

Q: The state equation of a second order system is

$\dot{x}(t) = Ax(t)$, $x(0)$ is the initial condition.

Suppose λ_1 and λ_2 are two distinct eigenvalues of A , and v_1 and v_2 are the corresponding eigenvectors. For constants α_1 and α_2 , the solution, $x(t)$, of the state equation is

- (A) $\sum_{i=1}^2 \alpha_i e^{\lambda_i t} v_i$
 (B) $\sum_{i=1}^2 \alpha_i e^{2\lambda_i t} v_i$
 (C) $\sum_{i=1}^2 \alpha_i e^{3\lambda_i t} v_i$
 (D) $\sum_{i=1}^2 \alpha_i e^{4\lambda_i t} v_i$

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Solution:

Variable	Description	Value
$x(t)$	state variable	-
$\dot{x}(t)$	derivative of $x(t)$ w.r.t t	$\frac{dx(t)}{dt}$
A	2x2 matrix	-
λ_i for $i = 1, 2$	eigen values of A	-
v_i for $i = 1, 2$	eigen vectors of A	-
α_i for $i = 1, 2$	component of $x(t)$ along v_i	-

TABLE I
INPUT PARAMETERS

Theories and Proofs:

$$A\mathbf{y} = \lambda\mathbf{y} \quad (1)$$

Eigen values of inverse of a matrix is reciprocal of eigen value of the given matrix

$$A^{-1}A\mathbf{y} = A^{-1}\lambda\mathbf{y} \quad (2)$$

$$\Rightarrow \mathbf{y} = \lambda A^{-1}\mathbf{y} \quad (3)$$

$$\Rightarrow A^{-1}\mathbf{y} = \frac{1}{\lambda}\mathbf{y} \quad (4)$$

Eigen value of a matrix shifts by the same amount as that of the matrix.

$$(A - \sigma I)\mathbf{y} = A\mathbf{y} - \sigma I\mathbf{y} \quad (5)$$

$$= \lambda\mathbf{y} - \sigma\mathbf{y} \quad (6)$$

$$= (\lambda - \sigma)\mathbf{y} \quad (7)$$

Sol:

Using Laplace transform:

Given Equation:

$$\dot{x}(t) = Ax(t) \quad (8)$$

$$\frac{dx(t)}{dt} = Ax(t) \quad (9)$$

Taking Laplace Transform:

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}(Ax(t)) \quad (10)$$

$$sX(s) - x(0) = AX(s) \quad (11)$$

$$(sI - A)X(s) = x(0) \quad (12)$$

$$X(s) = (sI - A)^{-1}x(0) \quad (13)$$

From Table ??, we can write $x(0)$ in terms of two linearly independent variables as

$$x(0) = \alpha_1 v_1 + \alpha_2 v_2 \quad (14)$$

$$= \sum_{i=1}^2 \alpha_i v_i \quad (15)$$

From (??), (??)

$$X(s) = (sI - A)^{-1} \left(\sum_{i=1}^2 \alpha_i v_i \right) \quad (16)$$

$$= \sum_{i=1}^2 (sI - A)^{-1} \alpha_i v_i \quad (17)$$

From (??), (??)

$$X(s) = \sum_{i=1}^2 \frac{1}{s - \lambda_i} (\alpha_i v_i) \quad (18)$$

Now take inverse Laplace Transform

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1} \left(\sum_{i=1}^2 \frac{1}{s - \lambda_i} (\alpha_i v_i) \right) \quad (19)$$

$$x(t) = \sum_{i=1}^2 e^{\lambda_i t} (\alpha_i v_i) \quad (20)$$

Hence the answer is option (A).