Lecture – 05

Monday, August 24, 2020 1:41 PM

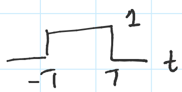
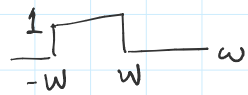
Last class – properties of Fourier Transform

Todays class – continue properties & LTI systems

(8) Duality :

Examples so far ...

Time : $x(t) \xleftrightarrow{FT} X(\omega)$: Frequency

$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$e^{j\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$
	$\dfrac{2\sin(\omega T)}{\omega}$
$\dfrac{\sin(Wt)}{\pi t}$	

In general : $x(t) \xleftrightarrow{FT} X(\omega)$ proof: HW

and $X(t) = g(t) \xleftrightarrow{FT} G(\omega) = 2\pi x(-\omega)$

(9) Parseval's Theorem :

$x(t) \xleftrightarrow{FT} X(\omega)$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

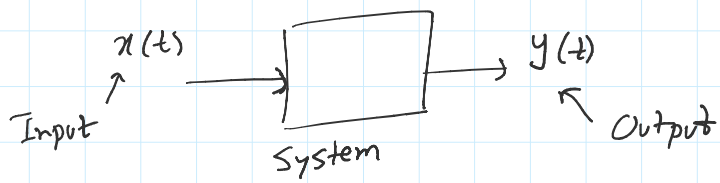
proof: HW

$\int_{-\infty}^{\infty} x(t)\,x^*(t)\,dt \rightarrow$ use IFT for $x^*(t)$ & simplify.

general : $x(t) \xleftrightarrow{FT} X(\omega)$ & $y(t) \xleftrightarrow{FT} Y(\omega)$

$$\int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,Y^*(\omega)\,d\omega$$

✩ LTI Systems (Linear & Time-invariant system)



Linearity: $x_1(t) \longrightarrow y_1(t)$ & $x_2(t) \longrightarrow y_2(t)$

$$\Rightarrow \alpha x_1(t) + \beta x_2(t) \longrightarrow \alpha y_1(t) + \beta y_2(t) \quad \forall\, \alpha, \beta \in \mathbb{C}$$

Time-Invariance: $x(t) \longrightarrow y(t)$

$$\Rightarrow x(t-t_0) \longrightarrow y(t-t_0) \qquad t_0 \in \mathbb{R}$$

Ex. all ckts made using R, L, C components

LTI systems can provide good approximation of many real-life systems

Important property: Any LTI system can be fully described/characterized using its impulse response

$$\delta(t) \longrightarrow h(t)$$
$\hookrightarrow$ impulse response

If any arbitrary input $x(t)$ is given to an LTI system with impulse response $h(t)$, the output is

Convolution: $$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Frequency Analysis of LTI systems

(A) LTI systems & complex sinusoid input

$x(t)$ [$h(t)$] $\longrightarrow y(t) = x(t) * h(t)$

(A)

$$x(t) \longrightarrow \underset{LTI}{\boxed{h(t)}} \longrightarrow y(t) = x(t) * h(t)$$

let $x(t) = e^{j\omega_0 t} \Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega_0 \tau}\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{j\omega_0 (t-r)}\, h(r)\, dr \qquad \ldots\ t-\tau = r,\quad d\tau = -dr$$

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega_0 r}\, h(r)\, dr = e^{j\omega_0 t} \int_{-\infty}^{\infty} h(r)\, e^{-j\omega_0 r}\, dr$$

$$\Rightarrow \quad y(t) = H(\omega_0)\, e^{j\omega_0 t}$$

$$e^{j\omega_0 t} \longrightarrow H(\omega_0)\, e^{j\omega_0 t} \qquad \text{for any } \omega_0$$

Eigen-functions

$$h(t) \xleftrightarrow{FT} H(\omega)$$

impulse response Frequency Response

$$x(t) \longrightarrow \underset{LTI}{\boxed{H(\omega)}} \longrightarrow y(t)$$

(B) LTI system with input as any periodic signal

Input : $x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$ I/P is periodic

Output : $y(t) = \sum_{-\infty}^{\infty} a_k H(k\omega_0)\, e^{jk\omega_0 t}$ O/P is periodic

$$x(t) \xleftrightarrow{FS} a_k \Rightarrow y(t) \xleftrightarrow{FS} a_k H(k\omega_0)$$

(c) general aperiodic input signal.
what is freq. domain characterization of o/p.

(10) Convolution Property :

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$h(t) \xleftrightarrow{FT} H(\omega)$$

Then $y(t) \xleftrightarrow{FT} Y(\omega) = \; ? \; = X(\omega)H(\omega)$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t}\, dt \right] d\tau = \int_{-\infty}^{\infty} x(\tau)\, H(\omega) e^{-j\omega \tau}\, d\tau$$

$$Y(\omega) = H(\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau = H(\omega)\, X(\omega)$$

$$Y(\omega) = H(\omega)\, X(\omega)$$

$$x(t) * h(t) \xleftrightarrow{FT} X(\omega) H(\omega)$$

$$x(t) \longrightarrow y(t)$$

$$X(\omega) \longrightarrow H(\omega) X(\omega)$$

Remark: LTI systems do NOT add any new frequencies.

using convolution property to find convolution (in time)

To Find: $x(t) * h(t) = y(t)$

① Find $X(\omega)$ & $H(\omega)$

② multiply $Y(\omega) = X(\omega)\, H(\omega)$

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③ inverse FT : $y(t) \xleftrightarrow{FT} Y(\omega)$

Ex. $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$ $h(t) = \delta(t-t_0)$

find $y(t)$

① direct convolution

$$y(t) = x(t) * \delta(t-t_0)$$

$$y(t) = x(t-t_0)$$

Shift / Delay system

② Fourier Transform

$$x(t) \longleftrightarrow X(\omega)$$

$$\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}$$

$$Y(\omega) = X(\omega)\,e^{-j\omega t_0}$$

$$\Rightarrow y(t) = x(t-t_0)$$

Ex. derivative system:

$x(t) \rightarrow \boxed{} \rightarrow \dfrac{dx}{dt}$

LTI.

find : $H(\omega)$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$\frac{dx}{dt} \xleftrightarrow{FT} j\omega\,X(\omega)$$

$$Y(\omega) = j\omega\,X(\omega) = H(\omega)\,X(\omega) \Rightarrow H(\omega) = j\omega$$

Ex. $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = ?$

$h(t) = e^{-at}u(t)$ $a, b > 0$ & $a \neq b$

$x(t) = e^{-bt}u(t)$, find $y(t)$

$x(t) = e^{-bt} u(t)$, find $y(t)$

$\rightarrow$ $e^{-at} u(t) \xleftrightarrow{FT} \dfrac{1}{a+j\omega}$

$e^{-bt} u(t) \longleftrightarrow \dfrac{1}{b+j\omega}$

$$Y(\omega) = \frac{1}{(a+j\omega)(b+j\omega)} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega}$$ find A, B

$$y(t) = A e^{-at} u(t) + B e^{-bt} u(t)$$