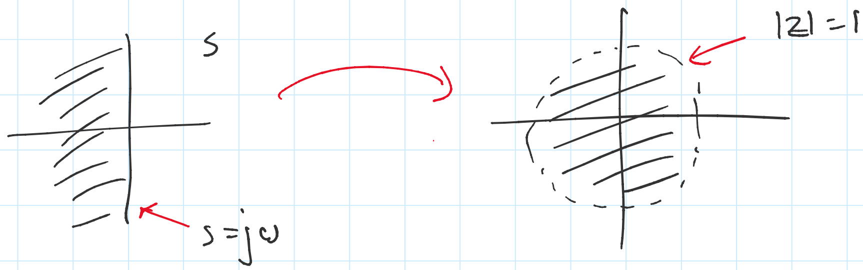




(s-plane) $\omega_a : 0 \to +\infty \Rightarrow$ (z-plane) $\omega_d : 0 \to +\pi$

(s-plane) $\omega_a : 0 \to -\infty \Rightarrow$ (z-plane) $\omega_d : 0 \to -\pi$

(b) The LHP of s-plane maps to inside the unit circle.



$$z = \frac{2/T + s}{\frac{2}{T} - s} \xrightarrow{s=\sigma+j\omega} \frac{\left(\frac{2}{T}+\sigma\right)+j\omega}{\left(\frac{2}{T}-\sigma\right)-j\omega}$$

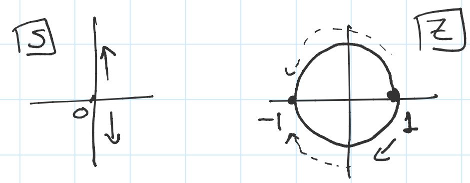
Show that: $\sigma < 0 \Rightarrow |z| < 1.$

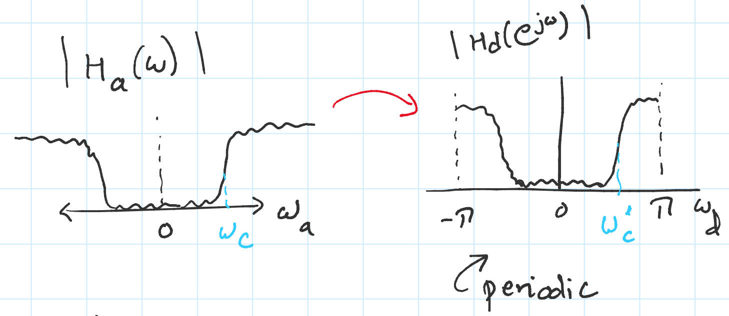
& $\sigma > 0 \Rightarrow |z| > 1$

& $\sigma = 0 \Rightarrow |z| = 1$ ... (from (a)).

$$z = \frac{\frac{2}{T} + s}{\frac{2}{T} - s}$$

s	z	
0	1	
$s \to +j\infty$	-1	
$s \to -j\infty$	-1	





The mapping $\omega_a : (0, \infty) \to \omega_d : (0, \pi)$

$\therefore$ There is frequency warping. Relation given by:

$$\omega_d = 2 \tan^{-1}\left(\frac{\omega_a T}{2}\right)$$