MA1150 Differential Equations

G.F. Simmons: Ordinary differential equations with applications and historical notes.

Differential Equations: An eq involving the derivatives of an unknown function w.r.t one or more independent variables.

$\boxed{y''+p(x)y=0}$ $y=y(x)$

y - dependent var

x - independent

ordinary diff eq.

defn $u=u(t,x)$ → independent variables

$u_t = c^2 u_{xx}$ — partial differential eq

Differential Equations

- Ordinary differential eq
  - 1st order
  - Higher order
  - systems (linear)
- partial diff eq

Order:

ODEs

t — independent var, x — dependent var or { x - independent, y - dependent }

$x' = \frac{dx}{dt}$, $x'' = \frac{d^2x}{dt^2}$ … $x^{(n)} = \frac{d^n x}{dt^n}$

$f(t,x,x')=0$

$f(t,x,x',x'')=0$

$f(t,x,x',x'',x''')=0$ .... $f(t,x,x',x'',\ldots,x^{(n)})=0$

$p=p(t,x)$

$$\begin{cases} p_t + p\,p_x = 0 \\ \boxed{p(t=0) = -x} \end{cases}$$

Burger's equation or the transport eq (non-linear eq)

Order: the highest order derivative

$y'+y^2=0$ order-1

$''+y'^2+y=0$ order-2

Degree: the power of the highest order derivative

$(y')+y^2=0$ order-1 degree-1

$(y')^2+y=0$ order-1 degree-2

$f(t,x,x',x'',\ldots,x^{(n)})=0$ n^th order diff eq.

⇕

$\phi(t,x,c_1,c_2,\ldots,c_n)=0$ n-parameter family of curves in x-t plane.

$c_1, c_2, \ldots, c_n$ → arbitrary constants

General solution will involve n-LI arbitrary constants equal to the order of d.eq

... A cos x + B sin x. → 2nd order ⇒ 2 arbitrary constants

General solution will involve n-LI arbitrary constants equal to the order of d.eq

$y'' + y = 0 \longrightarrow y = A\cos x + B\sin x$ → 2nd order ⇒ 2 arbitrary constants in sol. a: General sol.

$y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$= -y \Rightarrow y'' + y = 0$

Particular sol (Assign values to arbitrary consts)

$y'' + y = 0 \longrightarrow y = 2\cos x + 3\sin x$

$= 5\cos x - 2\sin x$

t=0

x=0, t=0 x=L

x=0, x=L

Singular solution it can't be obtained from the general sol by any choice of arbitrary parameters

$y'^2 - 4y = 0$ — $\boxed{y = (x+c)^2}$ is a sol.

$y' = 2(x+c)$

$y'^2 = 4(x+c)^2$ $y'^2 - 4y = 0$

$= 4y$

$\boxed{y = 0}$ is also a sol.

Recall: 1st order differential equations:

$f(t, x, x') = 0 \Longleftrightarrow \phi(t, x, c) = 0$ or $F(t, x) = c$

$$\frac{dx}{dt} = f(t, x)$$

Method of Variable-Seperable

① $F(t,x) = \dfrac{G(t)}{H(x)}$ or $\dfrac{G(x)}{H(t)}$

$\dfrac{dx}{dt} = \dfrac{G(t)}{H(x)} \Rightarrow \int H(x)\,dx = \int G(t)\,dt + K$

② $F(t,x) = f(ax + bt + c)$

$ax + bt + c = z$ diff w.r.t t

Get a diff eq

solve for z

$x' = f(t,x) \neq \dfrac{G(x)}{H(t)}$

$= f(ax + bt + c)$

$ax + bt + c = z$

$a x' + b = z'$

$x' = \dfrac{z' - b}{a}$

③ $F(t,x)$ is a homogeneous function of degree 0.

$F(t,x)$ $F(kt, kx) = k^n F(t,x)$ degree

$f = \dfrac{x^2 + y^2}{x}$ $\dfrac{k^2x^2 + k^2y^2}{kx} = (k)^1\left(\dfrac{x^2+y^2}{x}\right)$

$\dfrac{z' - b}{a} = f(z)$

$z = \phi(t, z, c) = 0$

$f = \sec x$ $\qquad \sec kx \neq k \sec x$. not a homogeneous function

$f(kx, ky) = f(x,y)$.

<u>Exact</u> Equations.

$$\frac{dy}{dx} = f(x,y) = -\frac{M(x,y)}{N(x,y)} \Rightarrow M(x,y)\,dx + N(x,y)\,dy = 0$$

sol. $u(x,y) = C$

$$M\,dx + N\,dy = du(x,y)$$

$\int \text{curl}\,\bar{F} = 0$

$\int_A^B \nabla\phi = \phi\Big|_A^B$.

Thm: $M\,dx + N\,dy = 0$ exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \text{ Necessary condition}$$

$M\,dx + N\,dy = 0$ & $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ holds. Q? Does there exist a function u s/t $du = M\,dx + N\,dy$

Sufficient condn.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\qquad$ for u. construct $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

sol

$$\frac{\partial u}{\partial x} = M \Rightarrow u = \int_{y\text{-const}} M\,dx + g(y) \Longrightarrow u \text{ has to be of this form}$$

$\hookrightarrow$ our goal is to construct $g(y)$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M\,dx + g'(y) = N$$

$$g'(y) = N - \frac{\partial}{\partial y}\int M\,dx.$$

$$g(y) = \int \underbrace{\left\{ N - \frac{\partial}{\partial y}\int M\,dx \right\}}\,dy \quad \text{function of } y \text{ alone.}$$

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int M\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

——— x ———

For the integrability of $M\,dx + N\,dy = 0$, the necessary and sufficient condition is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{or}) \quad \exists\, u(x,y) = C \text{ s/t } du = M\,dx + N\,dy = 0$$

In $\mathbb{R}^3$, we have $\bar{F}\cdot d\bar{r} = 0$, $\quad f_1\,dx + f_2\,dy + f_3\,dz = 0 \quad \exists\, U(x,y,z) = C$

$\Updownarrow$ $\qquad$ curl = '

$\bar{F}\cdot \text{curl}\,\bar{F} = 0$

$M(x,y)\,dx + N(x,y)\,dy = 0 \iff \int\limits_{y\text{-const}} M(x,y)\,dx + \int N(x,y)\,dy = K.$

(Remove terms involving x)

$e^y dx + (xe^y + 2y)\,dy = 0$ $\quad$ M $\quad$ N $\qquad \frac{\partial M}{\partial y} = e^y = \frac{\partial N}{\partial x} = e^y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

$u(x,y) = \int M\,dx + g(y) = \int e^y dx + g(y) = xe^y + g(y)$

$\frac{\partial u}{\partial y} = xe^y + g'(y) = N = xe^y + 2y$

$g'(y) = 2y \Rightarrow g(y) = y^2$

$u(x,y) = xe^y + y^2 = C$

or $\int\limits_{y\text{ cost}} e^y dx + \int (xe^y + 2y)\,dy$

(neglect x terms)

$xe^y + \cancel{xe^y} + y^2$

* What happens if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ in-exact / not exact { Introduce an integrating factor $\mu(x,y)$ s/t $\mu(M\,dx + N\,dy) = 0$ is exact.

$M\,dx + N\,dy = 0$ $\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$ $\Leftarrow\Rightarrow$ $(\underbrace{\mu M}_{P})\,dx + (\underbrace{\mu N}_{Q})\,dy = 0$ $\qquad \mu = \mu(x,y)$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$

$\int\limits_{y\text{ cst}} P\,dx + \int\limits_{w/o\ x\text{ terms}} Q\,dy$

$\mu(M\,dx + N\,dy) = 0$ is exact

its sol $f(x,y) = C$

$f_x\,dx + f_y\,dy = 0 \Rightarrow \frac{\partial f/\partial x}{M} = \frac{\partial f/\partial y}{N} = \mu.$

$\therefore \frac{\partial f}{\partial x} = \mu M$ & $\frac{\partial f}{\partial y} = \mu N$

$\rightarrow$ If there is a sol, then the eq has at least one integrating factor. In fact many! $\leftarrow$ why?

* f sol. $F(f)$

$\mu F(f)(M\,dx + N\,dy) = 0 = F(f)\,df$

$= d\int F(f)\,df$

How to construct μ? $\mu(M\,dx + N\,dy) = 0$ is exact

$\Rightarrow \frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$

$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}.$

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}.$$

$$\frac{1}{\mu}\left(N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y} \right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

(1) μ is a function of x alone.

$$\frac{\partial \mu}{\partial x} \Rightarrow \frac{d\mu}{dx}, \quad \frac{\partial \mu}{\partial y} = 0$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N} = g(x)$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = g(y) \Rightarrow \log \mu = \int g(x)\, dx$$

$$\mu = e^{\int g(x)\, dx}.$$

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Prob (1) $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0$

$M = x^2y - 2xy^2 \quad N = -x^3 + 3x^2y$

$M_y = x^2 - 4xy. \neq N_x = -3x^2 + 6xy.$

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2} = \mu(x, y)$$

$$\mu(M\,dx + N\,dy) = 0 \Rightarrow \underbrace{\left(\frac{1}{y} - \frac{2}{x}\right)}_{P} dx - \underbrace{\left(\frac{x}{y^2} - \frac{3}{y}\right)}_{Q} dy = 0$$

$$P_y = -\frac{1}{y^2} = Q_x = -\frac{1}{y^2} \qquad \underset{y \text{ const}}{\int P\, dx} + \underset{\text{remove } x \text{ term}}{\int Q\, dy} = k$$

$$\Rightarrow \frac{x}{y} - 2\log x + 3\log y = k$$

$M\,dx + N\,dy = 0$

(1) M and N are homogeneous functions of same degree

$\frac{1}{Mx + Ny}$ is an integrating factor.

(2) $M = y f_1(xy)$ and $N = x f_2(xy)$

$y f_1(xy)\, dx + x f_2(xy)\, dy = 0$

$\frac{1}{Mx - Ny}$ is an integrating factor.

$$\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

$(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$

$$M(M\,dx + N\,dy) = 0 \Rightarrow \left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0$$

exact

$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = K$

(3) we have $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Case (1) $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$

IF $\mu = e^{\int f(x)\,dx}$.

Case (2) $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$

then IF $\mu = e^{\int g(y)\,dy}$

(1) $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$

(2) $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

Linear Equations

$\frac{d}{dx}(\alpha u)$ (α scalar) $= \alpha \frac{du}{dx}$

$\int \alpha u\,dx = \alpha \int u\,dx$

$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

$\int (u+v) = \int u + \int v$

$\frac{dy}{dx} = f(x,y)$.

$\Rightarrow \frac{dy}{dx} + p(x)y = Q(x)$ Linear equation

$Q(x) \begin{cases} = 0 & \text{homogeneous} \\ \neq 0 & \text{non-homogeneous} \end{cases}$

$\underbrace{\left[\frac{d}{dx} + p(x)\right]}_{\mathcal{L}} y = Q(x)$

$\mathcal{L}$ – the differential operator

$\mathcal{L}[y]$ is Linear

$T : V \to W$

$\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$, $u, v \in V$.

$\mathcal{L}(\alpha u) = \alpha \mathcal{L}(u)$, $\alpha \in \mathbb{F}$.

$\overbrace{\left[\frac{d}{dx} + p(x)\right]}^{\mathcal{L}} (y_1 + y_2) = \left(\frac{dy_1}{dx} + py_1\right) + \left(\frac{dy_2}{dx} + py_2\right) = \mathcal{L}(y_1) + \mathcal{L}(y_2)$

$(\alpha u) = \alpha \mathcal{L}(u)$

$\left[\frac{d}{dx} + p(x)\right](y) \quad \left(\frac{d}{dx} + \ldots\right) \quad \frac{\ldots}{dx}$

$(\alpha u) = \alpha L(u)$

is a linear differential operator.

$y' + |y| = 0$ not a linear equation

why? $(y+z)' = y' + z'$ $\quad |y+z| \leq |y| + |z|$

$(y+z)' + |y+z| \leq y' + |y| + z' + |z|$

$\frac{dy}{dx} + p(x)y = Q(x)$ $\quad$ * $p(x)$ and $Q(x)$ are continuous on I

$\int p(x)\,dx$ is defined

$e^{\int p(x)dx}$ is also defined

$\underbrace{e^{\int p\,dx}}_{IF}\left(\frac{dy}{dx} + p(x)y\right) = e^{\int p\,dx} \cdot Q(x)$

$\frac{d}{dx}\left[y\, e^{\int p(x)\,dx}\right] = e^{\int p(x)\,dx} Q(x)$

$y\, e^{\int p(x)dx} = \int e^{\int p(x)dx} Q(x)\, dx.$

$y \cdot IF = \int (IF\; Q(x))\,dx + K$

If $Q(x) = 0$, ie $y' + p(x)y = 0$ $\longrightarrow$ $y \cdot IF = K$.

* observations $\quad y \cdot e^{\int p(x)\,dx} = K$ $\quad$ for homogeneous equations

$x \in I$ closed & bdd

$\int p\,dx$ bounded $\Rightarrow e^{\int p\,dx}$ is also bdd (bounded)

$y = K e^{-\int p\,dx} \Rightarrow y$ is bdd on I

$y' + p(x)y = Q(x)y^n$ not a linear eq.

$n > 1$

Bernoulli's equation

This can be converted into a linear differential eq.

↳ This can be converted into a linear differential eq.

$z' + a(x) z = b(x)$ solve.

z

$y' + P(x)\, y = Q(x)\, y^n \Longrightarrow y^{-n} y' + P y^{1-n} = Q.$

put $y^{1-n} = z \Longrightarrow (1-n)\, y^{-n} y' = z'$

$\downarrow$

$$y^{-n} y' = \frac{1}{1-n} z'$$

$\frac{1}{1-n} z' + P z = Q. \Longrightarrow \boxed{z' + (1-n) P z} = (1-n) Q.$

linear non-homogeneous eq

find z.

Then get the sol of original equation.

Orthogonal Trajectories:

Two families of curves such that every member of either family cuts each other at right angles.

$$\begin{tikzcd}
\phi(x,y,c) = 0 \arrow[r, leftrightarrow] \arrow[d, shift left] & f(x,y,y') = 0 \arrow[d, shift left, "\text{Replace } y' \text{ by } -\frac{1}{y'}"] \\
\psi(x,y,d) = 0 \arrow[u, shift left] \arrow[r, leftrightarrow] & f\left(x,y,-\frac{1}{y'}\right) = 0 \arrow[u, shift left]
\end{tikzcd}$$

Ex ① $(1+x)\frac{dy}{dx} - y = e^{3x}(1+x)^2.$

$\hookrightarrow \frac{dy}{dx} - \frac{1}{1+x}\, y = e^{3x} \frac{(1+x)^{\cancel{2}}}{\cancel{1+x}}$

$$p(x) = -\frac{1}{1+x}, \quad Q(x) = (1+x)e^{3x}.$$

$y \cdot IF = \int IF \cdot Q \, dx + K.$

$$IF = e^{-\int \frac{1}{1+x} dx} = \frac{1}{1+x}.$$

$y \cdot \frac{1}{1+x} = \int \frac{1}{\cancel{1+x}} \cancel{(1+x)} e^{3x} dx + K$

$= \frac{e^{3x}}{3} + K. \Longrightarrow y = (1+x)\left(\frac{e^{3x}}{3} + K\right)$

② $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$ $\qquad y' + P(x)y = Q(x)y^n$

$\hookrightarrow \dfrac{dy}{dx} = \dfrac{1+y^2}{\tan^{-1}y - x.}$ not liner

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$= \frac{-x}{1+y^2} + \frac{\tan^{-1}y}{1+y^2}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$ Liner in x.

$\qquad P(y) \qquad Q(y)$

$$x \cdot IF = \int IF \cdot Q(y)\,dy + K.$$

Ex ③

$$x\frac{dy}{dx} + y = x^3 y^6$$

$y' + \cancel{x^n} y = 0$

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$$

Take $y^{-5} = z \Rightarrow -5y^{-6}y' = z'$

$\Rightarrow y^{-6}y' = \frac{-1}{5}z'$

$$\frac{-1}{5}z' + \frac{z}{x} = x^2 \Rightarrow z' - \frac{5}{x}z = -5x^2$$

$IF = x^{-5}$

$$z \cdot x^{-5} = \int (-5x^2)x^{-5}\,dx + c$$

$$y^{-5}x^{-5} = \int (-5x^2)\,x^{-5}\,dx + c$$

$$= -5 \cdot \frac{x^{-2}}{-2} + c$$

$$\Rightarrow (10 + cx^2)\,x^3y^5 = 1$$

① $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y.$ $\longrightarrow \sec^2 y\,\dfrac{dy}{dx} + 2x\tan y = x^3$

$\tan y = z.$

$\varphi\quad x^2 + y^2 = a^2 \rightleftarrows 2x + 2y\,y' = 0 \qquad \psi \perp \varphi.$

$\varphi.\ x^2+y^2=a^2 \rightleftarrows 2x+2y\,y'=0 \qquad \psi - \varphi.$

$\psi(x,y,y')$ $\longleftarrow$ $2x+2y\left(\frac{-1}{y'}\right)=0$

$\frac{dy}{dx}+p(x)y=0 \qquad I.F. = e^{\int p\,dx}$

$\frac{1}{y}dy + p(x)y\,dx = 0. \qquad \frac{M_y - N_x}{N} = f(x) = p(x)$

$M = p(x)$

$N = \frac{1}{y}.$ $\qquad (x,y)\ \ \mu = e^{\int p(x)dx}$

Few more Problems:

Problems:

① $Ax^2+By^2=1$ i.e. $\phi(x,y,A,B)=1$

$Ax+By\,y'=0 \longrightarrow A+B[yy''+y'^2]=0$

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & yy' & 0 \\ 1 & yy''+y'^2 & 0 \end{vmatrix} = 0 \Rightarrow \boxed{xy'^2 + xyy'' - yy' = 0}.$$

② $y=\frac{a+x}{1+x^2} \longrightarrow (1+x^2)y=a+x$

$(1+x^2)y'+2xy=1 \longrightarrow y'+\frac{2x}{1+x^2}y=\frac{1}{1+x^2}$

$p(x) \qquad Q(x)$

$IF = e^{\int p(x)dx}$

③ Consider the family of parabolas having vertex at origin and foci on y-axis

$x^2=4ay \longrightarrow 2x=4ay' \Rightarrow 4a=\frac{2x}{y'}$

$x^2=\frac{2x}{y'}\cdot y \Rightarrow \boxed{xy'=2y.}$

④ $\frac{dy}{dx}=\frac{\sin x + x\cos x}{y(2\log y+1)}=\frac{F(x)}{G(y)}$

$\hookrightarrow \int(\sin x + x\cos x)dx = \int y(2\log y+1)dy + K.$

⑤ $\frac{dy}{dx}=(4x+y+1)^2$ put $4x+y+1=z \longrightarrow 4+y'=z' \longrightarrow y'=z'-4$

$z'-4=z^2 \longrightarrow z'=4+z^2 \Rightarrow \int\frac{dz}{4+z^2}=\int dx + K$

$\frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right)=x+C \qquad z = 4x+y+1$

⑥

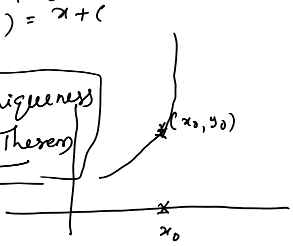
$(x^2+1)\frac{dy}{dx}+y^2+1=0$

$y(0)=1$

$\hookrightarrow \int\frac{dy}{1+y^2}+\int\frac{dx}{1+x^2}=K.$

$\tan^{-1}y+\tan^{-1}x=K.$

Existence-Uniqueness
Picard's Theorem



$y(0)=1$

$$\int \frac{dy}{1+y^2} + \int \frac{dx}{1+x^2} = K. \longrightarrow \tan^{-1}y + \tan^{-1}x = K.$$

$$y(0)=1 \implies K=\pi/4 \implies \tan^{-1}y + \tan^{-1}x = \pi/4$$

(7) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx \implies \frac{dy}{dx} = \frac{y+\sqrt{y^2+x^2}}{x}$ $F(x,y)$

$F(kx,ky) = k^{n=0}F(x,y)$ homogeneous func of deg. 0.

$y = vx \longrightarrow y' = v'x + v$

$$v'x + v = \frac{vx + \sqrt{v^2x^2+x^2}}{x} = \frac{vx + x\sqrt{1+v^2}}{x}$$

$$v'x + \cancel{v} = \cancel{v} + \sqrt{1+v^2}$$

$$\frac{dv}{dx}x = \sqrt{1+v^2} \implies \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} + C$$

$$f(v) = \log x + C$$

$v \to y/x$.

$$* \quad \frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

case (1) if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = K.$

$$\frac{dy}{dx} = K.$$

case (2) $\underbrace{\frac{a_1}{a_2} = \frac{b_1}{b_2}}_{K} \neq \frac{c_1}{c_2}$ $\quad \frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{K(\underbrace{a_1x+b_1y}) + c_2}$

put $z = a_1x + b_1y \longrightarrow dz = a_1 + b_1y'$

$$\frac{dz - a_1}{b_1}$$

case (3) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ $\quad \frac{dy}{dx} \longrightarrow \frac{dY}{dX}$

$x \longrightarrow X + h$

$y \longrightarrow Y + K.$

$$\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \implies \left.\begin{aligned} a_1h+b_1K+c_1 &= 0 \\ a_2h+b_2K+c_2 &= 0 \end{aligned}\right\}$$

$$\left.\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right\} \longrightarrow \left\{\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}\right.$$

homogeneous of deg 0.

put $Y = v\cancel{X}$:

$M\,dx + N\,dy = 0$ $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ✓

$$\int M\,dx + \int N\,dy = K.$$

(y-const) $\qquad$ (Remove all x-terms)

$$\overset{\neq 0}{\mu}(M\,dx + N\,dy) = 0 \longrightarrow P\,dx + Q\,dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\bar{X} \cdot \operatorname{curl} \bar{X} = 0 \iff (\mu \bar{X}) \cdot \operatorname{curl}(\mu \bar{X}) = 0$

$$\underbrace{\left(1 + e^{x/y}\right)}_{M} dx + \underbrace{e^{x/y}\left(1 - \frac{x}{y}\right)}_{N} dy = 0$$

$$M_y = e^{x/y}\left(\frac{-x}{y^2}\right) = N_x = e^{x/y}\left(\frac{-x}{y^2}\right) \quad \text{Exact}$$

$$\int \left(1 + e^{x/y}\right) dx + \int 0\, dy = K$$

(y const)

Result: For a linear homogeneous d.eq, if φ and ψ are two solutions, then their linear combination is also a solution.

* $\frac{dy}{dx} + p(x)\, y = 0$ — homogeneous, Linear.

ϕ is a sol. i.e. $\frac{d\phi}{dx} + p(x)\phi = 0$

ψ is another sol. $\frac{d\psi}{dx} + p(x)\psi = 0$

$\alpha\phi$ is also a sol. $\alpha \frac{d\varphi}{dx} + \alpha p(x)\varphi = 0$

scalar

ϕ, ψ are sols $\quad \tau = \underbrace{\alpha\phi + \beta\psi}$. Q? is this a sol. $\longrightarrow$ super position.

τ', $\quad \frac{d\tau}{dx} + p(x)\tau = 0$

① $xy = K$. Find the orthogonal trajectories

$$xy' + y = 0 \xleftrightarrow{\text{Replace } y' \text{ by } -\frac{1}{y'}} x\left(\frac{-1}{y'}\right) + y = 0$$

$$x\,dx - y\,dy = 0 \implies \boxed{x^2 - y^2 = d^2}$$

② $y^2 = 4a(x+a)$, a-parameter. Find the orthogonal trajectory?

$yy' = 2a \longrightarrow a = \frac{yy'}{2}$

$y^2 = 4 \cdot \frac{yy'}{2}\left(x + \frac{yy'}{2}\right)$

$= 2yy'\left(x + \frac{yy'}{2}\right) \longrightarrow \boxed{y(y')^2 + 2xy' - y = 0}$ (*)

Replace y' by $\frac{-1}{y'}$

$y \cdot \left(\frac{-1}{y'}\right)^2 + 2x\left(\frac{-1}{y'}\right) - y = 0$

$y \cdot \frac{1}{y'^2} - \frac{2x}{y'} - y = 0$

$y - 2xy' - yy'^2 = 0$

$\boxed{yy'^2 + 2xy' - y = 0}$ (#)

This family of curves intersect themselves orthogonally - ie Self-orthogonal

$y^2 = 4a(x+a)$

Self-orthogonal

(*) & (#) are same

This family of curves intersect themselves orthogonally - ie Self-orthogonal

$$\boxed{yy'^2 + 2xy' - y = 0}$$ Self-orthogonal

$\#$ $\boxed{* \ \& \ \# \text{ are same}}$

$*$ $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1 \longrightarrow f(x, y, \lambda) = 0$ Self-orthogonal

$\lambda \rightarrow$ parameter

singular

$$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1 \quad \lambda \text{ — parameter}$$

* Newton's law of cooling
 population growth
 Radioactive decay prob

$t \to \infty$ what happens

$\theta \to$ temp at t
$\theta_0 \to$ surrounding temp

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - \theta_0) \quad (>0)$$

$$\frac{dP}{dt} = k(P - P_0)$$

$$\int \frac{d\theta}{\theta - \theta_0} = \int -K\, dt + c$$

$$\log(\theta - \theta_0) = -Kt + c$$

$$\theta - \theta_0 = Me^{-Kt}$$

$$\theta = \theta_0 + Me^{-Kt}$$

?

$t = 0$ let θ_1 be the temp of the cup

$$\theta(0) = \theta_1$$

$$\theta_1 = \theta_0 + M \cdot e^{-K \cdot 0 \cdot t}$$

$$\Rightarrow M = \theta_1 - \theta_0$$

$$\therefore \theta = \theta_0 + (\theta_1 - \theta_0)e^{-Kt}$$

passes $(0, \theta_1)$.