Decremental Matching in General Graphs

Sepehr Assadi* **Aaron Bernstein*** **Aditi Dudeja***

*Rutgers University

The Dynamic Graph Model

- Process a sequence of edge insertion and deletions.
- Maintain a large matching with a small update time (amortized/worst case).
- Incremental Model: Only edge insertions allowed.
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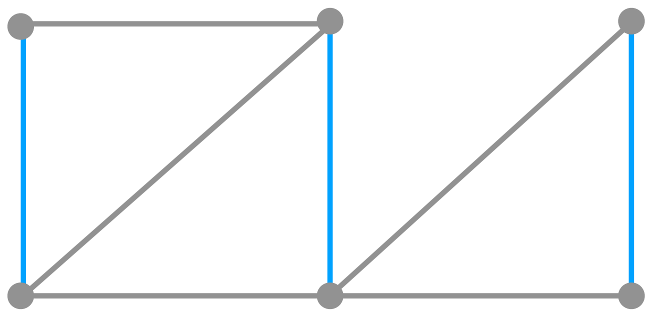

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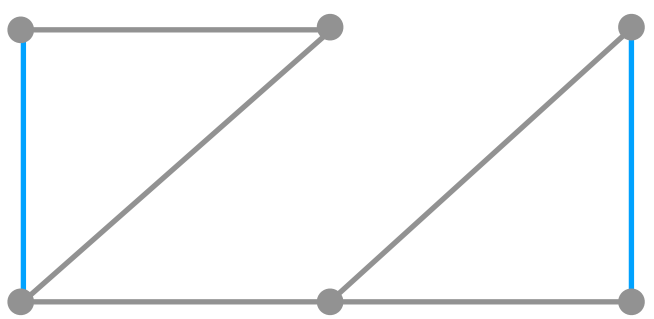

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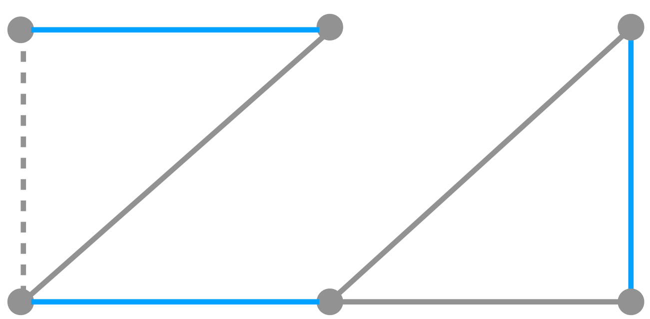

Background

Upper Bounds for $(1-\varepsilon)$ - approximation

	setting	update time	bipartite/general
[GP13]	fully dynamic	$O_\varepsilon(\sqrt{m})$	general
[GLSSS19]	incremental	$O_\varepsilon(1)$	general
[BGS20]	decremental	$O_\varepsilon(1)$	bipartite

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This Work: $(1-\varepsilon)$ - approximation in $O_\varepsilon(1)$ update time for general graphs.

Congestion Balancing [BGS20]

Let μ be the initial size of the matching. It is sufficient to solve the following problem in $\tilde{O}_{\varepsilon}(m)$ time:

1. Maintain a matching of size at least $\mu(1-\varepsilon)$ or,

2. Certify that maximum matching has dropped below $\mu(1-\varepsilon)$.

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Each time 2) happens start a new phase with new estimate $\mu(1-\varepsilon)$.

\# phases = $\log_{1+\varepsilon} n$

Congestion Balancing [BGS20]

- A lazy approach:

  1. Compute a $(1-\varepsilon)$- approximate maximum matching M in time $O_\varepsilon(m)$.

  2. Do nothing until the adversary reduces |M| by a $(1-\varepsilon)$ factor.

  3. Recompute M.

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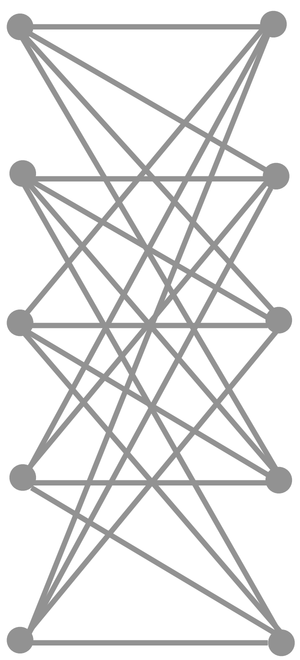

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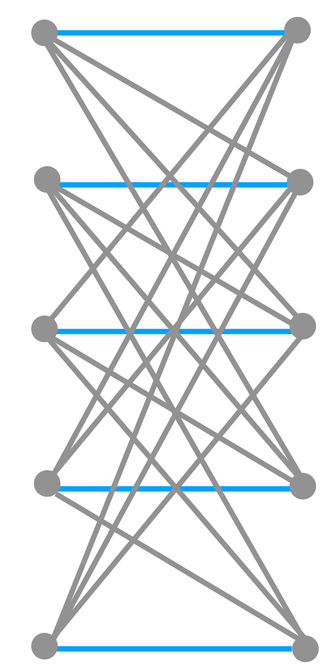

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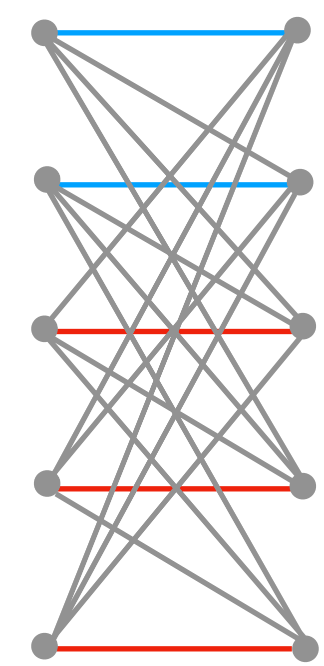

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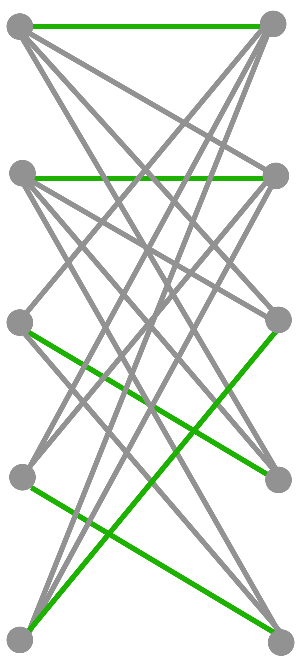

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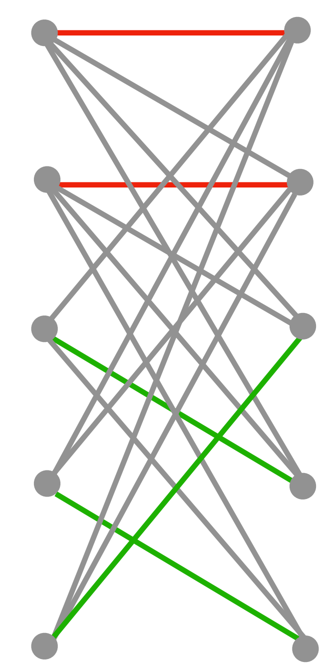

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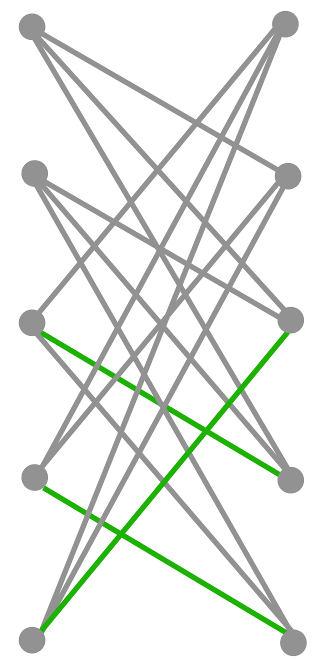

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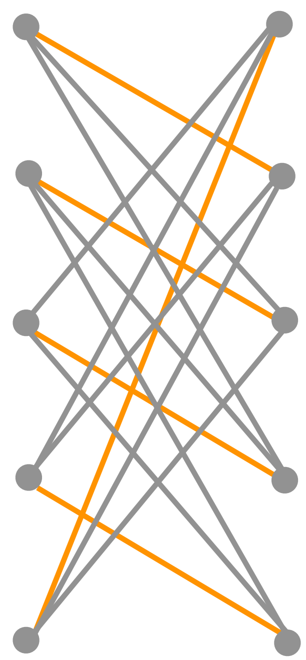

Main Issue: Too much congestion on edges.

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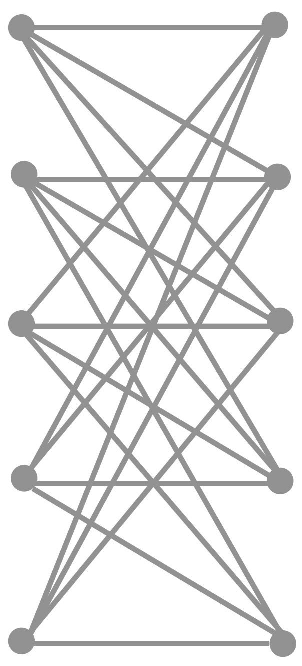

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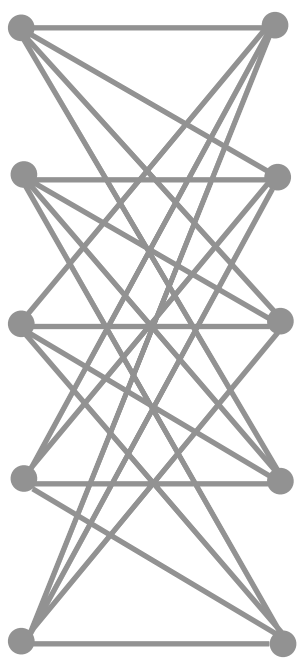

Put flow $\frac{1}{n}$ on every edge.

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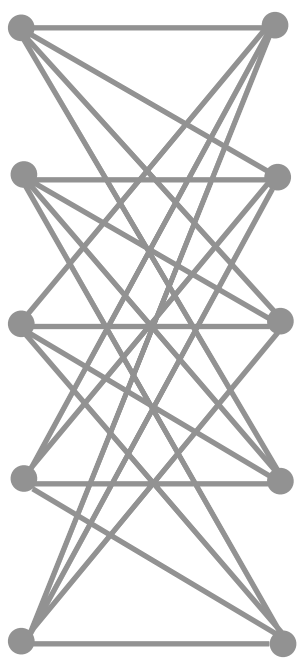

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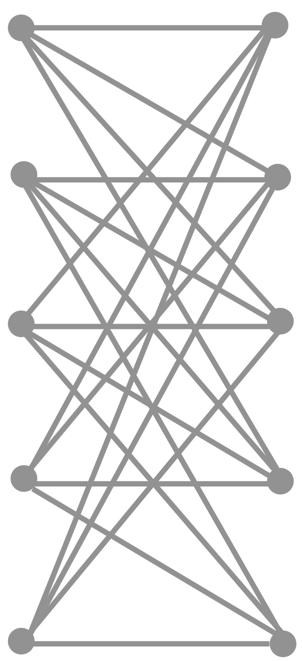

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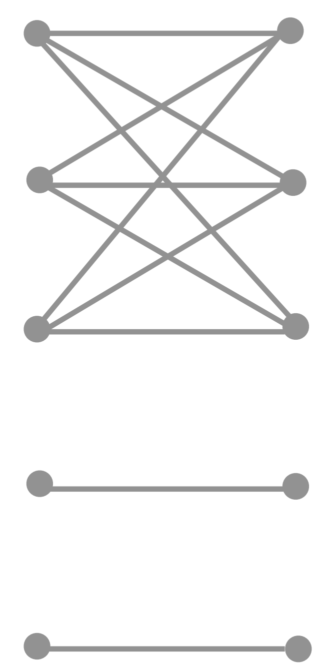

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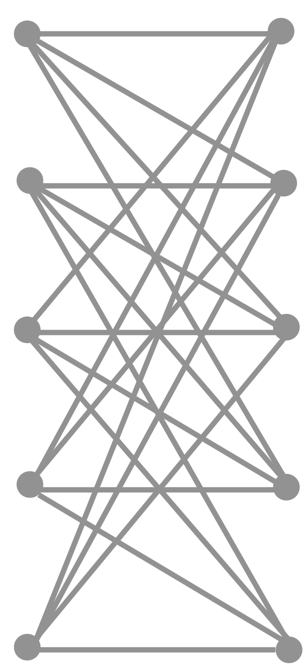

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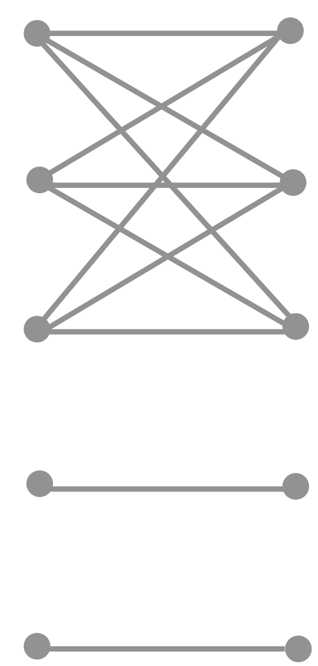

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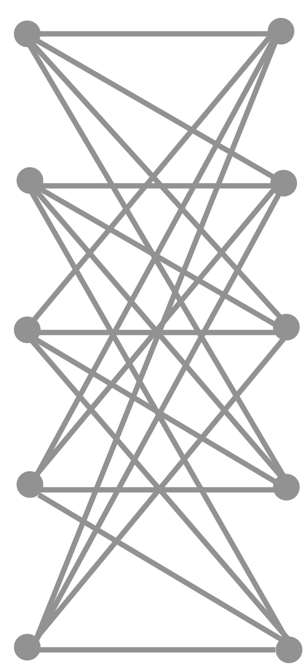

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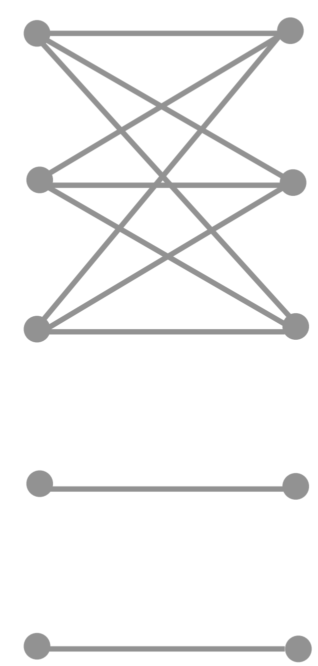

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Adversary can’t delete too many of them.

Congestion Balancing [BGS20]

1. Initially, set $c(e) = \dfrac{1}{n^2} \forall e \in E$.

2. **Initial phase:**

   - Estimate matching size. If smaller than $(1-\epsilon)\mu$ then terminate.

   - Compute fractional matching obeying capacities $\{c(e)\}_{e \in E}$.
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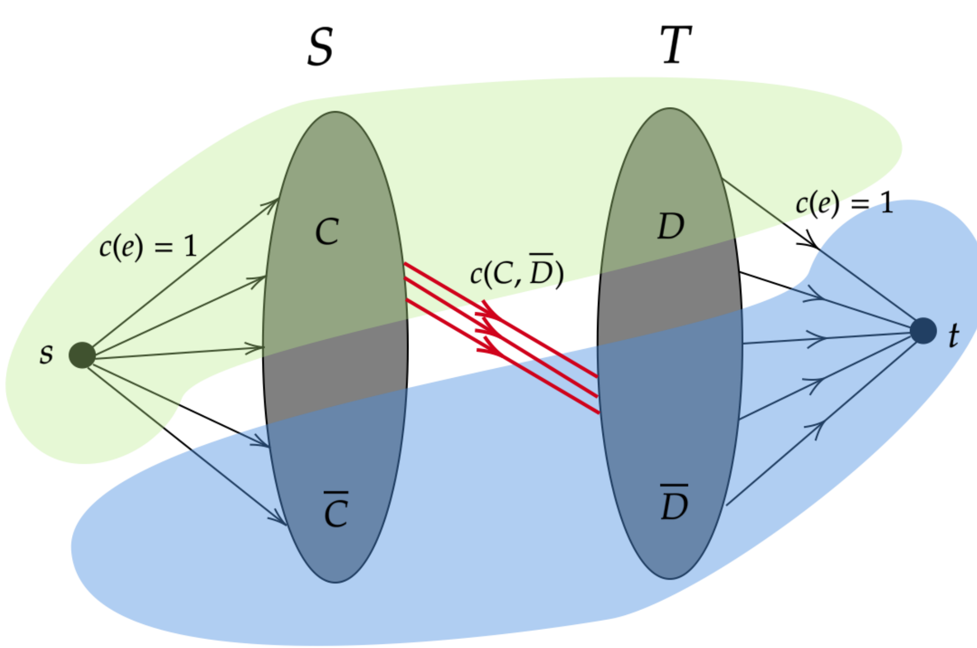

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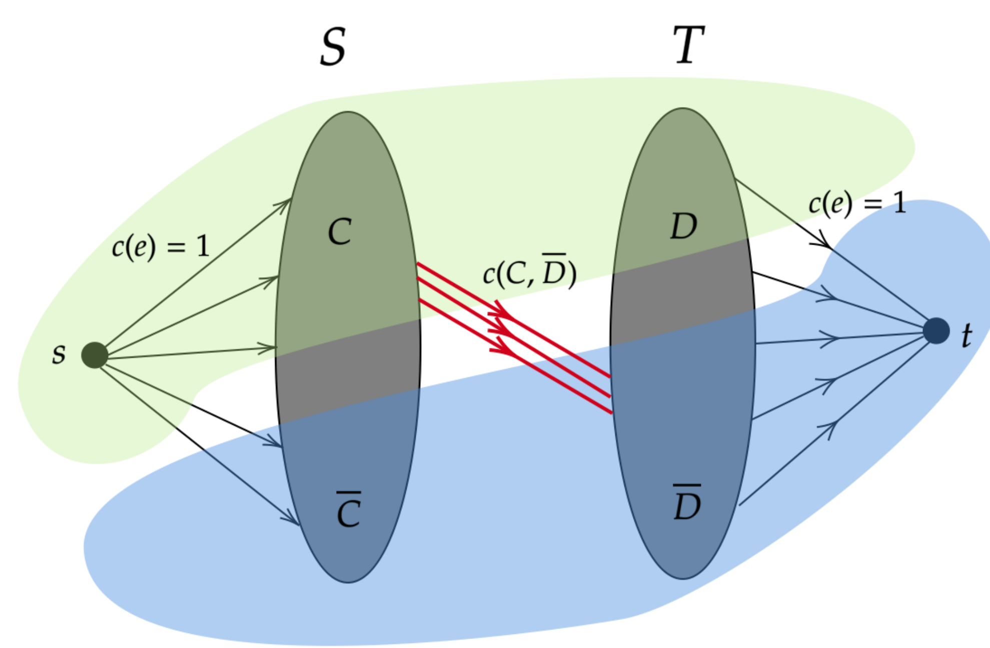

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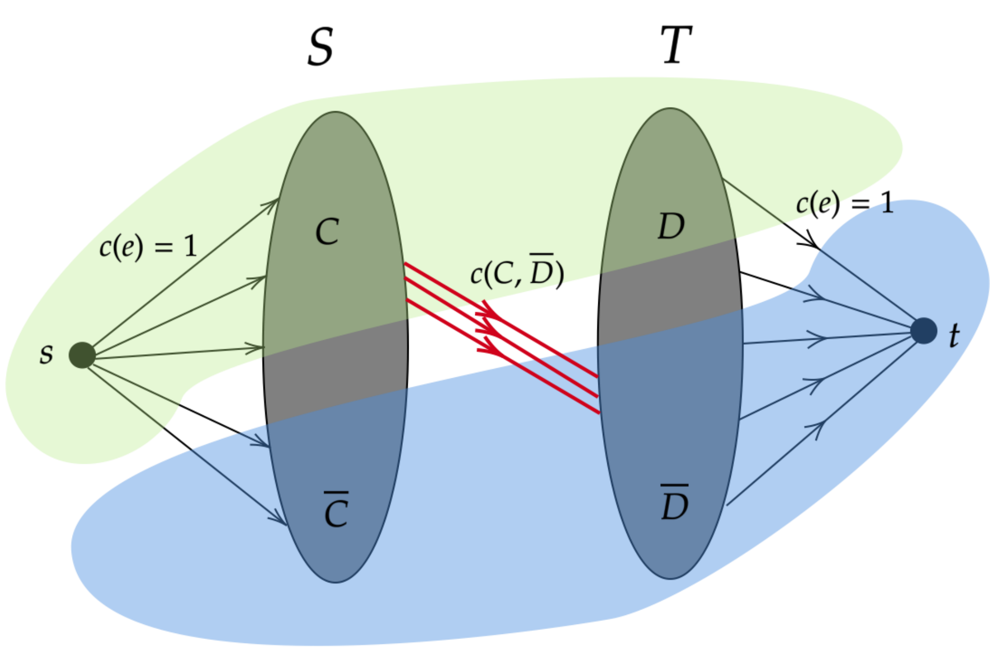
Runtime: $\tilde{O}(m)$



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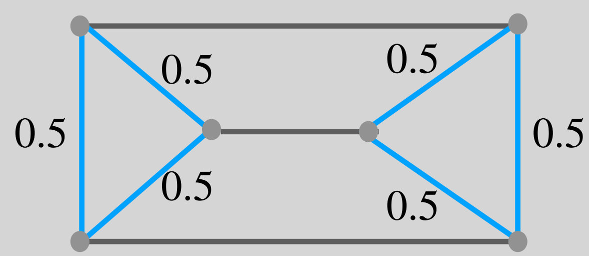

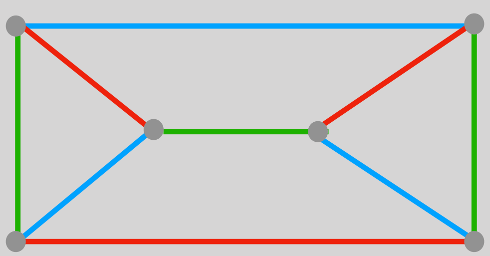

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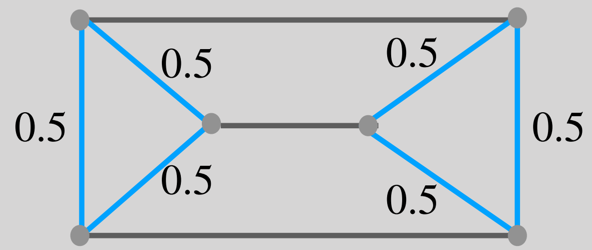

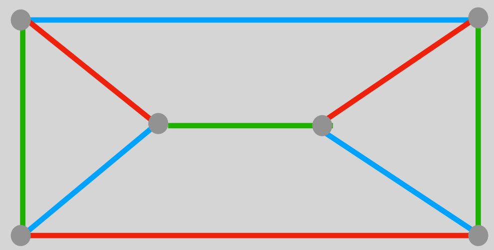

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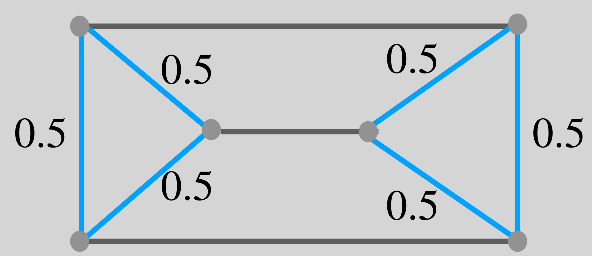

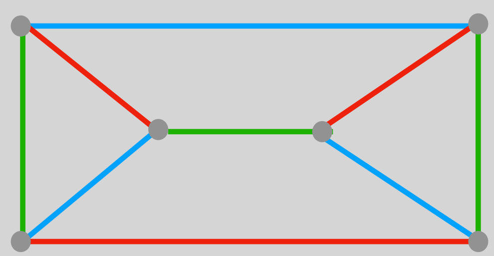

How to determine bottleneck edges?

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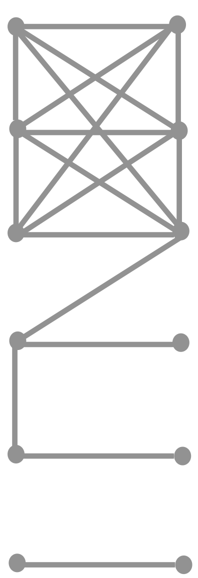

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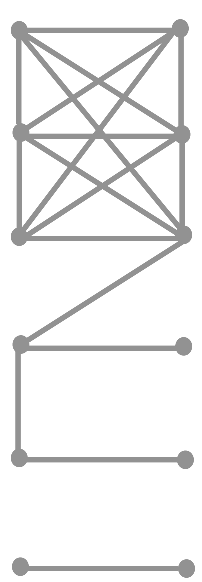

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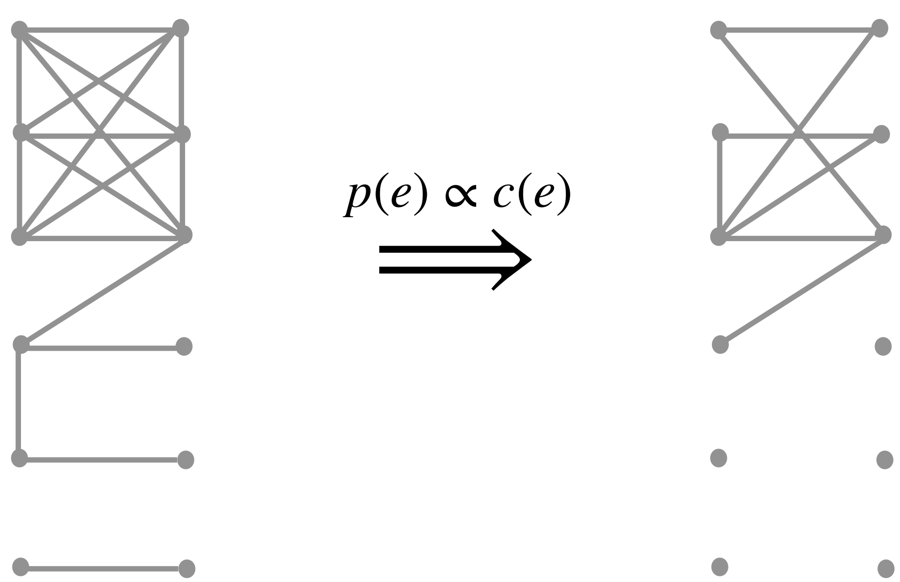

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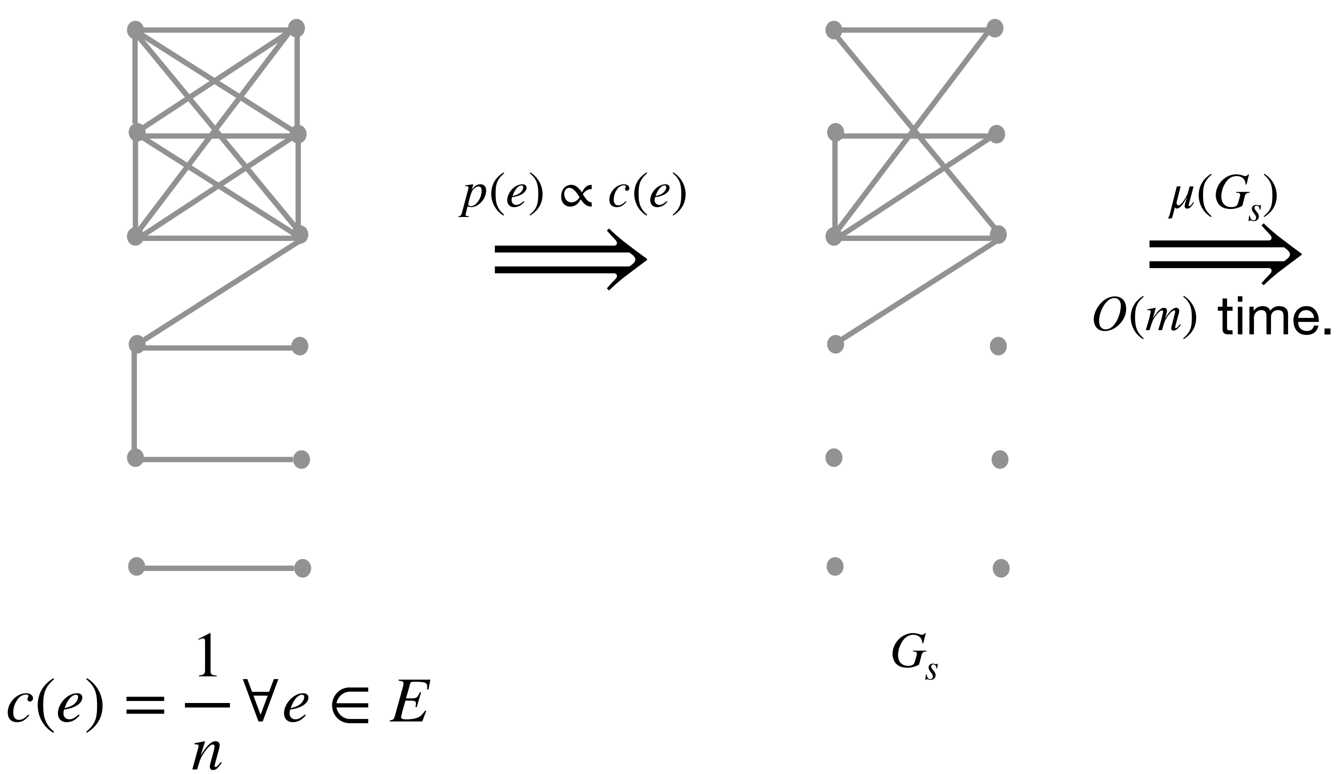

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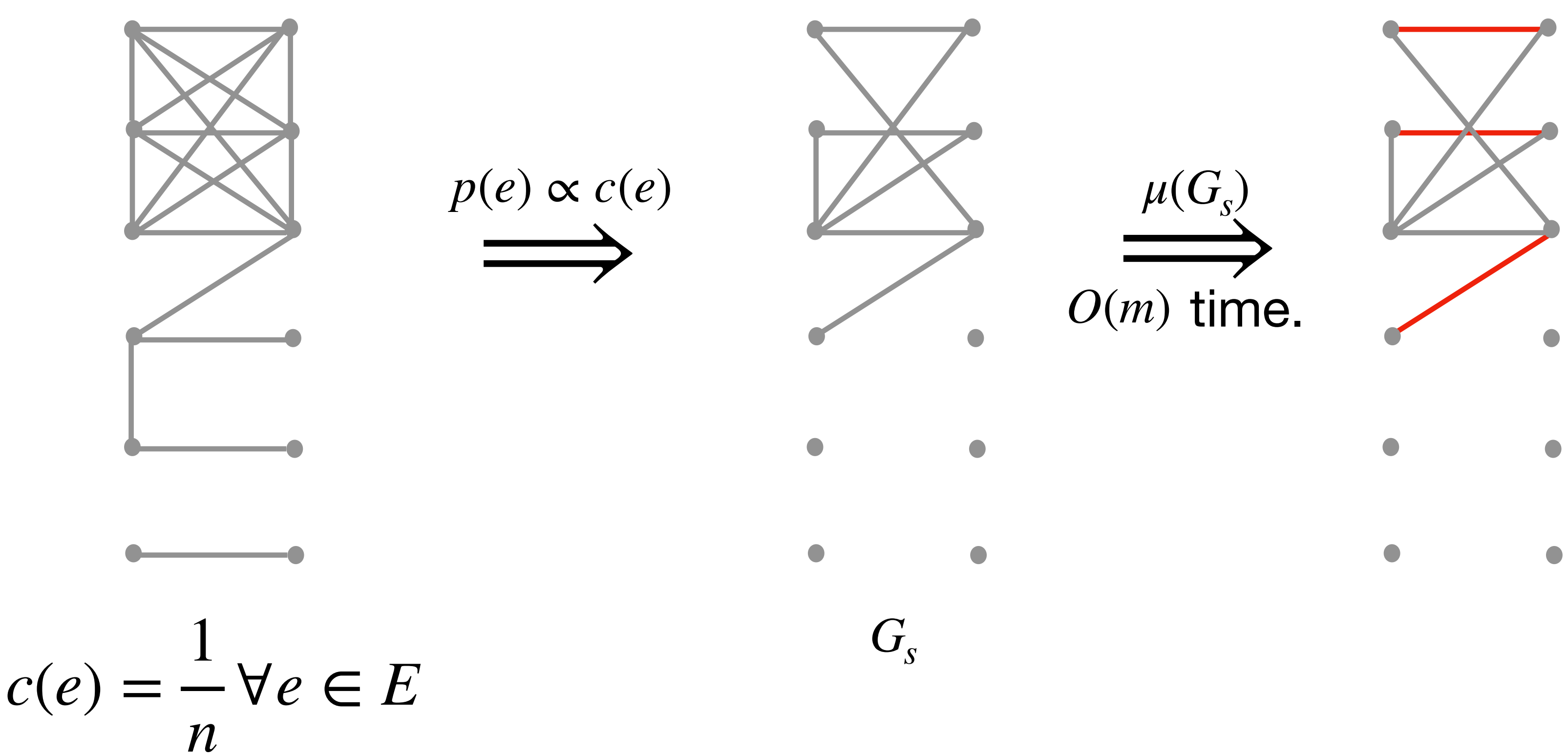

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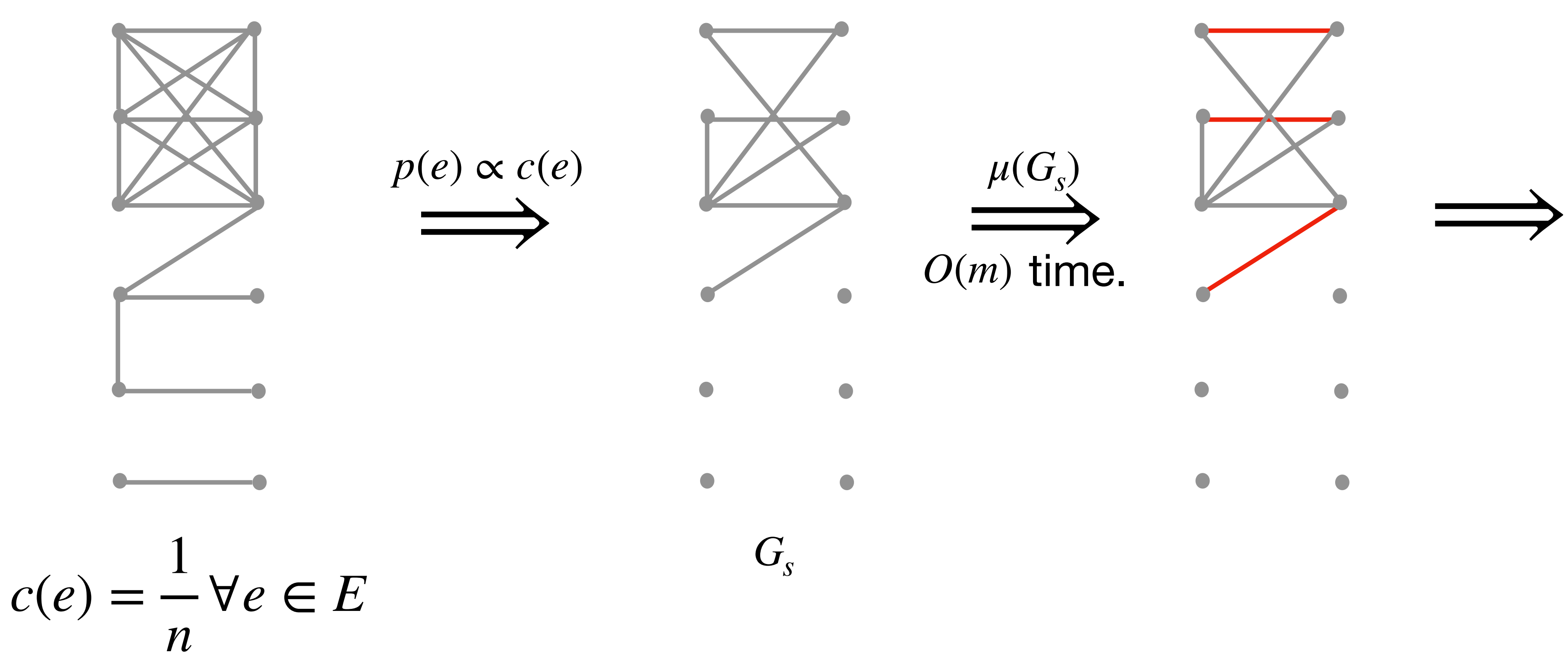

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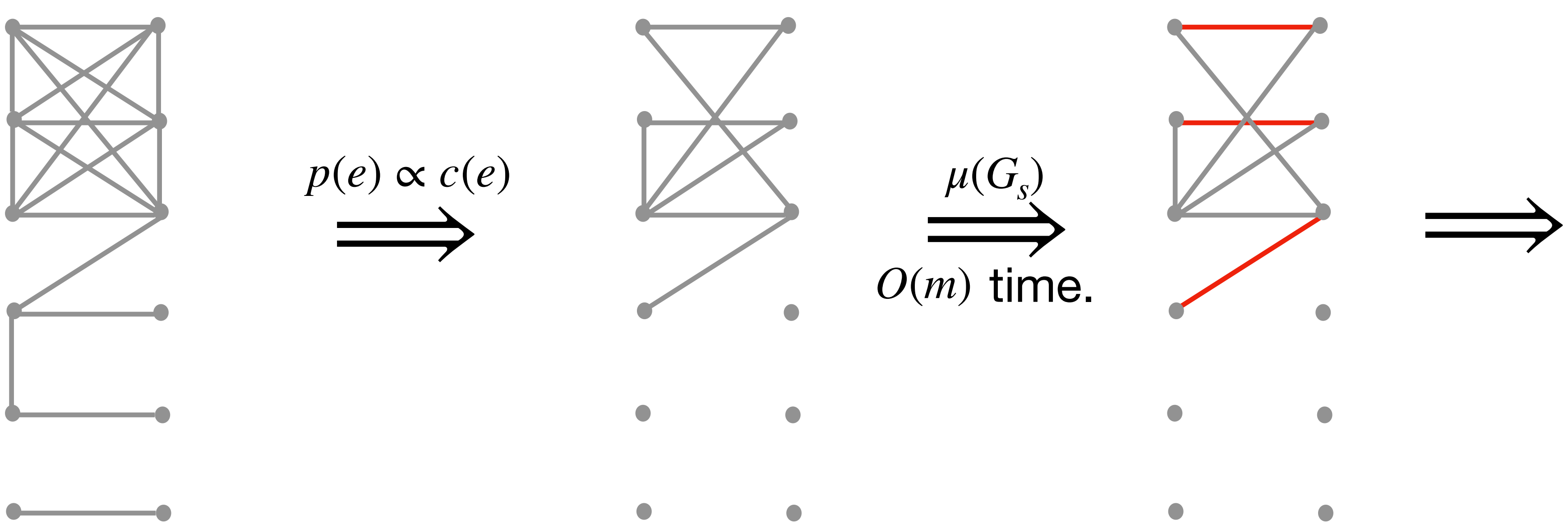
Else $\mu(G_s)$ is large and good fractional matching exists. Find it!



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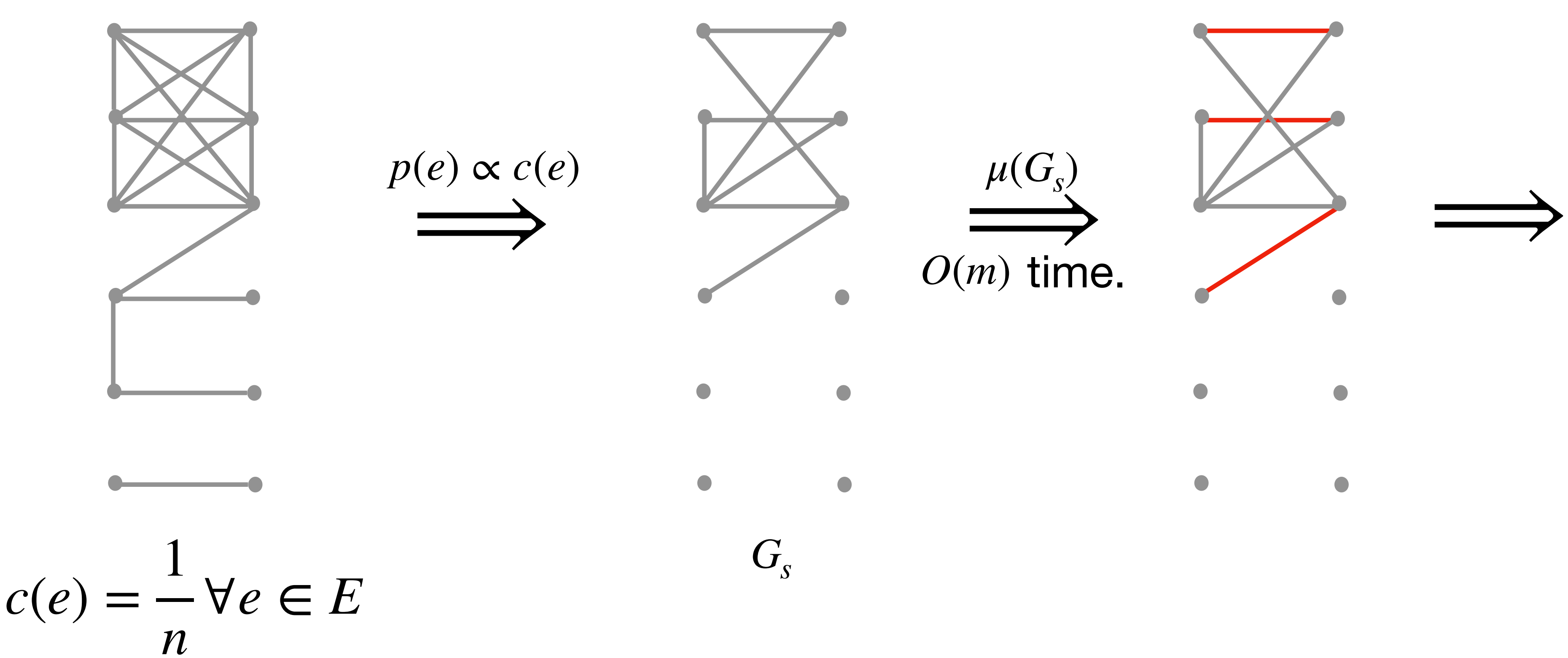

Else $\mu(G_s)$ is large and good fractional matching exists. Find it!

If $\mu(G_s)$ is small, then $\mu(G, c)$ is small. Increase capacity.

When $\mu(G_s)$ is large: Finding a good fractional matching

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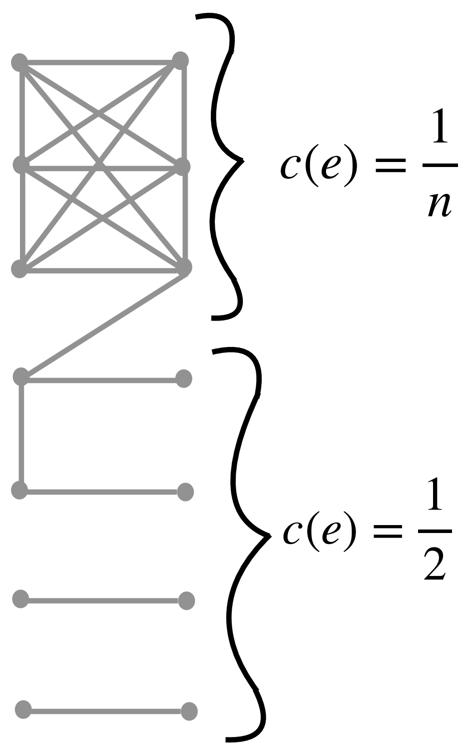

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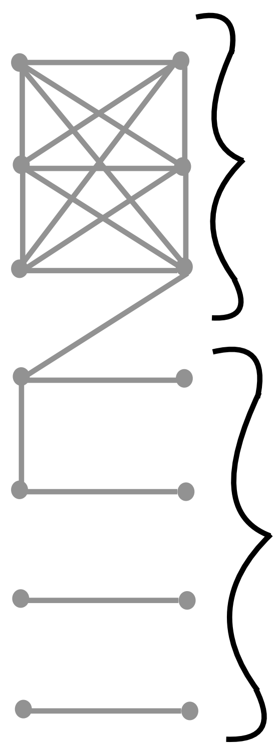

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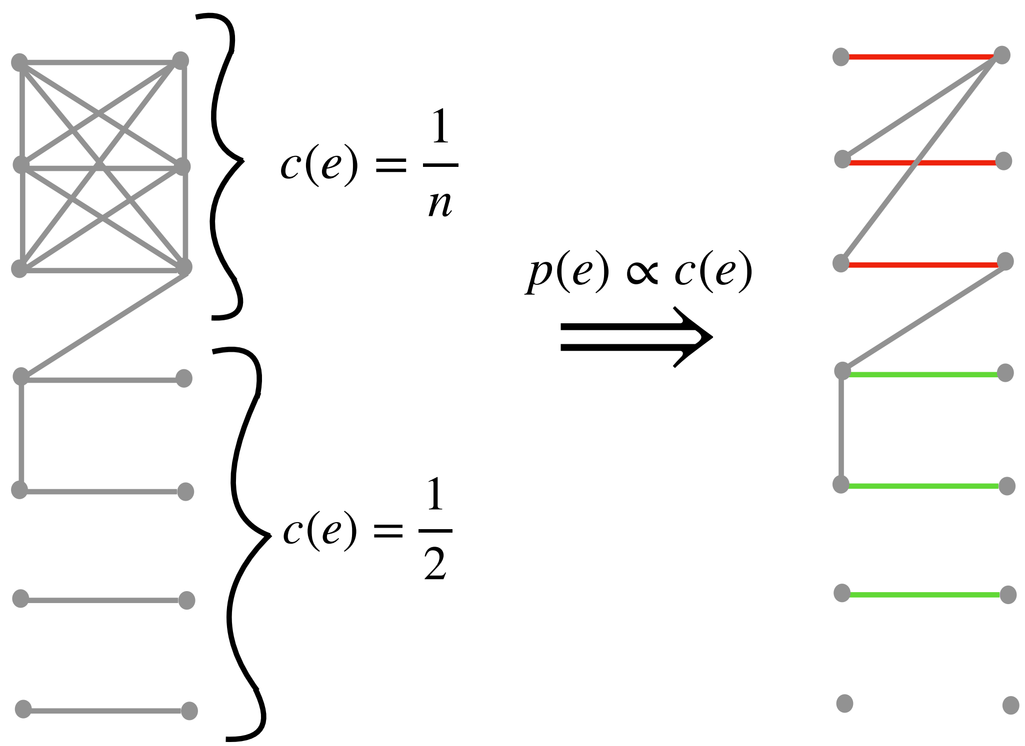

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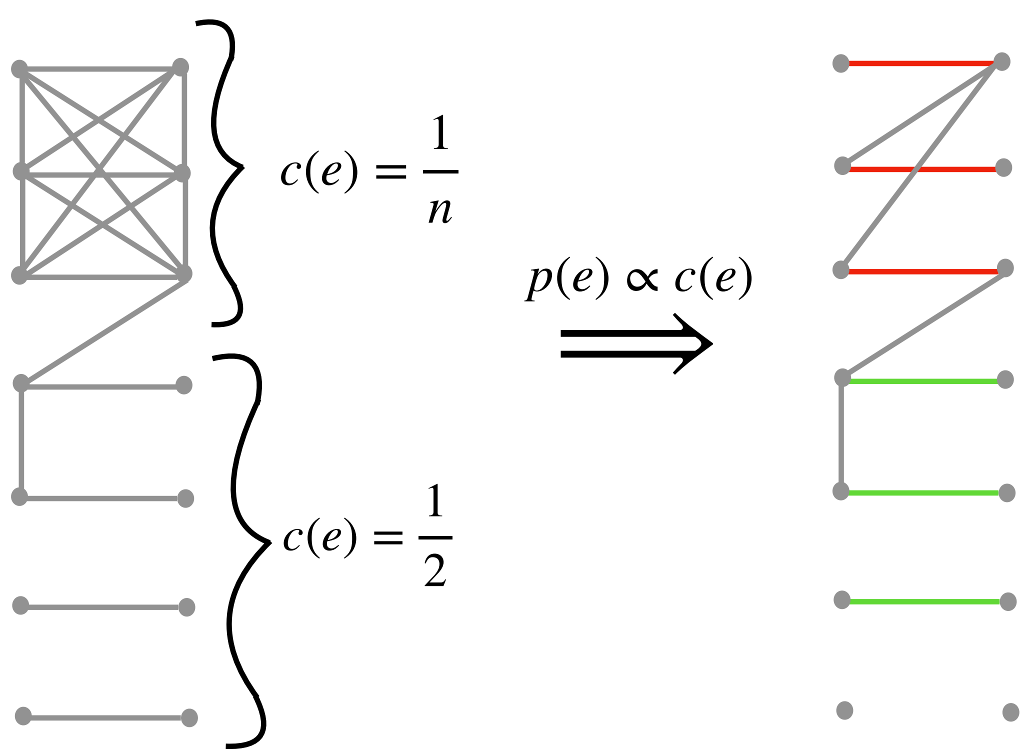

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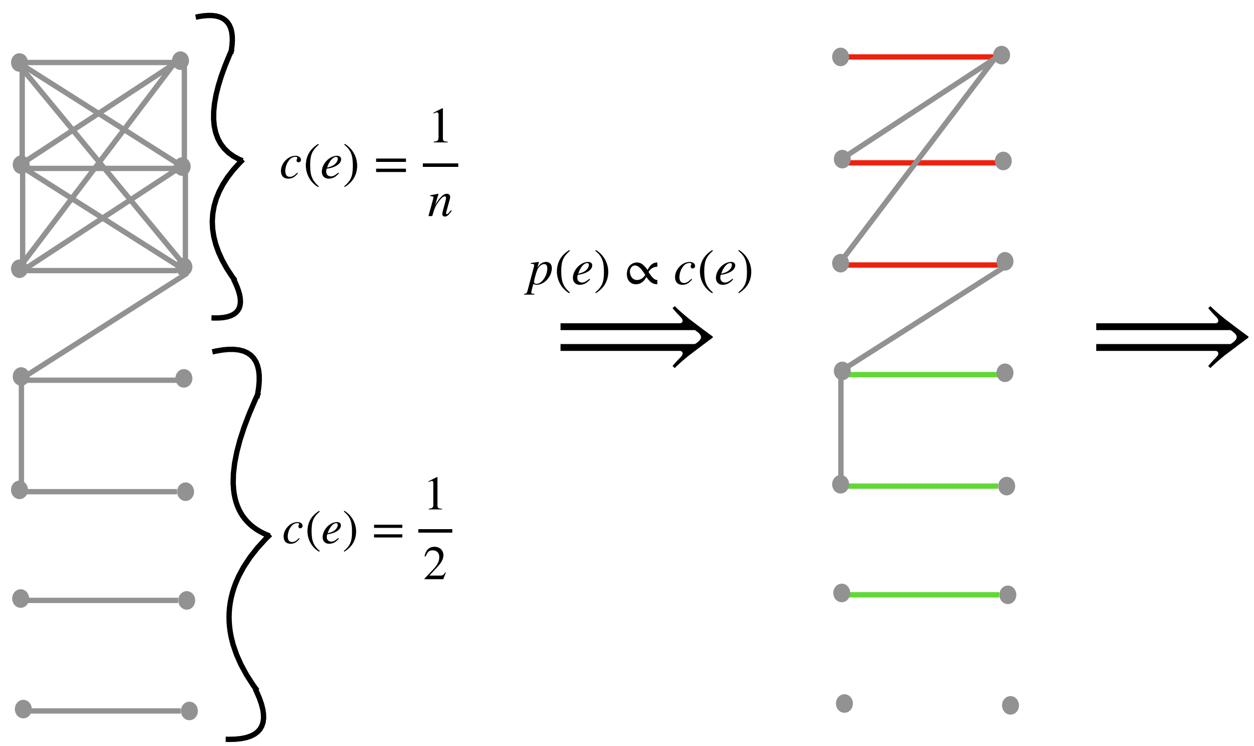

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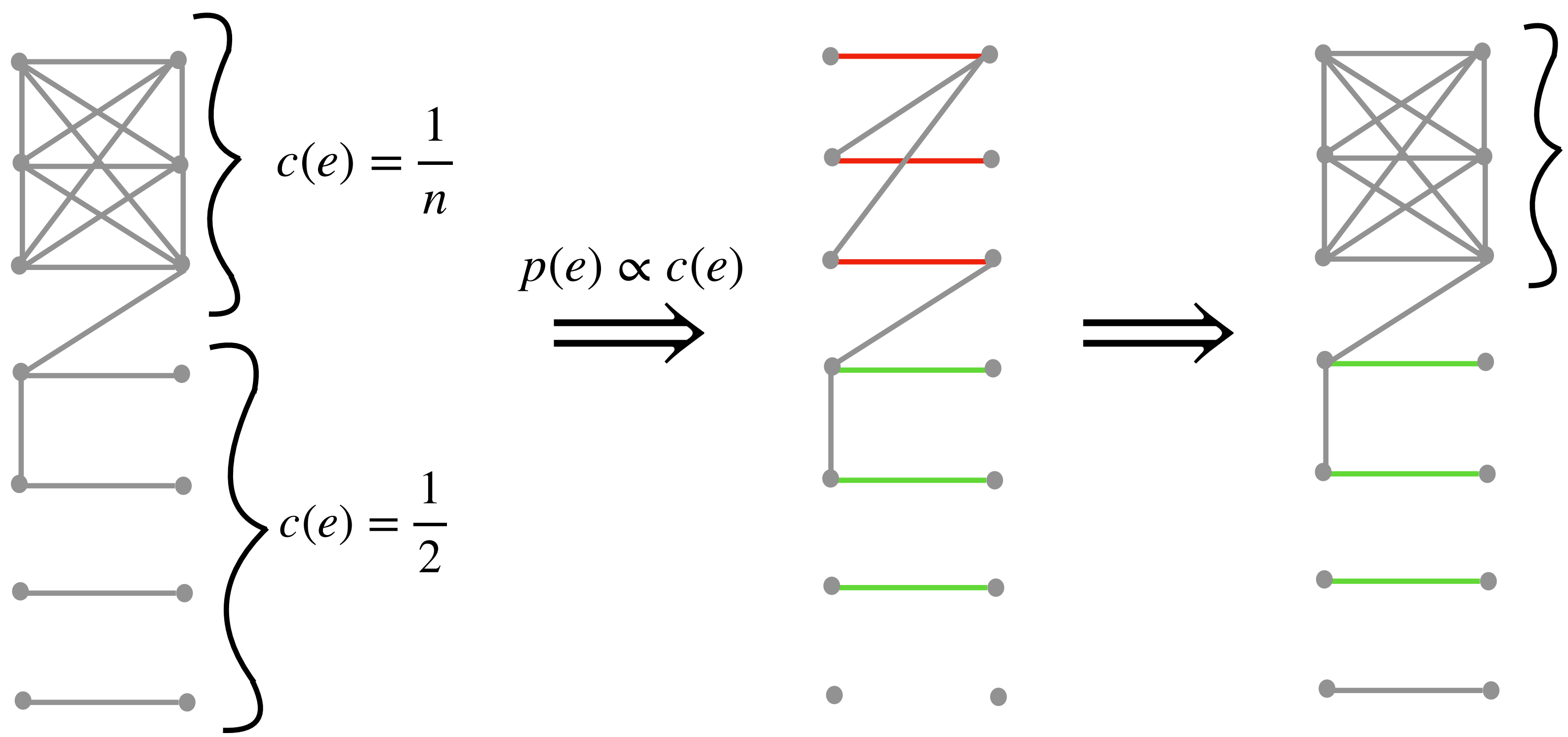

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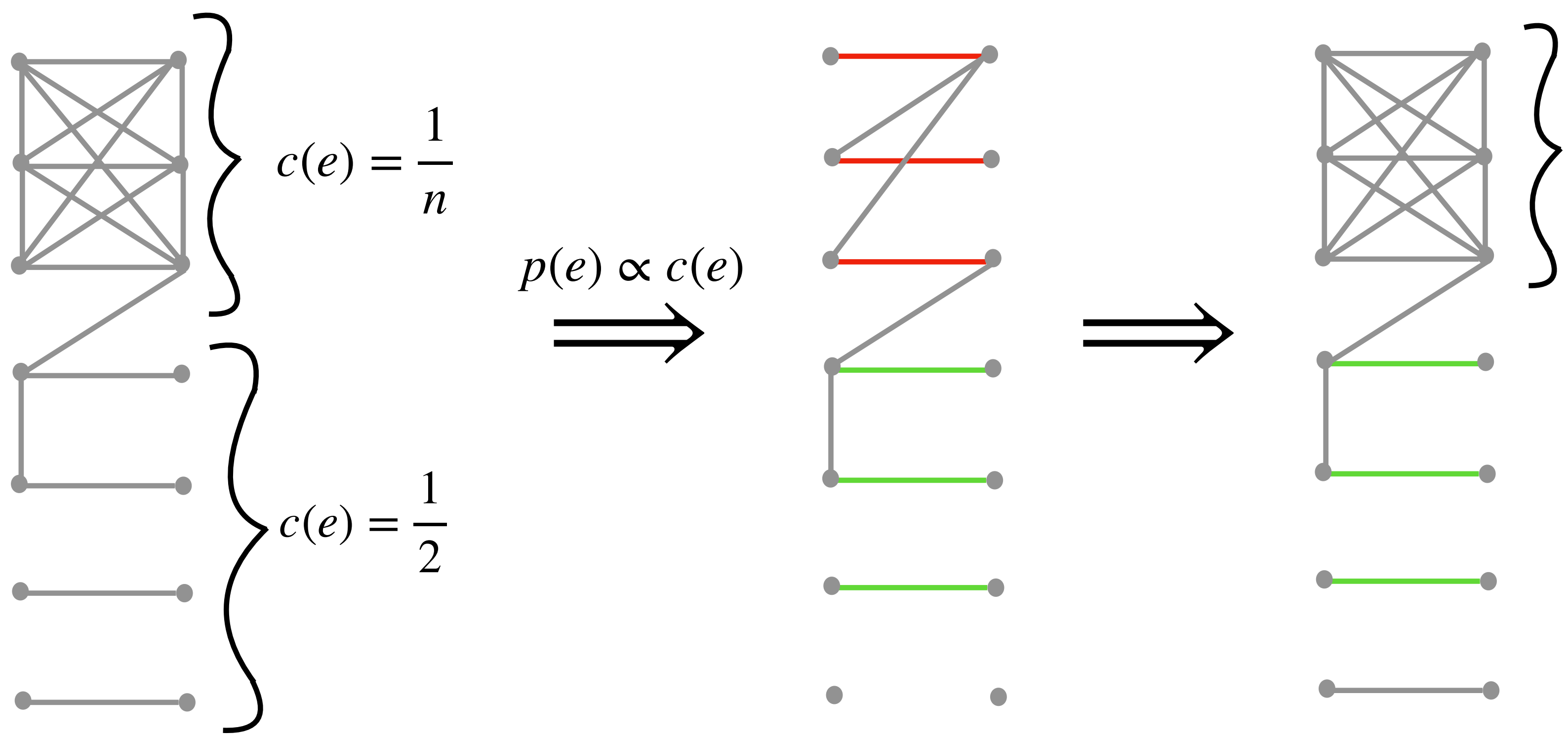
Find a fractional matching f using max flow.

When $\mu(G_s)$ is large: Finding a good fractional matching



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Theorem 2: f + green edges has value $\approx \mu(G_s)$ and satisfies blossom constraints.

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Theorem 3: Suppose we solve the dual problem on G_s, then using that solution, we can determine bottleneck edges.

Putting Things Together

1. Initially, set $c(e) = \frac{1}{n^2} \forall e \in E$.

2. **Initial phase:**

   - Estimate matching size. If smaller than $(1-\epsilon)\mu$ then terminate.

   - Compute value of fractional matching obeying capacities $\{c(e)\}_{e \in E}$ and blossom constraints.

   - If value is large, compute such a fractional matching.

   - If the matching is too small, then increase capacity along crucial edges until fractional matching is large enough.

3. **Process deletions.**

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Open Questions

1. Can the algorithm be derandomized?

2. Can we improve dependence on $\frac{1}{\varepsilon}$?