Power Series

Defn: A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where x is a variable and c_n's are constants

is called a power series in $(x-a)$ or a power series about a.

Thm: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$; there are only three possibilities

1) The series converges when $x = a$
2) The series converges for all $x \in \mathbb{R}$
3) There is a +ve number R such that the series converges if $|x-a| < R$

The no. R is called the radius of convergence of a power series

In case i) $R = 0$

In case ii) $R = \infty$

The interval of convergence is the interval consists of all values of x.

In case i) the interval of convergence is a single point a.

In case ii) the interval of convergence is $(-\infty, \infty)$ and

in case iii) $|x-a| < R \Rightarrow -R < (x-a) < R$

$$\Rightarrow a-R < x < a+R$$

$\therefore$ the interval is $(a-R, a+R)$

Representation of Functions as Power Series

Here we learn to represent certain functions as sum of power series by manipulating geometric series.

Ex: Find a power series representation for $f(x) = \tan^{-1}x$. Find its radius of convergence

Soln: We observe that

$$f'(x) = \frac{1}{1+x^2}$$

$$\Rightarrow \quad \tan^{-1}x = \int \frac{1}{1+x^2}\,dx \quad \text{——— } (1)$$

Geometric Series:

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}, \quad |x| < 1 \quad \left(\text{Radius of convergence is } 1\right)$$

Replace x by $-x^2$, we get

$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad \text{——— } (2)$$

Thus,

$$\tan^{-1}x = \int \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx \quad (\text{from } (1) \text{ \& } (2))$$

$$\Rightarrow \quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

Radius of convergence of the series for $\frac{1}{1+x^2}$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1.

Taylor and Maclaurin Series

Above we were able to find power series representations for a certain restricted class of functions.

Here we investigate general form: Which funtions have power series representations? How can we find such representations?

Let f be any function. Suppose f can be represented by a power series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \quad |x-a| < R \quad \text{— (1)}$$

We try to determine c_n in terms of f.

put $x=a$ in (1), we get

$$\boxed{f(a) = c_0}$$

Diff (1) wrt x,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad \text{— (2)}$$

put $x=a$ in (2),

$$\boxed{f'(a) = c_1}$$

Diff (2) wrt x,

$$f''(x) = 2c_2 + 3\cdot 2\, c_3(x-a) + 4\cdot 3\, c_4(x-a)^2 + \cdots \quad \text{— (3)}$$

put $x=a$ in (3),

$$f''(a) = 2c_2 \quad \text{or} \quad \boxed{c_2 = \frac{f''(a)}{2}}$$

Diff (3) wrt x,

$$f'''(x) = 3\cdot 2\cdot 1\, c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + \cdots \quad \text{— (4)}$$

put $x=a$ in (4),

$$f'''(x) = 3\cdot 2\cdot 1\, c_3$$

or $$\boxed{c_3 = \frac{f'''(a)}{3!}}$$

By now you can see a pattern. In general

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Theorem: If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$\Rightarrow \quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This series is called the **Taylor series of the function f at a** (or about a or centered at a).

When $a=0$ the Taylor series becomes

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called Maclaurin series

Consider the partial sums of a Taylor series

$T_1(x)$: $f(a) + \frac{f'(a)}{1!}(x-a)$ (is called 1^{st}-deg Taylor polynomial)

$T_2(x)$: $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$ (2^{nd}-deg Taylor polynomial)

$\vdots$

$T_n(x)$: $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Notice that T_n is a polynomial of degree n called the n^{th}-degree Taylor polynomial of f at a.

In general, $f(x)$ is the sum of its Taylor series,

That is, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x)$$

$$\Rightarrow f(x) = R_n(x) + T_n(x)$$

Then $R_n(x)$ is called the remainder of the Taylor series and if $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Ex1: Obtain the Maclaurin Series for the following functions

i) $f(x) = e^x$

Soln: Maclaurin series for $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots \quad \text{—} (*)$$

Given $f(x) = e^x$ $\qquad f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$ — (1)

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

Sub (1) in (*), we get

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Maclaurin series for e^x.

Radius of Convergence: $\quad$ Here $a_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

for all x.

$\therefore$ $R = \infty$ and interval of convergence is $(-\infty, \infty)$.

ii) $f(x) = \sin x$

Soln: $f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f'(0) = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -1$

(2)

$f^{IV}(x) = \sin x \qquad f^{IV}(0) = 0$

$f^{V}(x) = \cos x \qquad f^{V}(0) = 1$

$\vdots$

Sub (2) in (*),

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

is the Maclaurin series.

Radius of Convergence: Here $a_n = (-1)^{n-1} \dfrac{x^{2n-1}}{(2n-1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^n \dfrac{x^{2n+1}}{(2n+1)!}}{(-1)^{n-1} \dfrac{x^{2n-1}}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+1)\cdot(2n)} \right| = 0 < 1$$

for all x.

$\therefore$ $R = \infty$ and interval of convergence is $(-\infty, \infty)$.

iii) $f(x) = \cos(x)$

Soln:

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{IV}(x) = \cos x \qquad f^{IV}(0) = 1$

$f^{V}(x) = -\sin x \qquad f^{V}(0) = 0$

$\vdots$

(3)

Sub (3) in (*),

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!}$$

Radius of convergence $R = \infty$

interval of convergence $(-\infty, \infty)$ (check)

iv) $f(x) = \log(1+x)$

Soln: $f(x) = \log(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{1\cdot 2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{iv}(x) = -\frac{1\cdot 2\cdot 3}{(1+x)^4}$ $\qquad f^{iv}(0) = -6$

$f^{v}(x) = \frac{1\cdot 2\cdot 3\cdot 4}{(1+x)^5}$ $\qquad f^{v}(0) = 24$

$\vdots$

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Sub 4 in $\circledast$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{2x^3}{3!} - 6\frac{x^4}{4!} + 24\frac{x^5}{5!} - \cdots$$

$$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Radius of convergence: Here $a_n = (-1)^{n-1}\frac{x^n}{n}$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^n \frac{x^{n+1}}{n+1}}{(-1)^{n-1}\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x}{\frac{n+1}{n}} \right| = \lim_{n\to\infty} \left| \frac{x}{1+\frac{1}{n}} \right| = |x|$$

Series convergence when $|x| < 1$

$\therefore$ Radius of convergence $R = 1$,

interval of convergence: $\quad$ | $|x| < 1 \Rightarrow -1 < x < 1$

At $x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges

At $x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges

$\therefore$ interval is $(-1, 1]$.

Ex2: Obtain Taylor series for the function $f(x) = \cos x$ about $\pi/3$. Hence evaluate $\cos(61°)$.

Soln: Taylor series for $f(x)$ about a is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad (\#)$$

Here $f(x) = \cos x$ and $a = \pi/3$

$$\left.\begin{array}{ll} f(x) = \cos x & f(\pi/3) = 1/2 \\ f'(x) = -\sin x & f'(\pi/3) = -\sqrt{3}/2 \\ f''(x) = -\cos x & f''(\pi/3) = -1/2 \\ f'''(x) = \sin x & f'''(\pi/3) = \sqrt{3}/2 \\ f^{IV}(x) = \cos x & f^{IV}(\pi/3) = 1/2 \\ \vdots & \end{array}\right\} \quad (1)$$

Sub (1) in (#),

$$\cos(x) = 1/2 - \frac{\sqrt{3}}{2}(x - \pi/3) - \frac{1}{2.2!}(x-\pi/3)^2 + \frac{\sqrt{3}}{2.3!}(x-\pi/3)^3 + \cdots$$

$$= 1/2 - \sqrt{3}/2\,(x-\pi/3) - 1/4\,(x-\pi/3)^2 + \frac{\sqrt{3}}{12}(x-\pi/3)^3 + \cdots$$

is the required Taylor series.

To evaluate $\cos(x)$ at $61°$:

$$61° = 60° + 1° = \pi/3^{c} + \pi/180^{c}$$

put $x = \pi/3 + \pi/180$ in the above series.

$$\cos\left(\pi/3 + \pi/180\right) = 1/2 - \sqrt{3}/2 \left(\pi/180\right) - 1/4 \left(\pi/180\right)^2 + \cdots \simeq 0.4848$$

Binomial series

Ex 3: Obtain the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Soln: Maclaurin series of $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots \quad \text{— (1)}$$

Here

$$f(x) = (1+x)^k \qquad f(0) = 1$$

$$f'(x) = k(1+x)^{k-1} \qquad f'(0) = k$$

$$f''(x) = k(k-1)(1+x)^{k-2} \qquad f''(0) = k(k-1)$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \qquad f'''(0) = k(k-1)(k-2)$$

$$\vdots$$

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-(n-1))(1+x)^{k-n}$$

$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-(n-1))$$

$\therefore$ Maclaurin series of $f(x) = (1+x)^k$ is

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!} x^n.$$

This is called binomial series

If k is a non negative integer, then the terms are eventually 0 and so the series is finite.

when $k=2$

$$(1+x)^2 = 1+2x+x^2$$

$$\vdots$$

$k=n\in\mathbb{Z}^+$, $$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\cdots n-(n-1)}{n!}x^n$$

$= \frac{n!}{n!}$

Notation: If n is a +ve integer, then

$$\frac{n(n-1)(n-2)\cdots n-(r-1)}{r!} := \binom{n}{r} \quad \left(\text{or } {}^nC_r\right)$$

(read as n choose r)

Thus, $$(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r, \quad n\in\mathbb{Z}^+$$

Radius of convergence of a binomial series

$$(1+x)^k = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}x^n. \qquad k\in\mathbb{R}$$

Here $a_n = \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}$, $a_{n+1} = \frac{k(k-1)(k-2)\cdots(k-(n-1)).(k-n)}{(n+1)!}$

We use ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{\frac{\cancel{k(k-1)\cdots k-(n-1)}.(k-n)}{(n+1)!}x^{n+1}}{\frac{\cancel{k(k-1)\cdots k-(n-1)}}{n!}x^n}\right)$$

$$= \lim_{n\to\infty} \left| \frac{(k-n)}{(n+1)!} \cdot n! \; x \right|$$

$$= \lim_{n\to\infty} \left| \frac{k-n}{n+1} \cdot x \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{\frac{k}{n} - 1}{1 + \frac{1}{n}} \right|$$

$$= |x| \cdot 1$$

By ratio test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Radius of convergence is $R = 1$,

interval of convergence is $(-1, 1)$

Ex 4: Obtain Maclaurin Series for $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

Soln: $f(x) = \frac{1}{\sqrt{4-x}}$

$$= \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2} \cdot \frac{1}{\left(1-\frac{x}{4}\right)^{1/2}}$$

$$\Rightarrow f(x) = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-1/2}$$

Using binomial series with $k = -1/2$ and with x replaced by $-\frac{x}{4}$, we have

$$\frac{1}{\sqrt{4-x}} = 1 + \frac{(-1/2)}{1!}\left(-\frac{x}{4}\right) + \frac{(-1/2)(-1/2-1)}{2!}\left(-\frac{x}{4}\right)^2$$

$$+\frac{(-\tfrac{1}{2})(-\tfrac{1}{2}-1)(-\tfrac{1}{2}-2)}{3!}\left(-\frac{x}{4}\right)^3+\cdots$$

$$=1+\frac{1}{8}x+\frac{1\cdot 3}{2!\,8^2}x^2+\frac{1\cdot 3\cdot 5}{3!\,8^3}x^3+\cdots$$

$$+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!\cdot 8^n}x^n+\cdots$$

The Series converges when $\left|-\frac{x}{4}\right|<1 \Rightarrow |x|<4$

$\therefore$ Radius of convergence is $R=4$.

EX 5: Use M S to approximate the definite integral

$$\int_0^1 e^{\sin x}\,dx$$

Soln: We find M.S for $f(x)=e^{\sin x}$

$f(x)=e^{\sin x}$

$f'(x)=e^{\sin x}\cdot\cos x=f(x)\cdot\cos x$

$f''(x)=f'(x)\cdot\cos x-f(x)\sin x$

$f'''(x)=f''(x)\cos x-f'(x)\sin x-f'(x)\sin x-f(x)\cos x$

$\quad=f''(x)\cos x-2f'(x)\sin x-f(x)\cos x$

$f^{IV}(x)=f'''(x)\cos x-f''(x)\sin x-2f''(x)\sin x-2f'(x)\cos x$

$\quad-f'(x)\cos x+f(x)\sin x$

$f(0)=1,\quad f'(0)=1,\quad f''(0)=1,\quad f'''(0)=0,\quad f^{IV}(0)=-2-1$

$=-3$

$$\therefore\ e^{\sin x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} - 3\frac{x^4}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

$$\therefore \int_0^1 e^{\sin x}\,dx = \int_0^1 \left(1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right) dx$$

$$= \left. x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40} + \cdots \right]_0^1$$

$$= 1 + \frac{1}{2} + \frac{1}{6} - \frac{1}{40} + \cdots$$

$$\simeq 1.641$$

Ex 6: Obtain M.S for $f(x) = \sin^{-1} x$.

Soln: We have

$$\sin^{-1}x = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int (1-x^2)^{-1/2}\,dx$$

$\because f(x) = \sin^{-1}x$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

We find Maclaurin series for $g(x) = (1-x^2)^{-1/2}\,dx$.

We have

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Replace x by $-x^2$ and k by $-\frac{1}{2}$, we get

$$(1-x^2)^{-1/2} = 1 - \tfrac{1}{2}(-x^2) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(-x^2)^3 + \cdots$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2!}x^4 + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{3!}x^6 + \cdots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

$$\therefore \sin^{-1}x = \int (1-x^2)^{-1/2}\,dx$$

$$= \int \left(1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots\right) dx$$

$$\Rightarrow \boxed{\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots}$$

Ex 7: Use appropriate Taylor polynomial of deg. 3 to obtain an estimate for the foll.

a) $\sqrt[3]{1.3}$

Soln: Consider Maclaurin Series

$$(1+x)^{1/3} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \cdots$$

put $x = 0.3$

$$\Rightarrow (1+0.3)^{1/3} \approx 1 + \frac{0.3}{3} + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{2}(0.3)^2$$

$$= 1 + 0.1 - 0.01$$

$$= 1.09$$

$$\Rightarrow (1.3)^{1/3} \approx 1.09$$

b) $\log(1.1)$

Soln: We have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

put $x = 0.1$

$$\Rightarrow \log(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3}$$

$$= 0.09533$$

Ex8: Find Maclaurin series for the function $f(x) = e^{-x^2}$

Soln: We have

$$e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{(Show this)}$$

Replace x by $-x^2$

$$e^{-x^2} = 1 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

Ex1: Use appropriate Taylor polynomial of deg. 3 to obtain an estimate for the foll.

$(1.3)^{1/3}$.

Ex 2: Use MS to approximate the definite integral

$$\int_0^1 e^{\sin x}\, dx.$$

Ex 3: Obtain the series expansion of $\sqrt{1+\sin 2x}$ in ascending powers of x.

Ex4: Obtain MS expansion for $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Ex5: Obtain MS expansion for $\dfrac{x}{\sin x}$

Ex6: Obtain MS expansion for $\log(\sec x)$

Ex 7: Obtain Taylor series for $\log(\sin x)$ about 3.

Ex 8: Obtain MS for $\dfrac{x}{e^x-1}$

Ex 5: Obtain MS expansion for $\frac{x}{\sin x}$

Soln: $f(x) = \frac{x}{\sin x} = \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$

$$= \frac{1}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}$$

$$= \left(1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right)\right)^{-1}$$

$$= 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^4}{3!\cdot 3!} + \cdots$$

$$= 1 + \frac{x^2}{6} - \frac{x^4}{120} + \frac{x^4}{36} + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{(-3x^4 + 10x^4)}{360} + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \cdots$$

Ex 8: Obtain MS for $\dfrac{x}{e^x-1}$

Soln: $f(x) = \dfrac{x}{e^x-1} = \dfrac{x}{\cancel{1}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cancel{-1}}$

$$= \frac{x}{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$= \frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots}$$

$$= \left(1+\left(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)\right)^{-1}$$

$$= 1-\left(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)+\left(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)^2 - \left(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)^3+\cdots$$

$$= 1-\frac{x}{2!}-\frac{x^2}{3!}+\frac{x^2}{2!\,2!}+\cdots$$

$$= 1-\frac{x}{2}+\frac{1}{12}x^2+\cdots$$