Q: The state equation of a second order system is
$\dot{x}(t) = Ax(t), \quad x(0)$ is the initial condition.
Suppose λ_1 and λ_2 are two distinct eigenvalues of A, and v_1 and v_2 are the corresponding eigenvectors. For constants α_1 and α_2, the solution, $x(t)$, of the state equation is

(A) $\sum_{i=1}^{2} \alpha_i e^{\lambda_i t} v_{\mathbf{i}}$
(B) $\sum_{i=1}^{2} \alpha_i e^{2\lambda_i t} v_{\mathbf{i}}$
(C) $\sum_{i=1}^{2} \alpha_i e^{3\lambda_i t} v_{\mathbf{i}}$
(D) $\sum_{i=1}^{2} \alpha_i e^{4\lambda_i t} v_{\mathbf{i}}$

Solution:

Variable	**Description**	**Value**
$x(t)$	state variable	-
$\dot{x}(t)$	derivative of x(t) w.r.t t	$\frac{dx(t)}{dt}$
A	2x2 matrix	-
λ_i for $i = 1, 2$	eigen values of A	-
v_i for $i = 1, 2$	eigen vectors of A	-
α_i for $i = 1, 2$	constants for $x(0)$ i.e. initial conditions	-

TABLE I
INPUT PARAMETERS

Theories and Proofs:

$$Ax(t) = \lambda x(t) \tag{1}$$

Eigen values of inverse of a matrix is reciprocal of eigen value of the given matrix

$$A^{-1}Ax(t) = A^{-1}\lambda x(t) \tag{2}$$
$$\implies x(t) = \lambda A^{-1}x(t) \tag{3}$$
$$\implies A^{-1}x(t) = \frac{1}{\lambda}x(t) \tag{4}$$

Eigen value of a matrix shifts by the same amount as that of the matrix.

$$(A - \sigma I)x(t) = Ax(t) - \sigma Ix(t) \tag{5}$$
$$= \lambda x(t) - \sigma x(t) \tag{6}$$
$$= (\lambda - \sigma)x(t) \tag{7}$$

Sol:
Using Laplace transform:
Given Equation:

$$\dot{x}(t) = Ax(t) \tag{8}$$
$$\frac{dx(t)}{dt} = Ax(t) \tag{9}$$

Taking Laplace Transform:

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}\left(Ax(t)\right) \tag{10}$$

$$sX(s) - x(0) = AX(s) \tag{11}$$

$$(sI - A)X(s) = x(0) \tag{12}$$

$$X(s) = (sI - A)^{-1}x(0) \tag{13}$$

From Table I, we can write $x(0)$ in terms of two linearly independent variables as

$$x(0) = \alpha_1 v_1(t) + \alpha_2 v_2(t) \tag{14}$$

$$= \sum_{i=1}^{2} \alpha_i v_i(t) \tag{15}$$

From (13), (15)

$$X(s) = (sI - A)^{-1}\left(\sum_{i=1}^{2} \alpha_i v_i(t)\right) \tag{16}$$

From (4), (7)

$$X(s) = \sum_{i=1}^{2} \frac{1}{s - \lambda_i}\left(\alpha_i v_i(t)\right) \tag{17}$$

Now take inverse Laplace Transform

$$\mathcal{L}^{-1}\left(X(s)\right) = \mathcal{L}^{-1}\left(\sum_{i=1}^{2} \frac{1}{s - \lambda_i}\left(\alpha_i v_i(t)\right)\right) \tag{18}$$

$$x(t) = \sum_{i=1}^{2} e^{\lambda_i t}\left(\alpha_i v_i(t)\right) \tag{19}$$

Hence the answer is option (A).