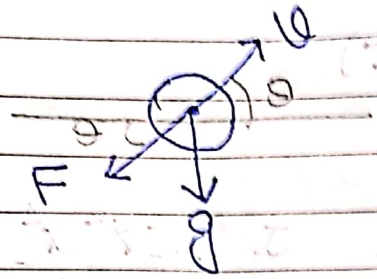
PH354 HW6

(85) $F = \frac{1}{2} \pi R^2 \rho C v^2$

for a ~~pt~~ ball in motion:



Along x axis: $\vec{a_x} = \frac{\vec{F_x}}{m}$

$$|\vec{F_x}| = |F| \cos\theta = \left(\frac{\pi R^2 \rho C v}{2}\right)(v\cos\theta)$$

$v_x = v\cos\theta$
$v_y = v\sin\theta$

$$\implies |\vec{F_x}| = \frac{\pi R^3 \rho C v}{2} (v_x) \quad \text{——(1)}$$

Slly $|\vec{F_y}| = |F|\sin\theta = \left(\frac{\pi R^2 \rho C v}{2}\right)(v\sin\theta) = \frac{\pi R^2 \rho C v}{2}(v_y) \quad \text{——(2)}$

Now $|\vec{F_x}| = m\ddot{x}$
$|\vec{F_y}| = m\ddot{y}$

$v_x = \dot{x}$
$v_y = \dot{y}$
$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$

substituting in (1), (2):

$$\ddot{x} = \frac{\pi R^2 \rho C \cancel{v}}{2m} \sqrt{\dot{x}^2 + \dot{y}^2}\, \dot{x}$$

$$\ddot{y} = \frac{\pi R^2 \rho C}{2m} \sqrt{\dot{x}^2 + \dot{y}^2}\, \dot{y}$$

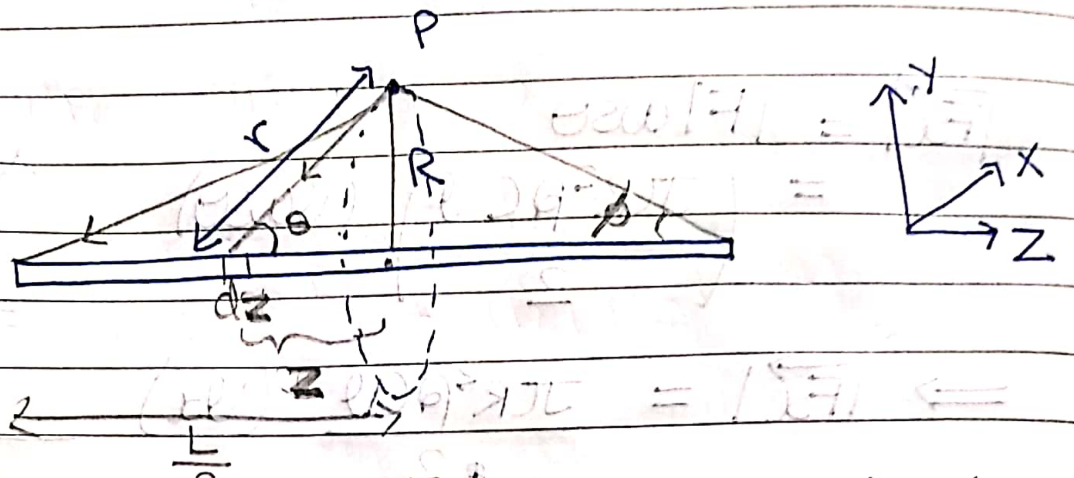
$\left(\frac{\ }{2m}\right)$

$$\frac{J^2 s^2}{m} \qquad J.\ \frac{\cancel{M} L^2 \cancel{T}^{-2} . \cancel{T}^2}{\pm \cancel{M}}$$

1st excited state = ~~At~~ 1

$n = 3$: GS

(Q6)



⇒ Force felt by ball at pt P due to dx element of rod:

$$\vec{dF} = -\frac{Gm.dM}{r^2}\,\hat{r} \qquad m = \text{mass of ball}$$

$$dM = \left(\frac{M}{L}\right) dz$$

$$= \cancel{\frac{Gm}{r^3}}\ \vec{dF_z} + \vec{dF_{centre}}$$

due to ball being symmetrically placed along z axis:

$$\vec{dF_z} = \cancel{dF_z}\ (dF).\cancel{\sin}\cos\theta.\hat{z} \quad \text{integrates to } 0$$

leaving behind only dF_{centre}

$$dF_{centre} = (dF)\sin\theta\ (\hat{R})$$

$$= \frac{-(Gm)\left(\frac{m}{L}\right)dz\ .\ \sin\theta}{r^2}$$

Now, $\sin\theta = \frac{R}{r}$

$$= -\frac{GMm}{Lr^2}\cdot\left(\frac{R}{r}\right)\hat{R}$$

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$$\therefore \vec{F}_{centre} = \int d\vec{F}_{centre}$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} -\frac{GMm}{L} \cdot R \frac{dz}{r^3} (\hat{R})$$

Now $R = \sqrt{x^2+y^2}$ $\quad r = \sqrt{x^2+y^2+z^2}$

$$\Longrightarrow \vec{F}_{center} = \int_{-\frac{L}{2}}^{\frac{L}{2}} -\frac{GMm\sqrt{x^2+y^2}}{L} \frac{dz}{(x^2+y^2+z^2)^{3/2}} (\hat{R})$$

$$= |F_{center}| (-\hat{R})$$

$$\Longrightarrow |F_{center}| = \frac{GMm}{L} \sqrt{x^2+y^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dz}{(x^2+y^2+z^2)^{3/2}}$$

(iv) $\frac{du}{dt} = 998u + 1998v$

$\frac{dv}{dt} = -999u - 1999v$

$(u(0), v(0)) = (1, 0)$

~~so~~ $X = \begin{bmatrix} u \\ v \end{bmatrix}$ $\dot{X} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

$\Rightarrow$ $\dot{X} = AX$ a linear ~~diffe~~ ODE where

$$A = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix}$$

EIGEN DECOMPOSITION:

the above matrix has eigenvalues:

~~eig~~ $|\det(A - \lambda \mathbb{1})| = 0$

$$\Rightarrow \begin{vmatrix} (998-\lambda) & 1998 \\ -999 & (-1999-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 + (1001)\lambda + (1998)(999) - (1999)(998) = 0$$

$$(\lambda^2) + (1001)\lambda + \cancel{(1000) = 0}\ (1000) = 0$$

$$\lambda(\lambda + 1000) + 1(\lambda + 1000) = 0$$

$$(\lambda + 1000)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -1000, -1$$

Now finding eigen vectors: let $\begin{pmatrix} x \\ y \end{pmatrix}$ be eigen vector:

for $\lambda = -1$:

$A\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-1)\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \cancel{\begin{pmatrix} -999 \\ 1998 \end{pmatrix}} \begin{pmatrix} 1998 \\ -999 \end{pmatrix} = \begin{pmatrix} -2 \\ +1 \end{pmatrix}$

$\lambda = -1000$

$A\begin{pmatrix} x \\ y \end{pmatrix} = -1000\begin{pmatrix} x \\ y \end{pmatrix}$

$998x + 1998y = -1000x$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \cancel{1998} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

∴ for same initial condition $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \alpha v_1 + \beta v_2$$

where v_1, v_2 are eigen vectors and λ_1, λ_2 are corresponding eigen vectors.

and $$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \alpha v_1 e^{\lambda_1 t} + \beta v_2 e^{\lambda_2 t}$$

New:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cancel{\alpha \begin{pmatrix} -999 \\ 1998 \end{pmatrix}} + \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\beta = \cancel{-1998\alpha}$ $-\alpha$ and. $-\overset{2}{\cancel{999}}\alpha - \beta = 1$

$\cancel{-999\alpha + 1998\beta = 1}$

$$\Rightarrow \alpha = \frac{1}{999}$$

$$\beta = -\frac{1998}{999}$$

∴ our analytic solution is:

$$u(t) = \frac{1}{999}(-999)e^{-1t} + -\frac{1998}{999}(-1)e^{-1000t}$$

$$= -e^{-1t} + \frac{1998}{999}e^{-1000t}$$

$$v(t) = \frac{1}{999}(1998)e^{-1t} - \left(\frac{1998}{999}\right)(1)e^{-1000t}$$

Required step size for stability in explicit method:

$$\begin{pmatrix}1&4\\3&8\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+\cancel{16}8\\3+16\end{pmatrix} = \begin{pmatrix}19\\19\end{pmatrix}$$

$$\begin{pmatrix}1&3\\4&8\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+6\\4+16\end{pmatrix} = \begin{pmatrix}7\\20\end{pmatrix}$$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\\cancel{9}11\end{pmatrix}$$

NOW: $\begin{pmatrix}1\\0\end{pmatrix} = \alpha\begin{pmatrix}-2\\1\end{pmatrix} + \beta\begin{pmatrix}-1\\1\end{pmatrix}$

$\alpha+\beta=0$ $\quad 1=-2\alpha-\beta \implies \cancel{\beta=2}$

$\alpha=-\beta$ $\quad \cancel{\beta=} \quad \implies \beta=1$

$\cancel{1=\beta}$ $\quad \alpha=-1$

$\therefore$ our analytic solutions are:

$$u(t) = (-1)(-2e^{-1t}) + (1)(-1e^{-1000t})$$

$$v(t) = (-1)(1e^{-1t}) + (1)(1.e^{-1000t})$$

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