Recall : $y'' + P(x) y' + Q(x) y = 0$, $x \in I$, P and $Q \in C(I)$

Let y_1 and y_2 be two sols.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Th ① : W is either identically zero or nowhere vanishes.

Th ② : y_1 and y_2 are LD iff $W \equiv 0$ on I

Remarks : ① if y_1 and y_2 have a common zero at $x_0 \in I$, then they are LD

② y_1 & y_2 can't have a max/min at the same point

* If y_1 & y_2 are LI sols, then y_2 vanishes exactly once between any two successive zeros of y_1. Vice-versa

* $\underline{y'' + P(x) y' + Q(x) y = R(x)}$ $(\neq 0)$ non-homogeneous. $P, Q, R \in C(I)$

$y = \underset{\downarrow}{y_c} + y_p \longrightarrow$ particular Integral.

complementary sol comes from $\boxed{y'' + P(x) y' + Q(x) y = 0}$

Let y_1 & y_2 are LI sols of $y'' + P y' + Q y = 0$

$$y_c = c_1 y_1 + c_2 y_2$$

Using y_c can we construct y_p? YES Method of Variation of parameters

Idea: Replace c_1 and c_2 by some unknown functions $u(x)$ & $v(x)$ s/t

$\underline{y_p = u y_1 + v y_2}$ is a sol to the non-homogeneous eq.

If y_p is a sol, we must have $y_p'' + P y_p' + Q y_p = R$.

$$y_p = u y_1 + v y_2 \implies y_p' = u y_1' + v y_2' + \boxed{u' y_1 + v' y_2}$$

set this to zero.

$$\boxed{u' y_1 + v' y_2 = 0} \text{ — } (*)$$

Then $y_p' = u y_1' + v y_2' \implies y_p'' = u y_1'' + v y_2'' + u' y_1' + v' y_2'$

$\therefore y_p'' + P y_p' + Q y_p = R \implies \cancel{u y_1''} + \cancel{v y_2''} + u' y_1' + v' y_2' + \cancel{P u y_1'} + \cancel{P v y_2'} + \cancel{Q u y_1} + \cancel{Q v y_2} = R$

$\implies u \underbrace{\cancel{(y_1'' + P y_1' + Q y_1)}}_{0} + v \overbrace{\cancel{(y_2'' + P y_2' + Q y_2)}}^{0}$ y_2 is a sol of homogeneous part

y_1 is a sol of homogeneous eq $\quad + \ u' y_1' + v' y_2' = R$.

$$\therefore \boxed{u' y_1' + v' y_2' = R} \text{ — } (\#)$$

or

$$\therefore \boxed{u'y_1' + v'y_2' = R} \quad \text{—— (\#)}$$

$$\left.\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= R \end{aligned}\right\} \qquad \underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} \begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{\underbrace{y_1y_2' - y_2y_1'}_{W(y_1,y_2)}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\left.\begin{aligned} u' &= \frac{-y_2R}{W} \implies u = +\int \frac{-y_2R}{W}\,dx. \\ v' &= \frac{y_1R}{W} \implies v = \int \frac{y_1R}{W}\,dx \end{aligned}\right\} \text{ use then } u \,\&\, v \text{ as } y_p.$$

* $(D^2 + a^2)\,y = \sin ax.$

$D^2 \to \frac{d^2}{dx^2}$

A.E $\lambda^2 + a^2 = 0 \implies \lambda = \pm ai$

$$y_c = c_1 \underbrace{\cos ax}_{y_1} + c_2 \underbrace{\sin ax}_{y_2}.$$

$$W(y_1, y_2) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

To construct y_p.

$$u = \int -\frac{y_2R}{W}\,dx = \int \frac{-\sin ax \sin ax}{a}\,dx.$$

$$v = \int \frac{y_1R}{W}\,dx = \int \frac{\cos ax \sin ax}{a}\,dx.$$

$$y_p = uy_1 + vy_2$$

$\therefore$ The general sol: $\underline{y = c_1y_1 + c_2y_2 + y_p}.$

<u>Q</u>: $y'' + P(x)y' + Q(x)y = 0$

y_1 is known

$y_2 = vy_1$

$$\underline{v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx}.$$

* $y'' + P(x)y' + Q(x)y = R(x)$ ✓ Method of variation of parameters

Method of undetermined Coefficients

$$y'' + Py' + Qy = 0 \longrightarrow y_c = c_1 y_1 + c_2 y_2$$

$R(x)$ is a known function

Q? y_p what

Based on the function $R(x)$, choose y_p.

$R(x)$	y_p
$a_0 + a_1 x + \cdots + a_n x^n$	$A_0 + A_1 x + \cdots + A_n x^n$
$K \sin ax$ or $K \cos ax$	$A \cos ax + B \sin ax$.
$K e^{\lambda x}$	$A e^{\lambda x}$.

Ex $R(x) = x^2 - 5x^4$

$y_p = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$

Ex $R(x) = 2 \sin 3x$

$y_p = A \sin 3x + B \cos 3x$

Ex $R(x) = e^{2x}$

$y_p = A e^{2x}$.

$R(x) = 2e^{2x} + 3\cos 2x$

what is y_p? $y_p = Ae^{2x} + b_1 \cos 2x + b_2 \sin 2x$.

* $y'' - y' - 2 = e^{+2x}$ $\quad \underbrace{+2, -1}$

$y_1 = e^{+2x} \quad y_1 = e^{-x}$.

Recall: $y'' + P(x)y' + Q(x)y = R(x)$

$y = y_c + y_p$

Variation of parameters: $y_c = c_1 y_1(x) + c_2 y_2(x)$

$\qquad\qquad \downarrow \qquad\qquad \downarrow$

$y_p = u(x) y_1(x) + v(x) y_2(x)$

Construct $u(x)$ & $v(x)$.

$$u(x) = \int \frac{-y_2(x) R}{W} dx \quad \& \quad v = \int \frac{y_1 R}{W} dx.$$

Prob $\underbrace{y'' + y} = \operatorname{cosec} x$.

$y'' + y = 0 \implies m^2 + 1 = 0 \quad m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$.

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$y'' + y = 0 \Rightarrow m^2 + 1 = 0$ $\quad W = \begin{vmatrix} \cdots \\ -\sin x & \cos x \end{vmatrix}$

$$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}.$$

$$u = \int -\frac{y_2 R}{W}\,dx = \int -\frac{\sin x \csc x}{1}\,dx = -x.$$

$$v = \int \frac{y_1 R}{W} = \int \frac{\cos x \csc x \,dx}{1} = \text{---}$$

$$y = c_1 \cos x + c_2 \sin x - x\cos x - \sin x \underline{\log(\sin x)}.$$

Method of undetermined Coefficients

Based on $R(x)$, chose the appropriate y_p.

$R(x)$	Choice of y_p
$a_0 + a_1 x + \dots + a_n x^n$	$A_0 + A_1 x + \dots + A_n x^n$
$K e^{\lambda x}$	$A e^{\lambda x}$.
$K \sin ax$ or $K \cos x$	$A_0 \cos ax + A_1 \sin ax$.

Q.1 $\quad \underbrace{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y}_{\downarrow} = x^2 \quad R(x)$

$\lambda^2 - 3\lambda + 2 = 0$ AE $\longrightarrow \lambda = 2, 1$

$\therefore y_c = c_1 e^{2x} + c_2 e^{x}.$ ✓

Choice of y_p. $\quad R(x) = x^2 \longrightarrow y_p = a_0 + a_1 x + a_2 x^2$

R $\qquad \downarrow$ solves the non-homogeneous equation

$y_p' = a_1 + 2a_2 x, \quad y_p'' = 2a_2$

$$y_p'' - 3y_p' + 2y_p = R \Rightarrow 2a_2 - 3(a_1 + 2a_2 x) + 2(a_0 + a_1 x + a_2 x^2) = x^2$$

$$2a_2 x^2 + (2a_1 - 6a_2)x + (2a_2 - 3a_1 + 2a_0) = x^2$$

$2a_2 = 1 \Rightarrow a_2 = \frac{1}{2}$

$2a_1 - 6a_2 = 0 \Rightarrow 2a_1 - 3 = 0 \Rightarrow a_1 = 3/2$

$2a_2 - 3a_1 + 2a_0 = 0 \Rightarrow 1 - 3\cdot\frac{3}{2} + 2a_0 = 0$

$1 - \frac{9}{2} + 2a_0 = 0 \Rightarrow a_0 = 7/4$

$\therefore \underline{y_p = \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2}$ $\qquad \underline{y = y_c + y_p}$

sol or not

Verification $\quad y_p' = \frac{3}{2} + x. \quad y_p'' = 1 \qquad \cancel{1 - \frac{9}{2}} - \cancel{3x} + \cancel{\frac{7}{2}} + \cancel{3x} + x^2 = R(x)$

Verification $y_p' = \frac{3}{2} + x$. $y_p'' = 1$ $1 - \cancel{\frac{9}{2}} - \cancel{3x} + \cancel{\frac{7}{2}} + \cancel{3x} + x^2 = R(x)$

* $y'' - 3y' + 2y = 5e^{2x}$ $R(x)$ $y_c = c_1 e^{2x} + c_2 e^{x}$, y_1, y_2

Choice $y_p = Ke^{2x}$.

$y_p' = 2ke^{2x}$, $y_p'' = 4ke^{2x}$.

$4ke^{2x} - 3ke^{2x} + 2ke^{2x} = \underbrace{(4k - 6k + 2k)}_{=0} e^{2x} \neq 0$

This choice of y_p will not solve the problem.

New $y_p = Kxe^{2x}$

* $\lambda = 2, 2 \longrightarrow y_c = c_1 e^{2x} + c_2 x e^{2x}$

$R(x) = 5e^{2x}$

$y_p = Kx^2e^{2x}$

* Limitations
  * $R(x)$ rational fun $\frac{f(x)}{g(x)}$ y_p?
  * log fun. y_p?

* ~~Homogeneous~~ Linear eq with Variable Coeffecients:

Euler – Cauchy

$$x^n \frac{d^n}{dx^n} y + p_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} y + \cdots + p_n y = R(x)$$

This can be converted into an eq with constant coefficients.

put $x = e^z \to \log x = z$, $x > 0$

$\frac{dz}{dx} = \frac{1}{x}$ ✓ $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$.

let $\frac{d}{dx} = D$, $\frac{d}{dz} = \theta$ $x\frac{d}{dx} \to \theta$

$D^2 = \frac{d^2}{dx^2}$ $x^2\frac{d^2}{dx^2} \to \theta(\theta - 1)$

$x^3\frac{d^3}{dx^3} \longrightarrow \theta(\theta-1)(\theta-2)$

$\vdots$ n $\theta(\theta-1)(\theta-2)\cdots(\theta-(n-1))$

$$x^3 \frac{d^3}{dx^3} \longrightarrow \theta \ldots$$

$$\vdots$$

$$x^n \frac{d^n}{dx^n} \longrightarrow \theta(\theta-1)(\theta-2)\cdots(\theta-(n-1))$$

$$\underbrace{x^2 \frac{d^2y}{dx^2}} - 3x\frac{dy}{dx} + 2y = 0 \qquad \theta = \frac{d}{dz}$$

$$\Big(\theta(\theta-1) - 3\theta + 2\Big)y = 0 \longrightarrow (\theta^2 - 4\theta + 2)y = 0$$

$$\lambda^2 - 4\lambda + 2 = 0 \qquad 2 \pm \sqrt{2}$$

$$y_c = c_1 e^{(2+\sqrt{2})z} + c_2 e^{(2-\sqrt{2})z}$$

$$= c_1 e^{(2+\sqrt{2})\ln x} + c_2 e^{(2-\sqrt{2})\ln x}$$

$$= c_1 x^{2+\sqrt{2}} + c_2 x^{2-\sqrt{2}}$$

Legendre's diff eq

$$(ax+b)^n \frac{d^ny}{dx^n} + p_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_n y = R(x)$$

$$ax+b = e^z \Rightarrow \log(ax+b) = z \qquad \frac{dz}{dx} = \frac{1}{ax+b}\, a.$$

$$\frac{dy}{dx} = ? \qquad (ax+b)\frac{dy}{dx} = (\quad)\frac{dy}{dz}$$