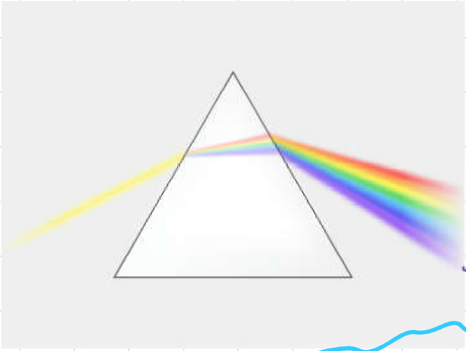
Thursday, August 13, 2020 2:17 PM



This a pictorial representation of Fourier Analysis

Spectrum of the signal.

Fourier Series : $\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$ $\boxed{T = \frac{2\pi}{\omega_0}}$ = period of $x(t)$

Synthesis Equation

How to find $a_k$ ?

Property : $\int_0^T e^{jm\omega_0 t}\, dt = \begin{cases} T, & m = 0 \\ 0, & m \neq 0 \end{cases}$

$$x(t)\, e^{-jn\omega_0 t} = \sum_{-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}$$ for some fixed integer 'n'

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \left( \sum_{-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} \right) dt = \sum_{-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = T a_n \Rightarrow \boxed{a_n = \frac{1}{T} \int_0^T x(t)\, e^{-j\omega_0 n t}\, dt}$$

Analysis Equation

$$\boxed{a_n = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-j\omega_0 n t}\, dt}$$

$a_k$ - Fourier Series coefficients
Spectral coefficients

F.S. is applicable to a broad class of periodic signals.

$$x(t) \xleftrightarrow{FS} a_k \qquad y(t) \xleftrightarrow{FS} b_k$$

Time-domain representation

Frequency-domain representation

**Remarks**

$a_k$ - contribution of $e^{jk\omega_0 t}$ in $x(t)$
└ corresponds to frequency of $k\omega_0$
higher $k \Rightarrow$ higher frequency

periodic signals have a discrete set or quantized frequencies.

periodic $x(t)$ : $t$ is continuous $\Rightarrow$ continuous-time signals

$a_k$ : $k$ is discrete $\Rightarrow$ discrete-frequency signals.

Ex. ① $x(t) = \sin(\omega_0 t)$ . Find $a_k$.

$$= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \Rightarrow a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}, \quad a_k = 0, k \neq \pm 1$$

H.W. $x(t) = \cos(\omega_0 t)$

② $x(t) = 1 + \sin(\omega_0 t + \pi/4) = 1 + \frac{1}{2j}\left(e^{j(\omega_0 t + \frac{\pi}{4})} - e^{-j(\omega_0 t + \frac{\pi}{4})}\right)$

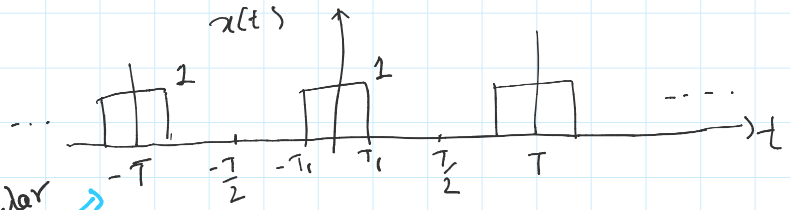
$$= \underset{\uparrow \atop a_0}{1} + \underbrace{\frac{e^{j\pi/4}}{2j}}_{a_1} e^{j\omega_0 t} - \underbrace{\frac{e^{-j\pi/4}}{2j}}_{a_{-1}} e^{-j\omega_0 t}$$

by comparing with synthesis eq.

H.W. $x(t) = 1 + \cos(\omega_0 t + \frac{\pi}{4}) + \sin(2\omega_0 t + \pi/3)$

③ $x(t)$ is periodic $(T)$ & $T_1 < T/2$

duty cycle $= \frac{2T_1}{T}$




$x(t)$ is discontinuous

rectangular pulse

$$a_k = ? = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\, \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{k\omega_0} \quad k \neq 0$$

$$a_k = \begin{cases} \frac{2T_1}{T}, & k = 0 \\ \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} & k \neq 0 \end{cases}$$

sinc function

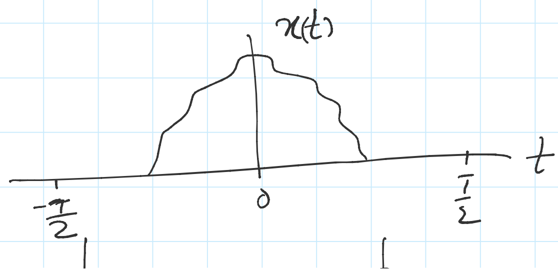
$$\text{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$a_0$ - average value of signal in one period.

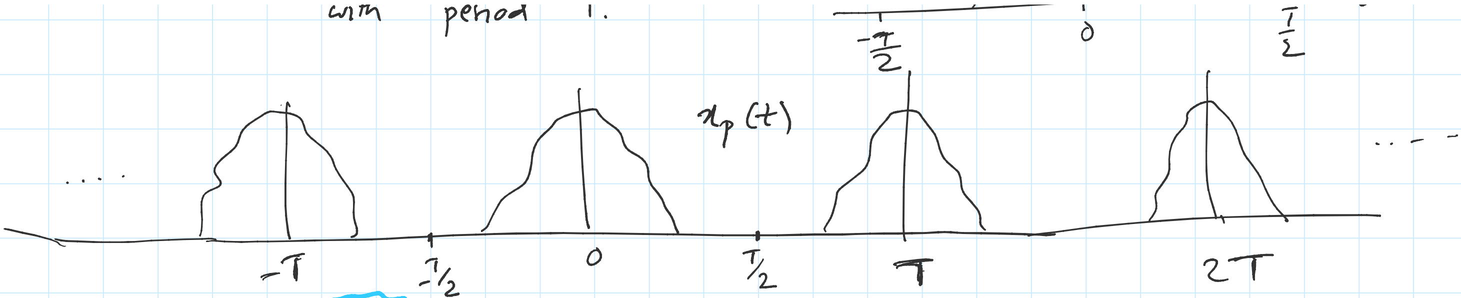
## Fourier Transform

aperiodic $x(t)$

consider a periodic extension of $x(t)$ with period $T$.



with period T.




intuition: do F.S. analysis of $x_p(t)$ as $T \to \infty$ i.e. $\omega_0 \to 0$

$$x_p(t) \xleftrightarrow{FS} a_k$$

F.S. Analysis

$$\hookrightarrow a_k = \frac{1}{T}\int_{\langle T \rangle} x_p(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt = a_k$$

Define: $\quad X(\omega) = \displaystyle\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \omega$ is continuous.

$$\Rightarrow a_k = \frac{1}{T} X(k\omega_0)$$

$a_k$ i.e. $X(k\omega_0)$ is samples from $X(\omega)$ at $\omega = k\omega_0$

$X(\omega)$ is "envelope" of samples $a_k$.

F.S. Synthesis:

$$\hookrightarrow x_p(t) = \sum_{-\infty}^{\infty} a_k\, e^{+jk\omega_0 t} = \sum_{-\infty}^{\infty} \frac{1}{T} X(k\omega_0)\, e^{jk\omega_0 t}$$

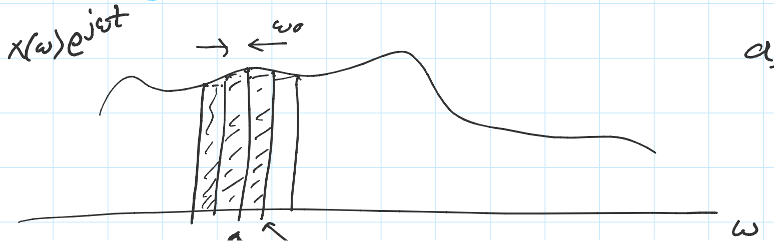
$$T = \frac{2\pi}{\omega_0} \qquad \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$x_p(t) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} X(k\omega_0)\, e^{jk\omega_0 t}\, \omega_0 \qquad (1)$$

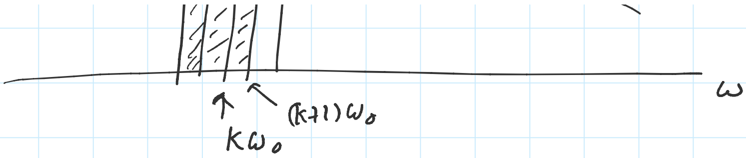
take limit $T \to \infty$ i.e. $\omega_0 \to 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad (2)$$

RHS of (1) is numerical integration of RHS of (2)




as $\omega_0 \to 0$ the sum in (1) converges to integral.




Inverse Fourier Transform: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$ Synthesis Eq.

Fourier Transform: $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$ Analysis Eq.

$$x(t) \overset{FT}{\longleftrightarrow} X(\omega)$$

Time-domain      Freq. domain.

$X(\omega)$ is spectrum of $x(t)$