Lecture - 18

Monday, October 12, 2020 2:03 PM

Last class: * properties of Z-Transform

* LTI systems & Z-Transform

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

LTI

$y[n] = x[n] * h[n]$

$Y(Z) = X(Z)\,H(Z)$ ---- convolution property

Transfer function

* Equivalent LTI system representation
  - * $h[n]$, * $H(e^{j\omega})$, * $H(Z)$ [+ ROC]
  - * pole-zero plot [+ ROC]

* System analysis / properties from pole-zero plot (& ROC)

  (A) $H(z)$ from pole-zero plot

  (B) $|H(e^{j\omega})|$ sketch from pole-zero plot (geometric intuition)

Today's class:

(C) When is the system causal?

$\rightarrow$ For causality: $h[n] = 0$ for $n < 0$

$y[n] = x[n] * h[n]$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\,x[n-m]$$

for a causal system

$y[n]$ depends on $x[m]$, $m \le n$

$\Rightarrow h[m] = 0$ for $m < 0$

$\Rightarrow$ Impulse response is right-sided

$$\Rightarrow H(Z) = \sum_{0}^{\infty} h[n]\,z^{-n}$$

$$= h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots$$

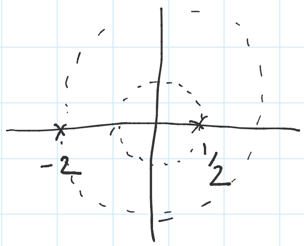
only negative powers of z.

$\Rightarrow$ ROC will be outside of some circle.

$\therefore$ For $H(z) = \dfrac{N(z)}{D(z)}$ (of this form)

* ROC of a causal system is outside the outermost pole and includes ∞ in the ROC.

Ex.



\# distinct systems = 3

for causal system: ROC will be $|z| > 2$.

(D) when is the system stable?

Bounded Input Bounded Output (BIBO) stability.

An LTI system is BIBO stable if and only if

$$\sum_{-\infty}^{\infty} |h[n]| < \infty$$

consider: $H(z) = \sum_{-\infty}^{\infty} h[n] z^{-n} \Rightarrow |H(z)| = \left| \sum_{-\infty}^{\infty} h[n] z^{-n} \right|$

$$\Rightarrow |H(z)| \leq \sum_{-\infty}^{\infty} |h[n] z^{-n}| = \sum_{-\infty}^{\infty} |h[n]| |z^{-n}|$$

for $|z| = 1$

$$\Rightarrow |H(z)| \leq \sum_{-\infty}^{\infty} |h[n]| \quad \text{for } |z| = 1$$

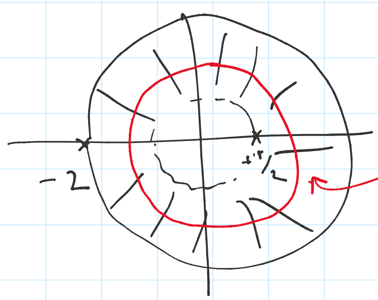
If the system is given to be BIBO stable

$\Rightarrow |H(z)| < \infty$ for $|z| = 1$. i.e. convergence.

$\Rightarrow$ Unit circle $(|z| = 1)$ must be part of ROC

\# An LTI system is BIBO stable if and only if the ROC of $H(z)$ includes the unit circle. \#

Ex.



$|z| = 1$

For stable system choose $\underbrace{\frac{1}{2} < |z| < 2}_{\text{ROC}}$

Ex. Classify following LTI systems as causal and/or Stable based on ROC:

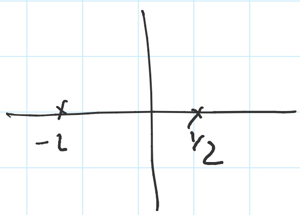
(a) $h[n] = \delta[n-1]$ (b) $h[n] = \delta[n+1]$ (c) $h[n] = 2^n u[n]$

(d) $2^n u[-n-1]$ (e) $\left(\frac{1}{2}\right)^n u[n]$

(E) when is a system both causal & Stable ?

☆ A causal LTI system is stable if and only if all the poles of $H(z)$ are inside the unit circle

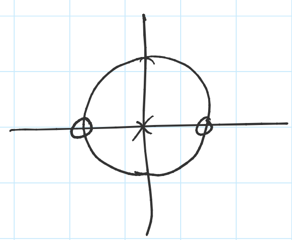
Ex.

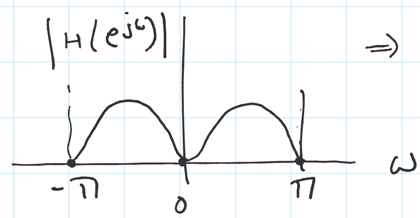


cannot be both causal & stable at same time.

☆ For given pole-zero plot, there can be at most one causal system & at most one stable system ☆

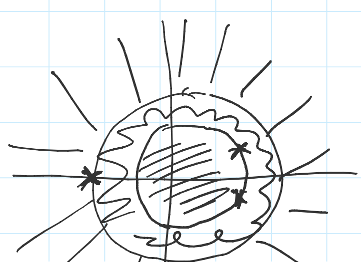
Answer of poll:





$\Rightarrow$ BPF

Ex.

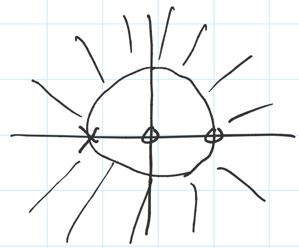


$$H(z) = \frac{A}{(z+1)(z^2 + az + b)}$$

It will have a causal system.

(z+1)(...) causal system.

Ex.



$$H(z) = \frac{A\, z(z-1)}{(z+1)}$$

RoC

① $|z| < 1$

② $1 < |z| < \infty$

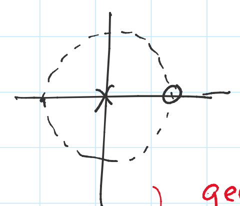
* LTI systems characterized by linear constant-coefficient difference equation *

Ex. $y[n] = x[n] - x[n-1]$

$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$

$h[n] = \delta[n] - \delta[n-1] \quad \Rightarrow y[n] = x[n] * h[n]$

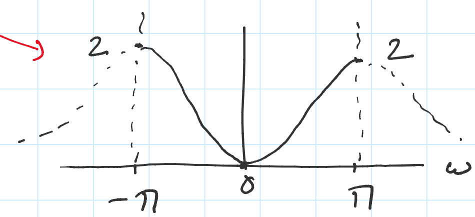
$H(z) = 1 - z^{-1} = \frac{z-1}{z} = \frac{N(z)}{D(z)}$ &



& RoC: $|z| > 0$

geometric intuition

Sketch $|H(e^{j\omega})|$



(high pass filter)

Equivalent representations $\begin{cases} h[n], \quad H(z), \quad H(e^{j\omega}) \\ \text{pole-zero plot (+RoC)}, \quad \text{difference-equation} \end{cases}$

Ex. $y[n] - \frac{1}{2} y[n-1] = x[n] + \frac{1}{3} x[n-1]$

$h[n]$ is not directly obvious

Take z-Transform on both sides:

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z) + \frac{1}{3} z^{-1} X(z)$$

$$Y(z)\left[1 - \tfrac{1}{2} z^{-1}\right] = X(z)\left[1 + \tfrac{1}{3} z^{-1}\right]$$

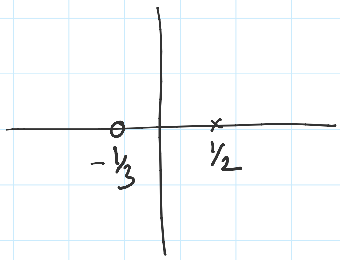
we know:

$Y(z) = H(z)\, X(z)$

$H(z) = \frac{Y(z)}{\ldots}$

$$Y(z)\left[1-\frac{1}{2}z^{-1}\right] = X(z)\left[1+\frac{1}{3}z^{-1}\right] \quad ; \quad H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{z+1/3}{z-1/2} = \frac{N(z)}{D(z)}$$



Two possible ROC : (a) $|z| < \frac{1}{2}$

(b) $|z| > 1/2$

Find the causal impulse response / system.

$\Rightarrow$ $|z| > 1/2$ should be chosen.

$$\Rightarrow \quad H(z) = \frac{z}{z-1/2} + \frac{1}{3}\frac{1}{z-1/2} \quad \& \quad |z| > 1/2$$

$$\Rightarrow \quad h[n] = \left(\frac{1}{2}\right)^{n} u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1] \quad \cdots \text{causal.}$$

Stability ? $\rightarrow$ yes, it is also stable.

In general, consider the difference equation (linear & constant-coefficients)

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{l=0}^{M} b_l\, x[n-l]$$

Taking z-Transform :

$$\sum_{k=0}^{N} a_k\, Y(z)\, z^{-k} = \sum_{l=0}^{M} b_l\, X(z)\, z^{-l}$$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}} = H(z)$$

Transfer function

$\hookrightarrow$ can convert to form $\frac{N(z)}{D(z)}$ ☆

only difference equation has no ROC information.

$D(z)$

only difference equation has no ROC information.

Typically assume system to be causal and/or stable.

From prev. examples:

$$\left.\begin{array}{l} ① \ y[n] = x[n] - x[n-1] \\ ② \ y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{1}{2}y[n-1] \end{array}\right\} \text{easy to implement}$$

whereas $y[n] = x[n] * h[n]$ ... may not be easy to implement.

$$\left.\{a_k\}_{k=0,1,\ldots N} \ \& \ \{b_l\}_{l=0,1,2,\ldots M}\right\} \text{finite parameters.}$$

① $h[n] = \delta[n] - \delta[n-1]$ finite impulse response (FIR)

② $h[n] = (\frac{1}{2})^n u[n] + \frac{1}{3}(\frac{1}{2})^{n-1} u[n-1]$ infinite impulse response (IIR)