GATE 23 EC

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Question 43

Q: The state equation of a second order system is
$\dot{x}(t) = Ax(t), \quad x(0)$ is the initial condition.
Suppose λ_1 and λ_2 are two distinct eigenvalues of A, and ν_1 and ν_2 are the corresponding eigenvectors. For constants α_1 and α_2, the solution, $x(t)$, of the state equation is

1. $\sum_{i=1}^{2} \alpha_i e^{\lambda_i t} \nu_{\mathbf{i}}$
2. $\sum_{i=1}^{2} \alpha_i e^{2\lambda_i t} \nu_{\mathbf{i}}$
3. $\sum_{i=1}^{2} \alpha_i e^{3\lambda_i t} \nu_{\mathbf{i}}$
4. $\sum_{i=1}^{2} \alpha_i e^{4\lambda_i t} \nu_{\mathbf{i}}$

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Input Parameter Table

Variable	Description	Value
$x(t)$	state variable	-
$\dot{x}(t)$	derivative of x(t) w.r.t t	$\frac{dx(t)}{dt}$
A	2x2 matrix	-
λ_i for $i = 1, 2$	eigen values of A	-
v_i for $i = 1, 2$	eigen vectors of A	-
α_i for $i = 1, 2$	component of $x(t)$ along v_i	-

Table: input parameters

Proofs

$$Ay = \lambda y \tag{1}$$

Eigen values of inverse of a matrix is reciprocal of eigen value of the given matrix

$$A^{-1}Ay = A^{-1}\lambda y \tag{2}$$

$$\implies y = \lambda A^{-1}y \tag{3}$$

$$\implies A^{-1}y = \frac{1}{\lambda}y \tag{4}$$

Proofs - Continued

Eigen value of a matrix shifts by the same amount as that of the matrix.

$$(A - \sigma I)\mathbf{y} = A\mathbf{y} - \sigma I\mathbf{y} \tag{5}$$

$$= \lambda\mathbf{y} - \sigma\mathbf{y} \tag{6}$$

$$= (\lambda - \sigma)\mathbf{y} \tag{7}$$

Solution

Using Laplace transform:
Given Equation:

$$\dot{x}(t) = Ax(t) \tag{8}$$

$$\frac{dx(t)}{dt} = Ax(t) \tag{9}$$

Solution - Continued

Taking Laplace Transform:

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}\left(Ax(t)\right) \tag{10}$$

$$sX(s) - x(0) = AX(s) \tag{11}$$

$$(sI - A)X(s) = x(0) \tag{12}$$

$$X(s) = (sI - A)^{-1}x(0) \tag{13}$$

Solution - Continued

From Table 1, we can write $x(0)$ in terms of two linearly independent variables as

$$x(0) = \alpha_1 v_1 + \alpha_2 v_2 \tag{14}$$

$$= \sum_{i=1}^{2} \alpha_i v_i \tag{15}$$

From (13), (15)

$$X(s) = (sI - A)^{-1} \left(\sum_{i=1}^{2} \alpha_i v_i \right) \tag{16}$$

$$= \sum_{i=1}^{2} (sI - A)^{-1} \alpha_i v_i \tag{17}$$

Solution - Continued

From (4), (7)

$$X(s) = \sum_{i=1}^{2} \frac{1}{s - \lambda_i} (\alpha_i v_i) \tag{18}$$

Now take inverse Laplace Transform

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\sum_{i=1}^{2} \frac{1}{s - \lambda_i} (\alpha_i v_i)\right) \tag{19}$$

$$x(t) = \sum_{i=1}^{2} e^{\lambda_i t} (\alpha_i v_i) \tag{20}$$