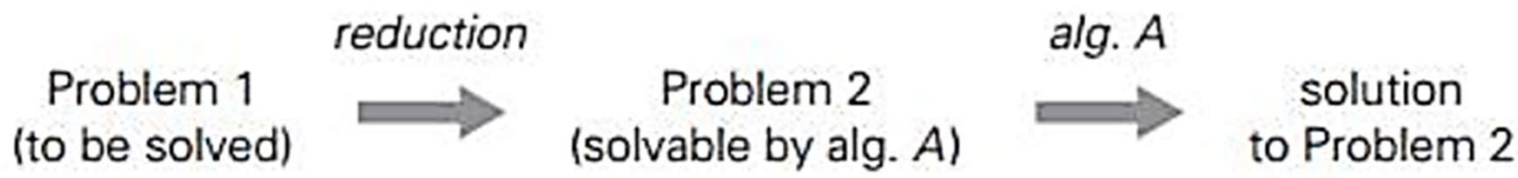
UNIT 3: Transform and Conquer

Transform and Conquer:

Problem reduction



Problem reduction strategy.

Problem Reduction

- The problem can be transformed to an easier problem to solve, i.e., transformed into an entirely different problem.

(very powerful idea, extensively used in complexity theory)

Example 1:
Computing the Least Common Multiple of two positive integers m and n : LCM(m, n)

- lcm(m,n) = the smallest integer that is divisible by both m and n.

Example: lcm(12, 15) = 60

lcm(50, 45) = 450

Middle school method:

Given the prime factorizations of m and n, lcm(m, n) can be computed as the product of all the common prime factors of m and n times the product of m 's prime factors that are not in n, n times n 's prime factors that are not in m.

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Example:

$$12 = 2 . 2 . 3$$

$$15 = 3. 5$$

$$\text{lcm}(12, 15) = (3) . 2 . 2 . 5 = 60$$

Example:

$$24 = 2 . 2 . 2 . 3$$

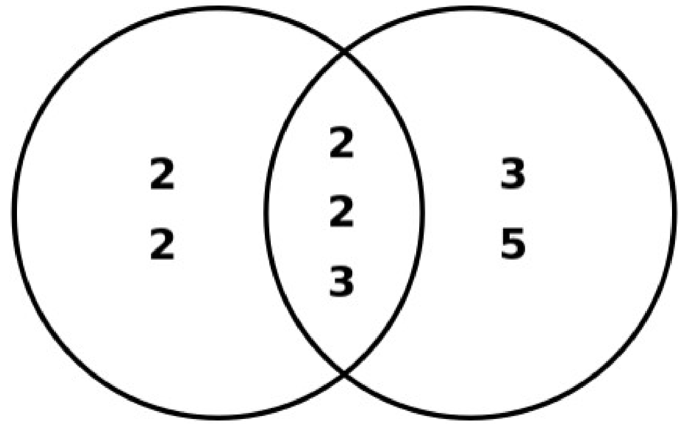
$$60 = 2 . 2 . 3 . 5$$

$$\text{lcm}(24, 60) = (2 . 2 . 3) . 2 . 5 = 120$$

Drawback: inefficient and requires a list of consecutive primes

Efficient algorithm using problem reduction:

$$\mathrm{lcm}(m, n) = \frac{m \cdot n}{\gcd(m, n)}$$



where gcd(m, n) can be computed very efficiently by Euclid's algorithm.

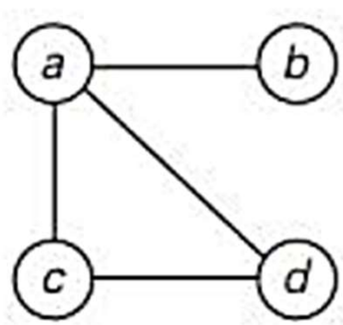
Example 2:
Counting paths between two vertices in a graph

Note:

The number of different paths of length $k > 0$ from the i^{th} vertex to the j^{th} vertex of a graph (undirected or directed) is equal to the $(i, j)^{th}$ element of A^k where A is the adjacency matrix of the graph.

Solution by problem reduction:

solved with an algorithm for computing an appropriate power of its adjacency matrix



$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad A^2 = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

A graph, its adjacency matrix A, and its square A^2. The elements of A and A^2 indicate the numbers of paths of lengths 1 and 2, respectively.

Example 3:
Solving optimization problems

Note:

If a problem asks to find a maximum of some function, it is said to be a **maximization problem**;

if it asks to find a function's minimum, it is called a **minimization problem**.

Example:

To find a minimum of some function f (x) and you know an algorithm for maximizing the function. How can you take advantage of the latter?

Solution by problem reduction:

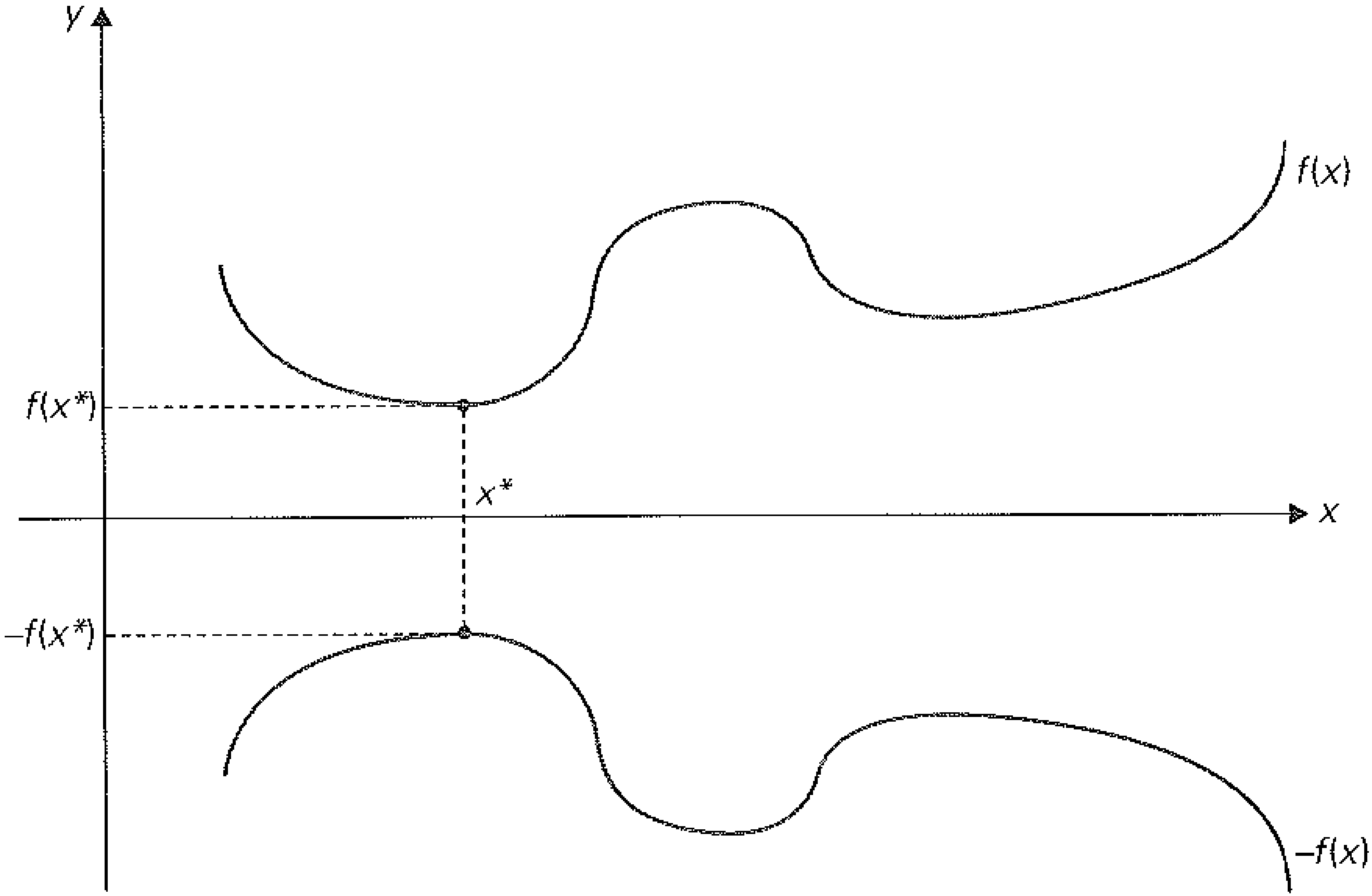
$$\min f(x) = -\max[-f(x)]$$

(to minimize a function, maximize its negative instead and, to get a correct minimal value of the function itself, change the sign of the answer)

Similarly, a maximization problem can be reduced to an equivalent minimization problem.

$$\max f(x) = -\min[-f(x)]$$

(to maximize a function, minimize its negative instead and, to get a correct maximal value of the function itself, change the sign of the answer)



Relationship between minimization and maximization problems: $\min f(x) = -\max[-f(x)]$

Example 4:
Linear Programming

Note:

Many problems of **optimal decision making** can be reduced to an instance of the **linear programming problem**, which is a problem of optimizing a linear function of several variables subject to constraints in the form of linear equations and linear inequalities.

Example:

Consider a university endowment that needs to invest \$100 million. This sum must be split between three types of investments: stocks, bonds, and cash. The endowment managers expect an annual return of 10%,7%, and 3% for their stock, bond, and cash investments, respectively.

- Since stocks are more risky than bonds, the endowment rules require the amount invested in stocks to be no more than one third of the moneys invested in bonds.
- In addition, at least 25% of the total amount invested in stocks and bonds must be invested in cash.

How should the managers invest the money to maximize the return?

Solution by problem reduction – linear programming:

Mathematical model:

Let x, y, and z be the amounts (in millions of dollars) invested in stocks, bonds, and cash, respectively. Then the optimization problem is:

$$
\begin{aligned}
\text{maximize} \quad & 0.10x + 0.07y + 0.03z \\
\text{subject to} \quad & x + y + z = 100 \\
& x \le \frac{1}{3}y \\
& z \ge 0.25(x + y) \\
& x \ge 0, \ \ y \ge 0, \ \ z \ge 0.
\end{aligned}
$$

Applications of Linear Programming (LP):

LP is flexible to model a wide variety of important applications, such as

- airline crew scheduling,
- Transportation and communication network planning,
- oil exploration and refining,
- Industrial production optimization

Example 5:
Reduction to graph problems

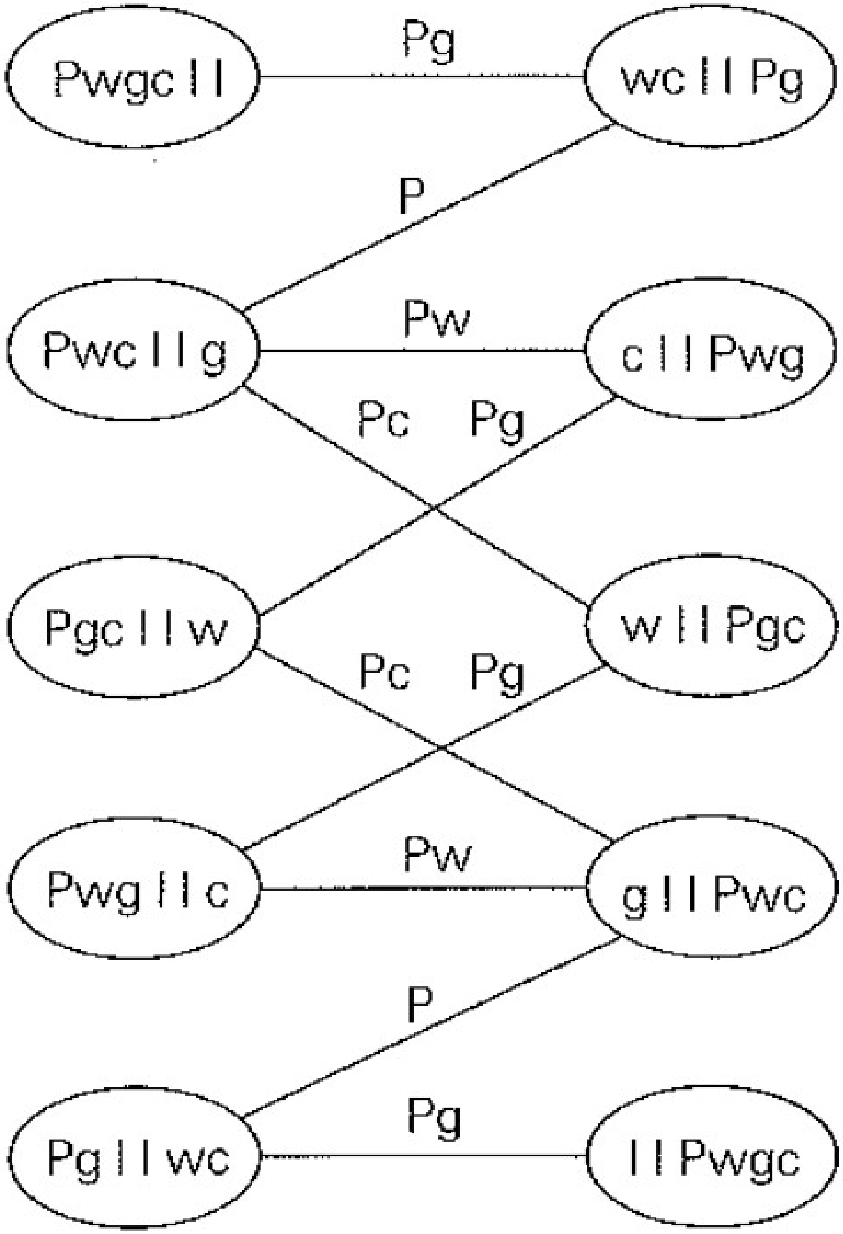
- variety of puzzles and games can be reduced to graph problems
- vertices of a graph typically represent possible states of the problem in question while edges indicate permitted transitions among such states.
- One of the graph's vertices represents an initial state, while another represents a goal state of the problem. (state-space graph)
- transformation reduces the problem to the question about a **path from the initial-state vertex to a goal-state vertex**

Example: River crossing puzzle

A peasant finds himself on a river bank with a wolf, a goat, and a head of cabbage. He needs to transport all three to the other side of the river in his boat.

However, the boat has room only for the peasant himself and one other item (either the wolf, the goat, or the cabbage). In his absence, the wolf would eat the goat, and the goat would eat the cabbage.

Find a way for the peasant to solve his problem or prove that it has no solution.



P, w, g, c stand for the peasant, the wolf, the goat, and the cabbage, respectively;

the two bars II denote the river

State-space graph for the peasant, wolf, goat, and cabbage puzzle

Let’s check our understanding

Puzzle

You are given a list of numbers for which you need to construct a min-heap.

How would you use an algorithm for constructing a max-heap to construct a min-heap?

Solution:

Inspired by Heapsort!!

Idea: Build min-heap in-place using array representing the max-heap

Puzzle

Consider a chocolate manufacturing company that produces only two types of chocolate – **A and B**. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:

- Each unit of A requires 1 unit of Milk and 3 units of Choco
- Each unit of B requires 1 unit of Milk and 2 units of Choco

The company kitchen has a total of **5 units of Milk and 12 units of Choco**. On each sale, the company makes a profit of

- Rs 6 per unit A sold
- Rs 5 per unit B sold.

Now, the company wishes to maximize its profit. **How many units of A and B should it produce respectively?**

	Milk	Choco	Profit per unit
A	1	3	Rs 6
B	1	2	Rs 5
Total	5	12	

Let the total number of units produced by A be = X

Let the total number of units produced by B be = Y

Now, the total profit is represented by Z

Profit: Maximize $Z = 6X+5Y$

Subject to:

$$X+Y \leq 5$$
$$3X+2Y \leq 12$$
$$X \geq 0 \ \& \ Y \geq 0$$

Puzzle

- You have **11, 8 and 5 litre** milk container.
- Each container has no markings except for that which gives you it's total volume.
- You have a 12 litre container full of milk.
- You must use the containers in such away as to exactly measure out 4 litre of milk.

How is this done?

Puzzle

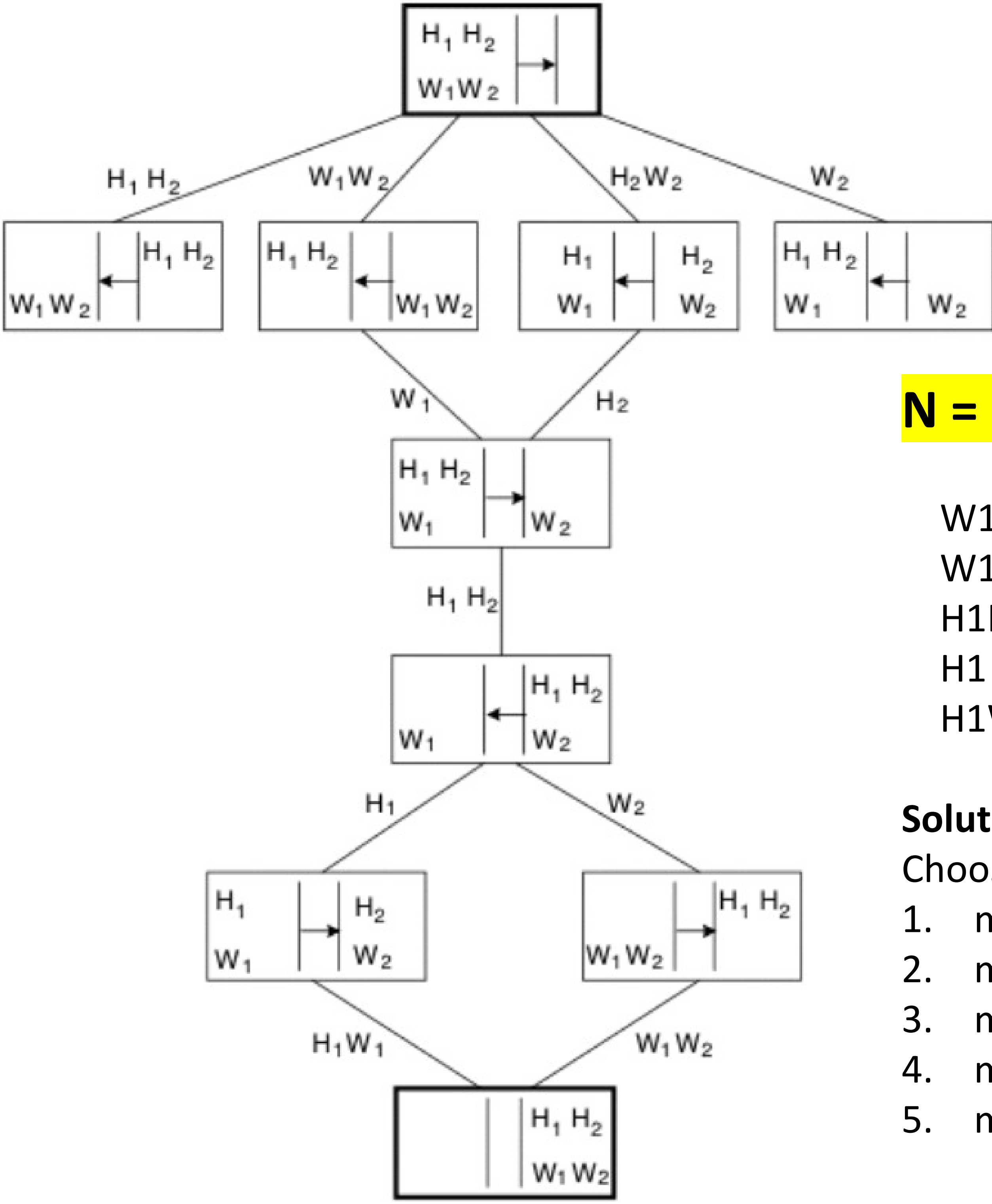
Jealous husbands:

There are **n (n ≥ 2)** married couples who need to cross a river. They have a boat that can hold no more than two people at a time. To complicate matters, all the husbands are jealous and will not agree on any crossing procedure that would put a wife on the same bank of the river with another woman's husband without the wife's husband being there too, even if there are other people on the same bank.

Can they cross the river under such constraints?

- Solve the problem for n = 2.
- Solve the problem for n = 3 (classical version of the problem)

Indicate how many river crossings they will take



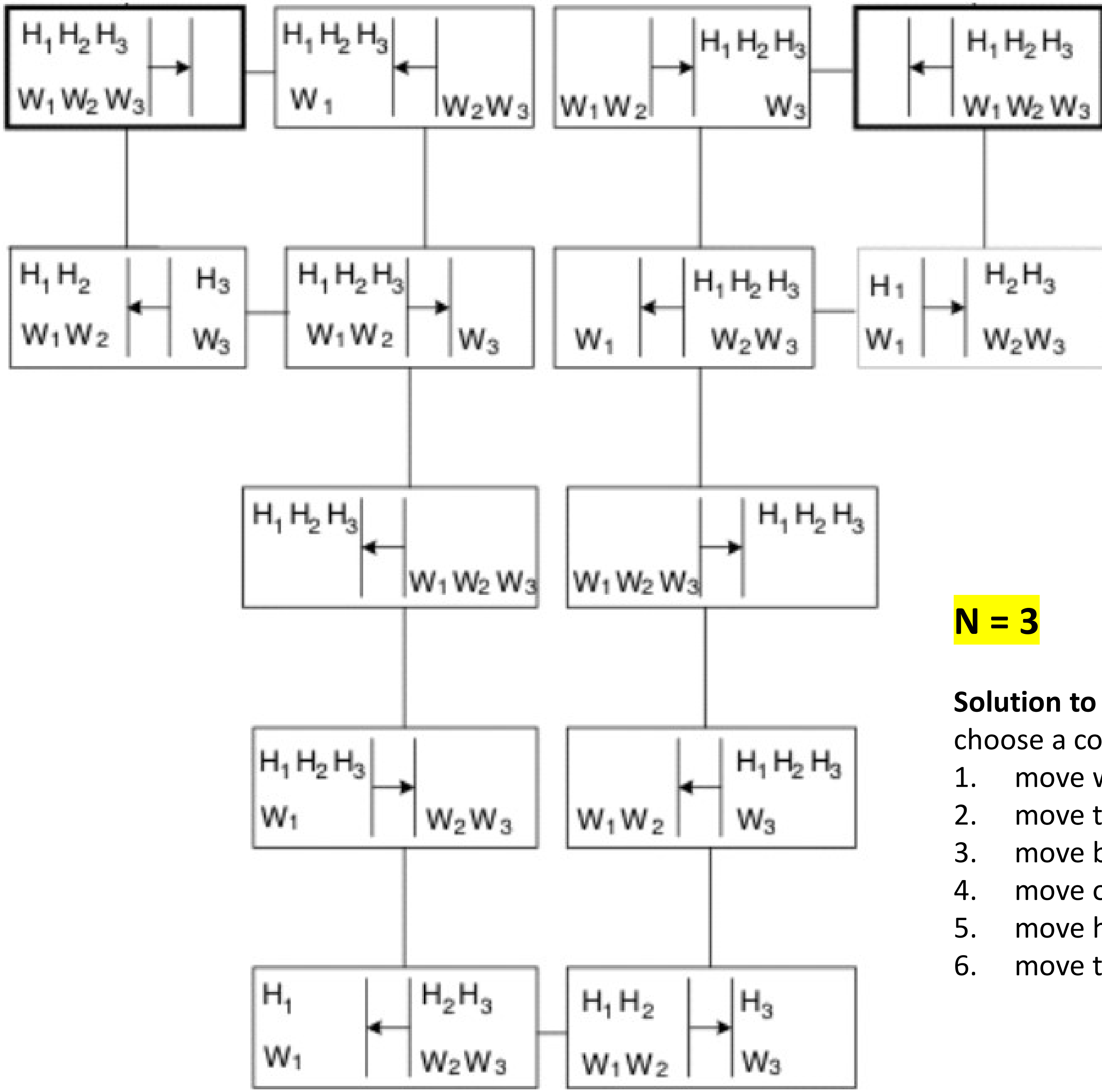
N = 2

W1W2
W1
H1H2
H1
H1W1

Solution to two-couple case:

Choose a couple to move first

1. move both wives across
2. move the other wife back
3. move both husbands across
4. move the other husband back
5. move the other couple



N = 3

Solution to three-couple case:

choose a couple h-i:w-i to move

1. move w-i and another wife across
2. move the other wife back
3. move both remaining wives across
4. move one of the other wives back
5. move h-i and another husband across
6. move the other couple back

Next session…

Dynamic Programming