AE 622 - Computing of high speed flows
Assignment 3: Report
Capturing a normal shock using Steger- Warming flux splitting method

Vinod Kumar Metla - 130010048
Aditi Taneja - 13D100026

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Introduction

Formulation for flux calculation using the Steger-Warming flux splitting and Lax-Friedrich methods are derived followed by Normal shock simulated using Lax Friedrich method in octave and Steger Warming Flux-Splitting Method in fortran with initial conditions as -

$$\begin{bmatrix} \rho_l \\ u_l \\ p_l \\ T_l \end{bmatrix} = \begin{bmatrix} 1.225 kg/m^3 \\ 3401.74 m/s \\ 101.325 KPa \\ 288 K \end{bmatrix},$$

$$\begin{bmatrix} \rho_r \\ u_r \\ p_r \\ T_r \end{bmatrix} = \begin{bmatrix} 7.0 kg/m^3 \\ 595.3 m/s \\ 11804.362 KPa \\ 5871.6 K \end{bmatrix}$$

Simulations are carried out by varying Grid size 200, 400, and 800. α in Lax Friedrich scheme is set to 0.5, and CFL number is set to 0.2. Their effects on ρ, u, p, T are shown in the graphs with corresponding relative error plots in u.

Dependencies

1. Pdflatex
2. Octave
3. Numpy
4. Matplotlib

Flux Formulation using Steger Warming Flux Splitting method and Lax Friedrich method

Introduction

The one-dimensional Euler Equations in conservative differential form are :-

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad \text{---} \ (1)$$

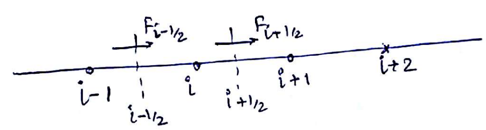
where $Q(x,t)$ is the vector of Dependent Variables.

$$Q = \begin{Bmatrix} \rho \\ \rho u \\ \rho e \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

ρ = density , u = velocity
e = total energy per unit mass.

The Flux vector $F(x,t)$ is,

$$F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} \rho u \\ \rho u u + p \\ \rho e u + p u \end{Bmatrix}$$

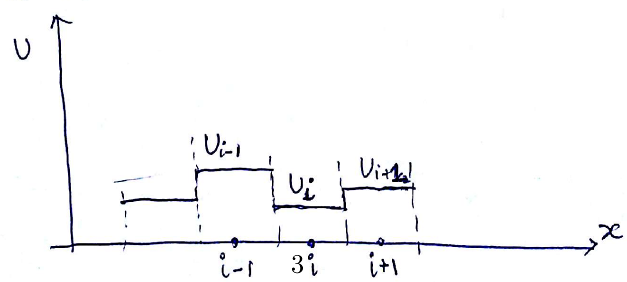


Integrating Equation (1) in i^{th} cell,

$$\int_{i-1/2}^{i+1/2} \frac{\partial Q}{\partial t}\, dx + \int_{i-1/2}^{i+1/2} \frac{\partial F}{\partial x}\, dx = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} Q\, dx + F_{i+1/2} - F_{i-1/2} = 0$$

Now, if Q is taken constant across the cell,



$$\frac{\partial Q_i}{\partial t} (x_{i+1/2} - x_{i-1/2}) + (F_{i+1/2} - F_{i-1/2}) = 0$$

$$\Rightarrow \frac{\partial Q_i}{\partial t} = \left[\frac{F_{i-1/2} - F_{i+1/2}}{x_{i+1/2} - x_{i-1/2}} \right] = \frac{F_{i-1/2} - F_{i+1/2}}{\Delta x} \quad \text{---} \ (2)$$

Equation ① can also be written as,

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial Q}\frac{\partial Q}{\partial x} = 0$$

Where $\partial F/\partial Q$ is called the Jacobian.

$$\frac{\partial F}{\partial Q} = \frac{\partial F}{\partial V}\frac{\partial V}{\partial Q}$$

$$V = \begin{Bmatrix} \rho \\ u \\ p \end{Bmatrix}$$

$$\frac{\partial F}{\partial V} = \begin{bmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} & \frac{\partial f_1}{\partial V_3} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} & \frac{\partial f_2}{\partial V_3} \\ \frac{\partial f_3}{\partial V_1} & \frac{\partial f_3}{\partial V_2} & \frac{\partial f_3}{\partial V_3} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial V_1} = \frac{\partial(\rho u)}{\partial \rho_{(u,p)}} = u \quad , \quad \frac{\partial f_1}{\partial V_2} = \frac{\partial(\rho u)}{\partial u_{(\rho,p)}} = \rho \quad , \quad \frac{\partial f_1}{\partial V_3} = \frac{\partial(\rho u)}{\partial p_{(\rho,u)}} = 0$$

$$\frac{\partial f_2}{\partial V_1} = \left.\frac{\partial(\rho u^2+p)}{\partial \rho}\right|_{p,u} = u^2 \quad , \quad \frac{\partial f_2}{\partial V_2} = \left.\frac{\partial(\rho u^2+p)}{\partial u}\right|_{\rho,p} = 2\rho u \quad , \quad \frac{\partial f_2}{\partial V_3} = \left.\frac{\partial(\rho u^2+p)}{\partial p}\right|_{\rho,u} = 1$$

$$\frac{\partial f_3}{\partial V_1} = \frac{\partial(\rho e u + p u)}{\partial \rho} = \left.\frac{\partial\left(\frac{pu}{\gamma-1} + \frac{1}{2}\rho u^3\right)}{\partial \rho}\right|_{u,p} = \frac{u^3}{2}$$

$$\frac{\partial f_3}{\partial V_2} = \left.\frac{\partial\left(\frac{pu}{\gamma-1} + \frac{1}{2}\rho u^3\right)}{\partial u}\right|_{p,\rho} = E + p + \rho u^2$$

$$\frac{\partial f_3}{\partial V_3} = \left.\frac{\partial\left(\frac{pu}{\gamma-1} + \frac{1}{2}\rho u^3\right)}{\partial p}\right|_{\rho,u} = \frac{\gamma}{\gamma-1}u$$

$$\therefore \frac{\partial F}{\partial V} = \begin{bmatrix} u & \rho & 0 \\ u^2 & 2\rho u & 1 \\ \frac{u^3}{2} & E+p+\rho u^2 & \frac{\gamma u}{\gamma-1} \end{bmatrix}$$

Similarly,

$$\frac{\partial V}{\partial Q} = \begin{bmatrix} \frac{\partial V_1}{\partial Q_1} & \frac{\partial V_1}{\partial Q_2} & \frac{\partial V_1}{\partial Q_3} \\ \frac{\partial V_2}{\partial Q_1} & \frac{\partial V_2}{\partial Q_2} & \frac{\partial V_2}{\partial Q_3} \\ \frac{\partial V_3}{\partial Q_1} & \frac{\partial V_3}{\partial Q_2} & \frac{\partial V_3}{\partial Q_3} \end{bmatrix}$$

$$\frac{\partial V_1}{\partial Q_1} = \left.\frac{\partial \rho}{\partial \rho}\right|_{\rho u, E} = 1 \quad , \quad \frac{\partial V_1}{\partial Q_2} = \left.\frac{\partial \rho}{\partial (\rho u)}\right|_{\rho, E} = 0 \quad , \quad \frac{\partial V_1}{\partial Q_3} = \left.\frac{\partial \rho}{\partial E}\right|_{\rho, \rho u} = 0$$

$$\frac{\partial V_2}{\partial Q_1} = \left.\frac{\partial u}{\partial \rho}\right|_{\rho u, E} = \left.\frac{\partial (\rho u/\rho)}{\partial \rho}\right|_{\rho u, E} = -\frac{\rho u}{\rho^2} = -u/\rho \, ,$$

$$\frac{\partial V_2}{\partial Q_2} = \left.\frac{\partial u}{\partial \rho u}\right|_{\rho, E} = \left.\frac{\partial (\rho u/\rho)}{\partial \rho u}\right|_{\rho, E} = 1/\rho \quad , \quad \frac{\partial V_2}{\partial Q_3} = \left.\frac{\partial u}{\partial E}\right|_{\rho, \rho u} = 0$$

$$\frac{\partial V_3}{\partial Q_1} = \left.\frac{\partial p}{\partial \rho}\right|_{\rho u, E} = \left.\frac{\partial \left((\gamma-1)E - \frac{(\gamma-1)}{2}\frac{(\rho u)^2}{\rho}\right)}{\partial \rho}\right|_{\rho u, E} = -\left(\frac{\gamma-1}{2}\right)\cdot -\frac{(\rho u)^2}{\rho^2} = \left(\frac{\gamma-1}{2}\right) u^2$$

$$\frac{\partial V_3}{\partial Q_2} = \left.\frac{\partial p}{\partial \rho u}\right|_{\rho, E} = \left.\frac{\partial \left((\gamma-1)E - \frac{(\gamma-1)}{2}\frac{(\rho u)^2}{\rho}\right)}{\partial \rho u}\right|_{\rho, E} = -(\gamma-1)u$$

$$\frac{\partial V_3}{\partial Q_3} = \left.\frac{\partial p}{\partial E}\right|_{\rho, \rho u} = \left.\frac{\partial \left((\gamma-1)E - \frac{(\gamma-1)}{2}\frac{(\rho u)^2}{\rho}\right)}{\partial \rho u}\right|_{\rho, E} = (\gamma-1)$$

$$\therefore \quad \frac{\partial V}{\partial Q} = \begin{bmatrix} 1 & 0 & 0 \\ -u/\rho & 1/\rho & 0 \\ \frac{\gamma-1}{2}u^2 & -(\gamma-1)u & (\gamma-1) \end{bmatrix}$$

$$A = \underbrace{\frac{\partial Q}{\partial V}}_{S^{-1}} \cdot \underbrace{\frac{\partial V}{\partial Q} \cdot \frac{\partial F}{\partial V}}_{T^{-1}\Lambda T} \cdot \underbrace{\frac{\partial V}{\partial Q}}_{S} = S^{-1} T^{-1} \Lambda T S^{-1} \quad \text{—— (3)}$$

$$\therefore \; S = \frac{\partial V}{\partial Q} \quad , \quad S^{-1} = \frac{\partial Q}{\partial V}$$

$$B = T^{-1}\Lambda T = \frac{\partial V}{\partial Q}\frac{\partial F}{\partial V} = \begin{bmatrix} 1 & 0 & 0 \\ -u/\rho & 1/\rho & 0 \\ \frac{(\gamma-1)}{2}u^2 & -(\gamma-1)u & (\gamma-1) \end{bmatrix} \begin{bmatrix} u & \rho & 0 \\ u^2 & 2\rho u & 1 \\ \frac{u^3}{2} & E+p+\rho u^2 & \frac{\gamma u}{\gamma-1} \end{bmatrix}$$

$$\therefore \quad B = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix}$$

Eigenvalues of B can be found out using $\det(B - \lambda I) = 0$

Steger Warming Method

$$\Rightarrow \begin{bmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \gamma p & u-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (u-\lambda)\left[(u-\lambda)^2 - \gamma p/\rho\right] = 0$$

$$\Rightarrow \lambda = u, \; u + \sqrt{\frac{\gamma p}{\rho}}, \; u - \sqrt{\frac{\gamma p}{\rho}}$$

where $\sqrt{\frac{\gamma p}{\rho}}$ can be denoted at a (Sound Speed)

$$\Rightarrow \Lambda = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix}$$

T^{-1} and T are left and right eigenvectors of B corresponding to Λ.

$$T = \begin{bmatrix} 1 & 0 & -1/a^2 \\ 0 & \rho a & 1 \\ 0 & -\rho a & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 1/2a^2 & -1/2a^2 \\ 0 & 1/2\rho a & -1/2\rho a \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Steger Warming Method

Basic concept of Flux-Splitting Method is to decompose flux vector into 2 parts -: $F = F^+ + F^-$

where F^+ represents the contribution to the flux associated with waves that move from left to right across the cell interface at $i+1/2$. Thus eigenvalues of $\frac{\partial F^+}{\partial Q}$ are positive.

Similarly, F^- represents the contribution to the flux associated with waves that move from right to left across the cell interface at $x_{i-1/2}$. Therefore, eigenvalues of $\frac{\partial F^-}{\partial Q}$ are negative.

From ③, $A = S^{-1} T^{-1} \Lambda T S^{-1}$

Assuming wave speed Splitting,

$$\lambda_i = \lambda_i^+ + \lambda_i^-$$

where $\lambda_i^+ = \frac{1}{2} (\lambda_i + |\lambda_i|)$

$$\lambda_i^- = \frac{1}{2} (\lambda_i - |\lambda_i|)$$

Thus, $\Lambda = \Lambda^+ + \Lambda^-$

where

$$\Lambda^+ = \begin{bmatrix} \lambda_1^+ & 0 & 0 \\ 0 & \lambda_2^+ & 0 \\ 0 & 0 & \lambda_3^+ \end{bmatrix}, \quad \Lambda^- = \begin{bmatrix} \lambda_1^- & 0 & 0 \\ 0 & \lambda_2^- & 0 \\ 0 & 0 & \lambda_3^- \end{bmatrix}$$

Using homogenity property,

$$F = AQ = A^+ U_i + A^- U_{i+1}$$

$$\therefore \quad A = S^{-1} T^{-1} (\Lambda^+ + \Lambda^-) T S^{-1}.$$

$$F = \underbrace{S_i^{-1} T_i^{-1} \Lambda_i^+ T_i S_i U_i}_{F^+} + \underbrace{S_{i+1}^{-1} C_{i+1}^{-1} \Lambda_{i+1}^- T_{i+1} S_{i+1} U_{i+1}}_{F^-}$$

For Supersonic Flows,

$$u > a$$

$$\therefore \quad \Lambda^+ = \text{diag}\,[u, u+a, u-a]$$

$$\Lambda^- = \text{diag}\,[0, 0, 0]$$

For Subsonic Flows,

$$u < a$$

$$\therefore \quad \Lambda^+ = \text{diag}\,[u, u+a, 0]$$

$$\Lambda^- = \text{diag}\,[0, 0, u-a]$$

Lax Friedrich Method

Using : Forward Difference in time in (1),

$$\frac{Q_i^{n+1} - Q_i^{n}}{\Delta t} + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} = 0 \quad \text{—— (4)}$$

If $F_{i+1/2} = \frac{F_i + F_{i+1}}{2}$ & $F_{i-1/2} = \frac{F_{i-1} + F_i}{2}$, the scheme becomes Unstable.

∴ For Stability, Fluxes at the interface are written as-:

$$F_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) - \frac{\alpha}{2}(Q_{i+1} - Q_i)$$

$$F_{i-1/2} = \frac{1}{2}(F_{i-1} + F_i) - \frac{\alpha}{2}(Q_i - Q_{i-1})$$

$$F_{i+1/2} - F_{i-1/2} = \frac{1}{2}(F_{i+1} - F_{i-1}) - \frac{\alpha}{2}(Q_{i+1} - 2Q_i + Q_{i-1})$$

Now, $\frac{Q_{i+1} - 2Q_i + Q_{i-1}}{2} = (\Delta x)^2 \frac{\partial^2 Q}{\partial x^2}$

∴ Equation (4) becomes,

$$Q^{n+1} = Q^n - \left[\frac{\Delta t}{2\Delta x}(F_{i+1} - F_{i-1}) + \frac{\alpha(\Delta x)^2}{2}\frac{\partial^2 Q}{\partial x^2}\right]$$

Therefore, modified equation becomes,

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = \alpha \Delta x \frac{\partial^2 Q}{\partial x^2}$$

where $\alpha \Delta x$ is called numerical viscocity.

Comparison Between Exact Solution and Approximate Methods

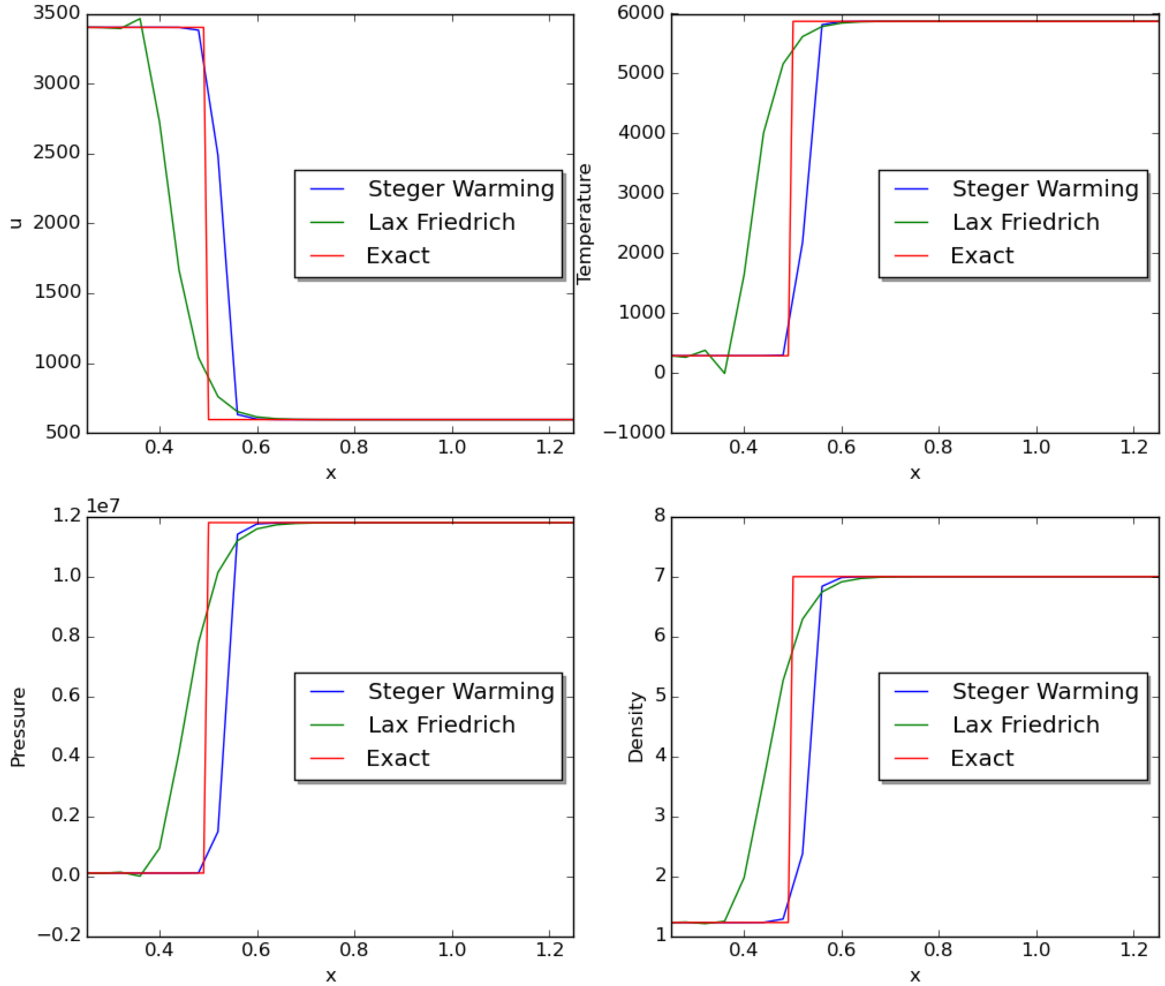


Figure 1: Plot for comparison of exact solution and approximate solutions for normal shock with 200 Grid points

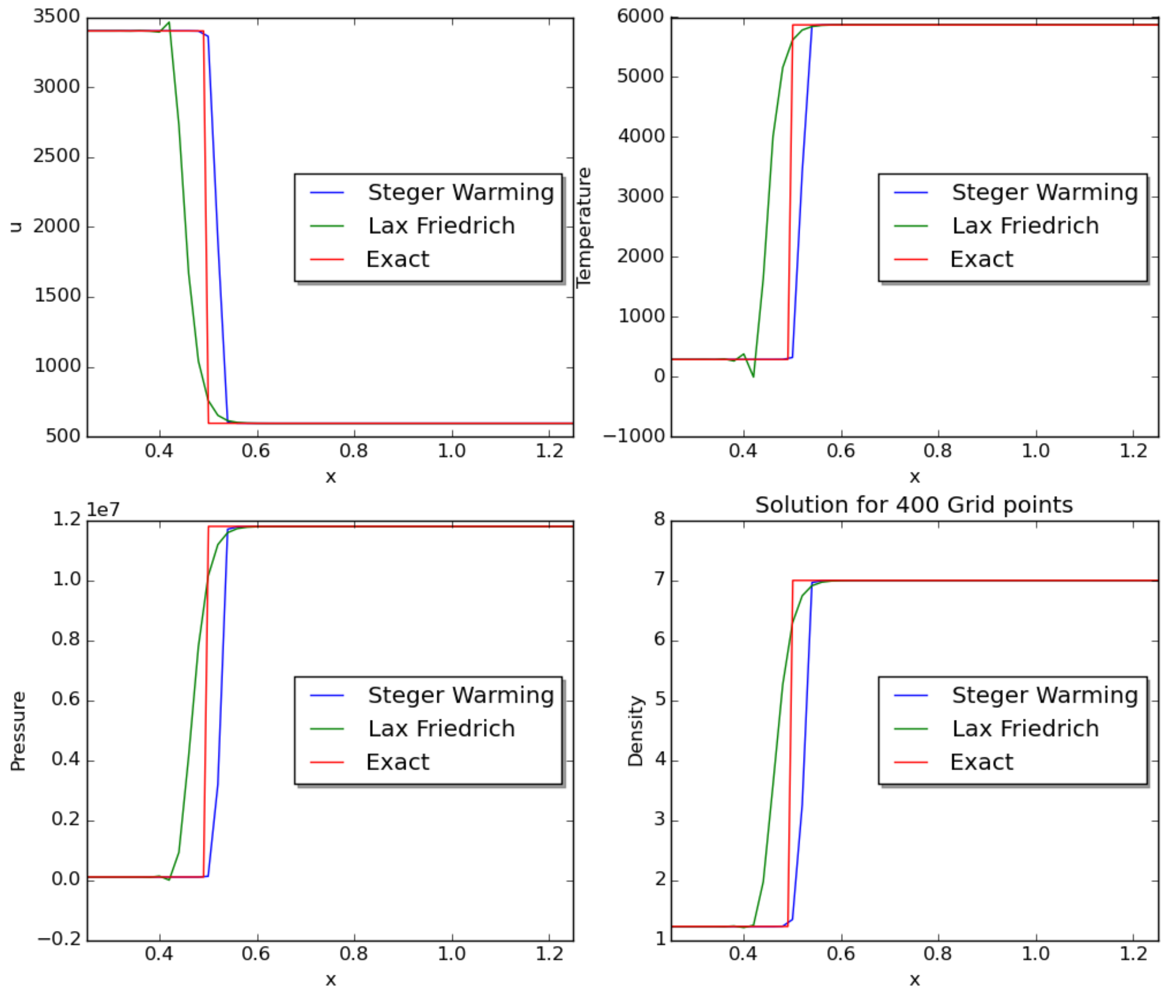


Figure 2: Plot for comparison of exact solution and approximate solutions for normal shock with 400 Grid points

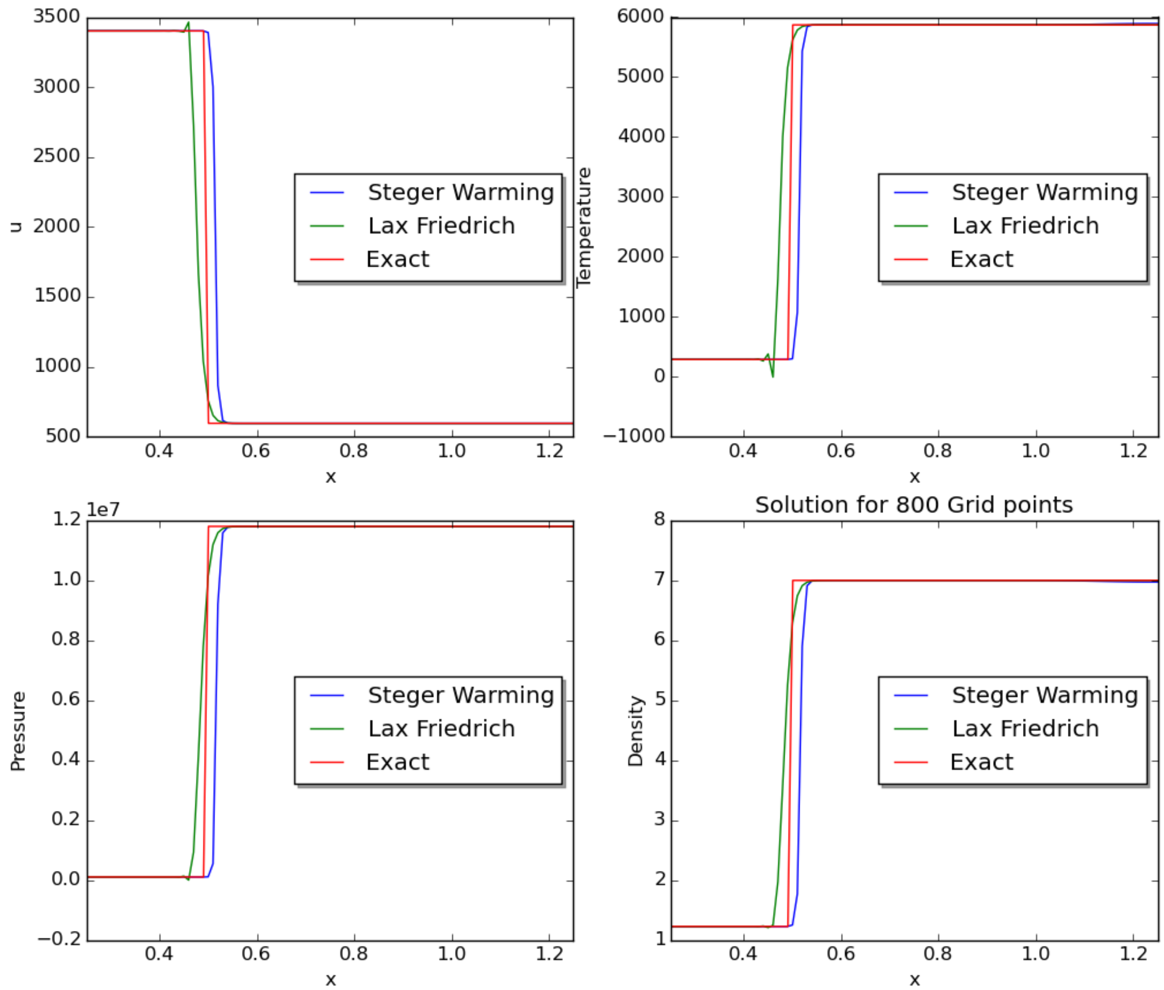


Figure 3: Plot for comparison of exact solution and approximate solutions for normal shock with 800 Grid points

Error Plots

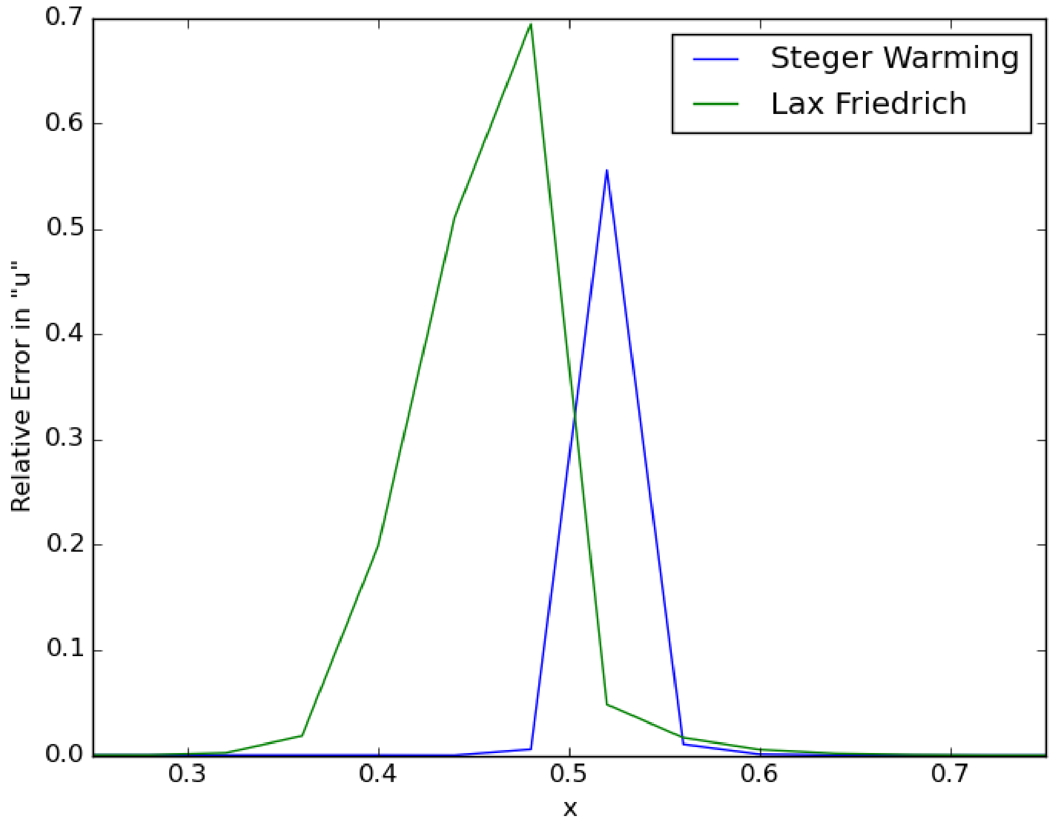


Figure 4: Error in the solution using different schemes with 200 Grid points

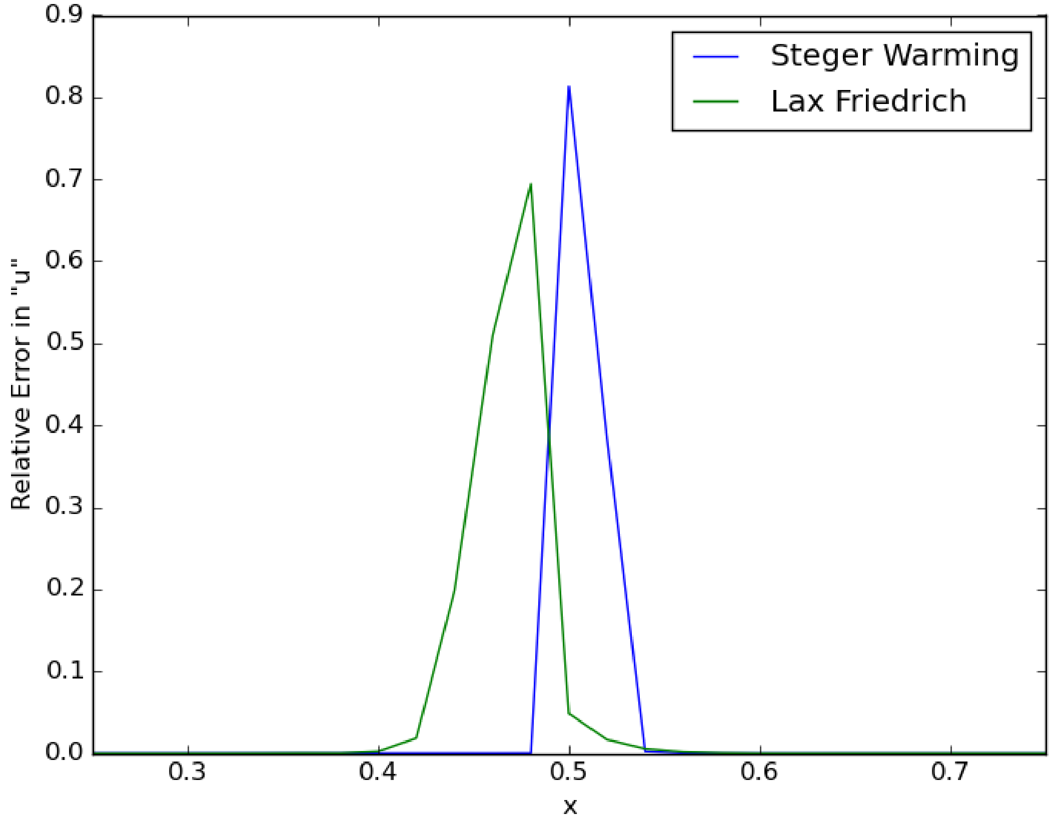


Figure 5: Error in the solution using different schemes with 400 Grid points

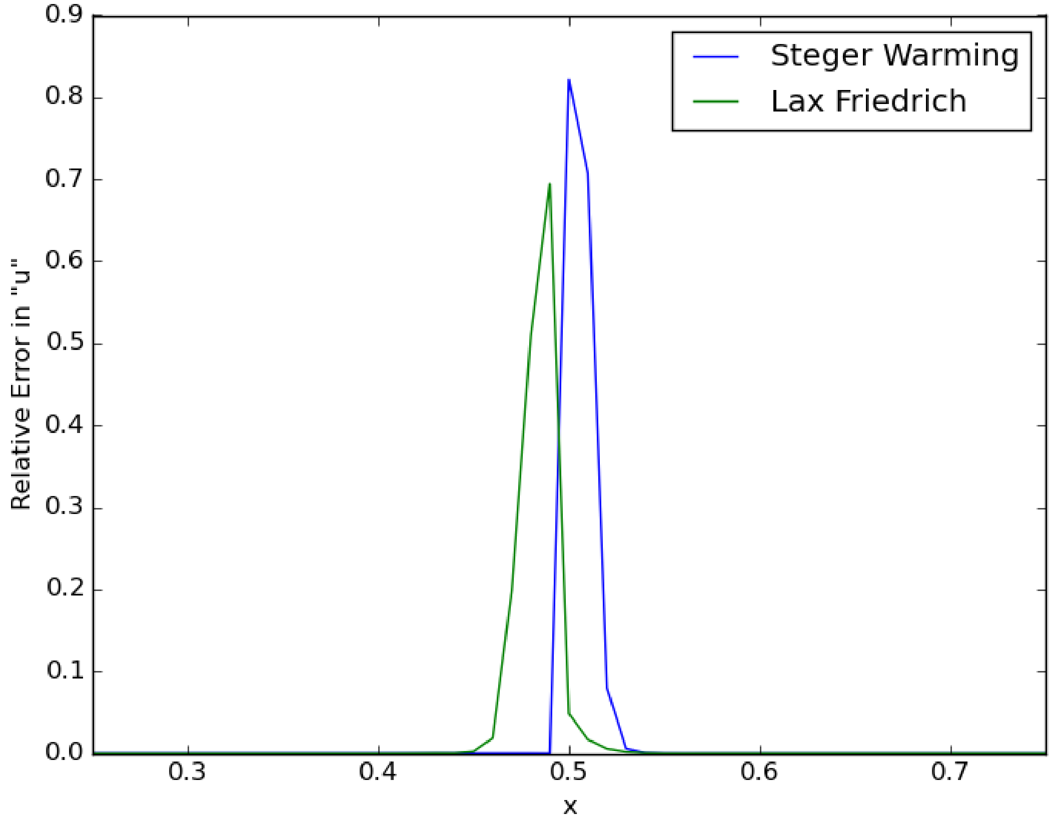


Figure 6: Error in the solution using different schemes with 800 Grid points

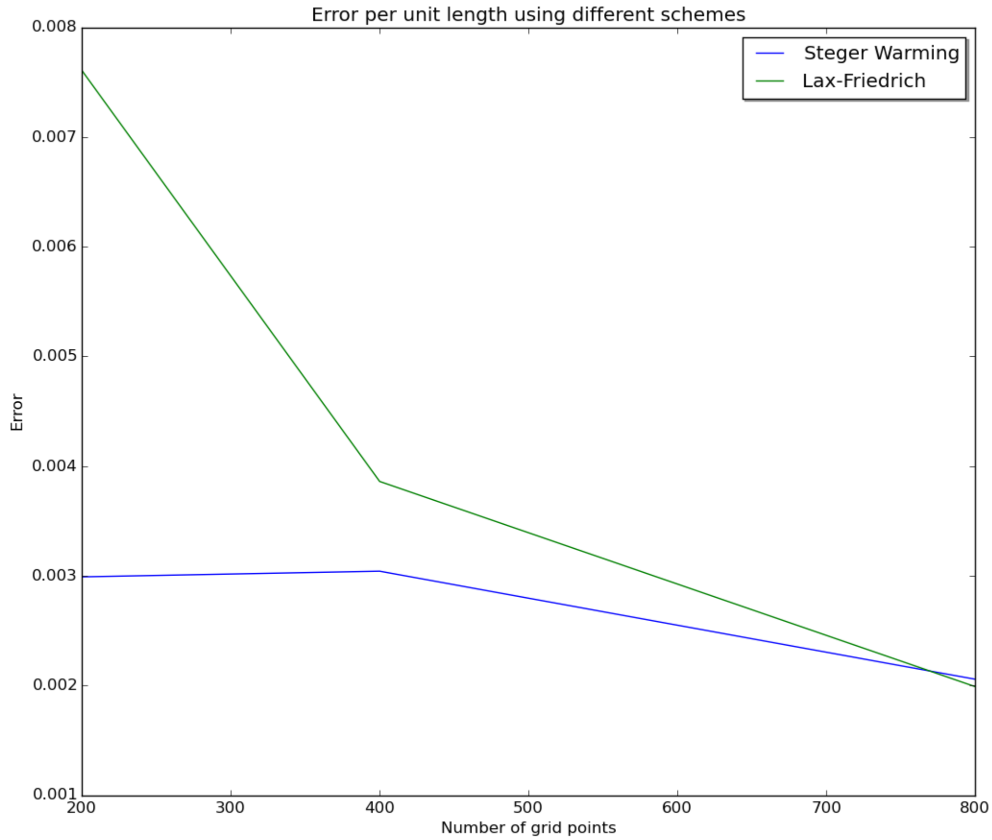


Figure 7: Error per unit length using different schemes with different grid points

Observations

1. As the number of grid points increases solution converges to the exact solution in both Lax Friedrich method and Steger Warming method.
2. Few fluctuations are observed in the solution obtained by Lax Friedrich method, but there are negligible fluctuations observed in Steger Warming method. Therefore, there is less dispersion effects in Steger Warming than Lax Friedrich method.
3. In Error plots, band is decreasing in both methods. There is only a small change observed in magnitude of error in Steger Warming method, whereas in Lax Friedrich method, the magnitude decreases as the number of grid points increase. So, for lower number of grid points Steger Warming method gives a better approximate solution compared to the Lax Friedrich method, hence reducing computation cost.

4. In Steger warming method, most of the error is present after the shock and concentrated in a very small region whereas in Lax Friedrich method, the error is diffused across the shock. Due to this reason, there is a drastic increase in the error in Steger Warming method just after the shock.

5. Error per unit length remains almost same in Steger warming method even though the number of grid points is increased, whereas for Lax Friedrich method, error decreases as number of grid points increase. Diffusion effect also decreases as number of grid points are increased in LF method.