Sunshine's Homepage - Welzl's Algorithm

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Computing the smallest enclosing disk in 2D

Download Applet with Source (26 kb)

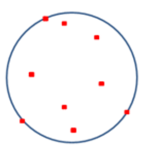
Instructions:
* By left-clicking in the blue area, you can set new points. * By right-clicking points you can remove them. * Note that they are also draggable. * If you don't want to set many points manually, you can create a specified number of random points automatically. * The Clear-Button removes all points.

* Finally pressing the Go-Button calculates and draws the minimal enclosing circle.

Notes:

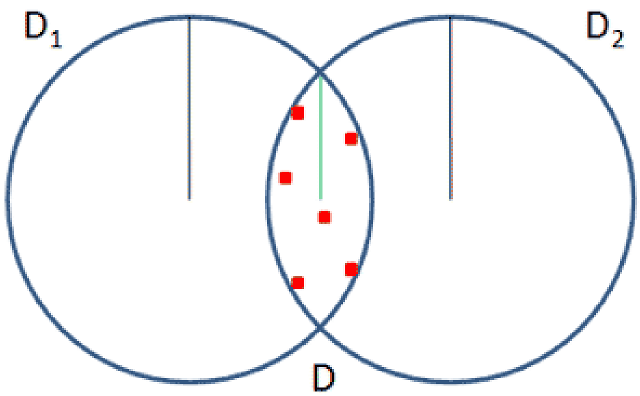
The applet you see above is my own implementation of Welzl's algorithm which computes the minimal enclosing circle. I came around this algorithm while learning for my oral exam in Computer Graphics and I thought why not trying to visualize it.
This algorithm was presented by Welzl in 1991 [1] and runs (in contrast to most other algorithms for solving this prolbem) in linear time!

So what it is all about?
Let's say we have a set ***P*** of n points in the plane, we want to calculate the closed disk of smallest radius ***D(P)*** which contains all points in ***P***.



Look at the picture left: we have a number of red points; the blue circle is the smallest one which contains all red points. So how to find this smallest circle (which means how to find its center and its radius) - that's the question!

First of all, it's easy to prove that such a *smallest enclosing disk* (sed) is *unique*. Assume we have two smallest disks D_1 and D_2. That means that each disk contains the points in P, so also the intersection D c D_1 u D_2 must contain them. But a circle around D has a smaller radius than the disks D_1 and D_2, so our assumption of two different smallest disks fails and we proved the uniqueness. The figure below should make it clear:



The algorithm works now step by step: assume we know already a sed (smalled enclosing disk) D for n-1 points $p_1,\dots,p_{n-1}$. Now there are two cases for the nth point.
1) p_n lies inside D. So nothing changes - the sed D for $p_1,\dots,p_{n-1}$ is the same for $p_1,\dots,p_n$!
2) p_n lies *NOT* inside D. So we have to compute a new sed. But we know (that's a fact) that pn must lie on the boundary of D, call it BD! So we have to calculate a sed D' for $p_1,\dots,p_{n-1}$ with p_n on the boundary of D'.

In fact, that's the main idea. This property along with the three following claims allows us to calculate such a smallest enclosing disk in an iterative way:
Let P again be a set of n points, P is not empty and p is a point in P. R is also a set of points, in fact these are the points on the boundary of the disk. Then the Lemma says:
(i) If there exists a disk containing P with R on its boundary, then $D(P,R)$ is well defined = unique.
(ii) If p lies not in $D(P - \{p\},R)$, then p lies on the boundary of $D(P,R)$, provided it exists, meaning:
$D(P,R) = D(P - \{p\}, R\ u\ \{p\})$.
(iii) If $D(P,R)$ exists, there is a set S of most max{0, 3 - |R|} points in P such that $D(P,R) = D(S,R)$. That means that P is determined by at most 3 points in P which lie on the boundary of $D(P)$.

With this information, one can implement this algorithm in a recursive way. Of course, as in all recursions, there are terminal or end cases in which you can calculate the solution directly. Here it is if we have 3 points only: a minimal circle in this case has the 3 points on the boundary and it's definitely determined (if the three points lie not on a common line!). In my implementation I also considered the cases with only 1 and 2 points. In fact I found it quite difficult to get the algorithm working - curius was that when I used arrays to hold the points the recursion works, but when I used vectors very strange things happened... Nevertheless, now it works and I hope you found my notes along the applet and its sources useful!

Pseudocode:

```
/*  * Calculates the sed of a set of Points. Call initially with R = empty set.  * P is the set of points in the plane. R is the set of points lying on the boundary of the current circle.

 */

function sed(P,R) {

    if (P is empty or |R| = 3) then

        D := calcDiskDirectly(R)

    else

        choose a p from P randomly;         D := sed(P - {p}, R);         if (p lies NOT inside D) then             D := sed(P - {p}, R u {p});

    return D;


}
```

References:

[1] Smallest enclosing disks (balls and ellipsoids), Emo Welzl, 1991

Sunshine, May 2008

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