PH354 HW3

Q1) a)

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \times \frac{-b \mp \sqrt{b^2-4ac}}{-b \mp \sqrt{b^2-4ac}}$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{(2a)\left[\left(-b \mp \sqrt{b^2-4ac}\right)\right]}$$

$$= \frac{b^2 - b^2 + 4ac}{(2a)\left(-b \mp \sqrt{b^2-4ac}\right)}$$

$$= \frac{2c}{-b \mp \sqrt{b^2-4ac}}$$

Q4) (a) $\frac{dx}{dt} = -x + ay + x^2y$ $\qquad$ $\frac{dy}{dt} = b - ay - x^2y$

$\frac{dx}{dt} = 0 \implies -x+ay+x^2y = 0$

$\frac{dy}{dt} = 0 \implies b - ay - x^2y = 0$

$(-x) + (b) = 0$

$\implies x = b$

and

$y(a+x^2) = x$

$\implies y = \frac{b}{a+b^2}$

(b) as $x = b$

$y = \frac{b}{a+b^2}$

$= \frac{b}{a+x^2}$

$\implies y = \frac{b}{a+x^2}$

$b = y(a+x^2) \quad (\because x = b)$

$\implies x = y(a+x^2)$

$\therefore x = y(a+x^2)$

$y = \frac{b}{a+x^2}$

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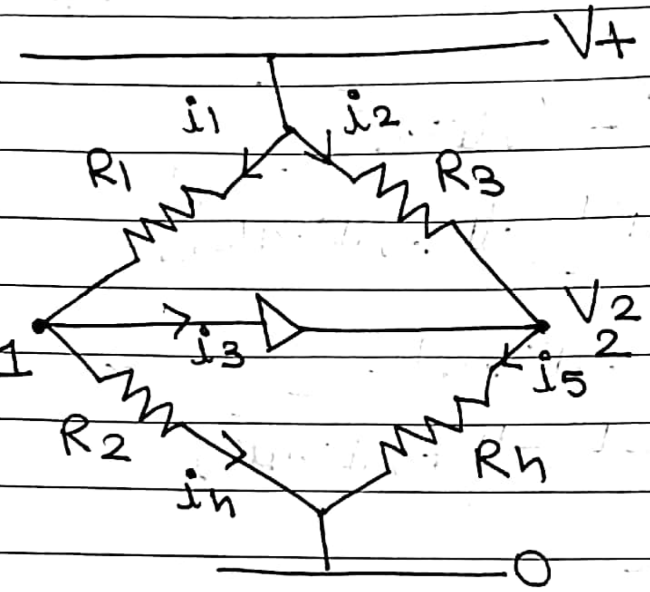
(c) modified rearrangement:

$$x = y(a+x^2)$$

$$y = \frac{x}{a+x^2}$$

Q9)



let $V_1 - V_2 = x$

$$i_3 = I_0\left(e^{x/V_T} - 1\right)$$

at pt 1:

$$\left(\frac{V_+ - V_1}{R_1}\right) = i_3 + \left(\frac{V_1 - 0}{R_2}\right)$$

at pt 2:

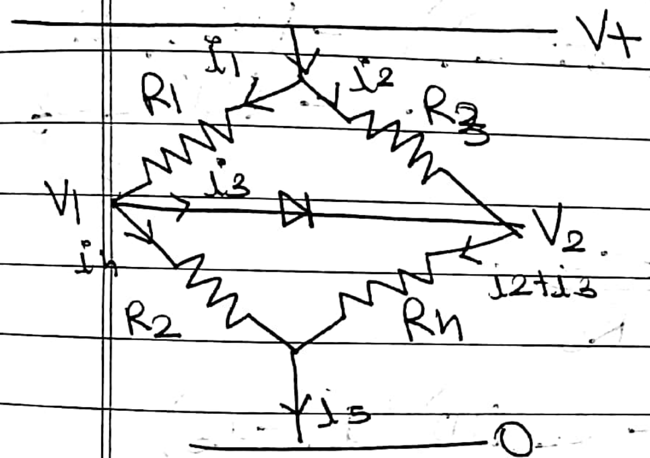
$$\frac{V_+ - V_2}{R_3} + i_3 = \frac{V_2 - 0}{R_4}$$

$$\cancel{\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}$$

$\rightarrow$ ~~$a_{nn} x_n = b_n$, $x_n = b_n / a_{nn}$, $a_{n-1,n-1} x_{n-1} + a_{n-1,n} x_n = b_{n-1}$~~

$\cancel{i_3}$: $V_1 = \dfrac{\dfrac{V_+}{R_1} - o\, i_3}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$

$$\frac{V_+}{R_1} = \left(I_0 e^{x/V_T} - 1\right) + V_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



$\forall$ at 1: $\left(\dfrac{V_+ - V_1}{R_1}\right) \cancel{+ i_3} = I_0\left(e^{V/V_T} - 1\right) + \dfrac{(V_1 - 0)}{R_2}$

$$\left(\frac{V_+ - V_2}{R_3}\right) + I_0\left(e^{V/V_T} - 1\right) = \frac{V_2}{R_4}$$

$V = V_1 - V_2 = x$

$V_2 = V_1 - V = (V_1 - x)$

$$\frac{V_+}{R_3} = V_2\left(\frac{1}{R_3} + \frac{1}{R_4}\right) - i_3$$

$$\frac{V_+}{R_3} = (V_1 - x)\left(\frac{1}{R_3} + \frac{1}{R_4}\right) - i_3$$

$V_+ - V_1$

$$\frac{V_+}{R_3} = \left[\frac{\dfrac{V_+}{R_1} - i_3}{\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)} - x\right]\left(\frac{1}{R_3} + \frac{1}{R_4}\right) - i_3$$

So we have

$$\frac{V_+}{R_3} = \left(\frac{\frac{V_+ - R_1 i_3}{R_1}}{\frac{R_1+R_2}{R_1R_2}} - x \right)\left(\frac{1}{R_3}+\frac{1}{R_4}\right) - i_3$$

$$\frac{V_+}{\cancel{R_3}} = \left(\frac{R_2(V_+ - R_1 i_3)}{R_1+R_2} - x\right)\frac{(R_3+R_4)}{R_3R_4} - i_3 \frac{R_3R_4}{\cancel{R_3}R_4}$$

$$\Rightarrow (R_4)(V_+) = \frac{\left(R_2(V_+ - R_1 i_3) - x(R_1+R_2)\right)(R_3+R_4) - i_3(R_3R_4)(R_1+R_2)}{(R_1+R_2)}$$

$$\cancel{(R_1+R_2)(R_4)(V_+)} = R_2(R_3+R_4)(V_+) - x(R_1+R_2)(R_3+R_4) - i_3\left[R_1R_2(R_3+R_4) + R_3R_4(R_1+R_2)\right]$$

$$\therefore f(x) = V_+\left[R_2(R_3+R_4) - R_4(R_1+R_2)\right] - x\left[(R_1+R_2)(R_3+R_4)\right] - i_3(x)\left[R_1R_2(R_3+R_4)+R_3R_4(R_1+R_2)\right]$$

$$= V_+\left[R_2R_3 - R_4R_1\right] - x\left[(R_1+R_2)(R_3+R_4)\right] - I_0\left(e^{x/V_T}-1\right)\left[R_1R_2(R_3+R_4)+R_3R_4(R_1+R_2)\right]$$

$$f'(x) = -1\left[(R_1+R_2)(R_3+R_4)\right] - i_3'(x)\left[R_1R_2(R_3+R_4) + R_3R_4(R_1+R_2)\right]$$

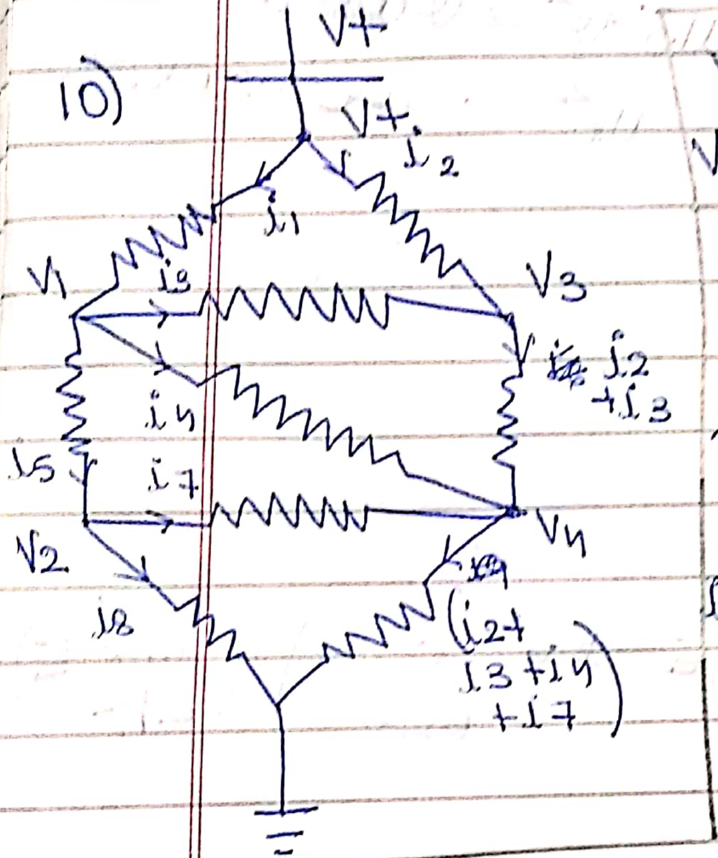
$$\therefore \text{if } \quad f(x) = \alpha V_+ + \beta x + \gamma i_3(x)$$

$$f'(x) = \beta + \gamma i_3'(x)$$

from x^*, V_1, V_2 values can be obtained

(3110)

10)



$$V^+ - i_1(R) = V_1$$
$$V^+ - i_2(R) = V_3$$

$$i_1 = \cancel{i_3 + i_4 + i_5}$$
$$\cancel{i_6 = i_2 + i_3}$$
$$\cancel{i_9 = i_6 + i_4 + i_7}$$
$$i_5 = i_7 + i_8$$
$$\cancel{i_3 = \left(\frac{V_1 - V_3}{R}\right)}$$
$$\cancel{i_4 = \left(\frac{V_1 - V_4}{R}\right)}$$

$$V^+ \cancel{- i_1 R - i_3 R + i_2 R = 0}$$
$$\cancel{(i_1 + i_3) - i_2 R = \frac{V^+}{R}}$$

$$i_3 R = (V_1 - V_3)$$
$$(i_2 - i_1)\cancel{R} = i_3 \cancel{R}$$

$$\left(\frac{V_1 - V_3}{R}\right) + \left(\frac{V_3 - V_4}{R}\right) - \left(\frac{V_1 - V_4}{R}\right) = 0$$

$$i_1 = i_3 + i_4 + i_5$$
$$\Rightarrow \left(\frac{V^+ - V_1}{\cancel{R}}\right) = \left(\frac{V_1 - V_3}{\cancel{R}}\right) + \left(\frac{V_1 - V_4}{\cancel{R}}\right) + \left(\frac{V_1 - V_2}{\cancel{R}}\right)$$

$$4V_1 - V_2 - V_3 - V_4 \cancel{- V^+}$$

$$\frac{V_+ - V_3}{\cancel{R}} + \frac{V_1 - V_3}{\cancel{R}} = \frac{V_3 - V_4}{\cancel{R}}$$

$$3V_3 - V_1 - V_4 = V_+$$

$$\frac{V_1 - V_2}{\cancel{R}} = \frac{V_2 - V_4}{\cancel{R}} + \frac{V_2 - 0}{\cancel{R}}$$

$$3V_2 - V_4 - V_1 = 0$$

$$\left(\frac{V_3 - V_4}{\cancel{R}}\right) + \left(\frac{V_1 - V_4}{\cancel{R}}\right) + \left(\frac{V_2 - V_4}{\cancel{R}}\right) = \frac{V_4}{\cancel{R}}$$

$$4V_4 - V_1 - V_2 - V_3 = 0$$

$$\therefore \quad 4V_1 - V_2 - V_3 - V_4 = V_+$$

$$-1.V_1 + 3V_2 + 0V_3 + -1.V_4 = 0$$

$$-1.V_1 + 0.V_2 + 3.V_3 - 1.V_4 = V_+$$

$$-1.V_1 + -1.V_2 + -1.V_3 + 4V_4 = 0$$

$$\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_+ \\ 0 \\ V_+ \\ 0 \end{pmatrix}$$

~~Gaussian elimin.~~

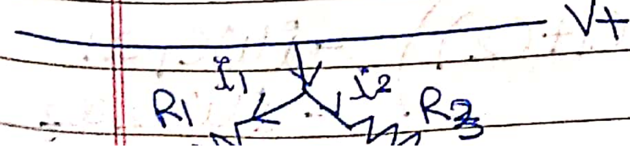
$$\cancel{a_{nn} x_n = b_n} \qquad \cancel{x_n = b_n / a_{nn}} \qquad \cancel{a_{n-1,n-1} x_{n-1} + a_{n-1,n} x_n = b_{n-1}}$$

$$\cancel{\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}}$$

$$V_1 = \frac{\frac{V_+}{R_1} + 0\, i_3}{\frac{1}{R_1} + \frac{1}{R_2}}$$

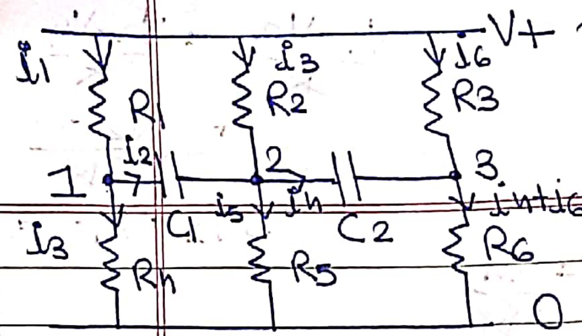
i_3

$$\frac{V_+}{R_1} = \left(I_o e^{x/V_T} - 1\right) + V_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



$$V_+ \text{ at } 1: \left(\frac{V_+ - V_1}{R_1}\right) - i_3 = I_o\left(e^{V/V_T} - 1\right)$$

Q3



$V_+ \sim x_+ e^{i\omega t}$

$V_1 = x_1 e^{i\omega t}$

$V_2 = x_2 e^{i\omega t}$

$V_3 = x_3 e^{i\omega t}$

$$\frac{V_+ - V_1}{R_1} = \cancel{V_2 - V_1}\; i_2 + \frac{V_1 - 0}{R_4}$$

$$i_2 = C.\frac{d}{dt}(V_2 - V_1) = C.(i\omega).(V_2 - V_1)$$
$$= C(i\omega)(x_2 - x_1)e^{i\omega t}$$

$$\frac{x_+ - x_1}{R_1} = C_1(i\omega)(x_2 - x_1) + \frac{x_1}{R_4}$$

$$\frac{x_+}{R_1} = x_1\left(\frac{1}{R_1} + \frac{1}{R_4} + C_1\omega i\right) - i\omega C_1 x_2$$

at 2: $i_2 = C_1(i\omega)(x_1 - x_2)e^{i\omega t}$

$i_4 = C_2(i\omega)(x_2 - x_3)e^{i\omega t}$

$$\frac{V_+ - V_2}{R_2} + i_2 = i_4 + \left(\frac{V_2 - 0}{R_5}\right)$$

$$\frac{x_+}{R_2} = x_2\left(\frac{1}{R_2} + \frac{1}{R_5}\right) - C_1(i\omega)(x_1 - x_2) + C_2(i\omega)(x_2 - x_3)$$

$$= x_2\left(\frac{1}{R_2} + \frac{1}{R_5} + i\omega(C_1 + C_2)\right) - i\omega C_1 x_1 - i\omega C_2 x_3$$

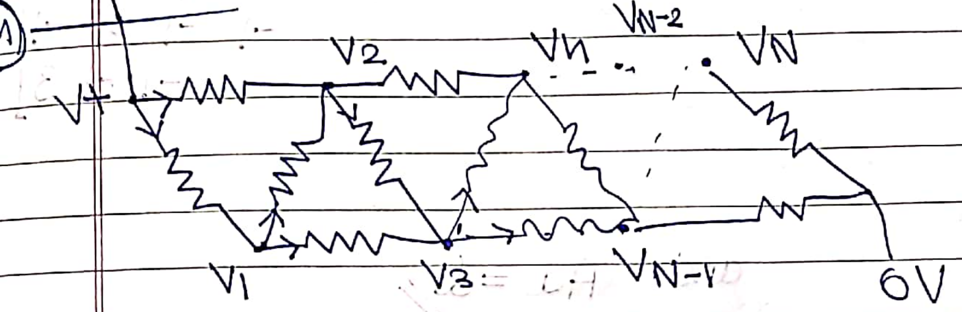
at 3: $\cancel{V_3}\; \frac{V_+ - V_3}{R_3} + i_4 = \frac{V_3 - 0}{R_6}$

$$\frac{x_+}{R_3} = x_3\left(\frac{1}{R_3} + \frac{1}{R_6}\right) - C_2(i\omega)(x_2 - x_3)$$

$$= x_3\left(\frac{1}{R_3} + \frac{1}{R_6} + i\omega C_2\right) - i\omega C_2 x_2$$

$$\begin{pmatrix} \left(\frac{1}{R_1}+\frac{1}{R_4}+i\omega C_1\right) & (-i\omega C_1) & 0 \\ (-i\omega C_1) & \left(\frac{1}{R_2}+\frac{1}{R_5}+i\omega C_1 + i\omega C_2\right) & -i\omega C_2 \\ 0 & -i\omega C_2 & \left(\frac{1}{R_3}+\frac{1}{R_6}+i\omega C_2\right)\end{pmatrix}\begin{pmatrix}x_1^+\\x_2^+\\x_3^+\end{pmatrix} = \begin{pmatrix}\frac{1}{R_1}\\ \frac{1}{R_2}\\ \frac{1}{R_3}\end{pmatrix} x^+$$

(14) V_+



$\cancel{(V_+ - V_1)}$ $(V^+ - V_1) = \cancel{V_2}(V_1 - V_2) + (V_1 - V_3)$

$(V_+ - V_2) + (V_1 - V_2) = (V_2 - V_4) + (V_2 - V_3)$

$3V_1 - V_2 - V_3 = V_+$

$V_+ = 4V_2 - V_1 - V_3 - V_4$

V_3: $(V_2 - V_3) + (V_1 - V_3) = (V_3 - V_4) + (V_3 - V_5)$

$-V_1 - V_2 + 4V_3 - V_4 - V_5 = 0$

V_4: $(V_2 - V_4) + (V_3 - V_4) = (V_4 - V_6) + (V_4 - V_5)$

$- V_2 - V_3 + 4V_4 - V_5 - V_6 = 0$

$- V_{N-3} - V_{N-2} + 4V_{N-1} - V_N - V_0 = 0$

V_N: $(V_{N-2} - V_N) + (V_{N-1} - V_N) = (V_N - 0)$

$-V_{N-2} + 3V_N \cancel{- V_N} - V_{N-1} = \cancel{0}$

$$A \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} V_+ \\ V_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ & -1 & -1 & 4 & -1 & -1 \\ & & -1 & -1 & 4 & -1 \\ & & & \cdots & -1 & -1 & 4 & -1 & -1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 3 & -1 & -1 & & & & & & \\ -1 & 4 & -1 & -1 & & & & & \\ -1 & -1 & 4 & -1 & -1 & & & & \\ & -1 & -1 & 4 & -1 & -1 & & & \\ & & \cdots & -1 & 4 & -1 & \cdots & & \\ & & & & & \ddots & & & \\ & & & & -1 & -1 & 4 & -1 & -1 \\ & & & & & -1 & -1 & 4 & -1 \\ & & & & & & -1 & -1 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \quad \text{and} \quad Ax = b$$

$$b = \begin{bmatrix} V+ \\ V+ \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Q(6) $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\left(\frac{n\pi x}{L}\right)$$

gn $\int_0^L \sin\frac{m\pi x}{L} \cdot \sin\frac{n\pi x}{L}\, dx = \begin{cases} \frac{L}{2} & ; m=n \\ 0 & , \text{otherwise} \end{cases}$

Now: $\hat{H}\psi(x) = E\psi(x)$ ~~considering LHS~~

$$\hat{H}\psi(x) = \hat{H}\left(\sum_{n=1}^{\infty} \psi_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty} \psi_n \, \hat{H}\left(\sin\frac{n\pi x}{L}\right)$$

Taking inner prod. wrt ψ_m:

Now $\langle \psi_m | \hat{H} | \psi \rangle = \int \sum_{n=1}^{\infty} \psi_n \left(\sin\frac{m\pi x}{L}\right) \hat{H}\left(\sin\frac{n\pi x}{L}\right)$

$$= \sum_{n=1}^{\infty} \psi_n \int \left(\sin\frac{m\pi x}{L}\right)\left(\frac{-\hbar^2}{2m}\right)\left(\frac{n\pi}{L}\right)^2 \cdot - \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi^2}{L^2}\right) \int_0^L \sin\left(\frac{m\pi x}{L}\right) \left(\sum_{n=0}^{\infty} \sin\frac{n\pi x}{L} \cdot n^2 \psi_n\right) dx$$

$$= \left(\frac{\hbar^2}{2m} \frac{\pi^2}{L^2}\right)\left(\frac{L}{2}\right) m^2 \psi_m$$

$$= \left(\frac{L}{2}\right)(E_m)\psi_m$$

$$E_m = \frac{\hbar^2}{2M}\frac{\pi^2 m^2}{L^2}$$

~~Thus, eq~~ $\therefore$ schrodinger eqn

$\Rightarrow$ the required condn to be proven

Thus, upon defining Hmn as a matrix element, we can say:

$$\hat{H}\psi_1 = \frac{1}{2} L.E_1\psi_1$$

$$\hat{H}\psi_2 = \frac{1}{2} L.E_2\psi_2$$

$$\vdots$$

taking $\psi_1, \psi_2, \ldots$ to be columns, above can be summed up as:

$$\hat{H}\left[\psi_1 \middle| \psi_2 \middle| \ldots \psi_n\right] = E\left[\psi_1 \middle| \psi_2 \middle| \ldots \middle| \psi_n\right]$$

(ii) $H_{mn} = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\sin\left(\frac{n\pi x}{L}\right) dx$

$\left(V(x) = \frac{ax}{L}\right)$ $= \frac{2}{L}\left\{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\left(\frac{\hbar^2}{2m}\right)\cdot\left(\frac{n\pi}{L}\right)^2 \times -\sin\left(\frac{n\pi x}{L}\right) dx\right.$

$\left. + \int_0^L \sin\left(\frac{m\pi x}{L}\right)\cdot\left(\frac{ax}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx\right\}$

$= \frac{2}{L}\left[\left(\frac{\hbar^2}{2m}\right)\left(\frac{n\pi}{L}\right)^2 \underbrace{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx}_{\frac{L}{2},\ m=n;\ =0\ \text{otherwise}}\right.$

$\left. + \left(\frac{a}{L}\right)\int_0^L x\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx\right]$

$$\int_0^L x\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m\neq n,\ \text{both even or both odd} \\ -\left(\frac{2L}{\pi}\right)^2 \frac{mn}{(m^2-n^2)^2} & m\neq n,\ \text{one even, other is odd} \\ \frac{L^2}{4} & \text{if } m=n \end{cases}$$

$H_{mn} =$

$$\begin{cases} \frac{2}{L}\left[\frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{L^2}\right)\cdot\left(\frac{L}{2}\right) + \left(\frac{a}{L}\right)\left(\frac{L^2}{4}\right)\right] & m=n \\ = \frac{2}{L}\left[-\left(\frac{2L}{\pi}\right)^2 \frac{mn}{(m^2-n^2)^2}\right] & m\neq n \text{ one is odd, other is even} \\ = 0 & m\neq n \text{ both are odd or both are even} \end{cases}$$