Operator Method to find the particular Integral.

* $D\varphi = \psi \longrightarrow \int \psi = \varphi$ $\qquad \frac{1}{D}\cos 3x = \frac{\sin 3x}{3}$

$\frac{d}{dx} \hookrightarrow \frac{1}{D}\psi = \varphi$ $\qquad D\left(\frac{\sin 3x}{3}\right) = \cos 3x$

$D^{-1}\psi = \varphi.$

* $f(D)\,\psi = Q. \Rightarrow \psi = (f(D))^{-1} Q. = \left(\frac{1}{f(D)}\right) Q.$

$f(D)$ ↳ differential operator.

Ex $\frac{1}{D^2+3D+2} e^{4x} = \frac{e^{4x}}{30}$ $\qquad$ Check $(D^2+3D+2)\frac{e^{4x}}{30} = e^{4x}.$

#①

* $\frac{1}{D-\alpha} Q = ?$ $\Rightarrow (D-\alpha)\,g(x) = Q.$

y_p ↓

$= e^{\alpha x}\int Q e^{-\alpha x}\,dx.$

$g(x)\cdot IF = \int Q\, e^{IF}\,dx.$

$g(x)\,e^{-\alpha x} = \int Q e^{-\alpha x}\,dx.$

$$\boxed{g(x) = e^{\alpha x}\int Q e^{-\alpha x}\,dx}$$

#②

$$\boxed{\frac{1}{D+\alpha} Q = ? \\ = e^{-\alpha x}\int Q e^{\alpha x}\,dx}$$

* $\frac{1}{(D-\beta)(D-\alpha)} Q = \frac{1}{D-\beta}\left\{\frac{1}{D-\alpha} Q\right\} = \frac{1}{D-\beta}\left\{e^{+\alpha x}\int Q e^{-\alpha x}\,dx\right\}$

$\alpha \neq \beta$

$= e^{+\beta x}\int\left[\int e^{\alpha x}\int Q e^{-\alpha x}\,dx\right] e^{-\beta x}\,dx.$

Problems : ① $\frac{1}{D^3}\cos x = ?$ ② $\frac{1}{D+1} x = ?$ *③ $\frac{1}{(D-2)(D-3)} e^{2x}.$

---

* $f(D)\,y = Q(x) \Rightarrow y = \frac{1}{f(D)} Q(x)$

If $f(D) = (D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$, $\alpha_i \neq \alpha_j \ \forall\ i \neq j$

$\frac{1}{(D-\alpha_1)\cdots(D-\alpha_n)} Q = \left\{\frac{A_1}{D-\alpha_1} + \cdots + \frac{A_n}{D-\alpha_n}\right\} Q.$

Example : $(D^2-5D+6)\,y = xe^{4x}$

A.E : $m^2-5m+6=0 \Rightarrow y_c = c_1e^{2x}+c_2e^{3x}.$

$y_p = \frac{1}{D^2-5D+6} xe^{4x}.$

$\frac{1}{\ldots} xe^{4x} = \left\{\frac{1}{D-3} - \frac{1}{D-2}\right\} xe^{4x}. = e^{4x}\frac{(2x-3)}{4}.$

$$y_p = \frac{1}{D^2-5D+6}\, xe^{4x}$$

$$= \frac{1}{(D-3)(D-2)}\, xe^{4x} = \left\{\frac{1}{D-3} - \frac{1}{D-2}\right\} xe^{4x} = e^{4x}\frac{(2x-3)}{4}.$$

$$\therefore y = y_c + y_p.$$

Exercise: Solve $(D^2+a^2)y = \sec ax$.

special cases of Q.

Case (1): $f(D)y = \varphi(x)$

$\varphi(x) = e^{ax}$. Then $y_p = \frac{1}{f(D)} e^{ax}$. $\quad De^{ax} = ae^{ax}$, $D \to a$

put $D = a$, then $y_p = \frac{1}{f(a)} e^{ax}$ Provided $f(a) \neq 0$

If $f(a)=0$, then $(D-a) \mid f(D) \Rightarrow f(D) = (D-a)F(D)$ where $F(a) \neq 0$

If a is a root of order k, then $f(D) = (D-a)^k F(D)$, $F(a) \neq 0$

$$y_p = \frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^k F(D)} e^{ax} = \frac{1}{F(a)} \cdot \frac{1}{(D-a)^k} e^{ax}.$$

$\frac{1}{D-a} e^{ax} = xe^{ax}$; or $(D-a)xe^{ax} = e^{ax}$.

$\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}, \ldots \quad \frac{1}{(D-a)^k} e^{ax} = \frac{x^k}{k!} e^{ax}$

Problems: (1) $y'' + 4y' + 3y = e^{2x}$ $\quad y_p = \frac{1}{D^2+4D+3} e^{2x} = \frac{1}{4+8+3} e^{2x} = \boxed{\frac{e^{2x}}{15}}$

(2) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

$= e^{-2x} + e^{x} - e^{-x}$.

$$y_p = \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^{x} - \frac{1}{(D+2)(D-1)^2} e^{-x} \Rightarrow \frac{e^{-x}}{(-1+2)(-1-1)^2} = \frac{e^{-x}}{4}$$

$f(-2)=0 \qquad f(1)=0$

$$\frac{1}{9}\frac{1}{D+2} e^{-2x} + \frac{1}{3}\frac{1}{(D-1)^2} e^{x}$$

$$= \frac{1}{9}\frac{x}{1!}e^{-2x} + \frac{1}{3}\frac{x^2}{2!}e^{x} = \frac{1}{9}xe^{-2x} + \frac{1}{6}x^2e^{x} - \frac{e^{-x}}{4}.$$

Ex: $y'' - 4y' + 3y = 4e^{3x}$ s/t $y(0) = -1$ & $y'(0) = 3$

$Q(x) = \sin bx$ or $\cos bx$.

Case 2

$d \sin bx = b\cos bx$. $\quad d\cos bx = -b\sin bx$

$Q(x) = \sin bx$ or $\cos bx$

Case 2

$d \sin bx = b \cos bx.$ $\quad d \cos bx = -b \sin bx$

$d^2 \sin bx = -b^2 \sin bx$ $\quad d^2 \cos bx = -b^2 \cos bx.$

$d^2 \to -b^2$ $\quad d^2 \to -b^2$

$$f(D^2) \longrightarrow f(-b^2)$$

Case $f(-b^2) \neq 0$, $\quad f(D^2) \sin bx = f(-b^2) \sin bx.$

$$\sin bx = \frac{1}{f(D^2)}\{f(-b^2)\sin bx\} \Rightarrow \frac{1}{f(D^2)} \sin bx = \frac{1}{f(-b^2)} \sin bx$$

* $f(-b^2)=0$, $\quad D^2+b^2 \mid f(D) \Rightarrow f(D) = (D^2+b^2)\underbrace{F(D^2)}$

$F(-b^2) \neq 0$

$$\frac{1}{\underbrace{D^2+b^2}_{f(D)}} \sin bx = \frac{1}{\underbrace{-b^2+b^2}_{=0}} \sin bx = \frac{-x \cos bx}{2b}$$

Check!

$$\frac{1}{D^2+b^2} \cos bx = \frac{x \sin bx}{2b}$$

Ex: (1) $(D^2+4)y = e^x + \sin 2x + \cos 2x$

(2) $(D^2-4D+3)y = \sin 3x \cos 2x$

(3) $(D^3+1)y = \cos(2x-1)$

$m = -1, \frac{1 \pm i\sqrt{3}}{2}$ $\quad y_c = ?$

$$y_p = \frac{1}{D^3+1} \cos(2x-1) = \frac{1}{\underbrace{D^2 \cdot D}+1} \cos(2x-1) = \frac{1}{(-2)^2 D+1} \cos(2x-1) = \frac{1}{1-4D} \cos(2x-1)$$

$$= \frac{1+4D}{\underbrace{(1-4D)(1+4D)}} \cos(2x-1)$$

$=$

(4) $(D^2+9)y = \cos 3x.$

$$y_p = \frac{1}{D^2+9} \underbrace{\cos 3x}_{\text{Re } e^{i3x}} \quad f(-3^2)=0$$

$(D+3i)(D-3i)$

$$= \frac{x}{2 \cdot 3} \sin 3x = \frac{x \sin 3x}{6}$$

How?

$$(D^2+9)\left(\frac{x}{6} \sin 3x\right) = \cos 3x$$

———— x ———— x ———— x ————

Case 3

Case $Q = x^m$, $\quad PI = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

↪ Expand.

Prob (1) $y'' + y' = x^2 + 2x + 4$ $\quad m^2 + m = 0$

$m = 0, -1$

$y'' + y' = 0 \Rightarrow y_c = c_1 + c_2 e^{-x}.$

$$y_p = \frac{1}{D^2+D} x^2+2x+4 = \frac{1}{D(1+D)} x^2+2x+4 = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$\ldots = \frac{1}{D}\{x^2+\cancel{2x}+4-\cancel{2x}-2+\ldots\}$$

$$= \frac{1}{D}\left(1 - D + D^2 - \cdots\right)\left(x^2 + 2x + 4\right) = \frac{1}{D}\left\{x^2 + \cancel{2x} + 4 - \cancel{2x} - \cancel{4} + \cancel{4}\right\}$$

$$= \int x^2 + 4 = \boxed{\frac{x^3}{3} + 4x} \; y_p.$$

Case: (4) $Q(x) = e^{ax} V(x)$

$$D(e^{ax}V) = e^{ax}DV + ae^{ax}V = e^{ax}(D+a)V$$

$$D^2(e^{ax}V) = e^{ax}(D+a)^2 V, \; \cdots ; \; D^n(e^{ax}V) = e^{ax}(D+a)^n V$$

$$\therefore f(D)(e^{ax}V) = e^{ax} f(D+a)V.$$

$$\frac{1}{f(D)} f(D)(e^{ax}V) = \frac{1}{f(D)}\left\{e^{ax} f(D+a)V\right\}$$

$$\boxed{e^{ax}V = \frac{1}{f(D)}\left\{e^{ax} f(D+a)V\right\}}$$

put $f(D+a)V = u$, $\Rightarrow V = \frac{1}{f(D+a)} u$ so that $e^{ax}\frac{1}{f(D+a)}u = \frac{1}{f(D)}(e^{ax}u)$

ie $$\boxed{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V}$$

Problem: $(D^2 - 2D + 4)y = e^x \cos x$.

$$y_p = \frac{1}{D^2 - 2D + 4} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x. = e^x \frac{1}{-1+3} \cos x = \boxed{\frac{e^x \cos x}{2}}$$

Note: There are other models as well.

But, these are simple modifications to the models discussed here.

In any case: these problems can also be solved easily using the method of Variation of parameters.