UNIT 5: Branch-and-Bound

Travelling Salesperson Problem

Travelling Salesperson Problem (TSP)

- mathematically formulated in the 1800s by the Irish mathematician **W.R. Hamilton** and by the British mathematician **Thomas Kirkman**
- one of the most intensively studied problems in optimization
- used as a benchmark for many optimization methods
- Even though the problem is computationally difficult, many heuristics and exact algorithms are known, so that some instances with tens of thousands of cities can be solved completely and even problems with millions of cities can be approximated within a small fraction of 1%

Travelling Salesperson Problem (TSP)

Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Travelling Salesperson Problem (TSP)

- is an NP-hard problem in combinatorial optimization
- important in theoretical computer science and operations research
- **travelling purchaser problem** and the **vehicle routing problem** are both generalizations of TSP

- **class NP-complete/NP-Hard** problems: It is possible that the worst-case running time for any algorithm for the TSP increases superpolynomially (but no more than exponentially) with the number of cities.

Travelling Salesperson Problem (TSP): Applications

- planning, logistics, and the manufacture of microchips, printed circuits
- scheduling of a route of the drill machine to drill holes in a PCB.
- robotic machining or drilling applications
- finding traffic collisions, one-way streets, and airfares for cities with different departure and arrival fees
- finding a Hamiltonian cycle with the least weight
- DNA sequencing
- travelling politician problem, travelling purchaser problem
- cutting stock problem
- routing data among data processing nodes; routes vary by time to transfer the data, but nodes also differ by their computing power and storage (Google)
- in astronomy (as astronomers observing many sources will want to minimize the time spent moving the telescope between the sources etc)

Travelling Salesperson Problem (TSP): Approaches

- Exhaustive search - Brute force: O(n!): becomes impractical even for only 20 cities
- Dynamic programming (Held–Karp algorithm): $O(n^2 2^n)$
- **Branch-and-bound**: process 40–60 cities
- Linear programming(LP): up to 200 cities
- Cutting plane method (based on LP): 15,112 cities
- Branch and cut: 85,900 cities (current record)

- Heuristic and approximation algorithms

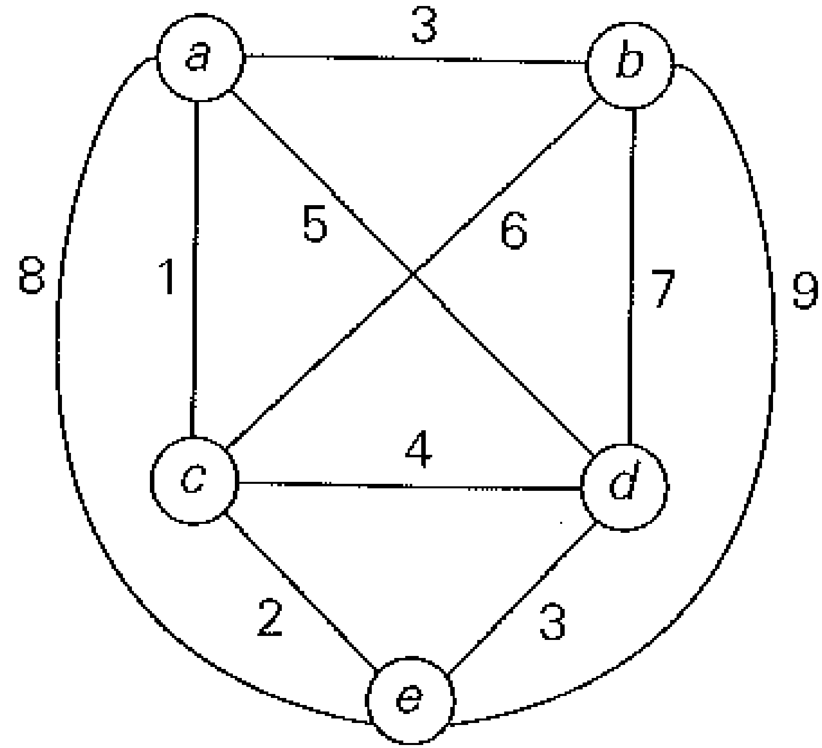
Branch and Bound: Travelling Salesperson Problem (TSP)

- **Compute the lower bound on tour lengths**

For each city i, $1 \leq i \leq n$, find the sum s_i of the distances from city i to the two nearest cities; compute the sum s of these n numbers; divide the result by 2; and, if all the distances are integers, round up the result to the nearest integer

$$lb = \lceil s/2 \rceil.$$

- consider only tours that start at “Origin city”
- (graph is undirected) Generate only tours in which b is visited before c city.

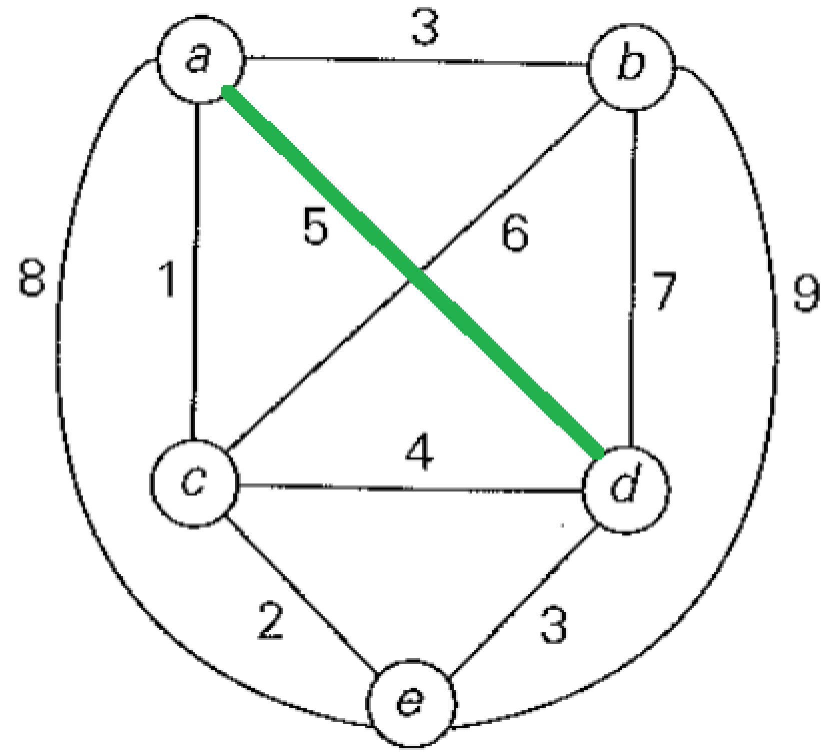


$$\boxed{lb = \lceil s/2 \rceil}$$

$$= [\overbrace{(1 + 3)}^{a} + \overbrace{(3 + 6)}^{b} + \overbrace{(1 + 2)}^{c} + \overbrace{(3 + 4)}^{d} + \overbrace{(2 + 3)}^{e}]/2$$

$$= 14$$

for any subset of tours that must include particular edges of a given graph, for example, must include edge (a, d), lower bound is computed by summing the lengths of the two shortest edges incident with each of the vertices, with the required inclusion of edges (a, d) and (d, a):

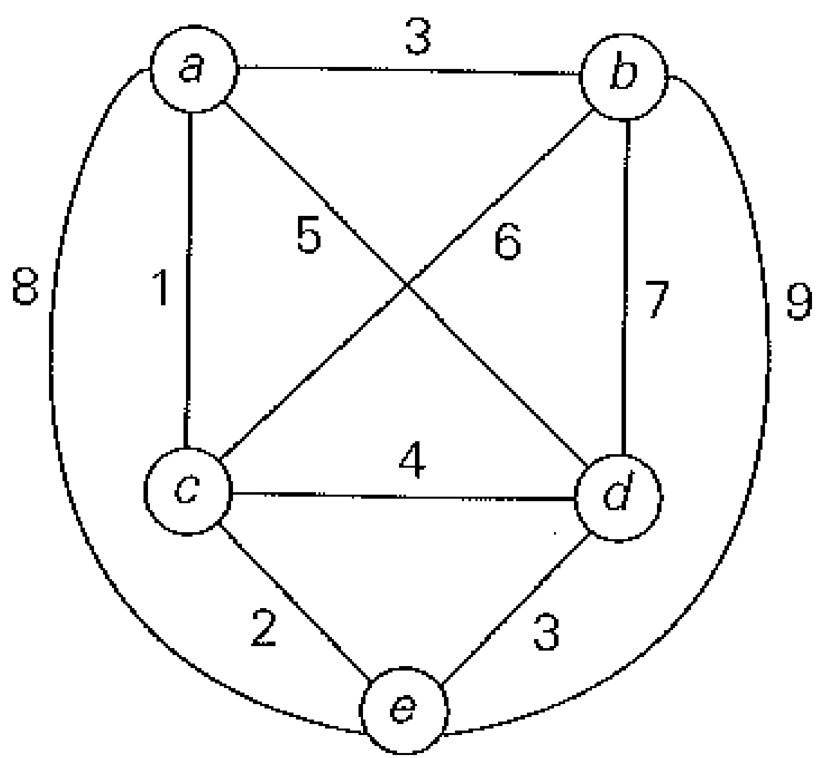


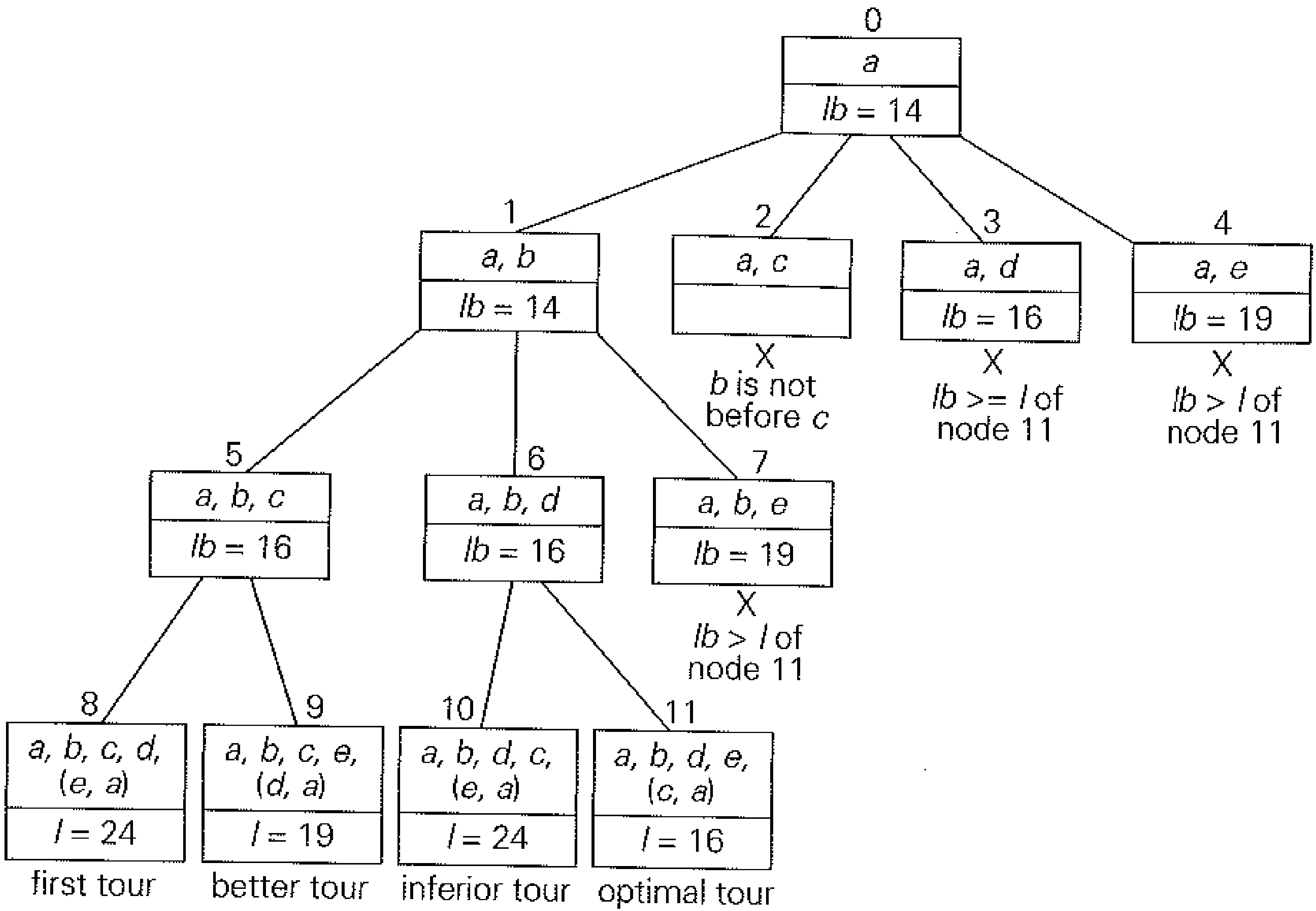
$$= [(1 + \mathbf{5}) + (3 + 6) + (1 + 2) + (\mathbf{5} + 3) + (2 + 3)]/2$$

$$= 16$$

Example:

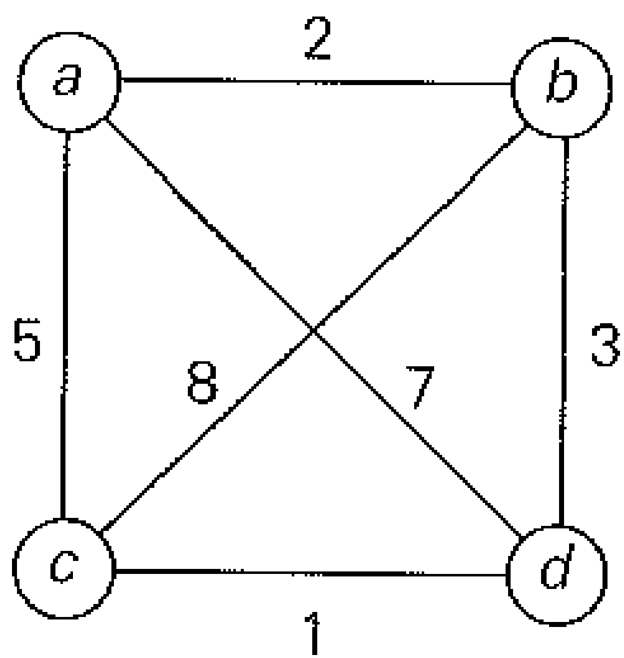
Apply the branch-and-bound algorithm to solve the traveling salesman problem for the following graph.

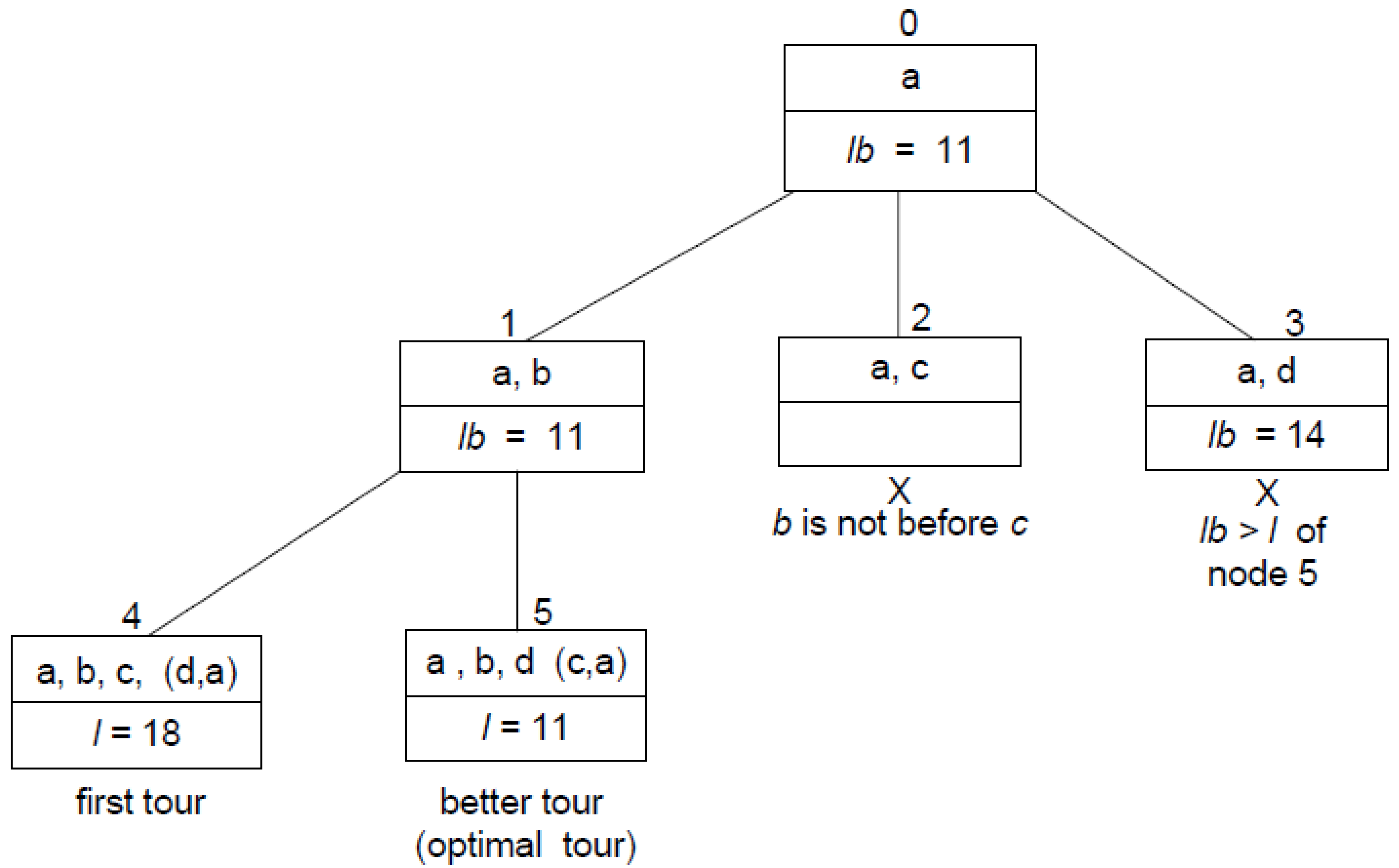



0
a
lb = 14
1
a, b
lb = 14
2
a, c
X
b is not before c
3
a, d
lb = 16
X
lb >= l of node 11
4
a, e
lb = 19
X
lb > l of node 11
5
a, b, c
lb = 16
6
a, b, d
lb = 16
7
a, b, e
lb = 19
X
lb > l of node 11
8
a, b, c, d, (e, a)
l = 24
first tour
9
a, b, c, e, (d, a)
l = 19
better tour
10
a, b, d, c, (e, a)
l = 24
inferior tour
11
a, b, d, e, (c, a)
l = 16
optimal tour

Let's check our understanding…

Apply the branch-and-bound algorithm to solve the traveling salesman problem for the following graph.

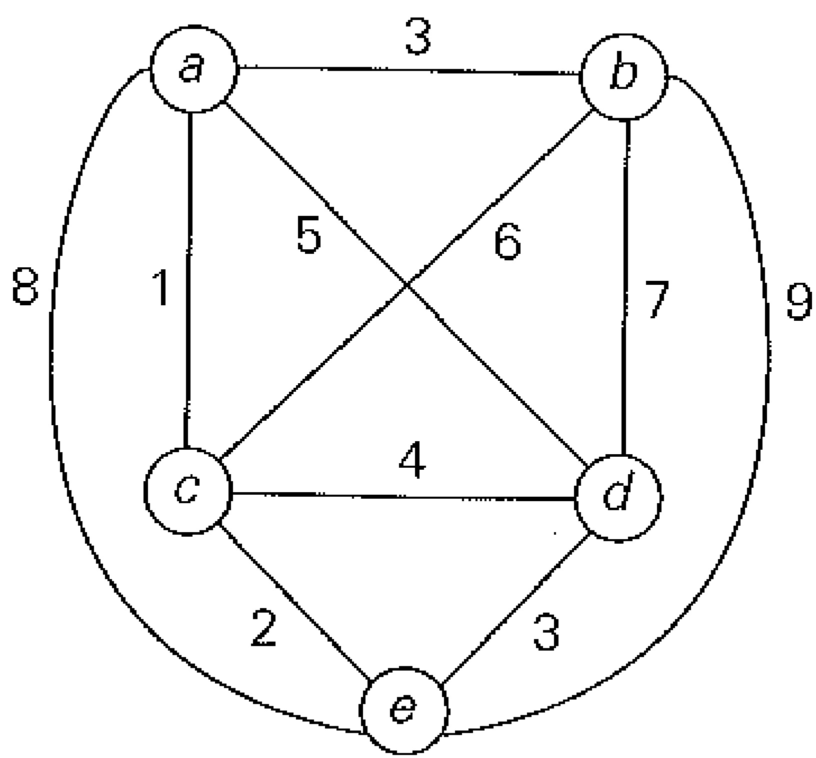




optimal tour is a, b, d, c, a of length 11.

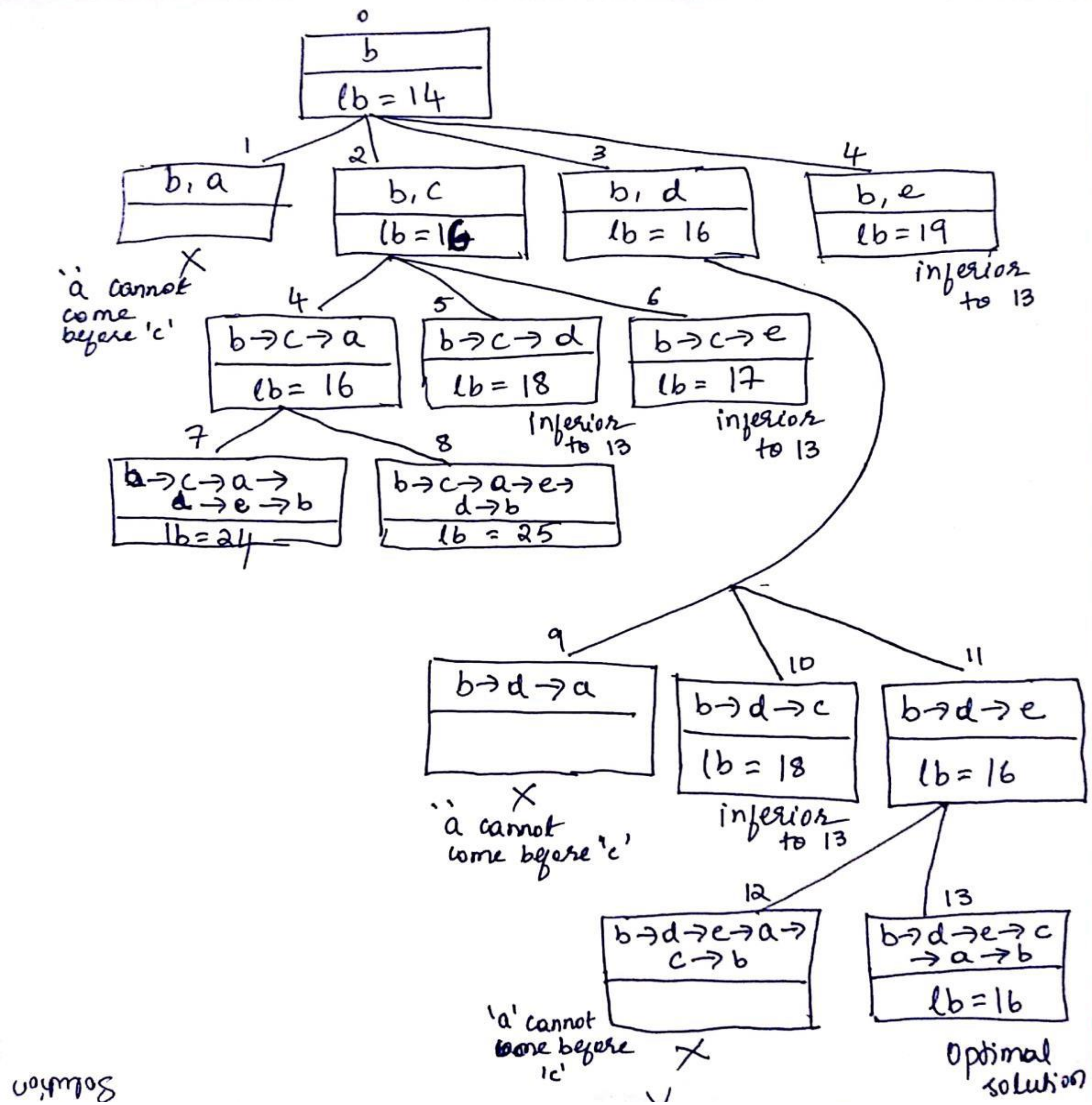
Solve:

Consider the graph given below representing an instance of TSP, where source vertex is “b”. Solve the problem instance using Branch and Bound technique and obtain the solution where “c” comes before “a”. Explain the procedure to compute lower bound. Write the state-space tree and number the nodes



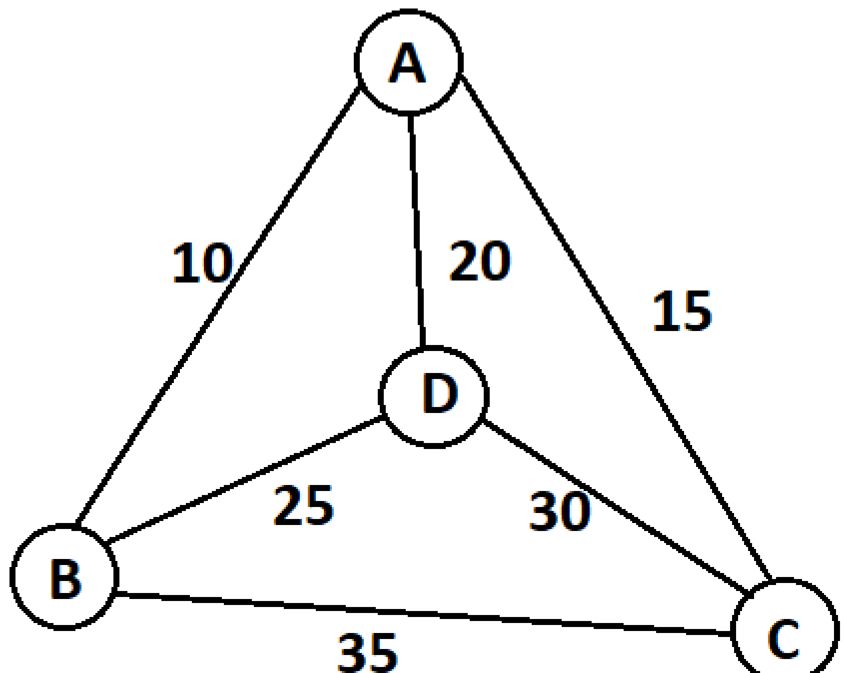
$$\boxed{lb = \lceil s/2 \rceil} = [\overset{a}{(1+3)} + \overset{b}{(3+6)} + \overset{c}{(1+2)} + \overset{d}{(3+4)} + \overset{e}{(2+3)}]/2$$

$$= 14$$



Let’s check our understanding…

Apply the branch-and-bound algorithm to solve the traveling salesman problem for the following graph.



Let’s check our understanding…

Apply the branch-and-bound algorithm to solve the traveling salesman problem.

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} &
\begin{bmatrix}
\infty & 4 & 12 & 7 \\
5 & \infty & \infty & 18 \\
11 & \infty & \infty & 6 \\
10 & 2 & 3 & \infty
\end{bmatrix}
\end{array}
$$