Lecture - 15

Thursday, October 1, 2020 12:59 PM

Last Class :

Z - Transform $\quad x[n] \xleftrightarrow{z} X(z) \quad$ z – complex

1. $X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$
2. Region of convergence (ROC)

Z-Transform uniquely specified by both (1) Expression & (2) ROC

★ DTFT is Z-Transform evaluated on the unit circle

i.e. $z = e^{j\omega}$ or $|z| = 1$

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}$$ if ROC includes unit circle.

★ For finite duration signals :

ROC is entire z-plane except possible $z=0$ & $z=\infty$.

★ In general the ROC has circular symmetry :

$r_1 < |z| < r_2$ where r_1 can be zero & r_2 can be infinity.

★ Examples :

$a^n u[n] \xleftrightarrow{z} \dfrac{1}{1-az^{-1}}$ & ROC : $|z| > |a|$ (right-sided)

$-a^n u[-n-1] \xleftrightarrow{z} \dfrac{1}{1-az^{-1}}$ & ROC : $|z| < |a|$ (left-sided)

Different signals can have same z-transform expression with different ROC.

Today's class :

We are interested in z-Transforms of the form :

$$X(z) = \frac{N(z)}{D(z)}$$ i.e. ratio of polynomials in z

$N(z)$ – numerator polynomial

$D(z)$ – denominator polynomial.

Zeros & Poles :

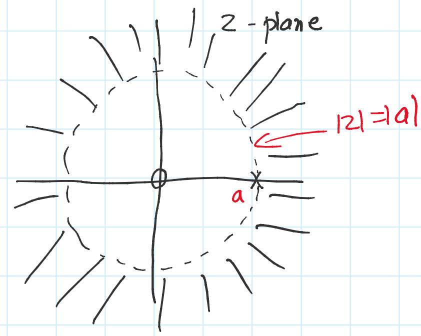
$N(z) = 0 \rightarrow X(z) = 0$ 'zeros' ★

Zeros & Poles :

$N(z) = 0 \Rightarrow X(z) = 0$ 'zeros' ✱

$D(z) = 0 \Rightarrow X(z) = \infty$ 'poles' ✱

Ex. ① $a^n u[n] \xleftrightarrow{z} \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a} = \dfrac{N(z)}{D(z)}$

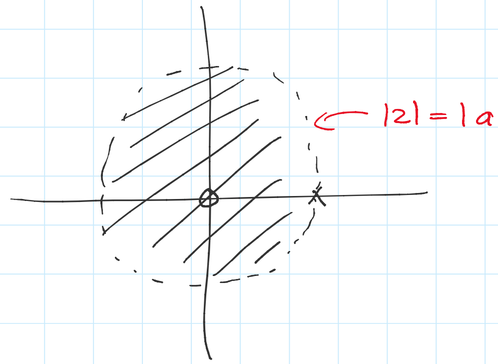


$N(z) = 0$ if $z = 0$ (zero)

$D(z) = 0$ if $z = a$ (pole)

ROC : $|z| > |a|$

② $-a^n u[-n-1] \xleftrightarrow{z} \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$



zero @ $z = 0$

pole @ $z = a$

ROC : $|z| < |a|$

✱ Poles play important role in deciding ROC
zeros do not play any role in ROC

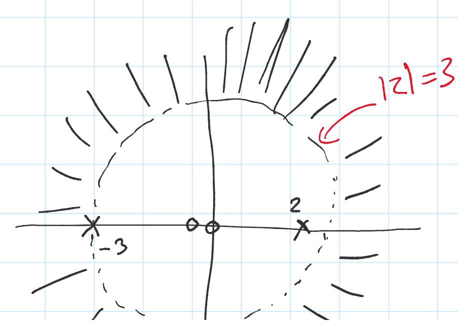
✱ Specifically, ROC can never contain any pole.

③ $x[n] = 2^n u[n] + (-3)^n u[n]$ find $X(z)$ & ROC.

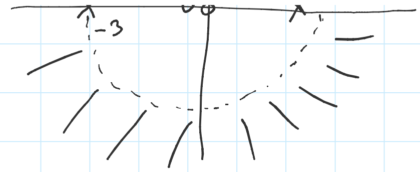
$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} = \sum_{0}^{\infty} 2^n z^{-n} + \sum_{0}^{\infty} (-3)^n z^{-n}$$

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{1}{1+3z^{-1}} \quad \Big\{ \; |z| > |2| \; \& \; |z| > |-3|$$

$$X(z) = \frac{z}{z-2} + \frac{z}{z+3} \quad \& \text{ ROC : } |z| > 3$$



$$X(z) = \frac{z(z+3) + z(z-2)}{\quad} = \frac{2z^2 + z}{\quad}$$



$$X(z) = \frac{z(z+3) + z(z-2)}{(z-2)(z+3)} = \frac{2z^2+z}{(z-2)(z+3)}$$

$$X(z) = \frac{z(2z+1)}{(z-2)(z+3)} \Rightarrow$$ zeros @ $z=0$ & $z=-\frac{1}{2}$

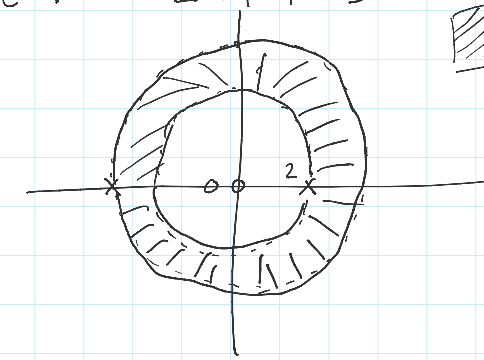
Poles @ $z=2$ & $z=-3$

(4) $x[n] = 2^n u[n] - (-3)^n u[-n-1]$, find $X(z)$ & ROC.

$$X(z) = \frac{z}{z-2} + \frac{z}{z+3}$$ & ROC $\{ |z| > 2$ & $|z| < |-3|$

$\Downarrow$

$$X(z) = \frac{z(2z+1)}{(z-2)(z+3)}$$ ROC: $2 < |z| < 3$



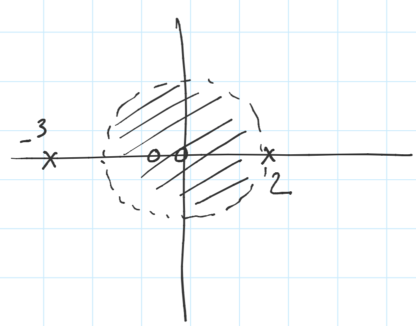
zeros @ $z=0$, $z=-\frac{1}{2}$

poles @ $z=2$, $z=-3$

(5) $x[n] = -2^n u[-n-1] - (-3)^n u[-n-1]$, find $X(z)$ & ROC.

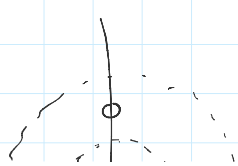
$$X(z) = \frac{z}{z-2} + \frac{z}{z+3}$$ & ROC: $\{ |z| < 2$ & $|z| < 3$

$$X(z) = \frac{z(2z+1)}{(z-2)(z+3)}$$ ROC: $|z| < 2$



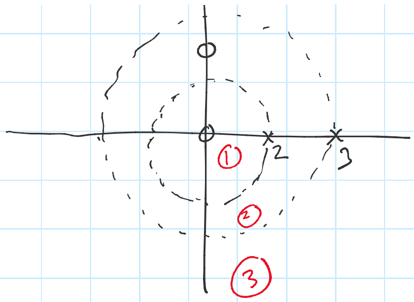
Compare examples (3), (4) & (5)
& their ROC.

(6)



Possible ROC:

(1) $|z| < 2$



Possible ROC:

(1) $|z| < 2$

(2) $2 < |z| < 3$

(3) $|z| > 3$

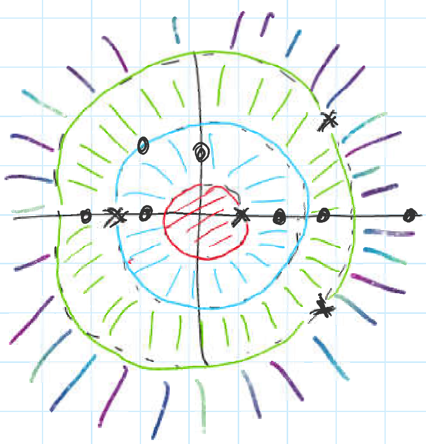
(7) $x[n] = -2^n u[-n-1] + (-3)^n u[n]$

$$-2^n u[-n-1] \rightarrow \frac{z}{z-2} \quad \text{if } |z| < 2$$

$$(-3)^n u[n] \rightarrow \frac{z}{z+3} \quad \text{if } |z| > 3$$

$\left.\begin{array}{l}\text{if } |z| < 2 \ \ \& \ \ \text{if } |z| > 3\end{array}\right\} \Rightarrow$ No ROC $\Rightarrow$ z-trasform doesn't exist.

(8)



possible ROC # 4

ROC cannot hae any poles

(9) $x[n] = a^n u[n] + b^n u[-n-1]$

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

$\frac{z}{z-a}$ if $|z| > |a|$, $\quad \frac{z}{z-b}$ if $|z| < |b|$

Since both have to converge:

Case (i) $|b| > |a| \Rightarrow |a| < |z| < |b|$

case (ii) $|b| < |a| \Rightarrow$ no feasible ROC.

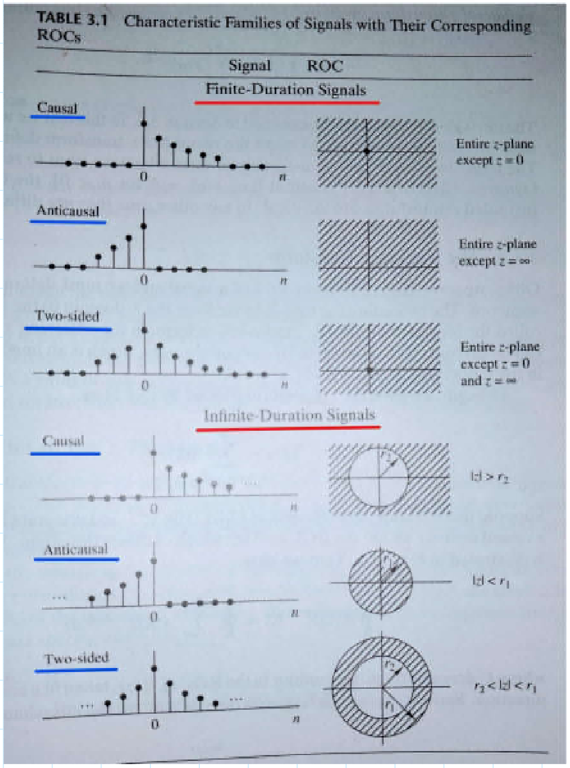


TABLE 3.1 Characteristic Families of Signals with Their Corresponding ROCs

Signal	ROC
Finite-Duration Signals	
Causal	Entire z-plane except $z = 0$
Anticausal	Entire z-plane except $z = \infty$
Two-sided	Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal	$\lvert z\rvert > r_2$
Anticausal	$\lvert z\rvert < r_1$
Two-sided	$r_2 < \lvert z\rvert < r_1$

Table from Proakis