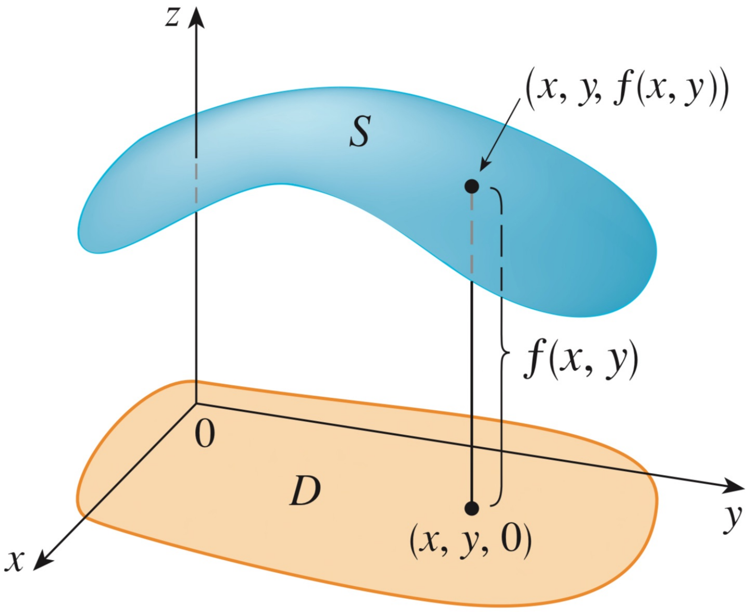
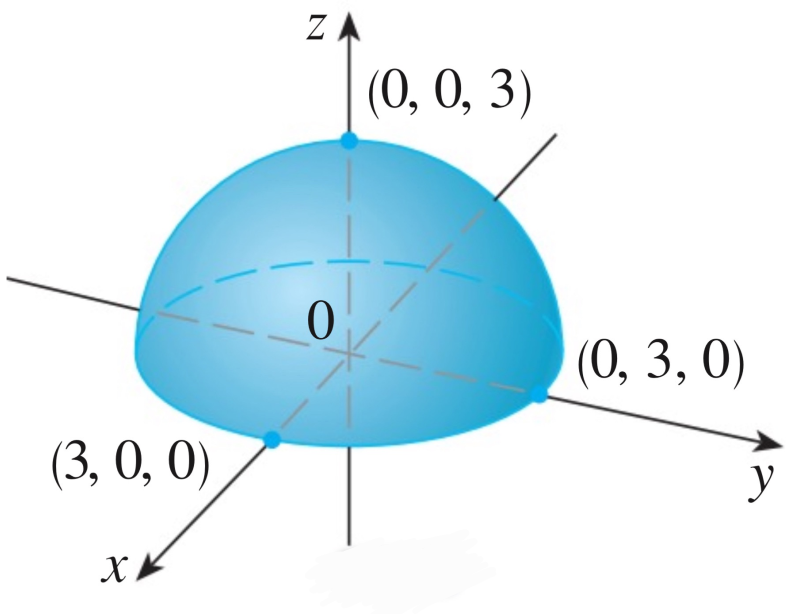
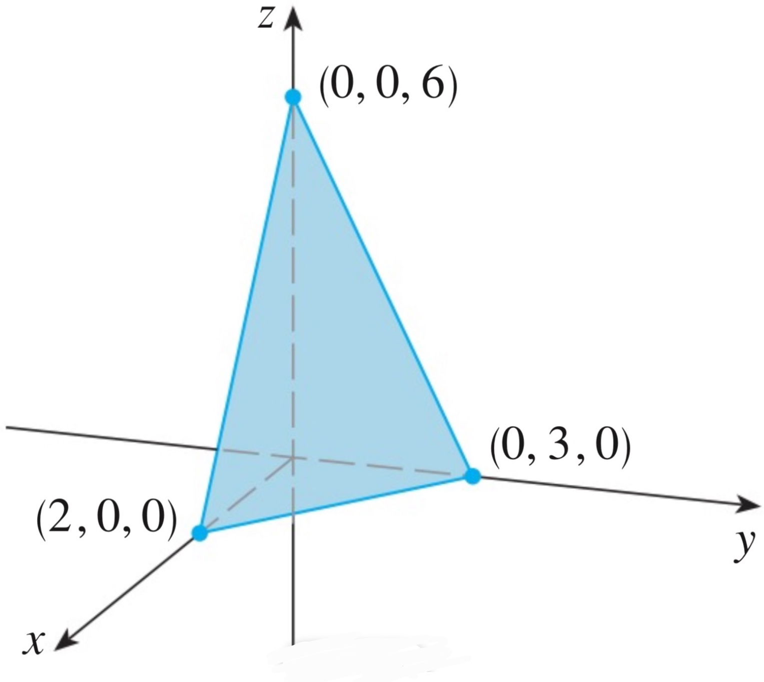


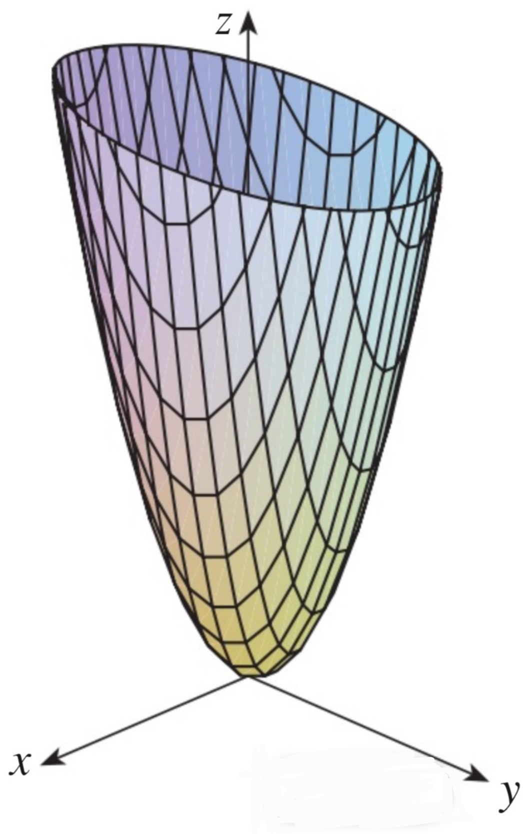
$z = f(x,y)$ (arbitrary surface)



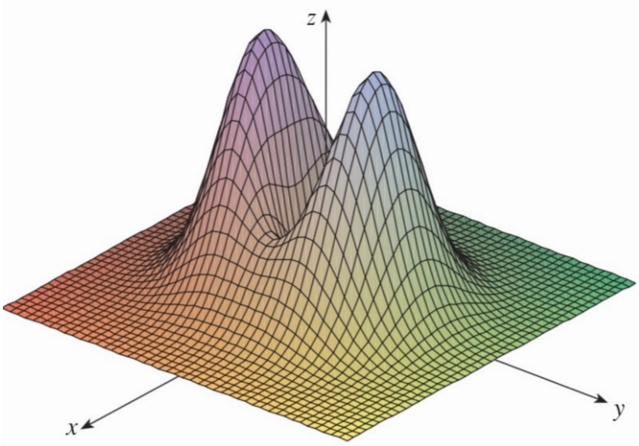
$z = \sqrt{9 - x^2 - y^2}$ (Hemisphere)



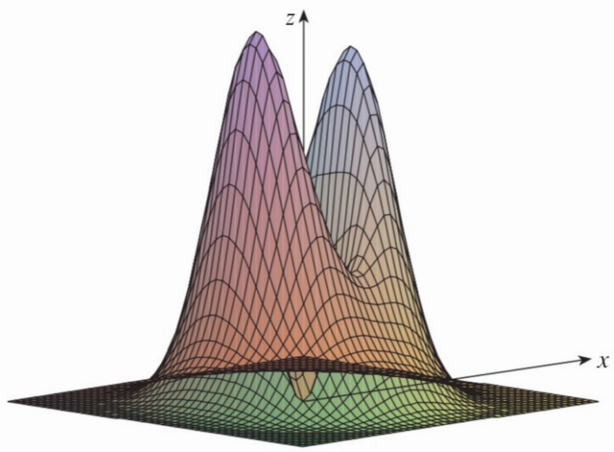
$z = 6 - 3x - 2y$ (plane)



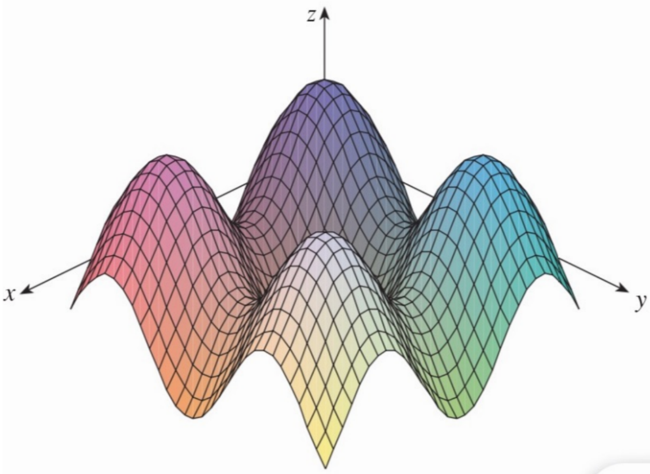
$z = 4x^2 + y^2$ (elliptic paraboloid)



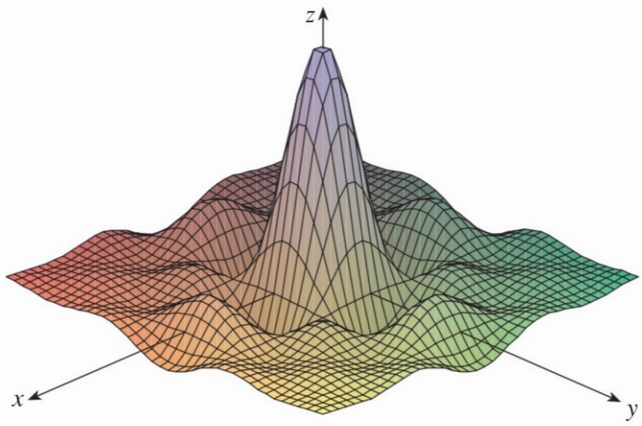
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

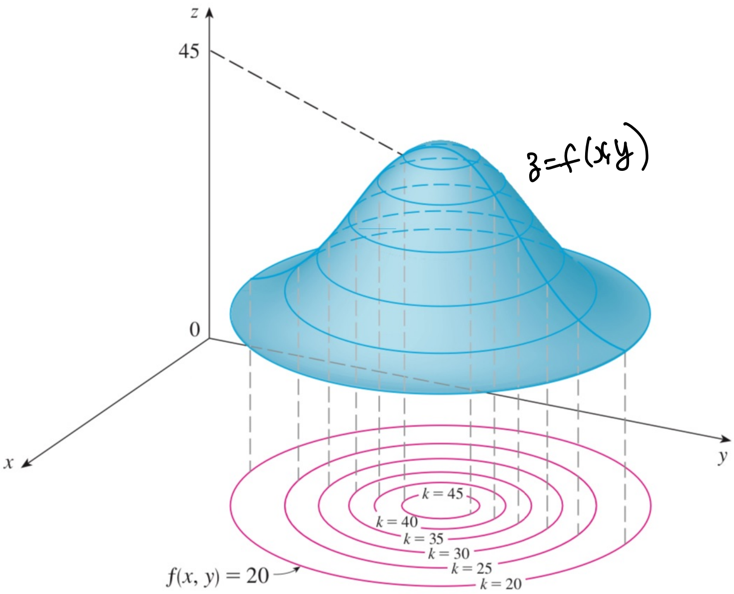


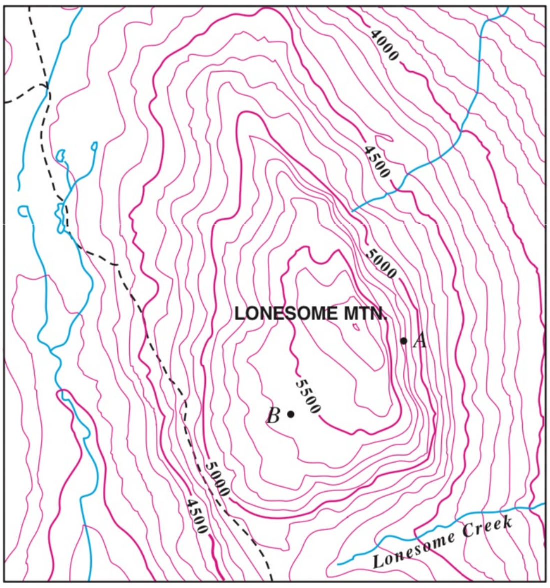
(c) $f(x, y) = \sin x + \sin y$

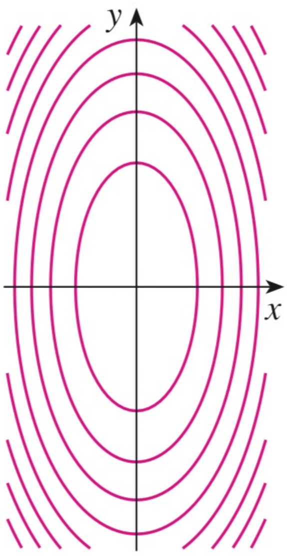


(d) $f(x, y) = \dfrac{\sin x \ \sin y}{xy}$

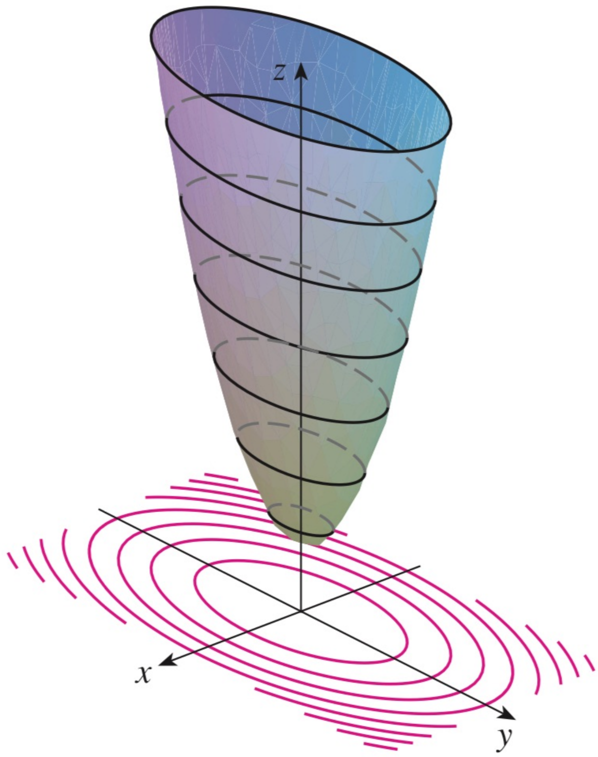
Level curves



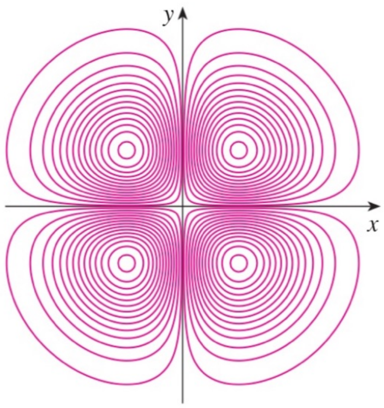




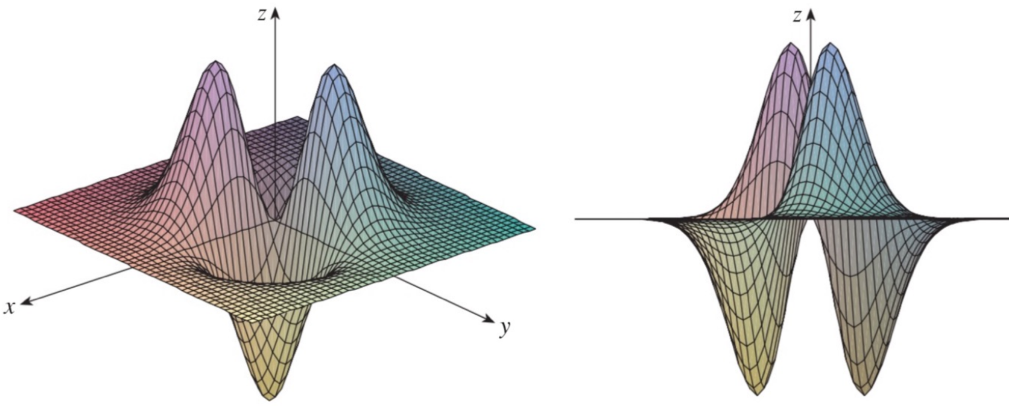
(a) Contour map



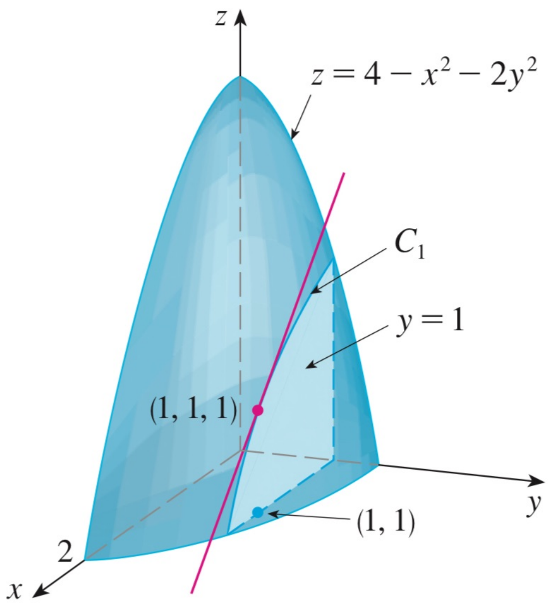
(b) Horizontal traces are raised level curves

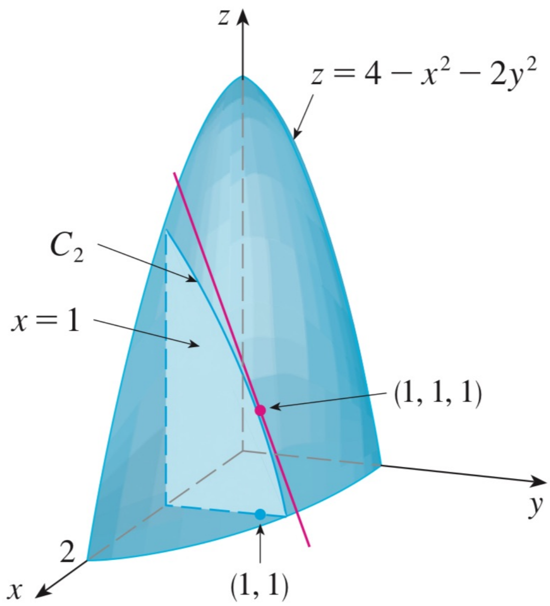


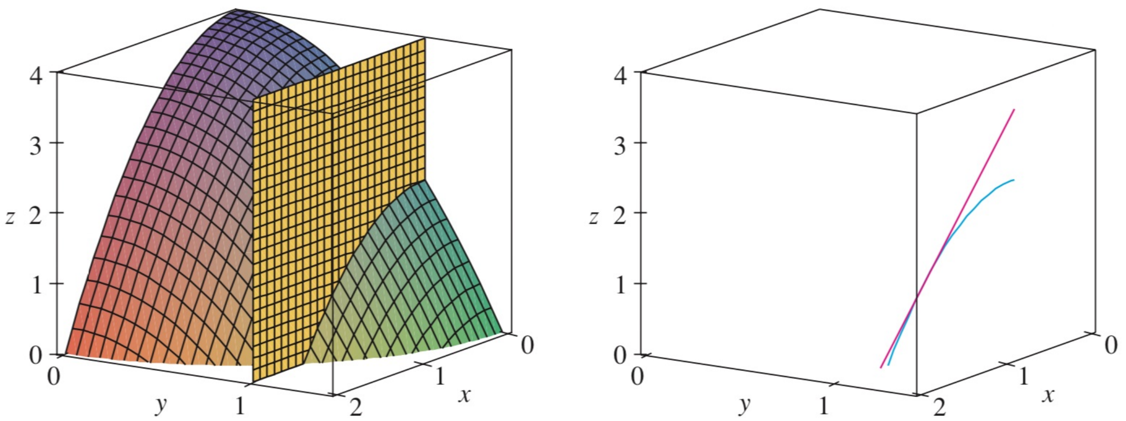
(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$



(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

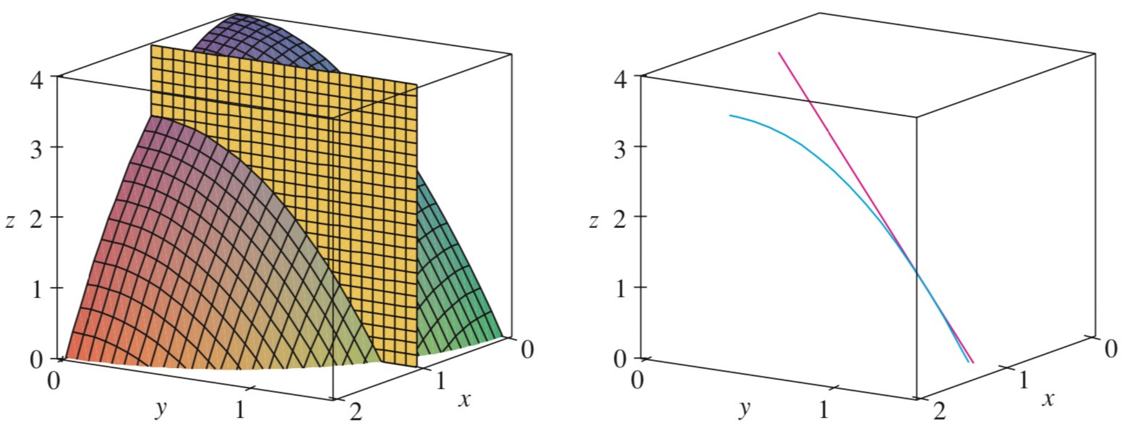

z
$z = 4 - x^2 - 2y^2$
C_1
$y = 1$
(1, 1, 1)
(1, 1)
2
x
y


z
$z = 4 - x^2 - 2y^2$
C_2
$x = 1$
(1, 1, 1)
(1, 1)
2
x
y


z
y
x
0
1
2
3
4

(a)

(b)


z
y
x
0
1
2
3
4

Ex 1) Calculate f_{xxyz} if $f(x,y,z) = \sin(3x + yz)$

Ex 2) Laplace Eqn is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. S.T $u(x,y) = e^x \sin y$

is a soln of Laplace Eqn.

Ex 3) Verify Clairaut's thm for the fn

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Ex 4) If $z = f(x+ct) + \phi(x-ct)$, PT

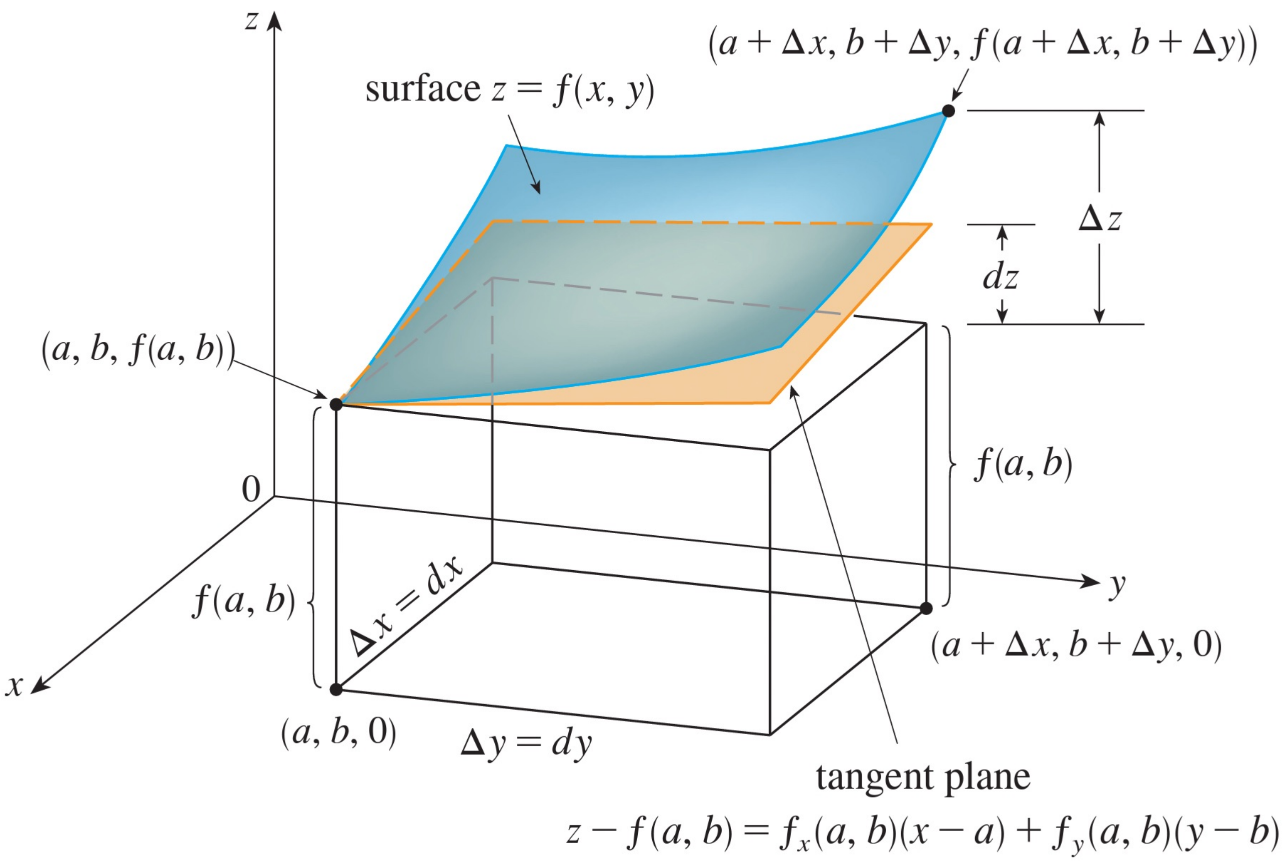
$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Ex 5) If $u = \log(x^3+y^3+z^3-3xyz)$.

$$\text{S.T} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Ex 6) If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$. S.T

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$



z
$(a + \Delta x, b + \Delta y, f(a + \Delta x, b + \Delta y))$
surface $z = f(x, y)$
Δz
dz
$(a, b, f(a, b))$
$f(a, b)$
0
$f(a, b)$
$\Delta x = dx$
y
$(a + \Delta x, b + \Delta y, 0)$
x
$(a, b, 0)$
$\Delta y = dy$
tangent plane
$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Total differentials

1) Find differentials of

i) $z = e^{-2x} \cos 2\pi t$

ii) $L = xz e^{-y^2-z^2}$

2) If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

EXAMPLE The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

EXAMPLE The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

EXAMPLE The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Recollection:

1) If $z = f(x,y)$; $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2) If $z = f(x,y)$; $x = x(s,t)$, $y = y(s,t)$, then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Chain rule:

1) If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

2) If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.

3) If $u = x^4y + y^2z^2$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

4) If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the Eqn $\quad t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$

5) If $z = f(x,y)$ has 2nd order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$

6) If $z = f(x,y)$ and $x = e^u \cos v$, $y = e^u \sin v$,
prove that i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$.

ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

7) If $u = F(x-y, y-z, z-x)$. Prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Implicit differentiation

Ex 1) Find $\frac{dy}{dx}$ if $x^3+y^3=6xy$

Ex 2) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$

Jacobians

Ex 1: Let $x=r\cos\theta$, $y=r\sin\theta$. Find $\frac{\partial(x,y)}{\partial(r,\theta)}$

Ex 2: If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Ex 3: Let $u=x+3y^2-z^3$, $v=4x^2yz$, $w=2z^2-xy$
Evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

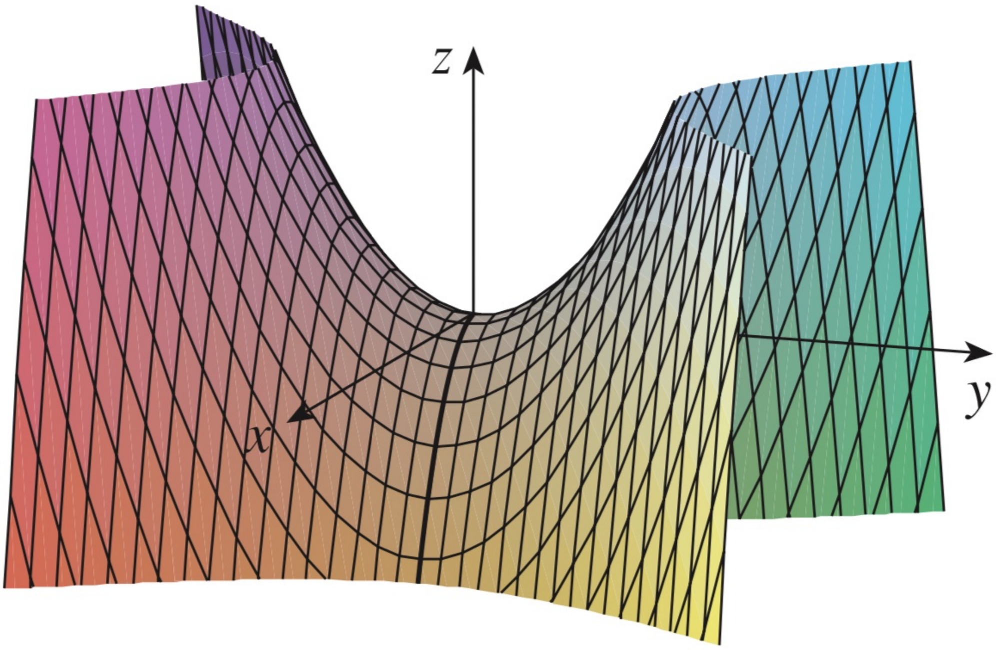
Ex 4: If $u=x^2-y^2$, $v=2xy$ and $x=r\cos\theta$, $y=r\sin\theta$,
find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Ex 5: If $u=x\sqrt{1-y^2}+y\sqrt{1-x^2}$, $v=\sin^{-1}x+\sin^{-1}y$. S.T
u and v are functionally related.

Ex 6: If $x=u(1-v)$, $y=uv$ Then Compute J and J^1
and verify that $JJ^1=1$

Ex 7: If $u+v=e^x\cos y$, $u-v=e^x\sin y$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Ans 1) r, 2) $r^2\sin\theta$ 3) 20 4) $4r^3$ 6) $-\frac{e^{2x}}{2}$


z
y
x

Recollection

Definition:

A function $f(x,y)$ is said to have local minimum at (a,b)

if $\quad f(a,b) \le f(x,y) \quad \forall (x,y)$ near (a,b)

A fn. $f(x,y)$ is said to have local maximum at (a,b)

if $\quad f(a,b) \ge f(x,y) \quad \forall (x,y)$ near (a,b)

Thm: If $f(x,y)$ has local maximum or local minimum at (a,b), then

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0.$$

Defn [critical points]:

A pt (a,b) is called critical pt of a fn. $f(x,y)$

if $\quad \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0 \quad$ and $\quad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$

$$\hat{u} = u_1\hat{i} + u_2\hat{j}$$

$$D_{\hat{u}}(f) = f_x u_1 + f_y u_2$$

$$D_{\hat{u}}(D_{\hat{u}}(f)) > 0$$

$$\Rightarrow \quad f_{xx} > 0 \quad \text{and}$$

$$\underbrace{f_{xx}f_{yy} - f_{xy}^2}_{D} > 0$$

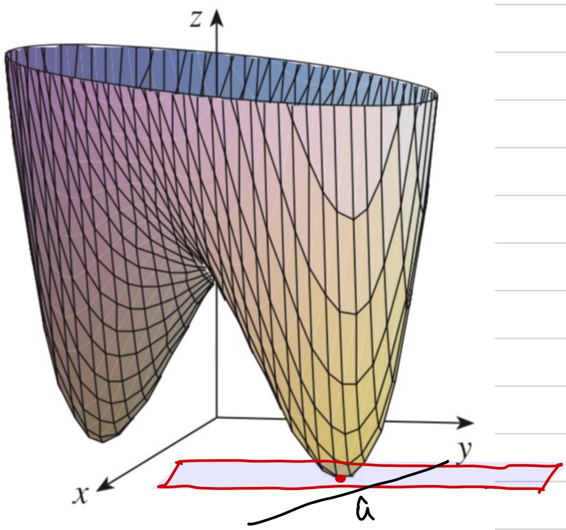


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

Second derivative test

Let (a,b) be a critical pt for a fn. $z=f(x,y)$.
If $f(x,y)$ has continuous 2^{nd} order partial derivatives.

Consider $D := f_{xx} f_{yy} - f_{xy}^2$ at (a,b)

i) If $f_{xx} > 0$ and $D > 0$, then $f(a,b)$ has min. value.

ii) If $f_{xx} < 0$ and $D > 0$, then $f(a,b)$ has max value.

iii) If $D < 0$, then (a,b) is called saddle pt
$f(a,b)$ is neither max nor min.

iv) $D = 0$, test fails.

Here $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ is called Hessian matrix.

Maximum and minimum Values

Ex 1) Find the local maximum and minimum values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

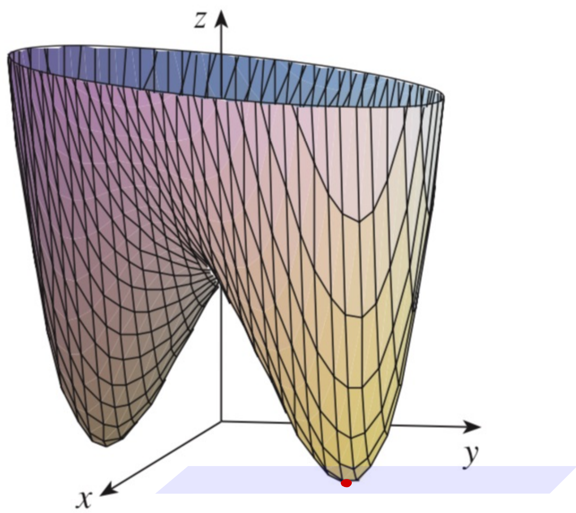


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

2) $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

3) $f(x,y) = \sin x + \sin y + \sin(x+y)$

4) A rectangular box open at the top to have a volume of 32 cubic units, what must be the dimension so that total surface area of the box is a minimum.

Lagrange multipliers

Ex1: A rectangular box without a lid is to be made from $12 m^2$ of cardboard. Find the maximum volume of such a box.

Ex 2: Find the max and min distance of the pt $(3,4,12)$ from the sphere $x^2+y^2+z^2=4$

Ex3: The temp. at any point (x,y,z) in space is $400xy^2z$. Find the highest temp. at the surface of unit sphere $x^2+y^2+z^2=1$ using Lagrange multipliers.

Ex 4: Find the vol. of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$