Filter Design

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1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for given filter number. This is a bandpass filter whose specifications are available below.

2 Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

2.1 The Digital Filter

1. Passband: The passband is from {4 + 0.6(j)}kHz to {4 + 0.6(j+2)}kHz.
   where

$$j = (r - 11000) \mod \sigma \tag{1}$$

   where σ is sum of digits of roll number and r is roll number.

$$r = 11016 \tag{2}$$
$$\sigma = 9 \tag{3}$$
$$j = 7 \tag{4}$$

   substituting $j = 7$ gives the passband range for our bandpass filter as 8.2 kHz - 9.4 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p1} = 8.2$ kHz and $F_{p2} = 9.4$ kHz.
   The corresponding normalized digital filter passband frequencies are

$$\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.34\pi \tag{5}$$
$$\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.39\pi \tag{6}$$

2. Tolerances: The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

3. Stopband: The transition band for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband.

$$F_{s1} = 8.2 - 0.3 = 7.9\text{KHz} \tag{7}$$
$$F_{s2} = 9.4 + 0.3 = 9.7\text{KHz} \tag{8}$$

$$\omega_{s1} = 2\pi \frac{F_{s1}}{F_s} = 0.329\pi \tag{9}$$
$$\omega_{s2} = 2\pi \frac{F_{s2}}{F_s} = 0.404\pi \tag{10}$$
$$\tag{11}$$

Parameter	Value	Description
F_s	48kHz	Sampling Frequency
F_{p_1}	8.2kHz	lower bound of pass band
F_{p_2}	9.4kHz	upper bound of pass band
ω_{p_1}	0.34π	corresponding normalised frequency($2\pi \frac{F_{p_1}}{F_s}$)
ω_{p_2}	0.39π	corresponding normalised frequency($2\pi \frac{F_{p_2}}{F_s}$)
F_{s_1}	7.9kHz	lower bound of stop band
F_{p_2}	9.7kHz	upper bound of stop band
ω_{s_1}	0.329π	corresponding normalised frequency($2\pi \frac{F_{s_1}}{F_s}$)
ω_{s_2}	0.404π	corresponding normalised frequency($2\pi \frac{F_{s_2}}{F_s}$)

Table 1: Parameter Table

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency(ω) :

$$\Omega = \tan \frac{\omega}{2} \tag{12}$$

Using this relation, we obtain the analog passband and stopband frequencies as: $\Omega_{p1} = 0.5913$, $\Omega_{p2} = 0.702$ and $\Omega_{s1} = 0.5662$, $\Omega_{s2} = 0.7361$ respectively.

Parameter	Value	Description
Ω_{p_1}	0.5913	corresponding analog frequency($tan(\frac{\omega_{p_1}}{2})$)
Ω_{p_2}	0.702	corresponding analog frequency($tan(\frac{\omega_{p_2}}{2})$)
Ω_{s_1}	0.5662	corresponding analog frequency($tan(\frac{\omega_{s_1}}{2})$)
Ω_{s_2}	0.7361	corresponding analog frequency($tan(\frac{\omega_{s_2}}{2})$)

Table 2: Parameter Table

3 The IIR Filter Design

Filter type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the Chebyschev approximation to design our bandpass IIR filter.

3.1 The Analog Filter

1. Low Pass Filter Specifications: Let $H_{a,BP}(j\Omega)$ be the desired analog bandpass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{13}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.644$ and $B = \Omega_{p2} - \Omega_{p1} = 0.1107$.
Substituting Ω_{s1} and Ω_{s2} in (13) we obtain the stopband edges of lowpass filter

$$\Omega_{Ls1} = \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = -1.502 \tag{14}$$

$$\Omega_{Ls2} = \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = 1.559 \tag{15}$$

And we choose the minimum of these two stopband edges

$$\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.502. \tag{16}$$

Parameter	Value	Description
Ω_0	0.644	natural frequency($\sqrt{\Omega_{p_1}\Omega_{p_2}}$)
B	0.1107	Damping coefficient($\Omega_{p_2} - \Omega_{p_1}$)
$\Omega_{L_{s_1}}$	-1.502	corresponding lowpass frequency($\frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}}$)
$\Omega_{L_{s_2}}$	1.559	corresponding lowpass frequency($\frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}}$)
Ω_{L_p}	1	cutoff frequency(chosen)

Table 3: Parameter Table

2. The Low Pass Chebyschev Filter Paramters: The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1+\epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{17}$$

The passband edge of the low pass filter is chosen as $\Omega_{Lp} = 1$. Therfore ,

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1+\epsilon^2 c_N^2(\Omega_L)} \tag{18}$$

Here c_N denote the chebyshev polynomials for a particular order N of the filter.

$$c_N(x) = \cosh(N \cosh^{-1} x), x = \Omega_L(\text{i.e. greater than 1 here}) \tag{19}$$

$$c_0(x) = 1 \tag{20}$$

$$c_1(x) = x \tag{21}$$

There exists a recurssive relation from which all the polynomials can be found out.

$$c_{N+2} = 2xc_{N+1} - c_N \tag{22}$$

Imposing the band restrictions on (17)

$$|H_{a,LP}(j\Omega_L)| < \delta_2 \text{ for } \Omega_L = \Omega_{Ls} \tag{23}$$

$$1-\delta_1 < |H_{a,LP}(j\Omega_L)| < 1 \text{ for } \Omega_L = \Omega_{Lp} \tag{24}$$

we obtain :

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1},$$

$$N \ge \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \tag{25}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling operator .

Parameter	Value
D_1	0.384
D_2	43.44
N	4
$c_4(x)$	$8x^4 - 8x^2 + 1$

Table 4: Parameter Table

we get $N \ge 4$ and $0.278 \le \epsilon \le 0.61$

The below code plots (17) for different values of ϵ .

H_a_lp_diff_epsilons.py

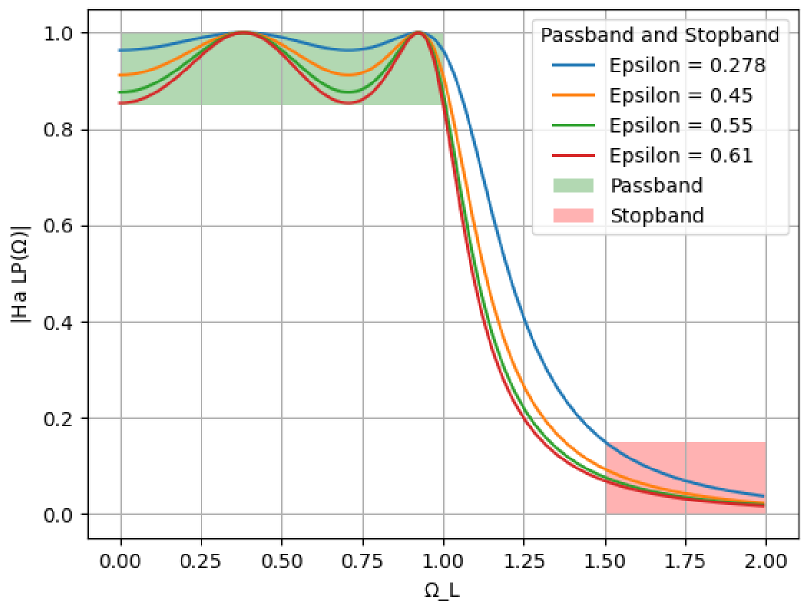

Figure 1: The Analog Low-Pass Frequency Response for $0.278 \le \epsilon \le 0.61$

In Fig. 1 we can observe the equiripple behaviour in passband and monotonic behaviour in stopband. As the value of ϵ increases the value of $|H_{a,LP}(j\Omega_L)|$ decreases.

3. The Low Pass Chebyschev Filter: The next step in design is to find an expression for magnitude response in s domain.

   Using $s = j\Omega$ or in this case $s_L = j\Omega_L$ we obtain:

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\frac{s_L}{j})} \tag{26}$$

   To find poles equate the denominator to zero:

$$1 + \epsilon^2 c_N^2\left(\frac{s_L}{j}\right) = 0 \quad \text{where} \quad c_N(x) = 8x^4 - 8x^2 + 1 \quad (N = 4) \tag{27}$$

   On solving (27) using the below code we obtain poles.

calc_poles.py

The poles obtained are formulated in the table below.

$Pole$	$Value$
s_1	-0.4604 - j0.4276
s_2	-0.4604 + j0.4276
s_3	-0.1907 + j1.0322
s_4	0.1907 + j1.0322
s_5	0.4604 + j1.4276
s_6	0.4604 - j0.4276
s_7	0.1907 - j0.0322
s_8	-0.1907 - j1.0322

Table 5: Values of poles

The below code plots the pole-zero plot.

poles_plot.py

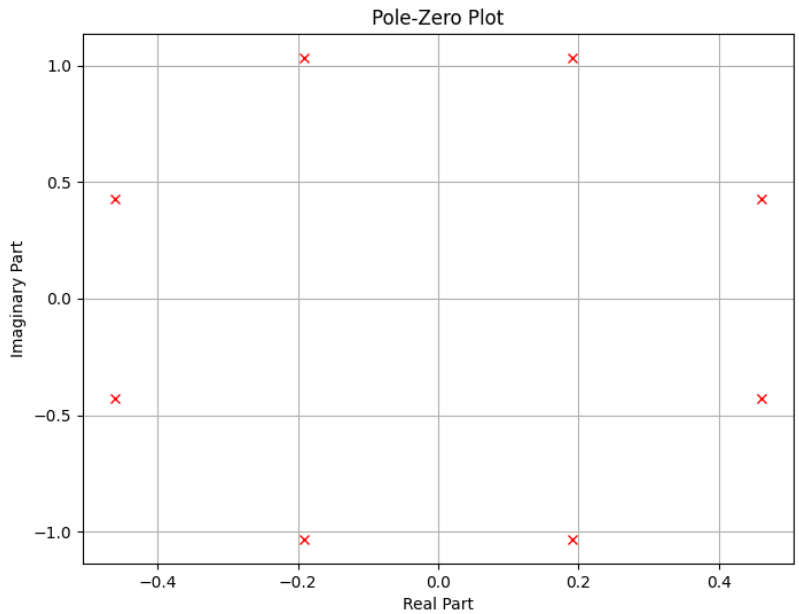

Figure 2: The Pole zero plot and all the poles lie on an ellipse.

The poles in the left half of the plane are considered in the design as we intend to design a stable system.
Therefore the magnitude response is written as :-

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - s_1)(s_L - s_2)(s_L - s_3)(s_L - s_8)} \tag{28}$$

where G_{LP} is the gain of the Low pass filter. Refer to Table 5 for pole's values.

We know that from (17):-

$$\left|H_{a,LP}(s_L)\right| = \frac{1}{\sqrt{1+\epsilon^2}} \text{at } \Omega_L = 1 \implies s_L = j \tag{29}$$

Substituting respective values in (29), using the below code, we get $G_{LP} = 0.4166$

```
G_lp.py
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$$H_{a,LP}(s_L) = \frac{0.4166}{(s_L - s_1)(s_L - s_2)(s_L - s_3)(s_L - s_8)} \tag{30}$$

Using the below code to simplify the polynomial, we get calc_polyno.py

$$H_{a,LP}(s_L) = \frac{0.4166}{s_L^4 + 1.3022 s_L^3 + 1.84781 s_L^2 + 1.16512 s_L + 0.435003} \tag{31}$$

The below code plots design v/s specification of our lowpass filter

verifications.py

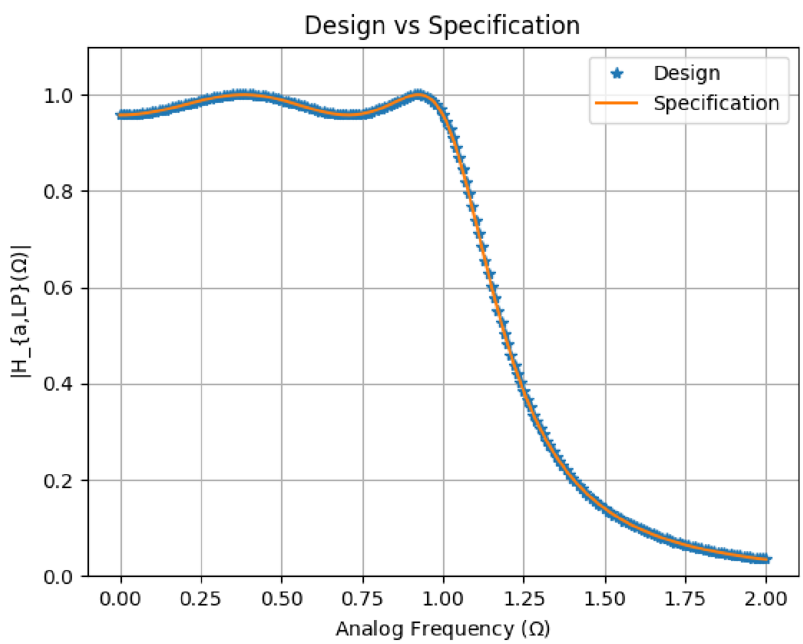

Figure 3: Design vs Specification corresponding to (31) and (18)

4. The Band Pass Chebyschev Filter: After verifying design with the required specifications the next step in design is to jump to required type of filter using frequency transformation.

$$s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{32}$$

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{33}$$

As there is one to one correspondence between the filters so $\Omega = \Omega_{p1}$ should correspond to Ω_{Lp}

$$s = j\Omega_{p1} \tag{34}$$

$$s_L = \frac{(j\Omega_{p1})^2 + \Omega_0^2}{B(j\Omega_{p1})} \tag{35}$$

$$\left|H_{a,BP}(j\Omega_{p1})\right| = 1 \tag{36}$$

$$G_{BP}\left|H_{a,LP}(s_L)\right| = 1 \tag{37}$$

Substituting (35) in (37) we obtain Gain of required bass pass filter using the below code:

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G_bp.py
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$$G_{BP} = 1.0370 \tag{38}$$

Thus the response in s domain (using substitutions, using the below code)

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H_in_s.py
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$$H_{a,BP}(s) = \frac{6.49 \times 10^{-5} s^4}{s^8 + 0.144s^7 + 0.1682s^6 + 0.1810s^5 + 1.05s^4 + 0.750s^3 + 0.289s^2 + 0.0102s + 0.029} \tag{39}$$

In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation. This is done using the below code:

```
H_a_bp.py
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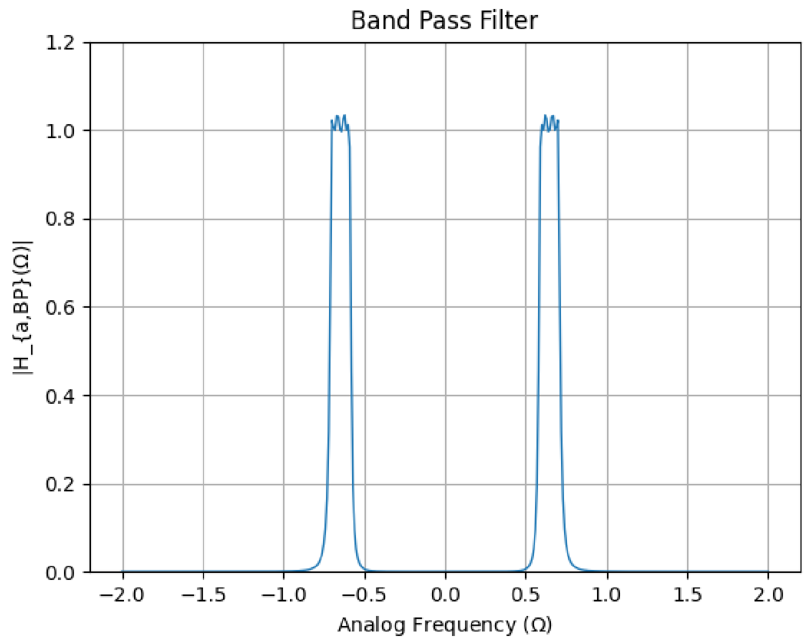

Figure 4: The Analog Bandpass Magnitude Response from (39).The filter design specifications are satisfied

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{40}$$

Substituting $s = \frac{1-z^{-1}}{1+z^{-1}}$ in (39) and calculating expression using the below python code we get :

analog_to_digital.py

$$H_{d,BP}(z) = \frac{G\left(1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8}\right)}{3.6214 - 6.9496z^{-1} + 26.7376z^{-2} - 60.1464z^{-3} + 73.758z^{-4} - 49.572z^{-5} + 26.144z^{-6} - 7.62z^{-7} + 1.451z^{-8}} \tag{41}$$

where $G = 6.49 \times 10^{-5}$

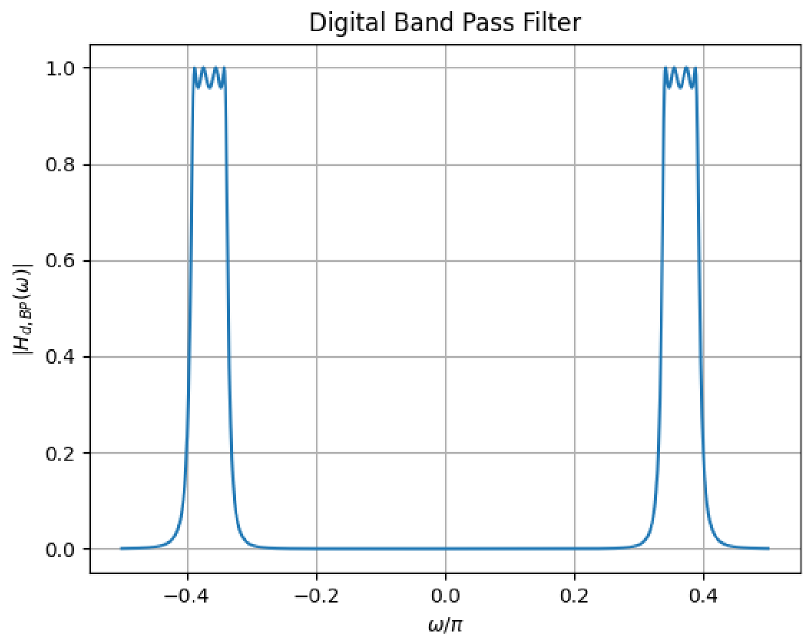

Figure 5: Digital Specifications are met. Passband and stopband frequencies are same

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is $\delta = 0.15$.The cutoff-frequency is given by :

$$\omega_l = \frac{B}{2} \tag{42}$$

$$= 0.025\pi \tag{43}$$

The below code plots this figure

freq_resp_imp_resp_ideal.py

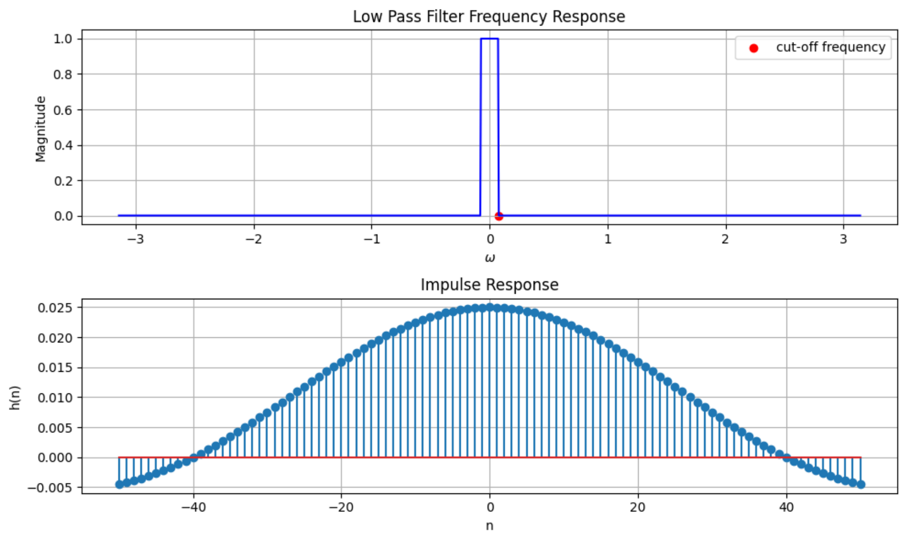

Figure 6: Frequency response and impulse response of an ideal Low Pass Filter

The impulse response of ideal Low Pass Filter is given by :

$$h(n) = \begin{cases} \frac{w_l}{\pi}, & \text{if } n = 0 \\ \frac{\sin(w_l n)}{n\pi}, & \text{if } n \neq 0 \end{cases} \tag{44}$$

From (44) we conclude that $h(n)$ for an ideal Low Pass Filter is not causal and can neither be made causal by introducing a finite delay. And $h(n)$ do not converge and hence the system is unstable.

4.2 The Kaiser Window

Therefore we move on windowing the impulse response.A window function is chosen and multiplied. The Kaiser window is defined as

$$w(n) = \begin{cases} \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, & -N \leq n \leq N, \quad \beta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

1. N is chosen according to

$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{45}$$

where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5849(A - 21)^{0.4} + 0.07886(A - 21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{46}$$

The window function is defined as :

$$w(n) = \begin{cases} 1, & \text{for } -48 \leq n \leq 48 \\ 0, & \text{otherwise} \end{cases} \tag{47}$$

Therefore the desired impulse response is :

$$h_{lp} = h_n w_n \tag{48}$$

$$h(n) = \begin{cases} \frac{\sin(w_l n)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

The below code plots this figure

fir_lp.py

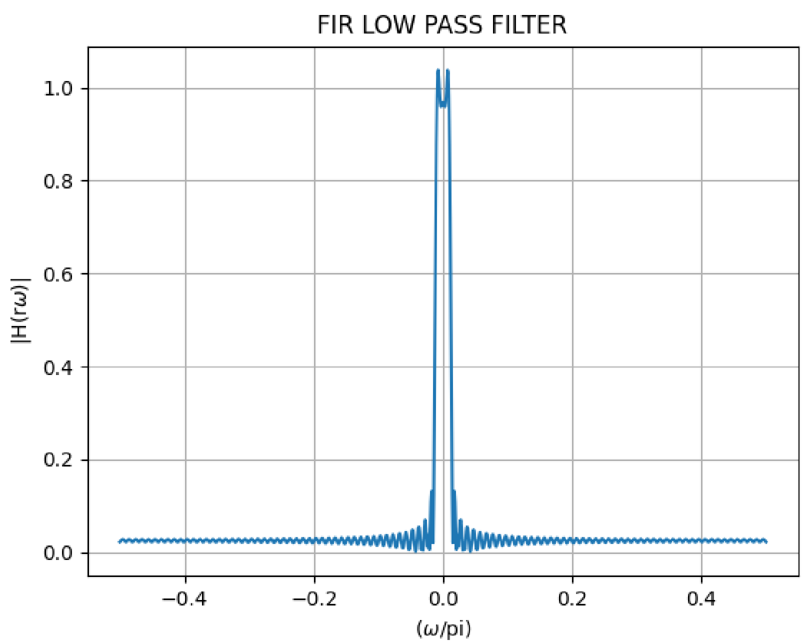

Figure 7: Magnitude Response of Low Pass Filter after using Kaiser Window

4.3 The Equivalent Band Pass Filter

A Band-Pass Filter (BPF) can be obtained by subtracting the magnitude response of a Low-Pass Filter (LPF) with cutoff frequency ω_{p1} from another LPF magnitude response with cutoff frequency ω_{p2}.

$$h_{BP}(n) = \begin{cases} \frac{\sin(w_{p2}n)}{n\pi} - \frac{\sin(\omega_{p1}n)}{n\pi}, & \text{for } n \neq 0 \\ \frac{\omega_{p2}-\omega_{p1}}{\pi} & \text{for} n = 0 \end{cases} \tag{50}$$

$$\frac{\sin(\omega_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi} = 2\cos\left(\frac{\omega_{p2}n + \omega_{p1}n}{2}\right)\sin\left(\frac{\omega_{p2}n - \omega_{p1}n}{2}\right) \tag{51}$$

$$= \frac{2\cos(0.365n\pi)\sin(0.025n\pi)}{n\pi} \tag{52}$$

Multipying by window function we get :

$$h_{BP}(n) = \begin{cases} \frac{2\cos(0.365n\pi)\sin(0.025n\pi)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

The below code plots this figure

fir_bp.py

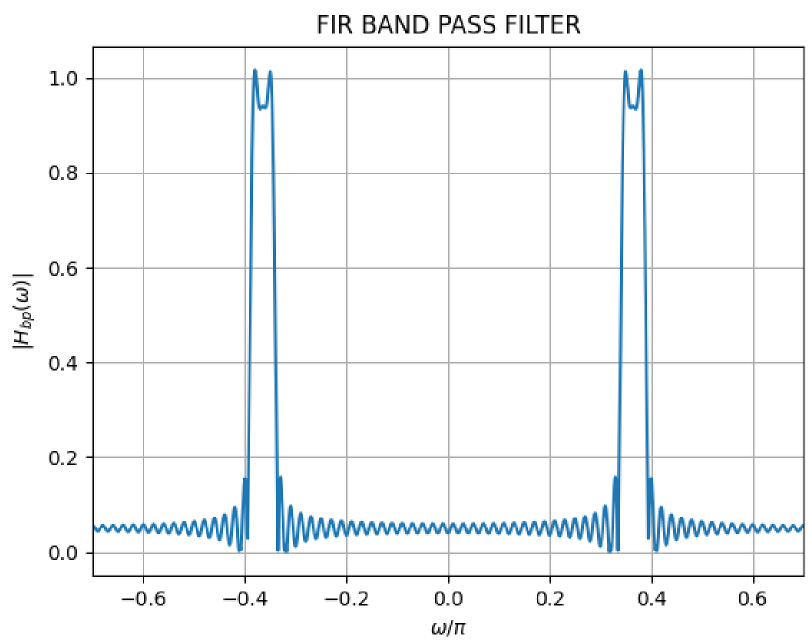

Figure 8: Magnitude Response of Band Pass Filter after using Kaiser Window