Finite differences:

The finite difference deals with change in value of function (dependent variable) due to changes in values of independent variable. The values of independent variable x are called arguments and corresponding values of dependent variables y are called entries. The difference b/w consecutive values of x is called interval of differencing (h)

Finite differences are used to obtain numerical solution to differential equations and approximating derivatives.

Forward differences:

If $y_0, y_1, y_2 \ldots y_n$ denote set of values of function $y = f(x)$. Then $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, ... $\Delta y_n = y_{n+1} - y_n$ are called first forward differences of $y_0, y_1 \ldots y_n$ where Δ is called forward difference operator. The differences of first forward differences are called second forward differences of $f(x)$.

i.e. $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$ is called second forward difference of y_0.

Similarly, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$, $\Delta^2 y_2 = \Delta y_3 - \Delta y_2$ .... $\Delta^n y_n = \Delta y_{n+1} - \Delta y_n$. Similarly we can define $3^{rd}, 4^{th}$ forward differences etc

In general n^{th} forward difference is defined by

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$$

Forward difference table :

x	$y=f(x)$	First diff. Δy	Second diff. $\Delta^2 y$	Third diff. $\Delta^3 y$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
$x_3 = x_0 + 3h$	y_3			

Here y_0 is called the leading term and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \dots$ are called leading differences.

The operator Δ has following properties :

1) If c is a constant, $\Delta c = 0$

Proof: Let $f(x) = c$

$$\therefore f(x+h) = c$$

$$\Delta f(x) = f(x+h) - f(x) = c - c = 0$$

2) $\Delta c f(x) = c \Delta f(x)$

Proof: $\Delta [c f(x)] = c f(x+h) - c f(x) = c [f(x+h) - f(x)]$

$$= c \Delta f(x)$$

3) $\Delta [a f(x) + b g(x)] = a \Delta f(x) + b \Delta g(x)$

4) The n^{th} difference of an n^{th} degree polynomial is a constant

$= (\text{coff. of } x^n)\, n!\, h^n$ and hence higher order differences are zero.

ex: $f(x) = x^2+1$

$$\Delta^2 f(x) = \Delta^2 y = \Delta^2 (x^2+1) = \Delta(\Delta(x^2+1))$$

$$\Delta(x^2+1) = \{(x+h)^2+1\} - \{x^2+1\} = \cancel{x^2}+2xh+h^2+\cancel{1}-\cancel{x^2}-\cancel{1}$$

$$\Delta(x^2+1) = 2xh+h^2$$

$$\Delta(\Delta(x^2+1)) = \Delta(2xh+h^2) = 2(x+h)h+h^2 - (2xh+h^2)$$

$$= \cancel{2xh}+2h^2+\cancel{h^2}-\cancel{2xh}-\cancel{h^2} = 2h^2$$

$$\Rightarrow \boxed{\Delta(2h^2) = 0}$$

Backward difference:

Backward difference operator ∇ is defined by

$$\boxed{\nabla y_n = y_n - y_{n-1}}$$

i.e. $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$ .... → first backward differences

$|||^{ly}$ $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$ .. → 2^{nd} –||–

In general n^{th} backward difference is given by

$$\boxed{\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}}$$

Backward difference Table :

x	$y=f(x)$	First diff. ∇y	Second diff. $\nabla^2 y$	Third diff. $\nabla^3 y$
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
$x_1 = x_0 + h$	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
$x_3 = x_0 + 3h$	y_3			

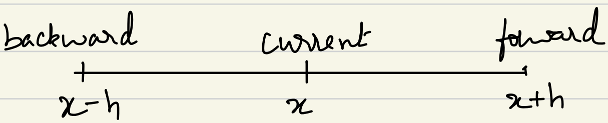
Note : 1) Relation b/w foward and backward operators : $\Delta^n y_r = \nabla^n y_{n+r}$

ex:- 1) $\Delta y_1 = \nabla y_2$ i.e. $\Delta y_1 = y_2 - y_1$

$\nabla y_2 = y_2 - y_1$

2) $\Delta^2 y_1 = \nabla^2 y_3$

2) $\Delta f(x) = f(x+h) - f(x)$ 3) $\nabla f(x) = f(x) - f(x-h)$



$\Delta f(x) \Rightarrow$ Forward - current

$\nabla f(x) \Rightarrow$ current - backward.

3) $\Delta \nabla = \Delta - \nabla$

4) It is observed from forward & backward diff tables that for a given table of values both the tables are same.

Problems:

1) Evaluate $\Delta(\log ax)$

Sol: $\Delta(\log ax) = \log(ax+ah) - \log(ax)$

$$= \log\left(\frac{ax+ah}{ax}\right) = \log\left(\frac{\cancel{a}(x+h)}{\cancel{a}x}\right)$$

$$= \log\left(\frac{x+h}{x}\right) = \log\left(1+\frac{h}{x}\right)$$

In general $\Delta \log f(x) = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$

2) Evaluate $\Delta^2 \sin(ax+b)$

Sol: $\Delta^2 \sin(ax+b) = \Delta(\Delta \sin(ax+b))$

$\Delta \sin(ax+b) = \sin\{a(x+h)+b\} - \sin\{ax+b\}$

$$= 2\cos\left(\frac{ax+ah+b+ax+b}{2}\right)\cdot \sin\left(\frac{\cancel{ax}+ah+\cancel{b}-\cancel{ax}-\cancel{b}}{2}\right)$$

$$= 2\cos\left(\frac{2ax+2b+ah}{2}\right)\cdot \sin\left(\frac{ah}{2}\right)$$

$$= 2\sin\left(\frac{ah}{2}\right)\cos\left(ax+b+\frac{ah}{2}\right)$$

$\Delta \sin(ax+b) = 2\sin\left(\frac{ah}{2}\right)\cdot \sin\left(\frac{\pi}{2}+ax+b+\frac{ah}{2}\right)$

$\Delta^2 \sin(ax+b) = \Delta\left[2\sin\left(\frac{ah}{2}\right)\cdot \sin\left(\frac{\pi}{2}+ax+b+\frac{ah}{2}\right)\right]$

$$= 2\sin\frac{ah}{2}\ \Delta\left[\sin\left(\frac{\pi}{2}+ax+b+\frac{ah}{2}\right)\right]$$

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot \sin\left(\frac{C-D}{2}\right)$

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$$= 2\sin\frac{ah}{2}\left[\sin\left\{\frac{\pi}{2}+a(x+h)+b+\frac{ah}{2}\right\} - \sin\left(\frac{\pi}{2}+ax+b+\frac{ah}{2}\right)\right]$$

$$= 2\sin\frac{ah}{2}\cdot 2\cos\left(\frac{\pi/2+ax+ah+b+\frac{ah}{2}+\frac{\pi}{2}+ax+b+\frac{ah}{2}}{2}\right)$$

$$\sin\left(\frac{\cancel{\pi/2}+\cancel{ax}+ah+\cancel{b}+\cancel{\frac{ah}{2}}-\cancel{\pi/2}-\cancel{ax}-\cancel{b}-\cancel{\frac{ah}{2}}}{2}\right)$$

$$= 4\sin^2\frac{ah}{2}\cdot\cos\left(\frac{\pi+2ax+ah+2b+ah}{2}\right)$$

$$= 4\sin^2\frac{ah}{2}\cdot\cos\left(\frac{\pi+2ax+2ah+2b}{2}\right)$$

$$= 4\sin^2\frac{ah}{2}\cdot\cos\left(ax+b+\frac{\pi+2ah}{2}\right)$$

$$= \left(2\sin\frac{ah}{2}\right)^2 \sin\left(\frac{\pi}{2}+ax+b+\frac{\pi+2ah}{2}\right)$$

$$\therefore \Delta^2\sin(ax+b) = \left(2\sin\frac{ah}{2}\right)^2 \sin\left[(ax+b)+2\frac{(\pi+ah)}{2}\right]$$

3) $\Delta^2\cos 2x$

$$\Delta^2\cos 2x = \Delta\left\{\cos 2(x+h)-\cos 2x\right\}$$

$$= \Delta\cos 2(x+h) - \Delta\cos 2x$$

$$= [\cos 2(x+h+h) - \cos 2(x+h)] - [\cos 2(x+h)-\cos 2x]$$

$$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$$

$$= -2\sin h[\sin(2x+3h)-\sin(2x+h)]$$

$$= -2\sin h[2\cos(2x+2h)\sin h]$$

$$= -4\sin^2 h\cos(2x+2h)$$

$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Practice problems: Evaluate

1) $\Delta \tan^{-1} ax$, $h=1$ Ans: $\tan^{-1} \dfrac{a}{1+a^2x+a^2x^2}$

2) $\Delta \left(\dfrac{e^x}{e^x+e^{-x}}\right)$, $h=1$ Ans: $\dfrac{e-e^{-1}}{(e^{x+1}+e^{-x-1})(e^x+e^{-x})}$

3) $\Delta^n a^{cx+d}$ Ans: $(a^{ch}-1)^2 a^{cx+d}$

4) Δe^{ax} Ans: $e^{ax}(e^{ah}-1)$

5) $\Delta^2 e^x$ Ans: $(e^h-1)^2 e^x$

6) $\Delta^2\left(\dfrac{1}{x}\right)$, $h=1$ Ans: $\dfrac{2}{x(x+1)(x+2}$

Differences of a polynomial

The n^{th} differences of a polynomial of n^{th} degree are constant and all higher order differences are zero.

i.e. Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$, then

$$\Delta^n f(x) = a_0\, n!\, h^n \quad \text{and for higher orders } \Delta^{n+1} f(x) = 0$$

1) Evaluate $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$ with $h=1,2$

Sol: $\Delta^{10}[abcd\, x^{10} + (\quad)x^9 + (\quad)x^8 + \dots + 1]$

$$= abcd\, \Delta^{10}(x^{10}) \qquad (\because \Delta^{10} x^n = 0 \text{ for } n<10)$$

$$= abcd\,(10!) \qquad \text{when } h=1$$

$$= abcd\,(10!)\,2^{10} \qquad \text{when } h=2$$

Q) Construct foward difference table from foll

x	1	2	3	4	5
y	2	5	10	20	30

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	2				
		3			
2	5		2		
		5		3	
3	10		5		-8
		10		-5	
4	20		0		
		10			
5	30				

Data points

Max. Foward diff is (N-1)

Given: no of data points = 5,

Max Order of diff = 4

Q) Construct finite difference table for function $f(x) = x^3 + x + 1$ where x takes the values 0, 1, 2, 3, 4, 5, 6. Identify leading foward & backward differences. Hence find $\Delta^2 y_1, \nabla^3 y_5$.

Sol:

x	y	First Difference	Second Diff	Third Diff	Fourth Diff
0	1				
		2			
1	3		6		
		8		6	
2	11		12 $\Delta^2 y_1$		0
		20		6	
3	31		18		0
		38		6 $\nabla^3 y_5$	
4	69		24		0
		62		6	
5	131		30		
		92			
6	223				

The leading forward differences are 2, 6, 6 and leading backward differences are 92, 30, 6

$$\Delta^2 y_1 = 12, \quad \nabla^3 y_5 = 6$$

(The 3rd differences are constants and higher order differences are zero as $f(x)$ is a polynomial of 3rd degree)

Interpolation

If the values of $f(x)$ are known for $x = x_0, x_1, \dots x_n$ then the process of finding the value of $f(x)$ for any other value of x in the interval (x_0, x_n) is called interpolation.

Extrapolation

The process of finding value of $f(x)$ for a value of x outside the interval (x_0, x_n) is called extrapolation.

Interpolation with equal intervals

* Newton – Gregory Forward interpolation formula

Let the function $y = f(x)$ takes the values $y_0, y_1, \dots y_n$ at the points where $x_i = x_0 + ih$. Then Newton's forward interpolation formula is

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

where $p = (x - x_0)/h$ is the index.

This formula is particularly used when $x_0 + ph$ is near the beginning or for interpolating values of $f(x)$ near the beginning of set of values given.

ex: If you have 5 data points, say $x = 2, 4, 6, 8, 10$. If you are looking for value of y at $x = 3$ which is near beginning of table then Newton's forward interpolation works best. (closer to x_0)

Newton-Gregory backward interpolation formula : Let $y = f(x)$ takes the values $y_n, y_{n-1}, \dots y_0$ at points $x_n, x_{n-1} \dots x_0$. Then Newton's backward interpolation formula is

$$f(x) = f(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

where $p = \dfrac{x - x_n}{h}$

This formula is used when $x_0 + ph$ is near the end or for interpolating values of $f(x)$ near the end of set of values given.

1) Find the cubic polynomial which takes following data

x	0	1	2	3
$f(x)$	1	2	1	10

Sol: The forward difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here $x_0 = 0$, $h = 1$, $p = \dfrac{x - x_0}{h} = \dfrac{x - 0}{1} = x$, $\Delta y_0 = 1$, $\Delta^2 y_0 = -2$, $\Delta^3 y_0 = 12$

By Newton - Gregory forward interpolation formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$f(x) = 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12) = 2x^3 - 7x^2 + 6x + 1$$

is the required polynomial

(Same polynomial is obtained if backward interpolation formula is used)

2) The following data gives melting point of an alloy of lead and zinc, where t is the temperature in °c and p is the percentage of lead in the alloy.

p(%)	60	70	80	90
t	226	250	276	304

Find melting point of alloy containing 84% of lead, using Newton's interpolation formula.

Sol: Here $x_n = 90$, $h = 10$, $p = \frac{x - x_n}{h} = \frac{x-90}{10} = \frac{84-90}{10} = -0.6$

The backward difference table is

p	t	∇t	$\nabla^2 t$	$\nabla^3 t$
60	226			
		24		
70	250		2	
		26		0
80	276		2	
		28		
90	304			

$$f(x) = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n$$

$$f(84) = y_3 + (-0.6) \nabla y_3 + \frac{(-0.6)(-0.6+1)}{2} \nabla^2 y_3$$

$$f(84) = 304 - (0.6)(28) - \frac{(0.6)(0.4)}{\cancel{2}} \cdot \cancel{2} = 286.96$$

3) From the following table, estimate no of students who obtained marks b/w 40 and 45.

Marks (x)	30-40	40-50	50-60	60-70	70-80
No. of students (y)	31	42	51	35	31

Sol: We prepare cumulative frequency table as:

Marks less than (x)	40	50	60	70	80
No. of students (y)	31	73	124	159	190

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

To find $y(45)$ i.e. no of students with marks less than 45.

$x_0 = 40, \quad x = 45, \quad h = 10, \quad p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5$

Using Newton's foward interpolation formula,

$$y(45) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \dots$$

$$= 31 + (0.5)(42) + \frac{0.5(-0.5)}{2}(9)$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{6}(-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(37)$$

$y(45) = 47.87$

$\therefore$ No. of students with marks less than 45 is $47.87 \simeq 48$

But no. of students with marks less than 40 is 31

Hence no. of students getting marks b/w 40 & 45 $= 48 - 31 = 17$

4) Details regarding marks scored by 280 candidates in an examination are given by the following table. Using Newton-Gregory interpolation formula estimate the no of candidates who scored marks between 45 and 65.

Marks	Below 30	30-40	40-50	50-60	60-70	70-80
No of students	35	49	62	74	40	20

Sol: Cumulative frequency table

Marks less than (x)	30	40	50	60	70	80
No of students (y)	35	84	146	220	260	280

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
30	35					
		49				
40	84		13			
		62		-1		
50	146		12		-45	
		74		-46		105
60	220		-34		60	
		40		14		
70	260		-20			
		20				
80	280					

$p = \dfrac{x - x_0}{h} = \dfrac{45-30}{10} = 1.5$ and $p = \dfrac{x - x_n}{h} = \dfrac{65-80}{10} = -1.5$

Using Newton's forward interpolation formula,

$$y(45) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$\therefore y(45) = 111.15 \approx 111$

Using Newton's backward interpolation formula, we get

$$y(65) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

$\therefore y(65) = 246.0117 \approx 246$

The no. of students with marks b/w 45 and 65 is

$246 - 111 = 135.$

Exercise:

1) A survey conducted in a locality reveals the following information as below:

Income per day (Rs.)	Below 500	500–1000	1000–2000	2000–3000	3000–4000
No of persons	6000	4250	3600	1500	650

Estimate the no of persons having income between 2000 and 2500. Ans : 934

Numerical Differentiation:

Derivatives using Newton's forward interpolation formula

By Newton's Interpolation Formula, we have

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \dots\dots\dots \quad (1)$$

where, $x = x_0 + Ph$ (or) $P = \frac{x - x_0}{h}$ ……… (2)

Differentiating (1) w.r.t. P, we get

$$\frac{dy}{dP} = \Delta y_0 + \frac{(2P-1)}{2!}\Delta^2 y_0 + \frac{3P^2 - 6P + 2}{3!}\Delta^3 y_0 + \frac{(4P^3 - 18P^2 + 22P - 6)}{4!}\Delta^4 y_0 + \dots\dots \quad (3)$$

Differentiating (2) w.r.t. x, we get

$\frac{dP}{dx} = \frac{1}{h}$………. (4) But, $\frac{dy}{dx} = \frac{dy}{dP} \cdot \frac{dP}{dx}$

Using (3) and (4) the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_0+Ph} = \frac{1}{h}\left[\Delta y_0 + \frac{(2P-1)}{2!}\Delta^2 y_0 + \frac{(3P^2-6P+2)}{3!}\Delta^3 y_0 + \frac{(4P^3-18P^2+22P-6)}{4!}\Delta^4 y_0 + \cdots\right] \dots (5)$$

At $x = x_0$, $P = 0$, the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \dots (6)$$

Formula (5) is used to compute y' at any point $x = x_0 + Ph$, whereas formula (6) is used to compute y' at any of the value of x when y is specified.

Similarly,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0+Ph} = \frac{1}{h^2}\left[\Delta^2 y_0 + (P-1)\Delta^3 y_0 + \frac{(6P^2-18P+11)}{12}\Delta^4 y_0 + \cdots\right] \dots (7)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\left(\frac{dp}{dx}\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right] \dots (8)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\right] \dots (9)$$

The formula (7) is used to compute y'' at any point $x = x_0$+Ph whereas formula (8) is used to compute y'' at any value of x where y is specified.

Derivatives using Newton's backward interpolation formula

By Newton's Backward Interpolation Formula, we have

$$y = y_n + \frac{P}{1!}\nabla y_n + \frac{P\,(P+1)}{2!}\nabla^2 y_n + \frac{P\,(P+1)(P+2)}{3!}\nabla^3 y_n + \ldots\ldots \qquad \ldots\ldots\ldots (1)$$

Where, $x = x_n + Ph$ (or) $P = \frac{x-x_n}{h}$ .... (2)

Differentiating (1) w.r.t. P, we get

$$\frac{dy}{dP} = \nabla y_n + \frac{(2P+1)}{2!}\nabla^2 y_n + \frac{(3P^2+6P+2)}{3!}\nabla^3 y_n + \ldots\ldots \qquad \ldots\ldots\ldots (3)$$

Differentiating (2) w.r.t. x, we get

$\frac{dP}{dx} = \frac{1}{h}$.... (4) But, $\frac{dy}{dx} = \frac{dy}{dP}\cdot\frac{dP}{dx}$

Using (3) and (4) the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_n+Ph} = \frac{1}{h}\left[\nabla y_n + \frac{(2P+1)}{2!}\nabla^2 y_n + \frac{(3P^2+6P+2)}{3!}\nabla^3 y_n + \frac{(4P^3+18P^2+22P+6)}{4!}\nabla^4 y_n + \ldots \qquad (5)\right.$$

At $x = x_n$, $P = 0$, the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \qquad \ldots. (6)$$

Formula (5) is used to compute y at any point $x = x_n$ + Ph whereas formula (6) is used to compute y' at any of the values of x when y is specified.

Similarly

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n+Ph} = \frac{1}{h^2}\left[\nabla^2 y_n + (P+1)\nabla^3 y_n + \frac{(6P^2+18P+11)}{12}\nabla^4 y_n + \cdots\right] \qquad \ldots. (7)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \qquad \ldots. (8)$$

The formula (7) is used to compute y'' at any point $x = x_n$+Ph whereas formula (8) is used to compute y'' at any of the values of x when y is specified.

Problems:

1.Given

x	1.0	1.2	1.4	1.6	1.8	2.0
y	2.72	3.32	4.06	4.96	6.05	7.39

Find y' and y'' at x = 1.2.

Solution: Here, the step-length is h = 0.2. We first form the following difference table.

x	y	first differences	Second differences	Third differences	Fourth differences
1.0	2.27				
		0.60			
1.2	3.32		0.14		
		0.74		0.02	
1.4	4.06		0.16		0.01
		0.90		0.03	
1.6	4.96		0.19		0.03
		1.09		0.06	
1.8	6.05		0.25		
		1.34			
2.0	7.39				

We have to compute $y'$and y''at x = 1.2, which is a specified value of x, for this purpose, we take x_0 = 1.2. Then we find from the table that,

Δy_0= 0.74, $\Delta^2 y_0$= 0.16, $\Delta^3 y_0$= 0.03, $\Delta^4 y_0$= 0.03

Using the formula we have,

$$\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

Then, $\left(\frac{dy}{dx}\right)_{(1.2)} = \frac{1}{(0.2)}\left[(0.74) - \frac{1}{2}(0.16) + \frac{1}{3}(0.03) - \frac{1}{4}(0.03)\right] = 3.3125$

Using the formula we have,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

And $\left(\frac{d^2y}{dx^2}\right)_{(1.2)} = \frac{1}{(0.2)^2}\left[(0.16) - (0.03) + \frac{11}{12}(0.03)\right] = 3.\ 9375$

2. Using appropriate interpolation formulas, find the values of $y' and\ y''$ when x = 4 using the following table.

x	1	2	3	4
y	4	12	20	36

Solution: Here x = 4 is a specified value of x which is at the end of the given table.
For this purpose we take $x_n = 4$.
The difference table is given by,

x	y	First differences	Second differences	Third differences
1	4			
		8		
2	12		0	
		8		**8**
3	20		**8**	
		16		
4	36			

Here $x_n = 4$, $\nabla y_n = 16$, $\nabla^2 y_n = 8$, $\nabla^3 y_n = 8$ and h=1

Then, $y'(4) = \frac{1}{h}\left[(\nabla y_n) + \frac{1}{2}(\nabla^2 y_n) + \frac{1}{3}(\nabla^3 y_n)\right] = \frac{1}{1}\left[16 + \frac{1}{2}(8) + \frac{1}{3}(8)\right]$

$= 16+4+\frac{8}{3} = 22.667$

And $y''(4) = \frac{1}{h^2}\left[(\nabla^2 y_n) + (\nabla^3 y_n)\right] = \frac{1}{1^2}[8 + 8] = 16$

3) A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of time t sec. Find velocity and acceleration of slider when $t = 0.3$ sec.

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Sol: Velocity at $t = 0.3$ is $f'(0.3)$ and acceleration at $t = 0.3$ is $f''(0.3)$ where $x = f(t)$.

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$
0	30.13						
		1.49					
0.1	31.62		−0.24				
		1.25		−0.24			
0.2	32.87		−0.48		0.26		
		0.77		0.02		−0.27	
t_0 ← 0.3	33.64		−0.46		−0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		−0.45		0.01		
		−0.14		0.02			
0.5	33.81		−0.43				
		−0.57					
0.6	33.24						

Here $t_0 = 0.3$, $h = 0.1$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$\Rightarrow \left(\frac{dx}{dt}\right)_{t=t_0} = \frac{1}{h}\left[\Delta x_0 - \frac{1}{2}\Delta^2 x_0 + \frac{1}{3}\Delta^3 x_0\right]$$

$$= \frac{1}{0.1}\left[0.31 - \frac{1}{2}(-0.45) + \frac{1}{3}(0.02)\right]$$

$$= \frac{1}{0.1}\left[0.31 + 0.225 + 0.007\right]$$

$$\left(\frac{dx}{dt}\right)_{t=t_0} = 5.42 \text{ cm/sec}$$

| 5.34 when $t_0 = 0.2$

$$\left(\frac{d^2x}{dt^2}\right)_{t=0} = \frac{1}{h^2}\left[\Delta^2 x_0 - \Delta^3 x_0\right] = \frac{1}{0.1^2}\left[(-0.45) - 0.02\right]$$

$$= -47 \text{ cm/s}^2$$

$$\left(\frac{dy}{dx}\right)_{x=x_n+ph} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2!}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{3!}\right)\nabla^3 y_n + \left(\frac{4p^3+18p^2+22p+6}{4!}\right)\nabla^4 y_n\right]$$

$$\left(\frac{d\theta}{dt}\right)_{(t=8)} = \frac{1}{2}\left[-6.2 + \frac{2(-.05)+1}{2}(0.3) + \frac{3(-0.5)^2+6(.5)+2}{6}(-0.7) + \frac{4(-0.5)^3+18(-0.5)^2+22(-.05)+6}{24}(1.6)\right]$$

$$= \frac{1}{2}[-6.2 + 0 + 0.029166 - 0.066666] = -3.11875$$

Thus the body cools at the rate of 3.11875 degree/second.

Practice problems:

1) Ordinates $f(x)$ of a normal curve in terms of standard deviation x are given as

x	1	1.02	1.04	1.06	1.08
y	0.2420	0.2371	0.2323	0.2275	0.2227

Find the ordinate for standard deviation $x = 1.025$ by using Newton - Gregory forward interpolation formula.

Ans: $f(1.025) = 0.23589$

2) The population of a town is as follows:

Year	1921	1931	1941	1951	1961	1971
Pop. in lakhs	20	24	29	36	46	51

Estimate the increase in population during the period 1955 to 1961 Ans: 6.017 lakhs

3) From the following table find the value of $\tan 17^\circ$

θ°	0	4	8	12	16	20	24
$\tan\theta$	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4452

Ans: $\tan 17^\circ = 0.3056$

4) For the frequency distribution

Marks obtained	0-19	20-39	40-59	60-79	80-99
No. of candidates	41	62	65	50	17

Estimate no of candidates who obtain less than 70 marks. Ans: 199

5) The following table gives values of $\sin\theta$ for different values of θ.

θ	0°	10°	20°	30°	40°
$\sin\theta$	0	0.1736	0.3420	0.5000	0.6428

Find value of $\cos 10^\circ$. Ans: $\cos 10^\circ = 0.9850$

6) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 54$ from the following table

x	50	51	52	53	54
y	3.6840	3.7084	3.7325	3.7563	3.7798

Ans: $\left.\frac{dy}{dx}\right|_{x=54} = 0.0232$, $\left.\frac{d^2y}{dx^2}\right|_{x=54} = 0.0003$

7) The following table gives corresponding values of pressure p and specific volume v of a superheated steam.

v	2	4	6	8	10
p	105	42.7	25.3	16.7	13

Find rate of change of p wrt v at $v = 2$.

Ans: -52.4

8) Following table gives census population of a state for the years 1961 to 2001.

Year	1961	1971	1981	1991	2001
Population (In million)	19.96	36.65	58.81	77.21	94.61

Find rate of growth of population in the year 2001

Ans: 2.08175

Interpolation with unequal intervals

Lagrange's formula for unequal intervals:

Let y = f(x) be a function whose values are y_0, y_1, y_2,......., y_n corresponding to x_0, x_1, x_2,......., x_n not necessarily equally spaced.

$$y \text{ or } f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}y_1 + \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}y_n$$

This formulais known asLagrange'sInterpolation formula.

Inverse interpolation:

The process of estimating the value of x for a given value of y is called Inverse interpolation. So far given a table of values of x and y, using one of the interpolation formulae we find the value of y corresponding to some value of x which is not in the table. On the other hand the process of estimating the value of x for some value of y which is not in the table is called inverse interpolation.

This method is used when the values of x are not necessarily equally spaced. Lagrange's interpolation formula can be simply viewed as a relation between two variables and any one of the variable can be taken as an independent variable. Therefore inverse interpolation formula can be obtained by interchanging the variables x and y in Lagrange's formula, we get.

$$x = \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}x_1 + \ldots\ldots + \frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})}x_n$$

Problems:

1) Find the polynomial by using Lagrange's formula and hence find for $x=3$

x	0	1	2	5
y	2	3	12	147

Solution:Here $n=3$.

Largange interpolation formula is given by

$$
\begin{aligned}
f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \\
f(x) &= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}2+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}3 \\
&+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}12+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}147 \\
f(x) &= 2\frac{(x-1)(x-2)(x-5)}{(-1)(-2)(-5)}+3\frac{x(x-2)(x-5)}{(1)(-1)(-4)}+12\frac{x(x-1)(x-5)}{(2)(1)(-3)}+147\frac{x(x-1)(x-2)}{(5)(4)(3)} \\
f(x) &= -\frac{2}{10}(x^3-8x^2+17x-10)-\frac{3}{4}(x^3-7x+10)-2(x^2-6x+5)+\frac{49}{20}(x^2-3x+2) \\
f(x) &= x^3+x^2-x+2.
\end{aligned}
$$

To approximate the value of $f(3)$ substitute $x=3$ in the interpolated polynomial $f(x)$.

$$f(3)=3^3+3^2-3+2=35$$

2.Find x for y = 7

x	1	3	4
y	4	12	19

Solution:

x_0=1 x_1=3 x_2=4

y_0= 4 y_1 = 12 y_2 = 14

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2$$

$$= \frac{(y-12)(y-19)}{(4-12)(4-19)}(1) + \frac{(y-4)(y-19)}{(12-4)(12-19)}(3) + \frac{(y-4)(y-12)}{(19-4)(19-12)}(4)$$

at y=7

$$x = \frac{60}{120} + \frac{108}{56} - \frac{60}{105}$$

$= 1.85714$

Practice problems:

1) Find $f(10)$ for the given data

x	5	6	9	11
$f(x)$	12	13	14	16

Ans: 14.6667

2) Using Lagrange's Interpolation, calculate profit in the year 2000 from the following data:

Year (x)	1997	1999	2001	2002
Profit (y in Lakhs)	43	65	159	248

Ans: 100 Lakhs

3) Apply Lagrange's formula inversely to find the root of equation $f(x) = 0$, given $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, $f(42) = 18$.

Ans: 37.2303