NI $\longrightarrow$ $M(\overset{W}{5}, \overset{R}{2})$ 2M ✓

SI $\longrightarrow$ $N(2, 5)$

$(34, 1)$ we need

(m, n) s/t

$xM + yN = (m, n)$

✓ $x = 5,\ y = 2$ $\rightarrow$ $\cancel{x = 3/2,\ y = 5/2}$

*

$$\begin{cases} xM + yN = (34, 1) \\ x(5, 2) + y(2, 5) = (34, 1) \\ \left.\begin{array}{l} 5x + 2y = 34 \\ 2x + 5y = 1 \end{array}\right\} \Rightarrow \cancel{x = 8,\ y = -3} \end{cases}$$

System of Linear equations

$x \in I$

$a_{ij}(x)$ & $b_i(x)$ are continuous

$$\begin{cases} y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + b_1(x) \\ y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n + b_2(x) \\ \vdots \\ y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + b_n(x) \end{cases}$$

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$Y' = A_{n \times n} Y + B$ — $\neq 0$ non-homogenous; $= 0$ homogeneous

$Y' = AY$ $A_{n\times n}$ constant matrix

Autonomous system

* Given n^{th} order ODE $\Longleftrightarrow$ we can get an equivalent system of n - 1^{st} order differential equations.

3^{rd}: $x''' - 6x'' + 2x' + x = 0 \Rightarrow$

let $x = x_1$

$$\left.\begin{array}{l} x_1' = x_2 \\ x_1'' = x_2' = x_3 \\ x_1''' = x_2'' = x_3' = 6x_1'' - 2x_1' - x_1 \end{array}\right\} \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 6x_3 - 2x_2 - x_1 \end{cases}$$

* $Y' = AY$, $A_{n\times n}$ constant matrix

How do we solve

$y' = ky \longrightarrow y(x) = c\, e^{kt}$

Let us try $Y = X e^{\lambda t}$ as a sol.
(X: const vector)

$Y' = \lambda X e^{\lambda t}$

$Y' = AY \rightarrow \lambda X \cancel{e^{\lambda t}} = AX\cancel{e^{\lambda t}} \Rightarrow AX = \lambda X$ eigenvalue prob.

$$Y' = \lambda x e^{\lambda t}$$

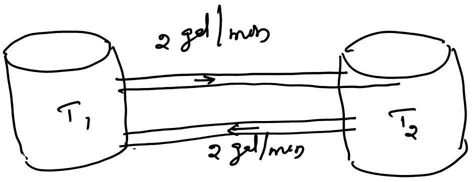
$$Y' = AY \implies \lambda \underline{x} e^{\lambda t} = A x e^{\lambda t} \implies Ax = \lambda x \quad \text{eigenvalue prob.}$$

$$\begin{pmatrix}\lambda_1 \\ | \\ x_1\end{pmatrix} \quad \begin{pmatrix}\lambda_2 \\ | \\ x_2\end{pmatrix} \quad \cdots \quad \begin{pmatrix}\lambda_n \\ | \\ x_n\end{pmatrix} \qquad (\lambda_i, x_i) \quad i = 1(1)n$$

$$Y_i = x_i e^{\lambda_i t} \qquad i = 1(1)n$$

Their Linear combination is again a sol,

Example :



T_1 contains 100 gal of pure water.
T_2 contains initially 100 gal of water in which 150 lb of fertilizer are dissolved.
Liquid circulates through the tanks at a const rate of 2 gal/min
and the mixture is kept uniform by stirring.
Find the fertilizer $y_1(t)$ & $y_2(t)$ in T_1 & T_2 respectively where t is any time.

For a single tank T_1: the time rate of change y_1' of y_1 equals to inflow - outflow.

$$\underline{y_1'} = \text{inflow/min} - \text{outflow/min} = \frac{2}{100} y_2 - \frac{2}{100} y_1$$

|||ly T_2: $\underline{y_2'} = \text{inflow/min} - \text{outflow/min} = \frac{2}{100} y_1 - \frac{2}{100} y_2$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} -2/100 & +2/100 \\ 2/100 & -2/100 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$A\underline{x} = \lambda \underline{x} \implies \det(A - \lambda I) = 0$$

$$\lambda(\lambda + 0.04) = 0 \implies \lambda = 0, \quad \lambda = -0.04$$

$$\lambda = 0 : \; x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda = -0.04, \; x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y_1 = x^{(1)} e^{0 \cdot t} = x^{(1)} \qquad y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

$$Y = c_1 x^{(1)} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

$$= c_1 \begin{bmatrix} 1 \\ \cdot \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \cdot \end{bmatrix} e^{-0.04t}$$

Adv Eng Math

$$Y = c_1 x \quad + c_2 \begin{bmatrix} \\ -1 \end{bmatrix} e$$
$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

Adv Eng Math
E. Kreyszig

$Y' = Ax + B$

$$\left\{ \begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0 \end{aligned} \right\} \Rightarrow ?$$

$(t_0, x_0) \in R$

R: rectangle in $\mathbb{R}^2$

$$\{ (t,x) \mid |t - t_0| \le a, \ |x - x_0| \le b, \ a>0, \ b>0 \}$$

f is continuous on R $\Rightarrow$ $|f| \le M$ $\forall (x,t) \in R$.

f is Lipschitz fn. on R

Integral equation

$$x(t) = x_0(t) + \int_{t_0}^{t} f(s, x(s))\, ds$$

Picard's method

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\, ds$$

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\, ds$$

$$x_3(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))\, ds$$

$x_0 \to x_1 \to x_2 \to x_3 \to \cdots \to x$

$\{x_n\} \longrightarrow \underbrace{x(t)}_{\text{sol}}$

$\vdots$

$$\boxed{x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))\, ds.}$$

Picard's iteration method

$n \to$

$$\left. \begin{aligned} X' &= \overset{A(t)}{A} X + B \\ X(t_0) &= X_0 \end{aligned} \right\} \Rightarrow X_n(t) = X_0 + \int_{t_0}^{t} A(s)\, X_{n-1}(s)\, ds.$$

?.

Note: $X' = AX$ — constant matrix

This is a coupled system
ie if we want one unknown, we should know other variables
and vice-versa.

Q? Can we decouple the system.

Use: If the system of equations are decoupled, then they can be solved independently.

Is it possible always? NO

BUT if A is diagonalizable, YES

$A_{n\times n}$ $\lambda_1, \lambda_2 - - - \lambda_n$ eigenvalues (need not be distinct)

$\downarrow \quad \downarrow \quad - - - - \downarrow$

$x_1 \quad x_2 \quad\quad x_n$ n LI ~~eigenvectors~~ eigenvectors exist (But)

Then $[x_1 \ x_2 \ --- \ x_n] = X$, clearly X^{-1} exists

$X^{-1}AX = D$

If X and D are known, we can construct A.

$XDX^{-1} = A.$

Idea: Solve the decoupled system, then go back to the original problem!