Alternative definition:

Scalar point function and scalar field: If to each point (x,y,z) of a region R in space there corresponds a definite scalar denoted by $\phi(x,y,z)$, then $\phi(x,y,z)$ is called scalar point function in R. The region R so defined is called a scalar field.

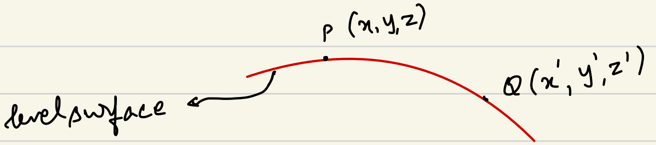
ex: Temperature $\phi(x,y,z)$ at any point P of unevenly heated body is scalar point function. The medium itself is scalar field.

Vector point function and vector field: If to each point (x,y,z) of a region R in space there corresponds a definite vector denoted by $\vec{f}(x,y,z)$, then $\vec{f}(x,y,z)$ is called vector point function in R. The region R so defined is called a vector field.

ex: The velocity of moving body in a certain region at time t is a vector point function.

Level surface: $\phi(x,y,z)=c$

represents family of surfaces in scalar field. If at each point on surface $\phi(x,y,z)=c$ has same value then surface is called level surface



$\phi(x,y,z)=c, \quad \phi(x',y',z')=c$

Vector differentiation

Vector functions: Functions whose values are vectors depending on points P in space.

$$v = v(P) = v_1(P)\hat{i} + v_2(P)\hat{j} + v_3(P)\hat{k} \text{ or } [v_1(P), v_2(P), v_3(P)]$$

(Domains may be 3-D or surface or a curve in space)

ex: velocity vector function of a rotating body.

Scalar functions: Functions whose values are scalars depending on the points P in space.

$$f = f(P)$$

ex: The distance $f(P)$ of any point P (x, y, z) from a fixed point P_0 (x_0, y_0, z_0) in space is a scalar function

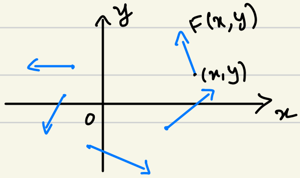
$$f(P) = f(x, y, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

A vector function defines a vector field and a scalar function defines a scalar field in the domain.

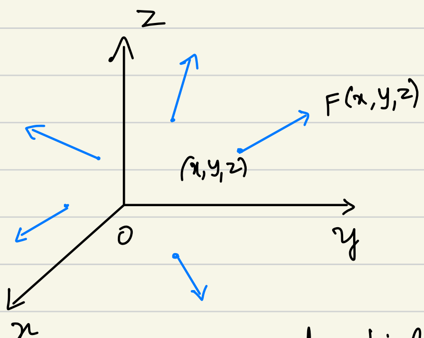
In general,

1) vector field is a function whose domain is a set of points in R^2 (or R^3) whose range is a set of vectors in V_2 (or V_3) or

Let D be a set in R^2 (a plane region). A vector field on R^2 is a function F that assigns to each point (x, y) in D a 2-dimensional vector $F(x, y)$



vector field on R^2



vector field on R^3

ex: 1) A magnetic field B in a region of space

$$B = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$$

2) Velocity field of water flowing in a pipe, $v(x,y,z)$

Scalar field : Let $D \subseteq R^2$ (a plane region)

A scalar field on R^2 is a function ϕ that assigns a scalar value $\phi(x,y)$ to each point (x,y) of D.

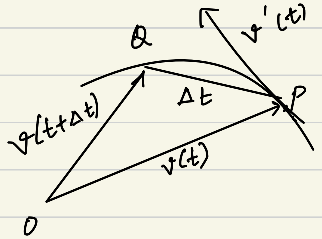
ex: Temperature, concentration, density etc

Derivative of vector function : A vector function is said to be differentiable at a point t if the following limit exists.

$$v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

$v'(t)$ is called derivative of $v(t)$

$v'(t)$ is obtained by differentiating each component separately.



Triangle law of add^n of vectors

In components wrt a given cartesian coordinate system

$$v'(t) = [v_1'(t),\ v_2'(t),\ v_3'(t)]$$

Velocity and Acceleration:

If $\vec{r}(t) = x(t)\,i + y(t)\,j + z(t)\,k$ is the position vector of a particle moving along a smooth curve in space, then $v(t) = \frac{d\vec{r}}{dt}$ is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $v(t)$ is the **direction of motion**, the magnitude of $v(t)$ is the particle's **speed**, and the derivative $a(t) = \frac{dv}{dt}$, when it exists, is the particle's **acceleration vector**.

In summary,

- Velocity is the derivative of position vector: $v(t) = \frac{d\vec{r}}{dt}$
- Speed is the magnitude of velocity: $speed = |v(t)|$
- Acceleration is the derivative of velocity: $a(t) = \frac{dv}{dt} = \frac{d^2 r}{dt^2}$
- Unit Tangent vector $\hat{T} = \frac{v(t)}{|v(t)|}$ is the direction of motion at time t.
- Component of velocity along a given vector $\vec{C}$ is $v(t) \cdot \hat{C}$
- Component of acceleration along a given vector $\vec{C}$ is $a(t) \cdot \hat{C}$

Differentiation rules for vector functions:

(1) $\frac{d}{dt}[\vec{a}(t) + \vec{b}(t)] = \frac{d}{dt}\vec{a}(t) + \frac{d}{dt}\vec{b}(t)$

(2) $\frac{d}{dt}[c\,\vec{a}(t)] = c\left[\frac{d}{dt}\vec{a}(t)\right]$, where c is constant

(3) $\frac{d}{dt}[\vec{a}(t) \cdot \vec{b}(t)] = \left[\frac{d}{dt}\vec{a}(t)\right] \cdot \vec{b}(t) + \vec{a}(t) \cdot \left[\frac{d}{dt}\vec{b}(t)\right]$

(4) $\frac{d}{dt}[\vec{a}(t) \times \vec{b}(t)] = \left[\frac{d}{dt}\vec{a}(t)\right] \times \vec{b}(t) + \vec{a}(t) \times \left[\frac{d}{dt}\vec{b}(t)\right]$

1) Find velocity, speed and acceleration of a particle whose motion in space is given by $r(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 5\cos^2 t\,\hat{k}$.

sol: $\vec{v}(t) = r'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} - 10\cos t\sin t\,\hat{k}$

$$\vec{v}(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} - 5\sin 2t\,\hat{k}$$

$$\vec{a}(t) = r''(t) = -2\cos t\,\hat{i} - 2\sin t\,\hat{j} - 10\cos 2t\,\hat{k}$$

speed is $|\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$

$$\text{speed} = \sqrt{4 + 25\sin^2 2t}$$

2) A particle moves such that its position vector at time t is $\vec{r} = e^{-t}\,i + 2\cos 3t\,j + 3\sin 3t\,k$. Determine its velocity, acceleration and their magnitude, direction at time $t = 0$.

Solution: velocity : $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\,i - 6\sin 3t\,j + 9\cos 3t\,k$

$\vec{v}(0) = -i + 9k$, magnitude $= \sqrt{82}$, direction is $\frac{1}{\sqrt{82}}(-i + 9k)$

acceleration: $\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\,i - 18\cos 3t\,j - 27\sin 3t\,k$, $\vec{a}(0) = i - 18j$, magnitude $= \sqrt{325}$,

direction is $\frac{1}{\sqrt{325}}(i - 18j)$.

2. For the curves whose equations are given below, find the unit tangent vectors:

(i) $x = t^2 + 1, y = 4t - 3, z = 2(t^2 - 3t)$ at $t = 0$.

(ii) $\vec{r} = a\cos 3t\,i + a\sin 3t\,j + 4at\,k$ at $t = \frac{\pi}{4}$

Solution: (i) In the vector form equation of the given curve is

$$\vec{r} = x\,i + y\,j + z\,k = (t^2+1)\,i + (4t-3)j + 2(t^2-3t)k$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 2t\,i + 4\,j + 2(2t-3)\,k$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4t^2+16+(4t-6)^2} = \sqrt{20t^2+52-48t} = 2\sqrt{5t^2+13-12t}$$

$\therefore$ Unit tangent vector to the given curve at a point '$\hat{n}$' is given by

$$\hat{n} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{2[t\,i+2\,j+(2t-3)\,k]}{2\sqrt{5t^2+13-12t}}$$

$$\text{At } t=0,\quad \hat{n} = \frac{(2j-3k)}{\sqrt{13}}$$

(ii) $\vec{r} = a\cos 3t\,i + a\sin 3t\;j + 4at\,k$

$$\frac{d\vec{r}}{dt} = -3\,a\sin 3t\;i + 3\,a\cos 3t\,j + 4a\,k$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9a^2\sin^2 3t + 9a^2\cos^2 3t + 16a^2} = 5a$$

$$\hat{n} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{a[-3\sin 3t\;i + 3\cos 3t\,j + 4\,k]}{5a}$$

$$\text{At } t=\frac{\pi}{4},\ \frac{1}{5}\left[\frac{-3}{\sqrt{2}}i - \frac{-3}{\sqrt{2}}j + 4\,k\right] = \frac{1}{5\sqrt{2}}\left[-3i-3j+4\sqrt{2}\,k\right]$$

3. Find the angle between the tangents to the curve $\vec{r} = t^2 i + 2t\, j - t^3 k$ at the points $t = \pm 1$.

Solution: $\dfrac{d\vec{r}}{dt} = 2t\, i + 2\, j - 3t^2\, k$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4t^2 + 4 + 9t^4}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\hat{n} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{2t\, i + 2 - 3t^2\, k}{\sqrt{4t^2 + 4 + 9t^4}}$$

$$\text{At } t = 1, \hat{n}_1 = \frac{2i + 2j - 3k}{\sqrt{17}}$$

$$\text{At } t = -1, \hat{n}_2 = \frac{-2i + 2j - 3k}{\sqrt{17}}$$

Angle between unit tangent vectors at the points $t = \pm 1$ is given by

$$\cos\theta = \hat{n}_1 . \hat{n}_2 = \frac{9}{17}$$

$$\theta = \cos^{-1}(9/17).$$

4. A particle moves along the curve $x = \cos(t-1)$, $y = \sin(t-1), z = at^3$ where a is a constant. Find a so that acceleration is perpendicular to position vectors at $t = 1$.

Solution: At time t position vector of particle is

$$\vec{r} = \cos(t-1)i + \sin(t-1)\, j + at^3 k$$

$$\bar{v} = \frac{d\bar{r}}{dt} = -\sin(t-1)i + \cos(t-1)\, j + 3at^2\, k$$

$$v'_{t=1} = j + 3a\, k$$

$$\text{Acceleration} = \frac{d\bar{v}}{dt} = -\cos(t-1)\, i - \sin(t-1)\, j + 6at\, k$$

$$\left.\frac{d\bar{v}}{dt}\right|_{t=1} = -i + 6a\, k,\quad \vec{r}_{t=1} = i + ak$$

Given acceleration is perpendicular to position vector.

$$\bar{r}.\frac{d\bar{v}}{dt} = 0$$

$$\Rightarrow -1 + 6a^2 = 0 \Rightarrow a^2 = \frac{1}{6} \Rightarrow a = \pm 1/\sqrt{6}$$

5. A particle moves along the curve $\vec{r} = 2t^2\,i + (t^2 - 4t)\,j + (3t - 5)\,k$. Find the component of velocity and acceleration in the direction of vector $c = i - 3j + 2k$ at $t = 1$.

Solution: Given $\vec{r} = 2t^2\,i + (t^2 - 4t)\,j + (3t - 5)\,k$

Velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = 4t\,i + (2t - 4)\,j + 3\,k$

Acceleration$= \dfrac{d\vec{v}}{dt} = 4\,i + 2\,j$

At $t = 1$, $\vec{v} = 4i - 2j + 3k$

$\dfrac{d\vec{v}}{dt} = 4i + 2j$

Also $c = i - 3j + 2k$

$|c| = \sqrt{14}$

$\hat{c} = \dfrac{\vec{c}}{|c|} = \dfrac{i - 3j + 2k}{\sqrt{14}}$

$\therefore$ Component of velocity at $t = 1$ along the given vector $\vec{c}$ is,

$\vec{v}.\hat{c} = \dfrac{1}{14}(4 + 6 + 6) = \dfrac{16}{\sqrt{14}}$

$\therefore$ Component of acceleration at $t = 1$ along the given vector $\vec{c}$ is,

$$\frac{d\vec{v}}{dt}.\hat{c} = \frac{1}{14}(4 - 6) = \frac{-2}{\sqrt{14}}.$$

6) The position vector of a particle at time t is
$$\vec{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3 \hat{k}$$
Find the condition imposed on α by requiring that at time $t=1$, the acceleration is normal to the position vector.

Sol.
$$\frac{d\vec{r}}{dt} = -\sin(t-1)\hat{i} + \cosh(t-1)\hat{j} + 3\alpha t^2 \hat{k}$$
$$\text{Acceleration} = \frac{d^2\vec{r}}{dt^2} = -\cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + 6\alpha t\hat{k}$$
At $t=1$, acceleration $= -\hat{i} + 6\alpha\hat{k}$
$$\vec{r} = \hat{i} + \alpha\hat{k}$$

If $\vec{r}$ and $\vec{a}$ are normal, then dot product is 0.
$$(-\hat{i} + 6\alpha\hat{k})\cdot(\hat{i} + \alpha\hat{k}) = 0$$
$$-1 + 6\alpha^2 = 0 \Rightarrow \alpha^2 = \frac{1}{6} \Rightarrow \alpha = \pm\frac{1}{\sqrt{6}}$$

Practice problems :

1) A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$, where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $2\hat{i} + 3\hat{j} + 6\hat{k}$.

Ans: Component of velocity along $2\hat{i}+3\hat{j}+6\hat{k} = \frac{24}{7}$

-11- -11- acceleration -11- -11- $= \frac{18}{7}$

2) A particle moves along the curve $\vec{r} = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$, where t is the time. Find the magnitude

of tangential components of its acceleration at $t=2$.

Ans: 16

3) Find angle between unit tangent vectors drawn to the curve $x = a\cos 2t$, $y = a\sin 2t$, $z = at$ at the points $t = \frac{\pi}{6}$ and $t = \frac{\pi}{4}$.

Ans: $\theta = \cos^{-1}\left(\frac{2\sqrt{3}+1}{5}\right)$

The vector differential operator is denoted by ∇ (read as del) and is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient of a scalar field :

If $\phi(x, y, z)$ is a scalar function then, the gradient of ϕ is denoted by grad(ϕ) or $\nabla\phi$, defined as

$$\text{grad}(\phi) = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

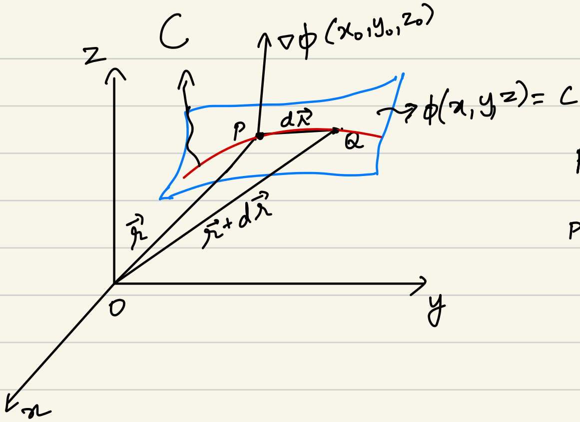
Note: The operator gradient is always applied on scalar field and the resultant is a vector i.e. the operator gradient converts a scalar field into a vector field. $\nabla\phi$ is a vector function. Beware ∇ cannot exist alone, it is an operator and must operate on scalar function.

Physical significance: $\nabla\phi$ gives the direction of maximum rate of change of ϕ at a particular point. $-\nabla\phi$ points in the direction of most rapid decrease of ϕ.

Geometrical interpretation of gradient : (Function of 3 variables)

Let $\phi(x, y, z) = c$ be the surface. Let $P(x_0, y_0, z_0)$ be a point on the surface. If the differentiable functions $x = f(t)$, $y = g(t)$, $z = h(t)$ are parametric equations of curve C on surface then derivative of $\phi(f(t), g(t), h(t)) = c$ wrt t is

$$\frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} = 0$$



Position vector of $P = \bar{r}$

Position vector of $Q = \vec{r} + d\vec{r}$

or $$\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0 \quad \rightarrow (1)$$

At $t = t_0$,

$$\nabla F(x_0, y_0, z_0) \cdot r'(t_0) = 0$$

$\Rightarrow \nabla F(x_0,y_0,z_0)$ is orthogonal to vector $r'(t_0)$.

$\Rightarrow \nabla F(x_0,y_0,z_0)$ is normal (perpendicular) to surface at P.

Note: $$\nabla\phi \cdot \overbrace{dr}^{\text{differential (actual change in } r)} = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\nabla\phi\cdot dr = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \underbrace{d\phi}_{\text{total differential of } \phi}$$

Properties: If f and g are continuous and differentiable scalar point functions then

1) $\text{grad}(f \pm g) = \text{grad} f \pm \text{grad} g$

or $\nabla(f \pm g) = \nabla f \pm \nabla g$

2) $$\text{grad}\left(\frac{f}{g}\right) = \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

3) $\nabla(fg) = g\nabla f + f\nabla g$

Unit normal vector : Since $\nabla\phi$ is normal vector to surface $\phi(x,y,z) = c$ then unit vector is denoted by $\hat{n}$ and is defined as $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\phi}{|\nabla\phi|}$

where $\hat{n} = \nabla\phi$ = normal vector.

Note. The angle b/w the normals to the 2 surfaces ϕ_1 and ϕ_2 is $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$ where $\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|}$, $\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|}$

Directional derivative :

From the definition of gradient it is observed the partial derivatives rate of change of ϕ along coordinate axes. The idea of extending this to arbitrary direction makes sense and lead to concept of directional derivative. Directional derivative measures variation of a function along a given direction

Definition: If f is a differentiable function of x and y, then the directional derivative of f in the direction of any unit vector, $\hat{u} = \langle a, b \rangle$ is

$$\boxed{D_u f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b = \nabla f \cdot \hat{u}}$$

nothing but projection of grad f on unit vector $\hat{u}$.

Note: 1) If the unit vector $\hat{u}$ makes an angle θ with positive x axis then we write $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ and $D_u \phi(x,y) = \phi_x(x,y)\cos\theta + \phi_y(x,y)\sin\theta$

2) Maximum value of directional derivative :

$$D_u f = \nabla f \cdot \hat{u} = |\nabla f|\,|\hat{u}| \cos\theta = |\nabla f| \cos\theta \quad (\because |\hat{u}| = 1)$$

The range of $\cos\theta$ is $[-1, 1]$ ie. $-1 \le \cos\theta \le 1$

$$\Rightarrow -|\nabla f| \le D_u f \le |\nabla f|$$

In other words,

The maximum value of directional derivative is $|\nabla f|$ and it occurs when $\hat{u}$ has same direction as ∇f (when $\cos\theta = 1$, θ is angle b/w $\hat{u}$ and $\nabla\phi$.)

The minimum value of directional directive is $-|\nabla f|$ and it occurs when $\hat{u}$ and f have opposite directions (when $\cos\theta = -1$).

ex: If you are standing in a hot room and thinking to move in a direction along which temperature change is quickest or maximum. Then gradient of temperature field calculated at your position points to the direction you need.

Directional derivative $D_u T$ measures the variation of temperature in that direction.

Problems:

1) Find gradient of scalar field $\phi(x,y,z) = xy^2 + 2x^2yz - 3yz^2$

sol:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (y^2 + 4xyz)\hat{i} + (2xy + 2x^2z - 3z^2)\hat{j} + (2x^2y - 6yz)\hat{k}$$

2) If $\phi = x^2yz^3 + xy^2z^2$, determine grad ϕ at $P = (1,3,2)$

sol:

$$\nabla\phi = (2xyz^3 + y^2z^2)\hat{i} + (x^2z^3 + 2xyz^2)\hat{j} + (3x^2yz^2 + 2xy^2z)\hat{k}$$

At $P = (1,3,2)$ we get

$$\nabla\phi = 84\hat{i} + 32\hat{j} + 72\hat{k}$$

3) Determine the directional derivative $D_u\phi(x,y)$ if $\phi(x,y) = x^3 - 3xy + 4y^2$ and $\hat{u}$ is the unit vector making angle $\theta = \frac{\pi}{6}$ with x-axis. What is $D_u\phi(1,2)$?

sol:

$$D_u\phi(x,y) = \phi_x(x,y)\cos\frac{\pi}{6} + \phi_y(x,y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$

$$= \frac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + (8 - 3\sqrt{3})y\right]$$

$$D_u\phi(1,2) = \frac{1}{2}\left[3\sqrt{3}\cdot 1^2 - 3(1) + (8 - 3\sqrt{3})2\right]$$

$$\therefore D_u\phi(1,2) = \frac{13 - 3\sqrt{3}}{2}$$

4) Obtain the directional derivative of a function $\phi(x,y,z) = x^2z + 2xy^2 + yz^2$ at the point $P(1,2,-1)$ in the direction of vector $2\hat{i}+3\hat{j}-4\hat{k}$.

Sol: Given: $\phi = x^2z+2xy^2+yz^2$, $\vec{v} = 2\hat{i}+3\hat{j}-4\hat{k}$

WKT $D_u\,\phi(x,y,z) = \nabla\phi \cdot \hat{u} \rightarrow (1)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (2xz+2y^2)\hat{i} + (4xy+z^2)\hat{j} + (x^2+2yz)\hat{k}$$

$$\nabla\phi_{(1,2,-1)} = 6\hat{i}+9\hat{j}-3\hat{k}$$

Here $\vec{v}$ is not a unit vector.

unit vector, $\hat{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{2\hat{i}+3\hat{j}-4\hat{k}}{|2\hat{i}+3\hat{j}-4\hat{k}|} = \dfrac{2}{\sqrt{29}}\hat{i}+\dfrac{3}{\sqrt{29}}\hat{j}-\dfrac{4}{\sqrt{29}}\hat{k}$

$(1) \Rightarrow$ $$D_u\,\phi(x,y,z) = (6\hat{i}+9\hat{j}-3\hat{k})\cdot\left(\frac{2}{\sqrt{29}}\hat{i}+\frac{3}{\sqrt{29}}\hat{j}-\frac{4}{\sqrt{29}}\hat{k}\right)$$

$$= \frac{2}{\sqrt{29}}(6) + \frac{3}{\sqrt{29}}(9) + \frac{-4}{\sqrt{29}}(-3)$$

$$= \frac{51}{\sqrt{29}} \approx 9.4704$$

5) Find unit normal to the surface $xyz=2$ at $(2,1,1)$

Sol. $$xyz-2=0$$

Let $\phi = xyz-2$

$$\nabla\phi = yz\hat{i}+xz\hat{j}+xy\hat{k}$$

At $(2,1,1)$: $\nabla\phi = \hat{i}+2\hat{j}+2\hat{k}$

$\therefore |\nabla\phi| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

Unit vector normal to surface ϕ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

6) Find the directional derivative of $\phi(x,y,z) = \frac{1}{x^2+y^2+z^2}$ at the point $(1,2,-3)$ in the direction of $2\hat{i}-3\hat{j}+\hat{k}$

sol:

$$\nabla\phi = \frac{-2x}{(x^2+y^2+z^2)^2}\hat{i} - \frac{2y}{(x^2+y^2+z^2)^2}\hat{j} - \frac{2z}{(x^2+y^2+z^2)^2}\hat{k}$$

At $(1,2,-3) \Rightarrow$ $\nabla\phi = \frac{-2}{(14)^2}\hat{i} - \frac{4}{(14)^2}\hat{j} + \frac{6}{14^2}\hat{k}$

$$\nabla\phi = \frac{-2}{14^2}\left[\hat{i}+2\hat{j}-3\hat{k}\right]$$

$$\vec{v} = 2\hat{i}-3\hat{j}+\hat{k} \Rightarrow |\vec{v}| = \sqrt{14}$$

Unit vector, $\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{14}}(2\hat{i}-3\hat{j}+\hat{k})$

The directional derivative is $\nabla\phi\cdot\hat{u} = \frac{-2}{(14)^2}(\hat{i}+2\hat{j}-3\hat{k})\cdot\frac{1}{\sqrt{14}}(2\hat{i}-3\hat{j}+\hat{k})$

$$= \frac{-2}{(14^2)\sqrt{14}}(2-6-3) = \frac{14}{14^2\sqrt{14}} = \frac{1}{14\sqrt{14}}$$

7) Find the angle b/w the normals to the surface $xy^3z^2 = 4$ at the points $(-1,-1,2)$ and $(4,1,1)$

Sol: Let $\phi = xy^3z^2 - 4$

$$\nabla\phi = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$$

Normal at $(-1,-1,2)$: $n_1 = \nabla\phi\Big|_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}$

Normal at $(4,1,-1)$: $n_2 = \nabla\phi\Big|_{(4,1,-1)} = \hat{i} + 12\hat{j} - 8\hat{k}$

Let θ be angle b/w 2 normals.

$$\therefore \cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{(-4\hat{i}-12\hat{j}+4\hat{k})\cdot(\hat{i}+12\hat{j}-8\hat{k})}{\sqrt{16+144+16}\ \sqrt{1+144+64}}$$

or $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$
where $\hat{n}_1 = \dfrac{\nabla\phi_{(-1,-1,2)}}{|\nabla\phi_{(-1,-1,2)}|}$

$$\cos\theta = \frac{-180}{\sqrt{176}\ \sqrt{209}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-180}{\sqrt{176}\ \sqrt{209}}\right) = 159.765^\circ$$

8) Find the angle b/w 2 surfaces $x^2+y^2+z^2 = 9$ and $x^2+y^2-z = 3$ at $(2,-1,2)$

Sol: The angle b/w 2 surfaces at common point is angle b/w the normals drawn to the surfaces at that point.

Given: $\phi_1 = x^2+y^2+z^2-9$, $\phi_2 = x^2+y^2-z-3$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}, \quad \nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla\phi_1\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}, \quad \nabla\phi_2\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{16+4-4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) = 54.41^\circ$$

9) Find the maximum directional derivative of $\phi(x,y,z) = x^3y^2z$ at $(1,-2,3)$.

Sol:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = 3x^2y^2z\,\hat{i} + 2x^3yz\,\hat{j} + x^3y^2\,\hat{k}$$

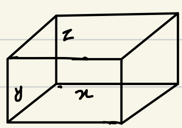
At $(1,-2,3) \Rightarrow \nabla\phi = 36\hat{i} - 12\hat{j} + 4\hat{k}$

The maximum directional derivative $= |\nabla\phi| = 4\sqrt{9^2+(-3)^2+1^2} = 4\sqrt{91}$

10) The temperature in a rectangular box is approximated by $T(x,y,z) = xyz(1-x)(2-y)(3-z)$, $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$. If a mosquito is located at $(\frac{1}{2}, 1, 1)$ in which direction should it fly to cool off as rapidly as possible?

Sol: $\nabla T(x,y,z) = yz(2-y)(3-z)(1-2x)\hat{i} + xz(1-x)(3-z)(2-2y)\hat{j}$
$+ xy(1-x)(2-y)(3-2z)\hat{k}$

$$\nabla T\Big|_{(\frac{1}{2},1,1)} = \frac{1}{4}\hat{k}$$



To cool off most rapidly, the mosquito should fly in the direction of $-\frac{1}{4}\hat{k}$

11) Find constants a and b so that surface $3x^2 - 2y^2 - 3z^2 + 8 = 0$ is orthogonal to surface $ax^2 + y^2 = bz$ at point $(-1, 2, 1)$

Sol:

$$\text{Let } \phi_1(x,y,z) = 3x^2 - 2y^2 - 3z^2 + 8$$

$$\Rightarrow \nabla\phi_1 = 6x\hat{i} - 4y\hat{j} - 6z\hat{k}$$

$$\text{At } (-1,2,1), \ \nabla\phi_1 = -6\hat{i} - 8\hat{j} - 6\hat{k}$$

$$\Rightarrow |\nabla\phi_1| = \sqrt{6^2+8^2+6^2} = \sqrt{136} = 2\sqrt{34}$$

$$\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{2\sqrt{34}}(-6\hat{i} - 8\hat{j} - 6\hat{k}) = -\frac{1}{\sqrt{34}}(3\hat{i} + 4\hat{j} + 3\hat{k})$$

$$\text{Let } \phi_2(x,y,z) = ax^2 + y^2 - bz \Rightarrow \nabla\phi_2 = 2ax\hat{i} + 2y\hat{j} - b\hat{k}$$

$$\text{At } (-1,2,1) \Rightarrow \nabla\phi_2 = -2a\hat{i} + 4\hat{j} - b\hat{k}$$

$$\Rightarrow |\nabla\phi_2| = \sqrt{4a^2+16+b^2}$$

$$\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{-2a\hat{i} + 4\hat{j} - b\hat{k}}{\sqrt{4a^2+b^2+16}}$$

Since surfaces intersect orthogonally

$$\hat{n}_1 \cdot \hat{n}_2 = 0 \Rightarrow -\frac{1}{\sqrt{34}}(3\hat{i} + 4\hat{j} + 3\hat{k}) \cdot \frac{1}{\sqrt{4a^2+b^2+16}}(-2a\hat{i} + 4\hat{j} - b\hat{k}) = 0$$

$$\Rightarrow (3\hat{i} + 4\hat{j} + 3\hat{k}) \cdot (-2a\hat{i} + 4\hat{j} - b\hat{k}) = 0$$

$$\Rightarrow -6a + 16 - 3b = 0 \quad \text{i.e. } 6a + 3b = 16 \rightarrow ①$$

Also the point $(-1, 2, 1)$ lies on surface $ax^2 + y^2 = bz$

$$\Rightarrow a + 4 = b \quad \text{i.e. } a - b = -4 \rightarrow ②$$

solving ① & ② $\Rightarrow a = \frac{4}{9}$ and $b = \frac{40}{9}$.

Practice problems :

1. Find the gradient of the given function at indicated point.

I. $\phi(x,y,z) = x^2yz + 4z^2$; $(1,-2,-1)$ Ans.: $8\vec{i} - \vec{j} + 10\vec{k}$

II. $\phi(x,y,) = x^2z^2 \sin 4y$; $(-2, \frac{\pi}{3}, 1)$

III. $\phi(x,y,z) = x^3yz + xy^2z^3$; $(1,2,3)$ Ans.: $126\vec{i} + 111\vec{j} + 110\vec{k}$

2. Find the directional derivative of the given function at the given point in the indicated direction.

I. $\phi(x,y,z) = 4xz^2 + x^2yz;(1,-2,1), 2\vec{i} + 3\vec{j} + 4\vec{k}$ Ans.: $\frac{27}{\sqrt{29}}$

II. $\phi(x,y,z) = x^2yz + 4xz^2$;$(1,-2,-1), 2\vec{i} - \vec{j} - 2\vec{k}$Ans.: $\frac{37}{3}$

3. Find the directional derivative of the function $\phi = xe^y + ye^z + ze^x$ at the point $(0,0,0)$ in the direction of the vector $\vec{v} = -\hat{i} - 2\hat{j} + 2\hat{k}$. Also find the maximal directional derivative.

Ans.: Directional derivative of ϕ is $\hat{a}.\,grad\ \phi = \frac{-1}{3}$. Maximal directional derivative is $|\nabla\phi| = \sqrt{3}$.

4. The temperature of a point in a space is given by$T(x,y,z) = (x^2 + y^2 + z^2)$. Determine unit vector when a honey bee located at $(2,1,2)$ desires to fly in such a direction that it will get heat as soon as possible.

Ans.:$\nabla T(2,1,2) = 2\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$. Unit vector along gradient is $\frac{i+j-2k}{3}$

5) Find the angle b/w normals to the surface $z^2 - xy = 0$ at the points $(4, 1, 2)$ and $(3, 3, -3)$ Ans: $\cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$

6) Find the constants a and b such that surface $x^2 + ayz = 3x$ and $bx^2y + z^3 = (b-8)y$ intersect orthogonally at point $(1, 1, -2)$
Ans: $a = -1$, $b = 2$

7) Find the angle b/w normals to the surface $x \log z = y^2 - 1$ at points $(1, 1, 1)$ and $(2, 1, 1)$
Ans: $\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

8) Find the directional derivative of function $\phi(x, y, z) = x^3 + y^3 + z^3$ at the point $(1, -1, 2)$ in the direction of normal to surface $x^3 + y^3 + z^3 = 8$. Ans: $9\sqrt{2}$

$\nabla\phi \cdot \hat{u}$
where $\hat{u} = \frac{\nabla\phi_2}{|\nabla\phi_2|}$

9) If $f(x, y) = xe^y$, find rate of change of f at point $(2, 0)$ in the direction of $\vec{PQ}$ with $Q(\frac{1}{2}, 2)$. In what direction does f have maximum rate of change? What is the maximum rate of change? Ans: Direction is $\nabla f(2, 0) = \langle 1, 2 \rangle$,
Max. rate of change is $|\langle 1, 2 \rangle| = \sqrt{5}$

10) Suppose that temperature at a point (x, y, z) in space is given by the following relationship:

$$T(x,y,z) = \frac{80}{1+x^2+2y^2+3z^2},$$

where T is measured in degree celsius and x, y, z in metres. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is maximum rate of change?

Ans: Direction is $\nabla T(1,1,-2) = \frac{5}{8}(-\hat{i} - 2\hat{j} + 6\hat{k})$

Max rate of change is $|\nabla T| = \frac{5\sqrt{41}}{8} \approx 4^\circ C/m$