Q: If $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$, $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in arithmetic progression (AP), prove that a, b, c are also in AP.

Solution: Common difference can be written as:

$$b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right)$$
$$(b\left(\frac{1}{c}+\frac{1}{a}\right)+1)-(a\left(\frac{1}{b}+\frac{1}{c}\right)+1)=(c\left(\frac{1}{a}+\frac{1}{b}\right)+1)-(b\left(\frac{1}{c}+\frac{1}{a}\right)+1)$$
$$\implies (b\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{b}{b})-(a\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{a}{a})=(c\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{c}{c})-(b\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{b}{b})$$
$$\implies (b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right))-(a\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right))=(c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right))-(b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right))$$
$$\implies b-a=c-b$$

Hence proved that a, b, c are in AP.

parameter	value	description
$x(0)$	$a\left(\frac{1}{b}+\frac{1}{c}\right)$	First Term of given AP
d	$(b-a)(\frac{1}{a}+\frac{1}{b}-\frac{1}{c})$	Common Difference of given AP
$x(n)$	$(x(0)+nd)u(n)$	General Term of given AP

TABLE I
INPUT PARAMETER TABLE

From table I

$$X(z)=x(0)\frac{1}{1-z^{-1}}+d\frac{z^{-1}}{(1-z^{-1})^2} \tag{1}$$
$$X(z)=a\left(\frac{1}{b}+\frac{1}{c}\right)(\frac{1}{1-z^{-1}})+(b-a)(\frac{1}{a}+\frac{1}{b}-\frac{1}{c})(\frac{z^{-1}}{(1-z^{-1})^2}) \quad |z|>1 \tag{2}$$

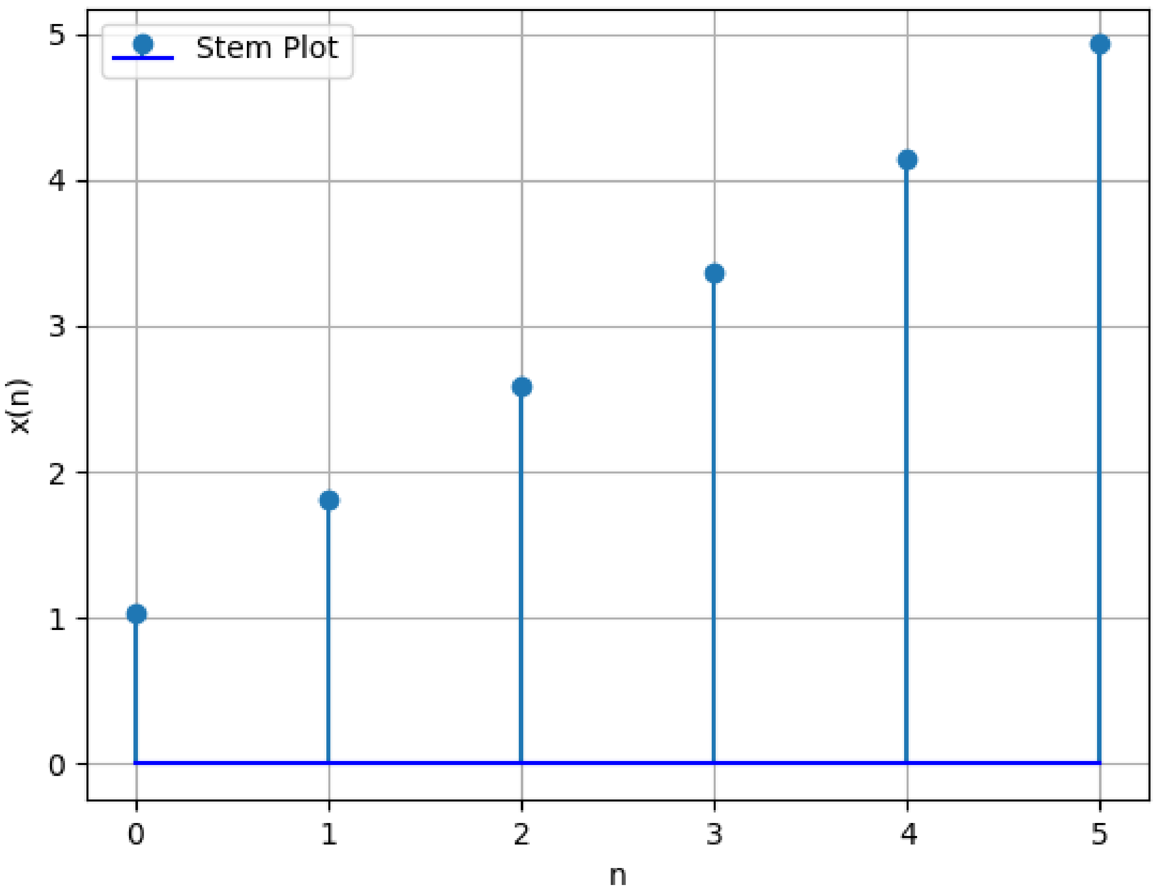

Fig. 1. graph with value of $a = 3, b = 5, c = 7$