Higher order differential equations

n^{th} order: $f(x, y, y', y'', \ldots, y^{(n)}) = 0$

2^{nd} order: $\underbrace{f(x,y,y',y'')}=0$

General form: $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)\,y = R(x)$

$P(x)$, $Q(x)$ and $R(x)$ are continuous on I.

$R(x)=0 \longrightarrow$ homogeneous

$\neq 0 \longrightarrow$ non homogeneous.

$y'' + P(x)y' + Q(x)y = R(x)$ Linear equation $\mathcal{L} : \dfrac{d^2}{dx^2} + Py + Q$

its sol $y(x) = \underbrace{C_F(y_c)}_{\text{complementary solution}} + \underbrace{y_p}_{\text{particular sol.}}$

y_c — complementary sol.? it is a sol of the associated homogeneous part

$\xrightarrow{\text{sol of}}$ ie $y'' + P(x)y' + Q(x) = 0$ All the arbitrary constants will appear only in the complementary sol.

y_p: a sol that solves the full eq: it doesn't contain the arbitrary constants

$\longrightarrow$ particular sol.

$\therefore y = y_c + y_p$. This is called the general sol.

Complementary functions.

$T = y_c + y_p$

$$\frac{d^2T}{dx^2} + P\frac{dT}{dx} + QT = R$$

$$\frac{d^2}{dx^2}(y_c+y_p) + P\frac{d}{dy}(y_c+y_p) + Q(y_c+y_p)$$

$$= \cancel{\left(\frac{d^2y_c}{dx^2} + P\frac{dy_c}{dx} + Qy_c\right)}^{\,0} + \underbrace{\left(\frac{d^2y_p}{dx^2} + P\frac{dy_p}{dx} + Qy_p\right)} = R.$$

Complementary sol: $y'' + P(x)y' + Q(x)y = 0 \rightarrow$ eq with variable coefficients.

with constant coefficients

Complementary sol: $y''+p(x)y'+q(x)y=0$ → eq with variable coefficients.

$\underline{y''+py'+qy=0}$ → equation with constant coefficients $p, q \in \mathbb{R} | \mathbb{C}$

↓

Linear & homogeneous.

Observation ① y_1 & y_2 are two sol's

Then $\alpha_1 y_1 + \alpha_2 y_2$ is also a sol. Verify

$y''+py'+qy=0$ How do we solve

$y'=0 \xrightarrow{sol} y=K$ → $y'=\lambda y$ → $y=e^{\lambda x}\cdot K$

$= K e^{\lambda x}$.

$y''=0$ → $y=Kx+m$

Assume sol is of the form $y=e^{\lambda x}$.

$y'=\lambda e^{\lambda x}$, $y''=\lambda^2 e^{\lambda x}$.

$y''+py'+qy=0$ → $(\lambda^2+p\lambda+q)$

$\boxed{\lambda^2+p\lambda+q=0}$ → $\lambda=\lambda_1$ & λ_2

Characteristic eq or

Auxilliary eq.

Case ① $\lambda_1 \neq \lambda_2 \in \mathbb{R}$.

$e^{\lambda_1 x}$ & $e^{\lambda_2 x}$ are two solutions.

↓ y_1 ↓ y_2

$c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = y$ Check this is a sol.

$\frac{d^2y}{dx^2}-4y=0 \Rightarrow \lambda^2-4=0 \Rightarrow \lambda=\pm 2$

e^{2x} & e^{-2x} are two sols.

$c_1 e^{2x} + c_2 e^{-2x} = y$ is the general sol.

Case ② $\lambda_1=\lambda_2 \in \mathbb{R}$

$e^{\lambda_1 x}$, $e^{\lambda_1 x}$ $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_1 x}$

$(c_1+c_2)e^{\lambda_1 x}$

$e^{\lambda_1 x}$, $e^{\lambda_1 x}$ $\quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_1 x}$

$y_1 \qquad y_2$

$= (c_1 + c_2) e^{\lambda_1 x}$

$= c_3 e^{\lambda_1 x}$ sol. not a general sol.

post-pone this analysis

bcz no. of arbitrary consts $\neq$ order of d.eq.

Case (3) $\lambda_1 \in \mathbb{C} \Rightarrow \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$

$$\begin{cases} \lambda_1 = \alpha + i\beta, & \alpha, \beta \in \mathbb{R} \\ \lambda_2 = \alpha - i\beta \end{cases}$$

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}.$$

$$= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left(c_1 \cos\beta x + i c_1 \sin\beta x + c_2 \cos\beta x - i c_2 \sin\beta x\right)$$

$$= e^{\alpha x}\left(\underbrace{(c_1 + c_2)}_{K_1} \cos\beta x + \underbrace{i(c_1 - c_2)}_{K_2} \sin\beta x\right)$$

$$= e^{\alpha x}\left(K_1 \cos\beta x + K_2 \sin\beta x\right)$$

Eg. $y'' + 4y = 0 \Rightarrow \lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$\lambda = 0 \pm 2i$

$= e^{0x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$

Recall $\lambda_1 = \lambda_2$ $\quad e^{\lambda_1 x}$ is the only sol.

How do we construct the second sol.

$$\begin{cases} y_1 = e^{\lambda_1 x} \\ y_2 = e^{\lambda_1 x} \end{cases} \quad \frac{y_1}{y_2} = \frac{2e^{\lambda_1 x}}{e^{\lambda_1 x}} = 2 \Rightarrow y_1 = 2y_2$$

$$\begin{cases} v_1, v_2, \ldots, v_n \in V, \text{ are LI if} \\ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \iff \alpha_i = 0 \quad i = 1(1)n \end{cases}$$

$\varphi_1, \varphi_2, \ldots, \varphi_n \in C[a,b]$

$C(\Omega)$

$\boxed{\text{Wronskian}}$

$\lambda_1 = \lambda_2$ ✓ How to construct second sol.

$y'' + P(x)y' + Q(x)y = 0 \qquad$ Let $y_1(x)$ is a sol.

$$y'' + p(x)y' + Q(x)y = 0$$

let $y_1(x)$ is a sol.

y_2 construct

$y_2 = K y_1$ is also a sol. $\quad$ y_2 LI to y_1

$\downarrow$

$v(x) y_1$

$y_2 = v y_1$ is a sol. (Assumption)

$\hookrightarrow$ construct v s/t y_2 is a sol.

Assumption
$$\begin{cases} y_2 = v y_1 \text{ is a sol.} \\ y_2' = v y_1' + v' y_1 \\ y_2'' = v y_1'' + 2v' y_1' + v'' y_1 \end{cases}$$

substitute

$$v y_1'' + 2v' y_1' + v'' y_1 + P(v y_1' + v' y_1) + Q v y_1 = 0$$

$$v\underbrace{\cancel{(y_1'' + P y_1' + Q y_1)}}_{0 \text{ bcz } y_1 \text{ is a sol.}} + \underbrace{v'' y_1 + v'(2y_1' + P y_1)}_{} = 0$$

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P.$$

integrate $\quad \log v' = -2\log y_1 - \int p(x)\,dx$

$$v' = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

$$\boxed{v = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx} + C$$

$\therefore \; y_2 = v y_1$

$$\frac{y_1}{y_2} = \frac{y_1}{v y_2} \neq K. \Rightarrow \boxed{y_2 \neq m\, y_1}$$

Ex: $x^2 y'' + x y' - y = 0$

$\hookrightarrow$ $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$ $\quad$ Check $y_1 = x$ is a sol.

Find other sol: $p(x) = \frac{1}{x}$. $\quad$ $y_1 = x$

$$v = \int \frac{1}{y_1^2} e^{-\int p\,dx}\,dx = \int \frac{1}{x^2} e^{-\int \frac{1}{x}dx}\,dx = \frac{x^{-2}}{-2}$$

$\therefore y_2 = v \cdot y_1 = \frac{x^{-1}}{-2} = \frac{-1}{2x}$. $y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x^{-1}$.

Go back to the original prob: $p, q \in \mathbb{R}$

$\lambda_1 = \lambda_2 = -p/2$, $y_1 = e^{(-p/2)x} = e^{\lambda_1 x}$

$y_2 = v y_1$

$$v = \int \frac{1}{y_1^2} e^{-\int p\,dx} dx$$

$$= \int \frac{1}{\cancel{e^{-px}}} \cdot \cancel{e^{-px}} dx = x. \quad \boxed{v = x}$$

$\lambda_1 x$

Recall: $y'' + p(x) y' + Q(x) y = 0$

$y'' + p y' + q y = 0$, $p, q \in \mathbb{R}$.

$y = e^{\lambda x}$, $\boxed{\lambda^2 + p\lambda + q = 0}$ A.Eq.

$$\lambda - \begin{cases} \lambda_1 \neq \lambda_2 \in \mathbb{R} & y(x) = A e^{\lambda_1 x} + B e^{\lambda_2 x}. \\ \lambda, \bar{\lambda} \in \mathbb{C}, \ \alpha \pm i\beta, \ \alpha, \beta \in \mathbb{R}; & y(x) = e^{\alpha x}(A \cos\beta x + B \sin\beta x) \\ \lambda_1 = \lambda_2 \in \mathbb{R}, & y(x) = A e^{\lambda_1 x} + B x e^{\lambda_1 x}. \end{cases}$$

? L.I. $\not\exists\, y_2 = e^{\lambda_1 x} \longrightarrow y: A e^{\lambda_1 x} + B e^{\lambda_1 x} = (A+B) e^{\lambda_1 x}$

$= K e^{\lambda_1 x}$. not a general sol.

Construct a new sol from old one (method of reduction of order.)

idea: $y_1 = e^{\lambda_1 x}$ sol. $\Rightarrow$ $y_1 = K e^{\lambda_1 x}$ sol.

replace k by $v(x)$

Construct v s/t $v y_1$ is a sol.

If $y_2 = v y_1$ is a sol.

then $y = c_1 y_1 + c_2 y_2$ is a general sol.

$= c_1 y_1 + c_2 v y_1$

$= y_1 (c_1 + c_2 v)$.

To construct v? $y_2 = v y_1$ is a sol., y_2', y_2''

$$\boxed{v = \int \frac{1}{y_1^2} \cdot e^{-\int p(x) dx} dx}$$

* $\lambda^2 + p\lambda + q = 0$

$\lambda = 0, 1 \longrightarrow y = c_1 e^{0 \cdot x} + c_2 e^{x} = c_1 + c_2 e^{x}$.

$\lambda = 1, 2 \longrightarrow y = c_1 e^{x} + c_2 e^{2x}$

$\lambda = 1, 1 \longrightarrow y = c_1 e^{x} + c_2 x e^{x}$

$\lambda = n, n \longrightarrow y = c_1 e^{nx} + c_2 x e^{nx}$.

$\lambda = 2 \pm i3 \longrightarrow y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

* $y'' + pxy' + qy = 0$ Now we know how to write the sol.

Homogeneous Linear equations with variable coefficients

(1) Euler – Cauchy equations

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + \overset{P_{n-1} x \frac{dy}{dx}}{} + P_n y = 0$$

$$x^2 \frac{d^2y}{dx^2} + P_1 \, x\frac{dy}{dx} + P_2 y = 0 \quad \checkmark \longrightarrow x^2 \frac{d^2y}{dx^2} + \boxed{P_1 \frac{dy}{dz} + P_2 y = 0}$$

?

$x = e^z \longrightarrow \log x = z$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Longrightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \text{ or}$$

$$\boxed{x \frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \boxed{\frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}} \checkmark$$

$$x^2 \frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (D^2 - D)y \quad \text{where } D = \frac{d}{dz}$$

$$= D(D-1)y.$$

$$\begin{cases} x\dfrac{dy}{dx} = Dy. \\ x^2 \dfrac{d^2y}{dx^2} = D(D-1)y \end{cases} \qquad D = \frac{d}{dz}$$

Induction

$$\begin{cases} x^3 \dfrac{d^3y}{dx^3} = D(D-1)(D-2)y \\ \vdots \\ x^n \dfrac{d^ny}{dx^n} = D(D-1)\cdots(D-(n-1))y. \end{cases}$$

$$x^2\frac{d^2y}{dx^2} + P_1 x\frac{dy}{dx} + Q_1 y = 0 \Longrightarrow D(D-1)y + P_1 Dy + Q_1 y = 0$$

$$= D^2y - Dy + P_1 Dy + Q_1 y = 0$$

$$\boxed{= D^2y + (P_1 - 1)Dy + Qy = 0} \rightarrow \text{independent variable is } z.$$

$y = y(z)$

$= y(\log x)$

(1) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$

$x = e^z$, $\log x = z$

(1) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

$x = e^z \longrightarrow \log x = z$

$D(D-1)y - Dy + y = 0$

$(D^2 - D - D + 1)y = 0$

$(D^2 - 2D + 1)\, y = 0$ $\quad y = e^{\lambda z}$

$\lambda^2 - 2\lambda + 1 = 0$ $\quad$ AE $\quad \lambda = 1, 1$

$y = c_1 e^{1\cdot z} + c_2 z \cdot e^{1\cdot z}$

$= c_1 e^{\log x} + c_2 \log x \cdot e^{\log x} = c_1 x + c_2 x \log x$

Legendre's eq: Assignment

$(a+bx)^n \frac{d^n y}{dx^n} + P_1 \cdot (a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_n y = 0, \quad P_1, P_2, \dots, P_n \in \mathbb{R}.$

$a + bx = e^z \qquad \frac{dy}{dx} \qquad \frac{d^2y}{dx^2}, \quad \dots$

— x —

$\varphi_1, \varphi_2, \dots, \varphi_n$ are LI $\iff c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n = 0$ $\quad c_1 = 0,\ c_2 = 0, \dots, c_n = 0$

* $\cos x, \sin x$ are they LI on I

$c_1 \cos x + c_2 \sin x = 0$
$-c_1 \sin x + c_2 \cos x = 0$
$\Rightarrow \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \longrightarrow c_1 = 0,\ c_2 = 0$

$\cos^2 + \sin^2 x = 1$

det $\neq 0$

$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \longrightarrow$ Wronskian $(\cos x, \sin x)$

* φ_1, φ_2 are LI or not?

$c_1\varphi_1 + c_2\varphi_2 = 0$
$c_1\varphi_1' + c_2\varphi_2' = 0$
$\Rightarrow \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$ $\neq 0$ for unique sol.

$\longrightarrow$ Wronskian $W(\varphi_1, \varphi_2)$

$\varphi_1\varphi_2' - \varphi_2\varphi_1' = \det$

$|\ \varphi_1 \quad \varphi_2 \quad \varphi_3\ |$

$\varphi_1\varphi_2' - \varphi_2\varphi_1$ $\quad W(\varphi_1, \varphi_2)$

$\varphi_1, \varphi_2, \varphi_3$
$$\begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{vmatrix}$$

$$W(\varphi_1, \varphi_2, \dots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}_{n\times n}$$

* $1, x, x^2, \dots, x^n$ on I are these LI

$c_1 + c_2 x + c_3 x^2 + \dots + c_n x^n = 0$ $\qquad c_1 = 0$

$c_2 + \dots + c_n n x^{n-1} = 0$

$2c_3 + \dots + c_n n(n-1) x^{n-2} = 0$ $\qquad c_{n-2} = 0 \qquad c_i\ (i = 1(1)n) = 0$

$c_{n-1} = 0$

$c_n \underbrace{n(n-1)\dots 1}_{n!} = 0 \implies c_n = 0$

LD

* e^{ix}, $\cos x$, $\sin x$ are these LI

$e^{ix} - \cos x - i\sin x = 0$

$c_1 = 1, \quad c_2 = -1, \quad c_3 = -i \ (\neq 0)$

$y'' + p(x)y' + Q(x)y = 0$, $\quad p(x), Q(x)$ are continuous on I

Assume y_1 & y_2 are two solutions. $\qquad ax + by = 0$

Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$\frac{d}{dx}(\ ) = y_1 y_2'' + \cancel{y_1' y_2'} - y_2 y_1'' - \cancel{y_1' y_2'}$

$y_2(y_1'' + p(x)y_1' + Q(x)y_1) = 0$ bcz y_1 is a sol.

$y_1(y_2'' + p(x)y_2' + Q(x)y_2) = 0$ $\quad - y_2 -$

$(y_2 y_1'' - y_1 y_2'') + p(x)\underbrace{(y_2 y_1' - y_1 y_2')} + Q(x)(\cancel{y_1 y_2 - y_2 y_1}) = 0$

$+\frac{dW}{dx} + p(x) W = 0 \implies \frac{dW}{dx} + p(x) W = 0$

$W = K e^{-\int p(x)\,dx}$.

$e^{-\int p(x)\,dx} \neq 0$

(never vanishes)

$$W = K e^{-\int P(x)\,dx} \neq 0$$

W is never zero (no where vanishes)

$\begin{cases} K \neq 0, & W \text{ never zero} \\ K = 0, & W \equiv 0 \text{ identically zero} \end{cases}$

Conclusion: $y'' + P(x) y' + Q(x) y = 0$, $P(x), Q(x)$ are continuous on I. y_1 & y_2 are two sol^{n}, then the Wronskian of y_1, y_2 is either identically zero or no where vanishes on I.

Linear Dependency / Independency.

$\varphi_1, \varphi_2 \to \frac{\varphi_1}{\varphi_2} = K \in \mathbb{R}$. tells that $\varphi_1 = K\varphi_2$ φ_1, φ_2 are LD

Wronskian $W(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$

$x^2, x|x|$, $x \in \mathbb{R}$ Q? Are they LD? or LI?

$c_1 x^2 + c_2 x|x| = 0$ $\begin{cases} c_1 x^2 + c_2 x^2 = 0 = (c_1 + c_2) x^2 & \text{for } x \geq 0 \\ c_1 x^2 - c_2 x^2 = 0 = (c_1 - c_2) x^2 & \text{for } x < 0 \end{cases}$

$c_1 + c_2 = 0$ & $c_1 - c_2 = 0 \implies c_1 = 0$ & $c_2 = 0$

on whole of $\mathbb{R}$, x^2 & $x|x|$ are LI

$x^2, x|x|$, $x \in \mathbb{R}^+ \subset \mathbb{R}$

$c_1 x^2 + c_2 x^2 = 0 \implies (c_1 + c_2) x^2 = 0$

$\hookrightarrow c_1 = -c_2$ ✗

Conclusion: (1) φ_1, φ_2 can be LI on some domain A but they can be LD on a some proper subdomain $D \subset A$

$x^2, x|x|$, $x \in \mathbb{R}^+$ LD

$[1, 2] \subset \mathbb{R}^+$

$x^2, x|x|$ are LD.

conclusion * if φ_1, φ_2 are LD on A, then they continue to be LD on any proper subdomain of A. $(D \subset A)$

Wronskian: $y'' + P(x) y' + Q(x) y = 0$, $P(x), Q(x)$ are continuous on I

$x \in I$.

let y_1 & y_2 be two solutions.

y_1 is a sol $\Rightarrow$ $y_2\left(y_1'' + P(x)y_1' + Q(x)y_1\right) = 0$

y_2 is a sol $\Rightarrow$ $y_1\left(y_2'' + P(x)y_2' + Q(x)y_2\right) = 0$

subtract $\left(y_2 y_1'' - y_1 y_2''\right) + P(x)\left(y_2 y_1' - y_1 y_2'\right) = 0$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\frac{dW}{dx} = y_1 y_2'' - y_2 y_1''$$

$$\frac{dW}{dx} + P(x)W = 0$$

$$\downarrow$$

$$W = K e^{-\int P(x)\,dx} \neq 0 \text{ for any } x.$$

If $K=0$, then $W \equiv 0$, ie Wronskian vanishes throughout the domain
$K \neq 0$, then $W \neq 0$ ie Wronskian nowhere vanishes.

Conclusion: $y'' + P(x)y' + Q(x)y = 0$, $P(x), Q(x)$ are continuous on I.
y_1 & y_2 are two sols. Then their Wronskian is either identically zero or nowhere vanishes on I.

Ex: $y_1 = x$, $y_2 = x^2$, $x \in [-1, 1]$

$$y = c_1 x + c_2 x^2$$

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$W\big|_{x=0} = 0.$$

$W \neq 0 \quad \forall x \neq 0,\ x \in I.$

$\cancel{y' = c_1 + 2c_2 x}$ ✓

$y'' = 2c_2 \Rightarrow c_2 = \frac{y''}{2}$

$y' - 2c_2 x = c_1$

$c_1 = y' - y'' x.$

$$y = (y' - y''x)\,x + \frac{y''}{2}x^2$$

$$= y'x - \frac{y''}{2}x^2$$

$$\Rightarrow \frac{y''}{2}x^2 - y'x + y = 0 \Rightarrow y''x^2 - 2y'x + 2y = 0$$

$$y'' - \frac{2x}{x^2}y' + \frac{2}{x^2}y = 0$$

$y'' + P(x)y' + Q(x)y = 0$

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0, \qquad x \in I$$

$\qquad P(x) \qquad Q(x)$

$[-1, 1]$

$\cancel{}$

$[\ \ ] \setminus \{0\}$

$x=0$ is a singular point

* Result $y'' + P(x)y' + Q(x)y = 0$, $P(x)$ & $Q(x)$ are continuous on I.
y_1 & y_2 are two sols.

* let us assume $y_1 \neq 0$ & $y_2 \neq 0$
Assume $y_1 = k y_2 \Rightarrow y_1$ & y_2 are LD

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ & \end{vmatrix} = \begin{vmatrix} y_1 & \frac{1}{k}y_1 \\ & \end{vmatrix} = \frac{1}{k}y_1 y_1' - \frac{1}{k}y_1 y_1' = 0$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & \frac{1}{k} y_1 \\ y_1' & \frac{1}{k} y_1' \end{vmatrix} = \frac{1}{k} y_1 y_1' - \frac{1}{k} y_1 y_1' = 0$$

y_1 & y_2 are LD sols $\Longrightarrow$ $W(y_1, y_2) = 0$ on $x \in I = [a, b]$
$\Longleftarrow$

* Assume $\underline{W(y_1, y_2) = 0}$. Further assume $y_1 \not\equiv 0$ on I

y_1 is conti $[c, d] \subset [a, b]$

$y_1 \neq 0 \ \forall x \in [c, d]$
$\Downarrow$
$y_1^2 \neq 0$

W on $[c, d]$. $\dfrac{y_1 y_2' - y_2 y_1'}{y_1^2} = d\left(y_2/y_1\right) = 0$

$\Downarrow$

$y_2/y_1 = K \Rightarrow y_2 = K y_1$

$\Rightarrow y_2$ & y_1 are LD. ✓

—x—

$y'' + P(x) y' + Q(x) y = R(x) \neq 0 \longrightarrow y = y_c + y_p$

(1) solve the homogeneous part $y'' + p(x) y' + Q(x) y = 0$

$\downarrow \uparrow$

$y_c = c_1 y_1(x) + c_2 y_2(x)$

(2) Construct y_p. method of undetermined coefficients ✓

Method of Variation of parameters

$y'' + P(x) y' + Q(x) y = R(x)$

$y'' + p(x) y' + Q(x) y = 0 \Longrightarrow y_c = c_1 y_1 + c_2 y_2$

idea Replace c_1 & c_2 by some functions u, & v
s/t $y_p = u y_1 + v y_2$ is a sol to $y'' + P y' + Q y = R$.

Then $y = y_c + y_p$ is a general sol.

$y_p = u y_1 + v y_2$ is a sol. find u & v

$y_p' = u y_1' + \boxed{u' y_1 + v' y_2} + v y_2'$

set this to 0.

$\boxed{u' y_1 + v' y_2 = 0}$ — (1)

$y_p' = uy_1' + vy_2'$

$y_p'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$

substitute y_p, y_p', y_p'' in the eq.

$\cancel{uy_1''} + \cancel{vy_2''} + u'y_1' + v'y_2' + \cancel{Puy_1'} + \cancel{Pvy_2'} + \cancel{Quy_1} + \cancel{Qvy_2} = R(x)$

$u(\cancel{y_1'' + Py_1' + Qy_1})^{0} + v(\cancel{y_2'' + Py_2' + Qy_2})^{0} + u'y_1' + v'y_2' = R(x)$

$\boxed{u'y_1 + v'y_2 = 0}$ — ②

$\boxed{u'y_1' + v'y_2' = R(x)}$ — ②

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{y_1 y_2' - y_2 y_1} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

$$= \frac{1}{W(y_1, y_2)} \begin{bmatrix} -y_2 R(x) \\ y_1 R(x) \end{bmatrix}$$

$u' = -\frac{y_2 R}{W} \Rightarrow u = -\int \frac{y_2 R}{W}\, dx$

$v' = \frac{y_1 R}{W} \Rightarrow v = \int \frac{y_1 R}{W}\, dx$

$$\boxed{y_p = -\left(\int \frac{y_2 R}{W} dx\right) y_1 + \left(\int \frac{y_1 R}{W} dx\right) \cdot y_2.}$$

General sol $y = c_1 y_1 + c_2 y_2 + y_p$

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① $(D^2 + a^2)\, y = \cancel{\sec ax}$ ~~sin ax~~ secax Variation of para

$(D^2 + a^2)\, y = 0 \Rightarrow \lambda^2 + a^2 = 0 \Rightarrow \lambda = \pm ai$

$y_c = c_1 \underbrace{\cos ax}_{y_1} + c_2 \underbrace{\sin ax}_{y_2}$

$y_1 = \cos ax, \quad y_2 = \sin ax \qquad W(y_1, y_2) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$

$= \underline{\underline{a}}$

$y_p = u\cos ax + v\sin ax.$

$$u = -\int \frac{y_2 \cdot R}{W}\,dx = -\int \frac{\sin ax \cdot \sec ax}{a}\,dx =$$

$$v = \int \frac{y_1 R}{W}\,dx = \int \frac{\cos ax \sec ax}{a}\,dx$$

$$\boxed{y = c_1 y_1 + c_2 y_2 + (u y_1 + v y_2)}$$

Recall: $y'' + P(x)y' + Q(x)y = 0, \quad x \in I, \; P$ and $Q \in C(I)$

Let y_1 and y_2 be two sols.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Th (1): W is either identically zero or nowhere vanishes. ✓

Th (2): y_1 and y_2 are LD iff $W \equiv 0$ on I

Remarks: (1) If y_1 and y_2 have a common zero at $x_0 \in I$, then they are LD

(2) y_1 & y_2 can't have a max/min at the same point

* If y_1 & y_2 are LI sols, then y_2 vanishes exactly once between any two successive zeros of y_1. Vice-versa

* $y'' + P(x)y' + Q(x)y = R(x)_{(\neq 0)}$ non-homogeneous. $P, Q, R \in C(I)$

$y = y_c + y_p \longrightarrow$ particular Integral.

y_c: complementary sol comes from $(y'' + P(x)y' + Q(x)y = 0)$

Let y_1 & y_2 are LI sols of $y'' + Py' + Qy = 0$

$$y_c = c_1 y_1 + c_2 y_2$$

Using y_c can we construct y_p? YES Method of Variation of parameters.

Idea: Replace c_1 and c_2 by some unknown functions $u(x)$ & $v(x)$ s/t $y_p = uy_1 + vy_2$ is a sol to the non-homogeneous eq.

If y_p is a sol. we must have $y_p'' + Py_p' + Qy_p = R$.

$y_p = uy_1 + vy_2 \implies y_p' = uy_1' + vy_2' + \boxed{u'y_1 + v'y_2}$

Set this to zero.

$\boxed{u'y_1 + v'y_2 = 0}$ — (*)

Then $y_p' = uy_1' + vy_2' \implies y_p'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$

$\ldots \; P\cancel{uy_1'} + P\cancel{vy_2'} + Q\cancel{uy_1} + Q\cancel{vy_2} = R.$

Then $y_p' = uy_1' + vy_2' \Rightarrow y_p'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$

$\therefore y_p'' + Py_p' + Qy_p = R \Rightarrow \cancel{uy_1''} + \cancel{vy_2''} + u'y_1' + v'y_2' + \cancel{Puy_1'} + \cancel{Pvy_2'} + \cancel{Quy_1} + \cancel{Qvy_2} = R$

$\Rightarrow u(\underbrace{y_1'' + Py_1' + Qy_1}_{0}) + v(\underbrace{y_2'' + Py_2' + Qy_2}_{0})$ y_2 is a sol of homogeneous part

y_1 is a sol of homogeneous eq

$+ u'y_1' + v'y_2' = R.$

$\therefore \boxed{u'y_1' + v'y_2' = R}$ — #

$$\left.\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= R \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{\underbrace{y_1y_2' - y_2y_1'}_{W(y_1,y_2)}}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\left.\begin{aligned} u' &= \frac{-y_2R}{W} \Rightarrow u = +\int \frac{-y_2R}{W}\,dx. \\ v' &= \frac{y_1R}{W} \Rightarrow v = \int \frac{y_1R}{W}\,dx \end{aligned}\right\}$$ use then u & v as y_p.

* $(D^2 + a^2)y = \sin ax.$

$D^2 \to \frac{d^2}{dx^2}$

$\to$ A.E $\lambda^2 + a^2 = 0 \Rightarrow \lambda = \pm ai$

$$y_c = c_1 \underbrace{\cos ax}_{y_1} + c_2 \underbrace{\sin ax}_{y_2}.$$

$$W(y_1, y_2) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

To construct y_p. $\quad u = \int -\frac{y_2R}{W}\,dx = \int \frac{-\sin ax \sin ax}{a}\,dx.$

$$v = \int \frac{y_1R}{W}\,dx = \int \frac{\cos ax \sin ax}{a}\,dx.$$

$$y_p = uy_1 + vy_2$$

$\therefore$ The general sol: $y = c_1y_1 + c_2y_2 + y_p.$

Q: $y'' + P(x)y' + Q(x)y = 0$

y_1 is known

$v = \int \frac{1}{u^2} e^{-\int P(x)\,dx}\,dx$

y_1 is known

$$y_2 = v y_1 \qquad v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx$$

* $y'' + P(x)y' + Q(x)y = R(x)$ ✓Method of variation of parameters

Method of undetermined Coefficients

$$y'' + Py' + Qy = 0 \longrightarrow y_c = c_1 y_1 + c_2 y_2$$

$R(x)$ is a known function

Q? y_p what

Based on the function $R(x)$, choose y_p.

$R(x)$	y_p
$a_0 + a_1 x + \dots + a_n x^n$	$A_0 + A_1 x + \dots + A_n x^n$
$K \sin ax$ or $K \cos ax$	$A \cos ax + B \sin ax$.
$K e^{\lambda x}$	$A e^{\lambda x}$.

Ex $R(x) = x^2 - 5x^4$

$y_p = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$

Ex $R(x) = 2 \sin 3x$

$y_p = A \sin 3x + B \cos 3x$

Ex $R(x) = e^{2x}$

$y_p = A e^{2x}$.

$R(x) = 2e^{2x} + 3\cos 2x$

what is y_p? $y_p = Ae^{2x} + b_1 \cos 2x + b_2 \sin 2x$.

* $y'' - y' - 2 = e^{+2x}$ $\quad +2, -1$

$y_1 = e^{+2x} \quad y_1 = e^{-x}$.

Recall : $y'' + P(x)y' + Q(x)y = R(x)$

$y = y_c + y_p$

Variation of parameters: $y_c = c_1 y_1(x) + c_2 y_2(x)$

$$y_p = u(x) y_1(x) + v(x) y_2(x)$$

Construct $u(x)$ & $v(x)$.

$$u(x) = \int \frac{-y_2(x) R}{W}\,dx \quad \& \quad v = \int \frac{y_1 R}{W}\,dx.$$

Prob $y'' + y = \operatorname{cosec} x$.

$y'' + y = 0 \Rightarrow m^2 + 1 = 0 \quad m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$.

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$y'' + y = 0 \Rightarrow m^2 + 1 = 0 \qquad m = \pm i \qquad W = \begin{vmatrix} & \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}.$$

$$u = \int -\frac{y_2 R}{W}\, dx = \int \frac{-\sin x \operatorname{cosec} x}{1}\, dx = -x.$$

$$v = \int \frac{y_1 R}{W} = \int \frac{\cos x \operatorname{cosec} x\, dx}{1} = - - .$$

$$y = c_1 \cos x + c_2 \sin x - x\cos x - \sin x \log(\sin x).$$

Method of undetermined Coefficients

Based on $R(x)$, chose the appropriate y_p.

$R(x)$	Choice of y_p
$a_0 + a_1 x + \cdots + a_n x^n$	$A_0 + A_1 x + \cdots + A_n x^n$
$K e^{\lambda x}$	$A e^{\lambda x}$.
$K \sin ax$ or $K \cos x$	$A_0 \cos ax + A_1 \sin ax$.

2; 1 $\quad \underbrace{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y}_{\downarrow} = x^2 \quad R(x)$

$\lambda^2 - 3\lambda + 2 = 0$ AE $\longrightarrow \lambda = 2, 1$

$\therefore y_c = c_1 e^{2x} + c_2 e^x.$

Choice of y_p. $\quad R(x) = x^2 \longrightarrow y_p = a_0 + a_1 x + a_2 x^2$

$R \qquad \downarrow$ solves the non-homogeneous equation

$y_p' = a_1 + 2a_2 x, \quad y_p'' = 2a_2$

$$y_p'' - 3y_p' + 2y_p = R \Rightarrow 2a_2 - 3(a_1 + 2a_2 x) + 2(a_0 + a_1 x + a_2 x^2) = x^2$$

$$2a_2 x^2 + (2a_1 - 6a_2)x + (2a_2 - 3a_1 + 2a_0) = x^2$$

$2a_2 = 1 \Rightarrow a_2 = \frac{1}{2}$

$2a_1 - 6a_2 = 0 \Rightarrow 2a_1 - 3 = 0 \Rightarrow a_1 = 3/2$

$2a_2 - 3a_1 + 2a_0 = 0 \Rightarrow 1 - 3\cdot\frac{3}{2} + 2a_0 = 0$

$1 - \frac{9}{2} + 2a_0 = 0 \Rightarrow a_0 = 7/4$

$\therefore y_p = \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2 \qquad y = y_c + y_p$

sol or not

Verification $\quad y_p' = \frac{3}{2} + x. \quad y_p'' = 1 \qquad \cancel{1} - \cancel{\frac{9}{2}} - \cancel{3x} + \cancel{\frac{7}{2}} + \cancel{3x} + x^2 = R(x)$

* $\quad y'' - 3y' + 2y = 5e^{2x} \quad R(x) \qquad y_c = c_1 \underset{y_1}{e^{2x}} + c_2 \underset{y_2}{e^{x}}$

* $y'' - 3y' + 2y = \underbrace{5e^{2x}}_{R(x)}$ $\qquad y_c = c_1 \underbrace{e^{2x}}_{y_1} + c_2 \underbrace{e^{x}}_{y_2}$

Choice $y_p = ke^{2x}$.

$y_p' = 2ke^{2x}, \quad y_p'' = 4ke^{2x}$.

$4ke^{2x} - 3 \cdot ke^{2x} + 2ke^{2x} = (\underbrace{4k - 6k + 2k}_{=0}) e^{2x} \neq 0$

This choice of y_p will not solve the problem.

New $y_p = K x e^{2x}$

* $\lambda = 2, 2 \longrightarrow y_c = c_1 e^{2x} + c_2 x e^{2x}$

$R(x) = 5e^{2x}$

$y_p = kx^2 e^{2x}$

* Limitations
  * $R(x)$ rational fun $\dfrac{f(x)}{g(x)}$ $\quad y_p$?
  * log fun. y_p ?

* Homogeneous Linear eq with Variable Coeffecients:

Euler - Cauchy

$$\underbrace{x^n \frac{d^n}{dx^n}}\, y + p_1 \underbrace{x^{n-1} \frac{d^{n-1}}{dx^{n-1}}}\, y + \dots + p_n y = R(x)$$

This can be converted into an eq with constant coefficients.

put $x = e^z \longrightarrow \log x = z, \quad x > 0$

$$\frac{dz}{dx} = \frac{1}{x} \qquad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Rightarrow x \frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x} \frac{dy}{dz}\right) \Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

let $\dfrac{d}{dx} = D, \quad \dfrac{d}{dz} = \theta$

$D^2 = \dfrac{d^2}{dx^2}$

$$x \frac{d}{dx} \rightarrow \theta$$

$$x^2 \frac{d^2}{dx^2} \rightarrow \theta(\theta - 1)$$

$$x^3 \frac{d^3}{dx^3} \rightarrow \theta(\theta-1)(\theta-2)$$

$$\vdots$$

$$x^n \frac{d^n}{dx^n} \rightarrow \theta(\theta-1)(\theta-2) \cdots (\theta - (n-1))$$

$$\underbrace{x^2 \frac{d^2y}{dx^2}} - 3x \frac{dy}{dx} + 2y = 0 \qquad \theta = \frac{d}{dz}$$

$$\downarrow$$

$$(\theta(\theta-1) - 3\theta + 2)y = 0 \longrightarrow (\theta^2 - 4\theta + 2)y = 0$$

$$\left(\theta(\theta-1) - 3\theta + 2\right)y = 0 \xrightarrow{u2} \left(\theta^2 - 4\theta + 2\right)y = 0$$

$$\lambda^2 - 4\lambda + 2 = 0 \qquad 2 \pm \sqrt{2}$$

$$y_c = c_1 e^{(2+\sqrt{2})z} + c_2 e^{(2-\sqrt{2})z}$$

$$= c_1 e^{(2+\sqrt{2})\ln x} + c_2 e^{(2-\sqrt{2})\ln x}$$

$$= c_1 x^{2+\sqrt{2}} + c_2 x^{2-\sqrt{2}}$$

Legendre's diff <u>eq</u>

$$(ax+b)^n \frac{d^n y}{dx^n} + p_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = R(x)$$

$$ax + b = e^z \Rightarrow \log(ax+b) = z \qquad \frac{dz}{dx} = \frac{1}{ax+b} a.$$

$$\frac{dy}{dx} = ? \qquad (ax+b)\frac{dy}{dx} = (\ \ )\frac{dy}{dz}$$