Q: If $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$, $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in arithmetic progression (AP), prove that a, b, c are also in AP.

Solution:

$$\text{Given } a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right) \text{ are in arithmetic progression (AP)}$$

$$\implies \text{On adding 1 to every term, using the property of AP, we get}$$

$$\implies a\left(\frac{1}{b}+\frac{1}{c}\right)+1,\ b\left(\frac{1}{c}+\frac{1}{a}\right)+1,\ c\left(\frac{1}{a}+\frac{1}{b}\right)+1$$

$$\implies a\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{a}{a},\ b\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{b}{b},\ c\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{c}{c}$$

$$\implies a\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\ b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\ c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\implies \text{Now cancel } \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \text{ from all terms, using the property of AP, we get}$$

$$\implies a,\ b,\ c \text{ are in AP}$$

Hence proved