Lecture - 20

Thursday, October 22, 2020 1:51 PM

**Last class** * Discrete Fourier Transform (DFT) : $x[n] \xleftrightarrow{N-DFT} X[k]$

$n = 0, 1, \cdots, N-1$ $\quad$ $k = 0, 1, \cdots, N-1$

* DFT as samples of DTFT spectrum at $\omega_k = \frac{2\pi}{N} k$, $k = 0, 1, \cdots N-1$.

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

N samples in interval $[0, 2\pi)$ spaced $\frac{2\pi}{N}$ apart.

* given N-length signal $x[n]$, $n = 0, 1, \cdots, N-1$ ; N-point DFT given by :

DFT : $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \qquad k = 0, 1, \cdots N-1.$$

IDFT : $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \qquad n = 0, 1, \cdots N-1$$

linear combination of periodic signals

* Note that above formula are valid for any integer k & n with the following periodicity

$$* \; X[k+N] = X[k] \quad \& \quad * \; x[n+N] = x[n]$$

* We can treat $x[n]$ & $X[k]$ as N-length vectors with the relation

$$X_N = F_N x_N \quad , \quad F_N - \text{DFT matrix.} \quad , \quad W_N = e^{-j\frac{2\pi}{N}}$$

**Today's Class :**

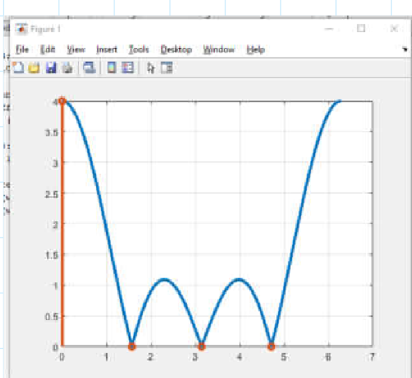
as function of discrete-time n, the set of signals

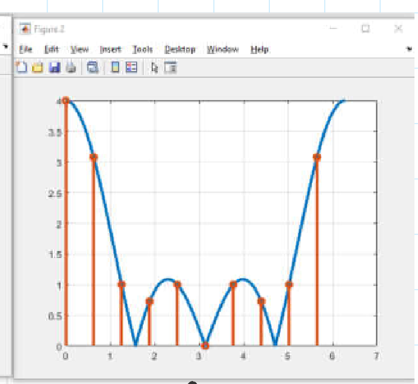
$$* \left\{ e^{j\left(\frac{2\pi}{N}k\right)n} \right\}, \; k = 0, 1, 2, \cdots N-1 \quad \text{are all periodic with period N.}$$

* Zero padding :

DTFT in blue.

DFT in red (stem plot)




$x[n] = 1$ , $n = 0, 1, 2, 3.$
$\uparrow$

$x(n) = 1 \ , \ n = 0,1,2,3.$

zero-padding is useful to get more DFT samples.

4-point DFT

$x(n) = \{1,1,1,1\}$

fft(x,4)

10-point DFT

$x[n] = \{1,1,1,1,0,0,0,0,0,0\}$

fft(x,10)

zero-padding.

## * Properties :

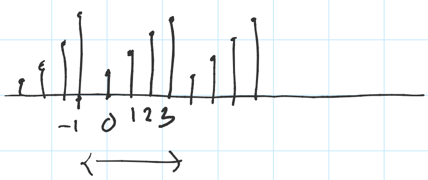
$$x[n] \xleftrightarrow{N\text{-DFT}} X[k]$$

① Linearity.

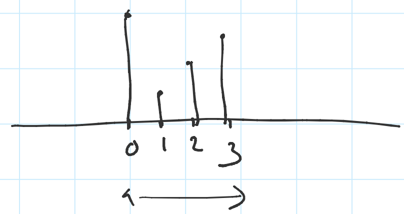
② Time Shift:
$$x[n-n_0] \xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}kn_0}\, X[k].$$

shift of the periodic signal / circular shift

$x[n]$



$x[n-1]$

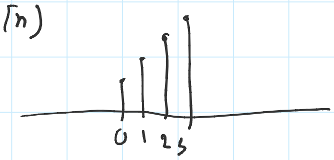


③ Time reversal :
$$x[-n] \xleftrightarrow{N\text{-DFT}} X[-k] = X[N-k]$$

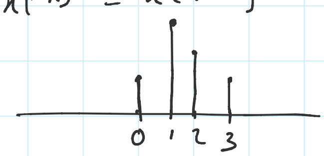
$x[N-n] \rightarrow$ (periodicity) $\qquad \hookrightarrow$ periodicity (N)

N=4

$x[n]$



$y[n] = x[-n] = x[N-n]$



④ Freq. Shift :
$$x[n]\, e^{j\frac{2\pi}{N}nk_0} = X[k-k_0]$$

⑤ Complex conjugate :
$$x^*[n] \longleftrightarrow X^*[-k] = X^*[N-k]$$

real signal : $X^*[N-k] = X[k].$

⑥ Parsevals' Theorem:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

⑦ Convolution:

$$x_1[n] \xleftrightarrow{N} X_1[k]$$
$$x_2[n] \xleftrightarrow{N} X_2[k]$$
N-point DFT

$$x_3[n] = x_1[n] \circledast x_2[n] \xleftrightarrow{N} X_3[k] = X_1[k]\, X_2[k]$$

circular convolution.

by IDFT

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\, X_2[k]\, e^{j\frac{2\pi}{N}kn}$$

$$x_3[n] = \frac{1}{N}\sum_k \left[\left(\sum_m x_1[m] e^{-j\frac{2\pi}{N}km}\right)\left(\sum_l x_2[l]\, e^{-j\frac{2\pi}{N}lk}\right)\right] e^{j\frac{2\pi}{N}kn}$$

$$x_3[n] = \frac{1}{N}\sum_m \sum_l x_1[m]\, x_2[l] \left(\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m-l)}\right)$$

$$\begin{cases} N, & n-m-l = pN \\ 0, & \text{otherwise.} \end{cases}$$

$$x_3[n] = \frac{1}{N}\sum_m\left(\sum_l x_1[m]\, x_2[l]\right)$$

$n-m-l=0$

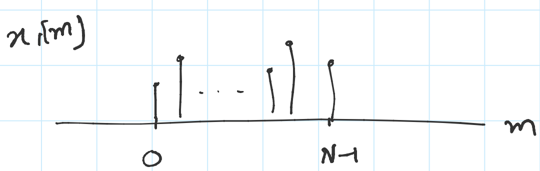
$\Rightarrow l = n-m$

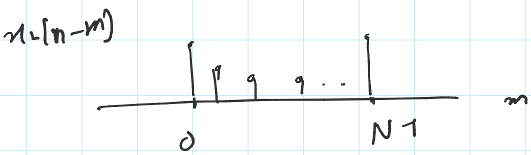
$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]$$

Circular convolution.

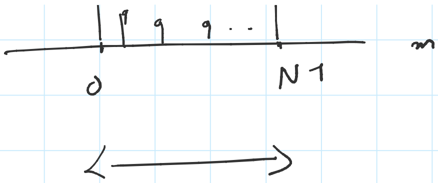
only uses one period (N)

define for some length signals




Ex. $x_1 = \{0, 1, 2, 3\}$




Ex. $x_1 = \{0, 1, 2, 3\}$

$x_2 = \{1, 2, 3, 4\}$

Find $x_1 \circledast x_2$. (HW)

$\Longrightarrow$

⑧ Multiplication property:

both N-length $\quad x_1[n]\, x_2[n] \xleftrightarrow{N} \frac{1}{N}\left(X_1[k] \circledast X_2[k]\right).$

We are interested in linear filtering i.e linear convolution.

$x[n] \qquad h[n]$

How to do linear/regular convolution using circular convolution?

ⓐ Circular convolution:

Ex. $N = 4$ & $\quad y[n] = x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m]$

$y[0] = x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1]$

$y[1] = x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2]$

$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[3]$

$y[3] =$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} h[0] & h[3] & h[2] & h[1] \\ h[1] & h[0] & h[3] & h[2] \\ h[2] & h[1] & h[0] & h[3] \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

↳ Circulant matrix.

(b) Linear convolution:

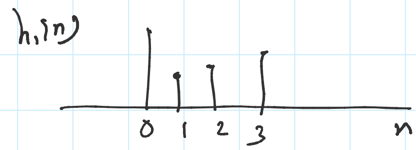
$$y_1[n] = x_1[n] * h_1[n]$$

$$y_1[n] = \sum_{m=0}^{2} x_1[m]\, h_1[n-m]$$

Ex. $x_1[n] = \{x_1[0], x_1[1], x_1[2]\}$

$h_1[n] = \{h_1[0], h_1[1], h_1[2], h_1[3]\}$

Other terms are zero.


$$y_1[0] = x_1[0]\,h_1[0]$$

$$y_1[1] = x_1[0]\,h_1[1] + x_1[1]\,h_1[0]$$

$$y_1[2] = x_1[0]\,h_1[2] + x_1[1]\,h_1[1] + x_1[2]\,h_1[0]$$

$$y_1[3] = x_1[0]\,h_1[3] + x_1[1]\,h_1[2] + x_1[2]\,h_1[1]$$

$$y_1[4] = x_1[1]\,h_1[3] + x_1[2]\,h_1[2]$$

$$y_1[5] = x_1[2]\,h_1[3]$$

putting in a matrix

$$\begin{bmatrix} y_1[0] \\ y_1[1] \\ \vdots \\ y_1[5] \end{bmatrix} = \begin{bmatrix} h_1[0] & 0 & 0 \\ h_1[1] & h_1[0] & 0 \\ h_1[2] & h_1[1] & h_1[0] \\ h_1[3] & h_1[2] & h_1[1] \\ 0 & h_1[3] & h_1[2] \\ 0 & 0 & h_1[3] \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \end{bmatrix}$$

$\hookrightarrow$ partly circulant

$$\begin{bmatrix} y_1[0] \\ y_1[1] \\ \vdots \\ \vdots \\ y_1[5] \end{bmatrix} = \begin{bmatrix} h_1[0] & 0 & 0 & h_1[3] & h_1[2] & h_1[1] \\ h_1[1] & h_1[0] & 0 & 0 & h_1[3] & h_1[2] \\ h_1[2] & h_1[1] & h_1[0] & 0 & 0 & h_1[3] \\ h_1[3] & h_1[2] & h_1[1] & h_1[0] & 0 & 0 \\ 0 & h_1[3] & h_1[2] & h_1[1] & h_1[0] & 0 \\ 0 & 0 & h_1[3] & h_1[2] & h_1[1] & h_1[0] \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Y = \tilde{H}_1 \cdot \tilde{x}_1$$

– circulant matrix

Circular convolution.

$$y_1[n] = \tilde{h}_1[n] \circledast \tilde{x}_1[n]$$

$$\tilde{h}_1[n] = [h_1[0], h_1[1], h_1[2], h_1[3], 0, 0]$$

$$\tilde{x}_1[n] = [x_1[0], x_1[1], x_1[2], 0, 0, 0]$$