Lecture - 04
Thursday, August 20, 2020 1:58 PM

last class - Standard examples of signals & their Fourier transform (FT)

Remark: Fourier Transform as special case of Laplace Transform:

☆ Laplace Transform

s – complex variable

$s = \sigma + j\omega$ } $\sigma = 0$ → $s = j\omega$ i.e. Imaginary axis

definition → $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

ROC – region of convergence
where the integral converges → ROC includes $j\omega$-axis.

☆ Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Remark: caution when either time-domain or freq-domain signal has impulses!

Ex
1. $\delta(t) \xleftrightarrow{FT} 1$
2. $\delta(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0}$
3. $e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\,\delta(\omega-\omega_0)$

} Integrals are well defined only in one direction!

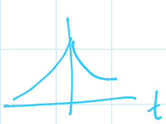
$$x(t) \xleftrightarrow{FT} X(\omega)$$

FT: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ & IFT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

1. $X(\omega) = 1$ & $x(t) = ?\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$

This integral isn't well defined

Ex. $x(t) = e^{-a|t|}$, $\boxed{a > 0}$ find $X(\omega)$

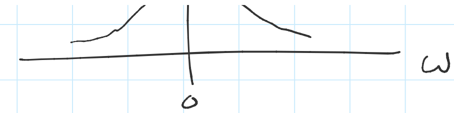


$$\to X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{+at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \left.\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$X(\omega) = \frac{1-0}{a-j\omega} + \frac{0-1}{-(a+j\omega)} = \frac{2a}{a^2+\omega^2}.$$

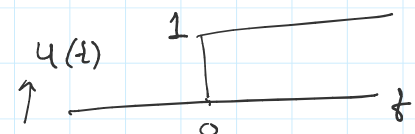
$$X(\omega) = \frac{\;}{a-j\omega} + \frac{\;}{-(a+j\omega)} = \frac{\;}{a^2+\omega^2}.$$



$$\frac{e^{at}\cdot e^{-j\omega t}}{(a-j\omega)}\Bigg|_{-\infty}^{0} = \frac{0-1}{(a-j\omega)}$$

Ex: $x(t) = e^{-at}u(t)$ & $a>0$, find $X(\omega)$.

$$\rightarrow \quad X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2} \quad \text{complex valued.}$$

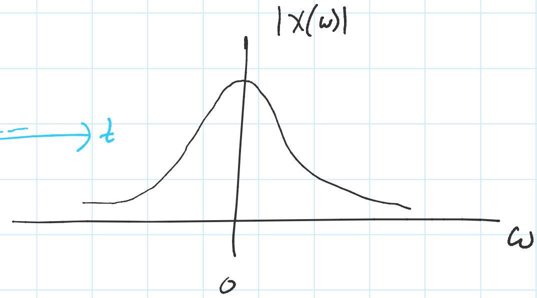


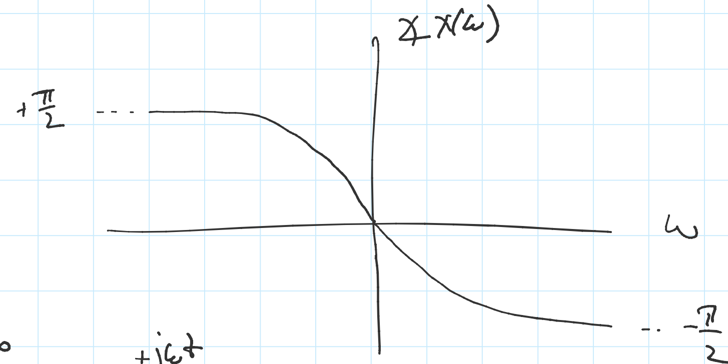
unit-step function

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$$

Sketch: magnitude: $|X(\omega)| = \dfrac{1}{\sqrt{a^2+\omega^2}}$

phase/angle: $\measuredangle X(\omega) = \tan^{-1}\left(\dfrac{-\omega}{a}\right) = -\tan^{-1}\left(\dfrac{\omega}{a}\right)$





★ $x(t) = \underbrace{e^{-at}u(t)}_{\text{real}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{a+j\omega}e^{+j\omega t}\,dt}_{\text{real}}$

★ Properties of Fourier Transform:

① Linearity: $x(t) \xleftrightarrow{FT} X(\omega)$ & $y(t) \xleftrightarrow{FT} Y(\omega)$

$$\Rightarrow \quad \alpha x(t) + \beta y(t) \xleftrightarrow{FT} \alpha X(\omega) + \beta Y(\omega)$$

② Time-Shift: $x(t) \xleftrightarrow{FT} X(\omega)$

} - magnitude is same

$\measuredangle Y(\omega) = \measuredangle X(\omega) - \omega t_0$

② Time-Shift :

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$\Rightarrow \quad x(t-t_0) \xleftrightarrow{FT} X(\omega)\, e^{-j\omega t_0}$$

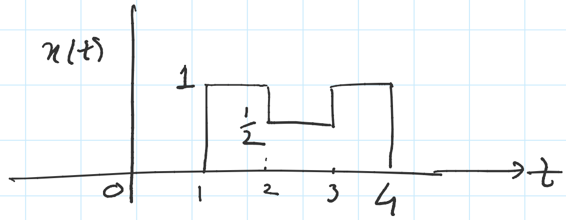
- magnitude is same
- $\measuredangle G(\omega) = \measuredangle X(\omega) - \omega t_0$

$x(t-t_0) \xleftrightarrow{FT}$

$$G(\omega) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(\gamma)\, e^{-j\omega(\gamma+t_0)}\, d\gamma \quad \text{.... put } \gamma = t - t_0$$

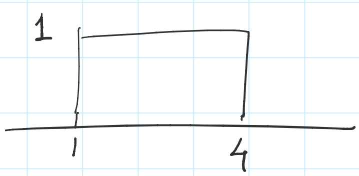
$$G(\omega) = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\gamma)\, e^{-j\omega\gamma}\, d\gamma = X(\omega)\, e^{-j\omega t_0} \quad \text{... proved.}$$

Ex. Find FT of $x(t)$ given as

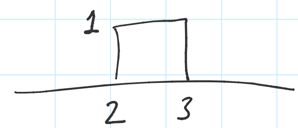


→ use time-shift, linearity & Standard examples to find $X(\omega)$

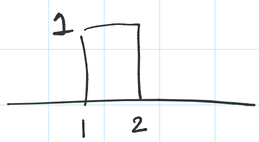
$x(t) =$ ①



$+ \left(-\frac{1}{2}\right)$



$x(t) =$ ②



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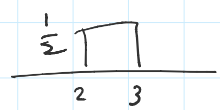

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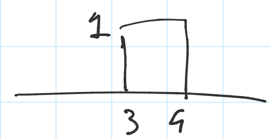

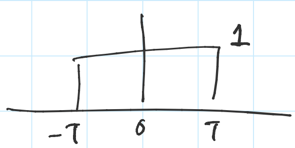

$$\xleftrightarrow{FT} \frac{2 \sin(\omega T)}{\omega}$$

③ Time & Frequency Scaling :

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$x(at) \xleftrightarrow{FT} \; ?\; G(\omega) \qquad a \in \mathbb{R} \quad \text{i.e. real constant}$$

case : $a > 0$

$$G(\omega) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt \qquad \text{put } \gamma = at$$

$$= \int_{-\infty}^{\infty} x(\gamma)\, e^{-j\omega\gamma/a}\, \frac{d\gamma}{a} = \frac{1}{a} \int_{-\infty}^{\infty} x(\gamma)\, e^{-j\frac{\omega}{a}\gamma}\, d\gamma = \frac{1}{a} X\!\left(\frac{\omega}{a}\right)$$

case : $a < 0$

Case : $a<0$

$$G(\omega) = \quad \cdots\cdots \quad = \frac{1}{(-a)} X\left(\frac{\omega}{a}\right)$$

Combining :

$$G(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$\Rightarrow \quad x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$a \equiv \begin{cases} \text{Stretching} & |a|<1 \\ \text{compression} & |a|>1 \\ \text{time-reversal} & a<0 \end{cases}$$

Ex. $x(t) = \sin(\omega_0 t)$

$a=2 \Rightarrow x(2t) = \sin(2\omega_0 t)$

$a=\frac{1}{2} \Rightarrow x(t/2) = \sin\left(\frac{\omega_0 t}{2}\right)$

$a=-1 \Rightarrow x(-t) = \sin(-\omega_0 t)$

④ Symmetry Properties :

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$x(-t) \xleftrightarrow{FT} X(-\omega) \quad \cdots \text{using scaling above.}$$

$x(t)$ is even

$x(-t) = x(t)$

$\Rightarrow X(-\omega) = X(\omega)$

$x(t)$ is odd.

$x(-t) = -x(t)$

$\Rightarrow X(-\omega) = -X(\omega)$

$$x^*(t) \xleftrightarrow{FT} X^*(-\omega) \quad \cdots \text{proof : HW}$$

If $x(t)$ is real : $x^*(t) = x(t)$

$X^*(-\omega) = X(\omega)$

If $x(t)$ is real & even

$X^*(-\omega) = X(\omega) = X(-\omega)$

$\Rightarrow X^*(\omega) = X(\omega)$

& $x(t)$ is real & odd

$X^*(-\omega) = X(\omega) = -X(-\omega)$

$\Rightarrow X^*(-\omega) = -X(-\omega)$

$\Rightarrow X^*(\omega) = X(\omega)$ $\qquad\qquad$ $\Rightarrow X^*(-\omega) = -X(-\omega)$

$\Rightarrow X(\omega)$ is real. $\qquad\qquad$ $\Rightarrow X(\omega)$ is imaginary

⑤ Differentiation (in time) : $x(t) \xleftrightarrow{FT} X(\omega)$

$$\Rightarrow \frac{dx}{dt} \xleftrightarrow{FT} ?$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$\frac{dx}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,\frac{d\left(e^{j\omega t}\right)}{dt}\, d\omega = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} \underbrace{j\omega X(\omega)}\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \frac{dx}{dt} \xleftrightarrow{FT} j\omega X(\omega)$$

HW. Ex. given : $\cos(\omega_0 t) \xleftrightarrow{FT} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

find FT of $\sin(\omega_0 t)$ using

(a) differentiation in time property

(b) time shift property

(c) verify ⓐ & ⓑ give same answer.

⑥ Frequency-Shift : $x(t) \xleftrightarrow{FT} X(\omega)$

$$x(t)\, e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0)$$ - proof HW

⑦ Differentiation (in freq) : $x(t) \longleftrightarrow X(\omega)$

$$(-jt)\, x(t) \xleftrightarrow{FT} \frac{dX}{d\omega}$$

$$t\, x(t) \xleftrightarrow{FT} j\frac{dX}{d\omega}$$

Proof : HW

Ex. Find FT of $x(t)$

$$x(t) = \begin{cases} t, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$