Topic: Fundamentals of Logic   Date:

Logic is the science dealing with the method of reasoning. A symbolic language has been developed over the past two centuries to express the principles of logic in precise and unambiguous terms. Logic expressed in such a language has come to be called "Symbolic logic" or "Mathematical logic".

Propositions

A proposition is a statement (declaration), which, in a given context, can be said to be either true or false, but not both.

eg Bengaluru is in karnataka. → true
Three is a prime number. → true
Seven is divisible by 3. → false
Every rectangle is a square. → false

Propositions are usually represented by small letters as $p, q, r, s, \ldots$ The truth or the falsity of a proposition is called its truth value.

If the proposition is true, we will indicate its truth value by the symbol 1 and if it is false by the symbol 0.

eg If "Three is a prime number" is denoted as p, then the truth value of p is 1.
If "Every rectangle is a square" is denoted as q, then the truth value of q is 0.

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Logical connectives

New propositions are obtained by starting with given propositions with the aid of words or phrases like 'not', 'and' 'if... then', and 'if and only if'. Such words or phrases are called logical connectives. The new propositions obtained by the use of connectives are called compound propositions. The original propositions from which a compound proposition is obtained are called the components or the primitives of the compound proposition. Propositions which do not contain any logical connective are called simple propositions.

Negation

A proposition obtained by inserting the work 'not' at an appropriate place in a given proposition is called the negation of the given proposition.

The negation of a proposition p is denoted by $\neg p$ ('not p'), the symbol $\neg$ denoting the word not.

ex If "3 is a prime number" is denoted by p, then $\neg p$ is "3 is not a prime number"

Conjunction:

A compound proposition obtained by combining two given propositions by inserting the word "and" in between them is called the conjunction of the given proposition.

The conjunction of two propositions p and q

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is denoted by $p \wedge q$ ("p and q"), the symbol $\wedge$ denoting the word and.

ex. If p is "$\sqrt{2}$ is an irrational number" and q is "9 is a prime number", then $p \wedge q$ is "$\sqrt{2}$ is an irrational number and 9 is a prime number".

Disjunction

A compound proposition obtained by combining two give propositions by inserting the word 'or' in between them is called the disjunction of the given propositions

The disjunction of two propositions p and q is denoted by $p \vee q$ ("p or q"), the symbol $\vee$ denoting the word or.

ex If p is "all triangles are equilateral", and q is "$2+5=7$", then $p \vee q$ is "all triangles are equilateral or $2+5=7$".

Conditional

A compound proposition obtained by combining two given propositions by using the words 'if' and 'then' at appropriate places is called a conditional.

The conditional of "If p, then q" is denoted $p \to q$ and the conditional "If q, then p" is denoted $q \to p$.

ex If p is "2 is a prime number" and q is "6 is a perfect square" then $p \to q$ is "If 2 is a prime number then 6 is a perfect square"

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Biconditional

A proposition obtained by inserting the words 'if and only if' ~~at an appropriate place in a between the~~ in between them is called the biconditional of the given propositions.

The biconditional ~~$p \leftrightarrow q$~~ 'p if and only if q' or 'q if and only if p' are denoted as $p \Leftrightarrow q$, or $q \leftrightarrow p$.

[The conjunction of the conditional $p \to q$ and $q \to p$ is called the biconditional of p and q].

ex. if p is "2 is a prime number" and q is "6 is a perfect square", then $p \Leftrightarrow q$ is "2 is a prime number if and only if 6 is a perfect square".

Truth Tables

Truth Tables are used to determine the truth value of compound propositions based on the truth values of their components.

Truth table for negation

p	$\neg p$
0	1
1	0

If p is true, then $\neg p$ is false

If p is false, then $\neg p$ is true.

~~Truth table for conjunction~~

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Truth Table for conjunction

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

$p \wedge q$ is true only when p is true and q is true, in all other cases it is false.

Truth Table for disjunction.

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

$p \vee q$ is false only when p is false and q is false, in all other cases it is true.

Truth Table for conditional

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

$p \to q$ is false only when p is true and q is false, in all other cases it is true.

Truth Table for biconditional

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

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* let p: A circle is a conic,
q: $\sqrt{5}$ is a real number
r: Exponential series is convergent.

Express the following compound propositions in words:
(i) $p \wedge (\neg q)$, (ii) $(\neg p) \vee q$, (iii) $q \to (\neg p)$, (iv) $\neg p \leftrightarrow q$.

Solⁿ (i) $p \wedge (\neg q)$: A circle is a conic and $\sqrt{5}$ is not a real number

(ii) $(\neg p) \vee q$: A circle is not a conic or $\sqrt{5}$ is a real number

(iii) $q \to (\neg p)$: If $\sqrt{5}$ is a real number then a circle is not a conic.

(iv) $\neg p \leftrightarrow q$: A circle is not a conic if and only if $\sqrt{5}$ is a real number.

* Construct the truth tables for the following compound propositions:
(i) $p \wedge (\neg q)$, (ii) $(\neg p) \vee q$ (iii) $p \to (\neg q)$.

Solⁿ

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \vee q$	$p \to (\neg q)$
0	0	1	0	1	1	1
0	1	0	0	1	1	1
1	0	1	1	0	0	1
1	1	0	0	0	1	0

* Let p and q be primitive statements for which the conditional $p \to q$ is false. Determine the truth value of the following compound propositions (i) $p \wedge q$, (ii) $(\neg p) \vee q$, (iii) $q \to p$, (iv) $(\neg q) \to (\neg p)$

Solⁿ Since $p \to q$ is false, p has to be true & q has to be false.

(i) $p \wedge q$ is 1 and 0 $\Rightarrow$ 0 is the truth value

(ii) $(\neg p) \vee q$ is 0 or 0 $\Rightarrow$ 0 is the truth value

(iii) $q \to p$ is $0 \to 1 \Rightarrow 1$ is the truth value

(iv) $(\neg q) \to (\neg p) \Rightarrow 1 \to 0 \Rightarrow 0$ is the truth value

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Q Find the possible truth values of p, q and r in the following cases:
(i) $p \rightarrow (q \vee r)$ is false. (ii) $p \wedge (q \rightarrow r)$ is true.

Sol (i) $p \rightarrow (q \vee r)$ can be false only when p is true and $q \vee r$ is false. Also $q \vee r$ is false only when both q and r are false. Hence the truth values of p, q, r are 1, 0, 0 respectively.
(ii) $p \wedge (q \rightarrow r)$ can be true, only when p is true and $q \rightarrow r$ is true. Also $q \rightarrow r$ is true, when q is false and r is false, q is false and r is true, q is true and r is true. Hence the possible values of p, q, r are (a) 1, 0, 0
(b) 1, 0, 1
(c) 1, 1, 1

Q Construct the truth tables for the following compound propositions: (i) $(p \vee q) \wedge r$, (ii) $p \vee (q \wedge r)$

Sol

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

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* Construct the truth tables for the following compound propositions:
(i) $(p \wedge q) \rightarrow (\neg r)$, (ii) $q \wedge ((\neg r) \rightarrow p)$

p	q	r	$p \wedge q$	$\neg r$	$p \wedge q \rightarrow (\neg r)$	$\neg r \rightarrow p$	$q \wedge ((\neg r) \rightarrow p)$
0	0	0	0	1	1	0	0
0	0	1	0	0	1	1	0
0	1	0	0	1	1	0	0
0	1	1	0	0	1	1	1
1	0	0	0	1	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	1	0	0	1	1

* If a proposition q has the truth value 1, determine all truth value assignments for the primitive propositions p, r and s for which the truth value of the following compound proposition is 1. $[q \rightarrow [(\neg p \vee r) \wedge \neg s]] \wedge [\neg s \rightarrow (\neg r \wedge q)]$

Solⁿ: let $u \equiv q \rightarrow [(\neg p \vee r) \wedge \neg s]$ and $v = [\neg s \rightarrow (\neg r \wedge q)]$
then given truth value of $u \wedge v$ is 1.
$\Rightarrow$ truth value of u is 1 and truth value of v is 1.
Since truth value of q is 1 and truth value of u is 1 it follows truth value of $(\neg p \vee r) \wedge \neg s$ is 1, consequently truth value of $\neg p \vee r$ is 1 and that of $\neg s$ is 1. consequently truth value of s is 0.
~~consequently~~ since truth value of $\neg s$ is 1 and truth value of v is 1, it follows that the truth value of $\neg r \wedge q$ is 1
since truth value of $\neg r \wedge q$ is 1 and that of q is 1, it follows truth value of $\neg r$ is 1, hence truth value of r is 0.
since truth value of $\neg p \vee r$ is 1 and truth value of r is 0, it follows truth value of $\neg p$ is 1, hence truth value of p is 0

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Tautology, Contradiction, Contrapositive

A compound proposition which is true for all possible truth values of its components is called a tautology (or a logical truth).

A compound proposition which is false for all possible truth values of its components is called a contradiction or an absurdity.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency. In other words, a contingency is a compound proposition which is neither a tautology nor a contradiction.

* Prove that, for any proposition p, the compound proposition $p \vee \neg p$ is a tautology and the compound proposition $p \wedge \neg p$ is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

From the table, since $p \vee \neg p$ is always true, it's a tautology and since $p \wedge \neg p$ is always false, it's a contradiction.

* Show that for any propositions p and q, the compound proposition $p \rightarrow (p \vee q)$ is a tautology and the compound proposition $p \wedge (\neg p \wedge q)$ is a contradiction.

Sol

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

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* Prove that, for any propositions p and q, the compound proposition $[(\neg q) \wedge (p \to q)] \to (\neg p)$ is a tautology.

sol)

p	q	$p \to q$	$\neg q$	$(\neg q) \wedge (p \to q)$	$\neg p$	$[(\neg q) \wedge (p \to q)] \to (\neg p)$
0	0	1	1	1	1	1
0	1	1	0	0	1	1
1	0	0	1	0	0	1
1	1	1	0	0	0	1

Since the last column has all true values, $[(\neg q) \wedge (p \to q)] \to (\neg p)$ is a tautology.

* Prove that, for any propositions p, q, r, the compound proposition $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

sol)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$(p \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

Tautology

* Prove that for any propositions p, q, r, $[(p \vee q) \wedge [(p \to r) \wedge (q \to r)]] \to r$ is a tautology

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$(p \vee q) \wedge [(p \to r) \wedge (q \to r)]$	$[(p \vee q) \wedge [(p \to r) \wedge (q \to r)]] \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Tautology

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Logical Equivalence

Two propositions u and v are said to be logically equivalent whenever u and v have the same truth value, or equivalently, the biconditional $u \leftrightarrow v$ is a tautology.
Then we write $u \Leftrightarrow v$. Here the symbol $\Leftrightarrow$ stands for "logically equivalent to". ~~when~~
When the propositions u and v are not logically equivalent, we write $u \not\Leftrightarrow v$.
Logically equivalent propositions are treated as identical propositions.

* Let x be a specified positive integer. Consider the following propositions:
p: x is an odd integer, q: x is not divisible by 2
Are p and q logically equivalent?
Soln. We note that p and q have the same truth values. As such p and q are logically equivalent, i.e., $p \Leftrightarrow q$.

* For any two propositions p, q prove that $(p \to q) \Leftrightarrow (\neg p) \vee q$.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

as $p \to q$ and $\neg p \vee q$ have the same truth values for all possible truth values of p and q, $p \to q \Leftrightarrow (\neg p \vee q)$.

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* Prove that $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.

sol

p	q	r	$p \to q$	$(p \to q) \to r$	$\neg p$	$\neg p \vee q$	$(\neg p \vee q) \to r$
0	0	0	1	0	1	1	0
0	0	1	1	1	1	1	1
0	1	0	1	0	1	1	0
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	0	1
1	1	0	1	0	0	1	0
1	1	1	1	1	0	1	1

as $(p \to q) \to r \iff [(\neg p \vee q) \to r]$, we can say $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.

* Prove that, for any three propositions p, q, r, $[p \to (q \wedge r)] \iff [(p \to q) \wedge (p \to r)]$

sol

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1

$\because$ $p \to (q \wedge r)$ and $(p \to q) \wedge (p \to r)$ have the same truth values, $(p \to (q \wedge r)) \iff [(p \to q) \wedge (p \to r)]$

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The laws of logic

The following results, known as the laws of logic, follow from the definition of logical equivalence. In these laws, T_0 denotes a tautology and F_0 denotes a contradiction.

1. Law of Double negation: $(\neg\neg p) \Leftrightarrow p$
2. Idempotent law: (a) $(p \vee p) \Leftrightarrow p$, (b) $(p \wedge p) \Leftrightarrow p$
3. Identity law: (a) $(p \vee F_0) \Leftrightarrow p$, (b) $(p \wedge T_0) \Leftrightarrow p$
4. Inverse law: (a) $(p \vee \neg p) \Leftrightarrow T_0$, (b) $(p \wedge \neg p) \Leftrightarrow F_0$
5. Domination law: (a) $(p \vee T_0) \Leftrightarrow T_0$, (b) $(p \wedge F_0) \Leftrightarrow F_0$
6. Commutative law: (a) $(p \vee q) \Leftrightarrow (q \vee p)$, (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$
7. Absorption law: (a) $[p \vee (p \wedge q)] \Leftrightarrow p$, (b) $[p \wedge (p \vee q)] \Leftrightarrow p$
8. DeMorgan's law: (a) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$, (b) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
9. Associative law: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$, (b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
10. Distributive law: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
    (b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

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Prove the following logical equivalences using the laws of logic.

(i) $p \to q \Leftrightarrow \neg q \to \neg p$.

Sol^n Consider $p \to q \Leftrightarrow \neg p \vee q$ (conditional)

$\Leftrightarrow q \vee \neg p$ (commutative)

$\Leftrightarrow \neg(\neg q) \vee \neg p$ (double negation)

$\Leftrightarrow \neg q \to \neg p$. (conditional.

(ii) $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

Sol^n consider $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ (biconditional)

$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$ (conditional)

$\Leftrightarrow [\neg p \wedge (\neg q \vee p)] \vee [q \wedge (\neg q \vee p)]$ (distributive)

$\Leftrightarrow [(\neg p \wedge \neg q) \vee (\neg p \wedge p)] \vee [(q \wedge \neg q) \vee (q \wedge p)]$ (distributive)

$\Leftrightarrow [(\neg p \wedge \neg q) \vee F_0] \vee [F_0 \vee (q \wedge p)]$ (Inverse)

$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge p)$ (Identity)

$\Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge q)$ (commutative)

$\Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$ (commutative)

(iii) $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

Sol^n Consider $\neg(p \leftrightarrow q) \Leftrightarrow \neg((p \to q) \wedge (q \to p))$ (Biconditional)

$\Leftrightarrow \neg(p \to q) \vee \neg(q \to p)$ (De Morgan's)

$\Leftrightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$ (Conditional)

$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$ (De Morgan's)

$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$ (Commutative)

(iv) $(p \vee q) \wedge [\neg(\neg p \vee q)] \Leftrightarrow p \wedge \neg q$

Consider $(p \vee q) \wedge \neg(\neg p \vee q) \Leftrightarrow (p \vee q) \wedge (p \wedge \neg q)$ (De Morgan's)

$\Leftrightarrow [(p \vee q) \wedge p] \wedge (\neg q)$ (Associative)

$\Leftrightarrow [p \wedge (p \vee q)] \wedge (\neg q)$ (Commutative

$\Leftrightarrow p \wedge (\neg q)$ (Absorption)

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(v) $[(\neg p \vee \neg q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

Sol^n consider $(\neg p \vee \neg q) \to (p \wedge q \wedge r)$

$\Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$ [Conditional]

$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$ [De Morgan's]

$\Leftrightarrow (p \wedge q) \vee ((p \wedge q) \wedge r)$ (Associative)

$\Leftrightarrow p \wedge q$ (Absorption)

(vi) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg(q \vee p)$

Sol^n Consider $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\Leftrightarrow (p \to q) \wedge [\neg q \wedge (\neg q \vee r)]$ (commutative)

$\Leftrightarrow (p \to q) \wedge (\neg q)$ (Absorption)

$\Leftrightarrow \neg[\neg(p \to q) \vee q]$ [De Morgan's]

$\Leftrightarrow \neg[(p \to q) \to q]$ [Conditional]

$\Leftrightarrow \neg[\neg(p \to q) \vee q]$ [Conditional]

$\Leftrightarrow \neg[\neg(\neg p \vee q) \vee q]$ [Conditional]

$\Leftrightarrow \neg[(p \wedge \neg q) \vee q]$ [De Morgan's]

$\Leftrightarrow \neg[q \vee (p \wedge \neg q)]$ [Commutative]

$\Leftrightarrow \neg[(q \vee p) \wedge (q \vee \neg q)]$ [Distributive]

$\Leftrightarrow \neg[(q \vee p) \wedge T_0]$ [Inverse]

$\Leftrightarrow \neg(q \vee p)$ [Identity]

(vii) $\neg[[(p \vee q) \wedge r] \to \neg q] \Leftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$

Consider $\neg[[(p \vee q) \wedge r] \to \neg q]$ ①

$\Leftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q]$ (conditional) ②

$\Leftrightarrow \neg\neg[[(p \vee q) \wedge r] \wedge q]$ (De Morgan's)

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$ (double negation)

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ (Associative)

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ (Commutative)

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ (Associative)

$\Leftrightarrow [q \wedge (p \vee q)] \wedge r$ (Commutative)

$\Leftrightarrow [q \wedge (q \vee p)] \wedge r$ (Commutative)

$\Leftrightarrow (q \wedge r)$ (Absorption) ③

① $\Leftrightarrow$ ② $\Leftrightarrow$ ③ is the required proof

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(viii) $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$

Consider $\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow [(\neg p \wedge \neg q) \wedge r] \vee (q \wedge r) \vee (p \wedge r)$ (Associative)

$\Leftrightarrow [r \wedge (\neg p \wedge \neg q)] \vee (q \wedge r) \vee (p \wedge r)$ (Commutative)

$\Leftrightarrow [r \wedge \neg(p \vee q)] \vee (q \wedge r) \vee (p \wedge r)$ (De Morgans)

$\Leftrightarrow [r \wedge \neg(p \vee q)] \vee (r \wedge q) \vee (r \wedge p)$ (Commutative)

$\Leftrightarrow [r \wedge \neg(p \vee q)] \vee [r \wedge (q \vee p)]$ (Distributive)

$\Leftrightarrow r \wedge [\neg(p \vee q) \vee (p \vee q)]$ (Distributive)

$\Leftrightarrow r \wedge T_0$ (Inverse)

$\Leftrightarrow r$ (Identity)

4. Prove that $[(p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology.

Sol: Let w be the given proposition.

Then $w = u \vee v$,

where $u = (p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]$

and $v = (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

Consider $u = (p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]$

$\Leftrightarrow (p \vee q) \wedge (p \vee \neg(\neg q \vee \neg r))$ (De Morgans)

$\Leftrightarrow (p \vee q) \wedge (p \vee (q \wedge r))$ (De Morgan's)

$\Leftrightarrow p \vee [q \wedge (q \wedge r)]$ (Distributive)

$\Leftrightarrow p \vee [(q \wedge q) \wedge r]$ (Associative)

$\Leftrightarrow p \vee (q \wedge r)$ (Idempotent)

Consider $v = (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$\Leftrightarrow \neg(p \vee q) \vee \neg(p \vee r)$ (DeMorgan's)

$\Leftrightarrow \neg[(p \vee q) \wedge (p \vee r)]$ (De Morgan's)

$\Leftrightarrow \neg[p \vee (q \wedge r)]$ (Distributive)

$\Rightarrow v = \neg u$.

$\therefore w = u \vee v \Rightarrow w = u \vee \neg u$

$w = T_0$ (Inverse)

$\therefore$ The given proposition is a Tautology

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Duality

Suppose u is a compound proposition that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurence of $\wedge$ and $\vee$ in u by $\vee$ and $\wedge$ respectively. Also, if u contains T_0 and F_0 as components, suppose we replace each occurence of T_0 and F_0 by F_0 and T_0 respectively. Then the resulting compound proposition is called the dual of u and is denoted by u^d.

eg If $u : p \wedge (q \vee \neg r) \vee (s \wedge T_0)$

then $u^d = p \vee (q \wedge \neg r) \wedge (s \vee F_0)$

Note

(i) $(u^d)^d \Leftrightarrow u$

(ii) If $u \Leftrightarrow v$, then $u^d \Leftrightarrow v^d$

(This is known as the Principle of Duality)

* Write the dual of the following propositions.

(i) $\neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$

sol^n The dual of given proposition is

$\neg(p \wedge q) \vee [p \wedge \neg(q \vee \neg s)]$

(ii) $u = [(p \vee T_0) \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

sol^n $u^d = [(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$

(iii) $u = p \to q$

sol^n $u = p \to q \Leftrightarrow \neg p \vee q$

$\therefore u^d = \neg p \wedge q$

(iv) $u = (p \to q) \to r$

sol^n $u = (p \to q) \to r \Leftrightarrow \neg(p \to q) \vee r \Leftrightarrow \neg(\neg p \vee q) \vee r$

$\therefore u^d = \neg(\neg p \wedge q) \wedge r$

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* Verify the principle of duality for the logical equivalence

$$[\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)] \Leftrightarrow (\neg p \vee q)$$

Sol^n

Let $u = [\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)]$ & $v = (\neg p \vee q)$

Also

$u = [\neg\neg(p \wedge q) \vee \neg p \vee (\neg p \vee q)]$ conditional

$= (p \wedge q) \vee \neg p \vee (\neg p \vee q)$ (double negation)

Then $u^d = (p \vee q) \wedge \neg p \wedge (\neg p \wedge q)$

$\Leftrightarrow (p \vee q) \wedge (\neg p \wedge \neg p) \wedge q$ (Associative)

$\Leftrightarrow (p \vee q) \wedge (\neg p \wedge q)$ (Idempotent)

$\Leftrightarrow [p \wedge (\neg p \wedge q)] \vee [q \wedge (\neg p \wedge q)]$ (Distributive)

$\Leftrightarrow [(p \wedge \neg p) \wedge q] \vee [q \wedge (\neg p \wedge q)]$ (Associative)

$\Leftrightarrow (F_0 \wedge q) \vee [q \wedge (\neg p \wedge q)]$ (Inverse)

$\Leftrightarrow F_0 \vee [q \wedge (\neg p \wedge q)]$ (Domination)

$\Leftrightarrow F_0 \vee [q \wedge (q \wedge \neg p)]$ (Commutative)

$\Leftrightarrow F_0 \vee [(q \wedge q) \wedge \neg p]$ (Associative)

$\Leftrightarrow F_0 \vee (q \wedge \neg p)$ (Idempotent)

$u^d \Leftrightarrow q \wedge \neg p$ (Identity

Also $v^d = \neg p \wedge q$

$v^d \Leftrightarrow q \wedge \neg p$.

$\therefore u^d \Leftrightarrow v^d$

This verifies the principle of duality.

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Converse, Inverse and Contrapositive; logical Implication

Consider the conditional $p \to q$.

Then ① $q \to p$ is called the converse of $p \to q$.

② $\neg p \to \neg q$ is called the inverse of $p \to q$.

③ $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

ex p: 2 is an integer, q: 9 is a multiple of 3.

Then $p \to q$: If 2 is an integer, then 9 is a multiple of 3.

$q \to p$: If 9 is a multiple of 3, then 2 is an Integer.

$\neg p \to \neg q$: If 2 is not an integer, then 9 is not a multiple of 3.

$\neg q \to \neg p$: If 9 is not a multiple of 3, then 2 is not an integer.

The following table gives the truth value of $(p \to q)$, $(q \to p)$, $(\neg p \to \neg q)$, $(\neg q \to \neg p)$ for all possible truth values of two arbitrary propositions p and q.

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

From this table, it is evident that $p \to q$ and $\neg q \to \neg p$ have the same truth values in all possible situations. Also, $q \to p$ and $\neg p \to \neg q$ have the same truth values in all possible situations.

We have two important results:

① A conditional and its contrapositive are logically equivalent: that is, for any propositions p and q,

$$(p \to q) \Leftrightarrow (\neg q \to \neg p)$$

② The converse and the inverse of a conditional are logically equivalent; that is for any propositions p and q,

$$(q \to p) \Leftrightarrow (\neg p \to \neg q)$$

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logical implications

- Consider two propositions,

p: 6 is a multiple of 2, and q: 3 is a prime number

Then $p \to q$: If 6 is a multiple of 2 then 3 is a prime number

Note that p is true and q is true, hence $p \to q$ is true.

This conditional makes no sence but it is logically true.

- Consider the propositions,

p: 4 is an odd number, and q: Bengaluru is not in Karnataka

Here both p and q are false,

The conditional $p \to q$: If 4 is an odd number, then Bengaluru is not in Karnataka.

This conditional makes no sense but it is logically true.

We do not deal with conditionals such as the ones considered in the above two examples.

Our major interest lies in conditionals $p \to q$ where p and q are related in some way so that the truth value of q depends upon the truth value of p or vice versa. Such conditions are called hypothetical statements.

When a hypothetical statement $p \to q$ is such that q is true whenever p is true, we say that p (logically) implies q. This is symbolically written as $p \Rightarrow q$, the symbol $\Rightarrow$ denoting the word implies.

When a hypothetical statement $p \to q$ is such that q is not necessarily true whenever p is true, we say that p does not imply q. This is symbolically written as $p \not\Rightarrow q$, the symbol $\not\Rightarrow$ denoting the phrase does not imply.

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Necessary and Sufficient Conditions

Consider two propositions p and q whose truth values are interrelated. Suppose that $p \Rightarrow q$. Then in order that q may be true it is sufficient that p is true. Also if p is true then it is necessary that q is true. In view of this interpretation, all of the following statements are taken to carry the same meaning: (i) $p \Rightarrow q$, (ii) p is sufficient for q, (iii) q is necessary for p.

For two propositions p and q, the following situations are possible:

(i) $p \Rightarrow q$, but $q \not\Rightarrow p$.
(ii) $p \not\Rightarrow q$, but $q \Rightarrow p$.
(iii) $p \Rightarrow q$, and $q \Rightarrow p$

In the first of the above cases, p is a sufficient but not a necessary condition for q.
In the second case, p is a necessary but not a sufficient condition for q.
In the last case, p is necessary and sufficient condition for q, and vice-versa.

eg:
Let A denote a specified city.
Consider p: The city A is in Karnataka.
q: The city A is in India.
Here $p \Rightarrow q$, but $q \not\Rightarrow p$.
Accordingly, p is a sufficient but not a necessary condition for q, and q is a necessary but not a sufficient condition for p.

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* ~~Ex~~ Consider a geometric object the quadrilateral.

let p: A quadrilateral is a rectangle.

q: A quadrilateral is a square

Here $p \not\Rightarrow q$, but $q \Rightarrow p$.

Accordingly, p is a necessary but not a sufficient condition for q, and q is a sufficient but not a necessary condition for p.

* Consider a specified integer x.

let p: The integer x is even.

q: The integer x is divisible by 2.

Here, $p \Rightarrow q$ and $q \Rightarrow p$.

Thus, here p is a necessary and sufficient condition for q and vice-versa.

* State the converse, inverse and contrapositive.

q: If a quadrilatoral is a parallelogram, then its diagonals bisect each other.

Sol^n converse: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

inverse: If a quadrilateral is not a parallelogram, then its diagonals donot bisect each other.

contrapositive: If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.

* Write the converse, inverse and contrapositive of $p \to (q \to r)$, with no occurence of the connective $\to$

Sol^n converse: $(q \to r) \to p \Leftrightarrow (\neg q \vee r) \to p \Leftrightarrow \neg(\neg q \vee r) \vee p$

inverse: $\neg p \to \neg(q \to r) \Leftrightarrow \neg\neg p \vee \neg(q \to r) \Leftrightarrow p \vee \neg(\neg q \vee r)$

contrapositive $\neg(q \to r) \to \neg p \Leftrightarrow \neg\neg(q \to r) \vee \neg p \Leftrightarrow (q \to r) \vee \neg p$

$\Leftrightarrow (\neg q \vee r) \vee \neg p$

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Prove the following

(i) $[p \wedge (p \to q)] \Rightarrow q$, (ii) $[(p \to q) \wedge \neg q] \Rightarrow \neg p$

(i)

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

(i) we find that when both p and $p \to q$ are true then q is true.
Hence $[p \wedge (p \to q)] \Rightarrow q$

(ii)

p	q	$p \to q$	$\neg q$	$\neg p$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	1	0	0

From the table, when both $p \to q$ and $\neg q$ are true then $\neg p$ is true
Hence $[(p \to q) \wedge \neg q] \Rightarrow \neg p$

(iii) $[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

p	q	r	$p \to q$	$p \vee q$	$(p \vee q) \to r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

From the table, when p and $p \to q$ and r are true, then $(p \vee q) \to r$ is true, Hence
$[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

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(iv) $[[p \vee (q \vee r)] \wedge \neg q] \Rightarrow p \vee r$

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p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\neg q$	$p \vee r$
0	0	0	0	0	1	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	0	1
1	1	1	1	1	0	1

From the table when $p \vee (q \vee r)$ and $\neg q$ are true, then $p \vee r$ is true, hence
$[[p \vee (q \vee r)] \wedge \neg q] \Rightarrow p \vee r$.

(v) $\{[(p \wedge q) \rightarrow r] \wedge (\neg q) \wedge (p \rightarrow \neg r)\} \Rightarrow \neg p \vee \neg q$

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p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$\neg q$	$\neg r$	$p \rightarrow \neg r$	$\neg p$	$\neg p \vee \neg q$
0	0	0	0	1	1	1	1	1	1
0	0	1	0	1	1	0	1	1	1
0	1	0	0	1	0	1	1	1	1
0	1	1	0	1	0	0	1	1	1
1	0	0	0	1	1	1	1	0	1
1	0	1	0	1	1	0	0	0	1
1	1	0	1	0	0	1	1	0	0
1	1	1	1	1	0	0	0	0	0

From the table when $(p \wedge q) \rightarrow r$, $\neg q$ and $p \rightarrow \neg r$ are true, then $\neg p \vee \neg q$ is true, hence
$[[p \vee (q \vee r)] \wedge \neg q] \Rightarrow p \vee r$.

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Rules of Inference-

Consider a set of propositions $p_1, p_2, \ldots p_n$ and a proposition q. Then a compound proposition of the form $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is called an argument. Here $p_1, p_2, \ldots, p_n$ are called the premises of the argument and q is called a conclusion of the argument. It is a practice to write the above argument in the following tabular form:

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

The preceding argument is said to be valid if whenever each of the premises $p_1, p_2, \ldots, p_n$ is true, then the conclusion q is likewise true. In other words, the argument $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is valid when $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow q$

It is to be emphasized that in an argument, the premises are always taken to be true whereas the conclusion may be true or false. The conclusion is true only in the case of a valid argument.

There exists rule of logic which can be employed for establishing the validity of arguments. These rules are called the Rules of inference.

1. Rule of Conjuctive Simplification

This rule states that, for any two propositions p and q, if $p \wedge q$ is true, then p is true.

ie, $(p \wedge q) \Rightarrow p$

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2. **Rule of Disjunctive Amplification**
This rule states that, for any two propositions p and q, if p is true then $p \vee q$ is true.
i.e, $p \Rightarrow p \vee q$

3. **Rule of Syllogism**
This rule states that, for any three propositions p, q, r, if $p \to q$ is true and $q \to r$ is true, then $p \to r$ is true.
i.e, $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$
and is expressed in the following tabular form

$$\begin{array}{c} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

4. **Modus Ponens (Rule of Detachment)** (Method of affirming)
This rule states that if p is true and $p \to q$ is true, then q is true.
i.e, $[p \wedge (p \to q)] \Rightarrow q$
In tabular form,

$$\begin{array}{c} p \\ \underline{p \to q} \\ q \end{array}$$

5. **Modus Tollens** (Method of denying)
This rule states that if $p \to q$ is true and q is false, then p is false.
i.e, $[(p \to q) \wedge \neg q] \Rightarrow (\neg p)$
In tabular form,

$$\begin{array}{c} p \to q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

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6. Rule of Disjunctive Syllogism

This rule states that if $p \vee q$ is true and p is false, then q is true.

i.e. $[(p \vee q) \wedge \neg p] \Rightarrow q$

In tabular form,

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

7. Rule of Contradiction

This rule states that if $\neg p \to F_0$ is true, then p is true.

i.e. $(\neg p \to F_0) \Rightarrow p$

* Using the Rules of Inference, test whether the following are a valid argument.

(i) If sachin hits a century, then he gets a free car.
Sachin hits a century.
∴ Sachin gets a free car.

Solⁿ let p: Sachin hits a century.
q: Sachin gets a free car.

Then the given argument reads:

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

In view of Modus Ponens rule, this is a valid argument.

(ii) If Sachin hits a century, he gets a free car.
Sachin doesnot get a free car.
∴ Sachin has not hit a century.

Solⁿ let p: sachin hits a century } Then,
q: Sachin get a free car }.

$$\left.\begin{array}{l} p \to q \\ \underline{\neg q} \\ \therefore \neg p \end{array}\right\}$$ By Modus Tollens rule, it is a valid argument

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(iii) I will become famous or I will not become a musician
I will become a musician
∴ I will become famous.

Sol: Let p: I will become famous
q: I will become a musician.

Then, we have
$$\begin{array}{l} p \vee \neg q \\ \underline{q} \\ \therefore p \end{array}$$

~~but~~ $\cancel{p \vee \neg q \Rightarrow}$

but $p \vee \neg q \Leftrightarrow \neg q \vee p \Leftrightarrow q \rightarrow p$

∴ the argument is:
$$\begin{array}{l} q \rightarrow p \\ \underline{q} \\ \therefore p \end{array}$$

In view of the Modus Pones rule, the argument is valid.

(iv) If I study, then I do not fail in the examination
If I do not fail in the examination, my father gifts a two-wheeler to me.

∴ If I study then my father gifts a two-wheeler to me.

Soln Let p: I study.
q: I do not fail in the examination
r: My father gift a two-wheeler to me.

Then, the argument is
$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r . \end{array}$$

∴ In view of the Rule of syllogism, this is a valid argument

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(v) If it snows, then I will stay home
If I stay home, I will watch a movie
It is snowing
∴ I will watch a movie

Sol^n: let p: It snows.
q: I will stay home
r: I will watch a movie

∴ The argument is
$$\begin{array}{c} p \to q \\ q \to r \\ \underline{\quad p \quad} \\ \therefore r. \end{array}$$

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \quad \text{(by Rule of Syllogism)} \\ \underline{\quad p \quad} \\ \therefore r \quad \text{(by Modus Pones)} \end{array}$$

∴ The argument is valid.

(vi) If the meeting is cancelled, then I will go to the gym.
If I go to the gym, I will feel tired.
I do not feel tired. ∴ The meeting was not cancelled.

Sol^n let p: The meeting is cancelled
q: I ~~will~~ go to the gym
r: I feel tired.

∴ The argument is
$$\begin{array}{c} p \to q \\ q \to r \\ \underline{\neg r} \\ \therefore \neg p. \end{array}$$

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \text{ (by Rule of Syllogism)} \end{array} \qquad \begin{array}{l} p \to r \\ \underline{\neg r} \\ \therefore \neg p \text{ (by Modus Tollens)} \end{array}$$

∴ The argument is valid.

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(vii) If Sachin hits a century, he gets a free car.
Sachin gets a free car.
∴ Sachin has hit a century.

Sol: Let p: Sachin hits a century,
q: Sachin gets a free car

Then, the argument is:
$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

We note that if $p \to q$ and q are true, there is no rule which asserts that p must be true.
Therefore the given argument is not valid.

(viii) I will get grade A in this course or I will not graduate.
If I do not graduate, I will join the army
I got grade A
∴ I will not join the army.

Sol: let p: I ~~will~~ get grade A in this course
q: I do not graduate
r: I join the army.

Then the argument is?
$$\begin{array}{l} p \vee q \\ q \to r \\ p \\ \hline \therefore \neg r \end{array}$$

$p \vee q \Leftrightarrow q \vee p \Leftrightarrow \neg q \to p$ (conditional)
$q \to r \Leftrightarrow \neg r \to \neg q$ (contrapositive)
∴ The argument is
$$\begin{array}{l} \neg q \to p \\ \neg r \to \neg q \\ \hline \therefore \neg r \to p \end{array}$$ (by Rule of Syllogism)

$$\begin{array}{l} \neg r \to p \\ p \\ \hline \therefore \neg r \end{array}$$ ✗ There is no rule which asserts that $\neg r$ must be true. ∴ The given argument is not valid.

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(ix)

p
$p \to \neg q$
$\neg q \to \neg r$
$\therefore \neg r$

Sol^n $p \wedge (p \to \neg q) \wedge (\neg q \to \neg r)$
$\Leftrightarrow p \wedge (p \to \neg r)$ (Rule of Syllogism)
$\Leftrightarrow \neg r$ (Modus Pones).
$\therefore$ The given argument is valid.

(x)

$p \wedge q$
$p \to (q \to r)$
$\therefore r$.

Sol^n $(p \wedge q) \wedge (p \to (q \to r))$
$\Leftrightarrow p \wedge (p \to (q \to r))$ (Rule of Conjuctive Singlification)
$\Leftrightarrow q \to r$ (Modus Pones)
$\Leftrightarrow q \wedge (q \to r)$ [$\because p \wedge q \Rightarrow q$]
$\Leftrightarrow r$ [Modus Pones]
$\therefore$ The given argument is valid.

(xi)

$p \to r$
$q \to r$
$\therefore (p \vee q) \to r$

Sol^n $(p \to r) \wedge (q \to r)$
$\Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)$ (Conditional)
$\Leftrightarrow (r \vee \neg p) \wedge (r \vee \neg q)$ (Commutative)
$\Leftrightarrow r \vee (\neg p \wedge \neg q)$ (Distributive)
$\Leftrightarrow (\neg p \wedge \neg q) \vee r$ (Commutative)
$\Leftrightarrow \neg(p \vee q) \vee r$ (De Morgan's)
$\Leftrightarrow (p \vee q) \to r$ (Conditional)
By logical equivalence, the given argument is valid.

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(xii) $p \vee q$
$\neg p$
$q \to r$
$r \to s$
s

sol^n $(p \vee q) \wedge (\neg p) \wedge (q \to r) \wedge (r \to s)$
$\Leftrightarrow q \wedge (q \to r) \wedge (r \to s)$ (Disjunctive Syllogism)
$\Leftrightarrow r \wedge (r \to s)$ (Modus Pones)
$\Leftrightarrow s$ (Modus Pones)
$\therefore$ the given argument is valid

(xiii) $p \to q$
$r \to s$
$p \vee r$
$\therefore q \vee s$

sol^n $(p \to q) \wedge (r \to s) \wedge (p \vee r)$
$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\neg p \to r)$ (Conditional)
$\Leftrightarrow (p \to q) \wedge (\neg p \to r) \wedge (r \to s)$ (Commutative)
$\Leftrightarrow (p \to q) \wedge (\neg p \to s)$ (Rule of Syllogism)
$\Leftrightarrow (\neg q \to \neg p) \wedge (\neg p \to s)$ (Contrapositive)
$\Leftrightarrow (\neg q \to s)$ (Rule of Syllogism)
$\Leftrightarrow q \vee s$ (Conditional)
$\therefore$ the given argument is valid

(xiv) $\neg q$ follows logically from the premises:
$p \to (q \to r)$, $\neg r$, p.
sol^n $[p \to (q \to r)] \wedge (\neg r) \wedge p$
$\Leftrightarrow [p \to (q \to r)] \wedge p \wedge (\neg r)$ (Commutative)
$\Leftrightarrow (q \to r) \wedge (\neg r)$ (Modus Pones)
$\Leftrightarrow \neg q$ (Modus Tollens)
$\therefore$ the given argument is valid.

(xv) p follows from $(\neg p \vee \neg q) \to (r \wedge s)$, $r \to t$, $\neg t$
sol^n $[(\neg p \vee \neg q) \to (r \wedge s)] \wedge (r \to t) \wedge (\neg t)$
$\Leftrightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge (\neg r)$ (Modus Tollens)
$\Leftrightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge (\neg r \vee \neg s)$ (Disjunctive Amplification)
$\Leftrightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge [\neg (r \wedge s)]$ (De Morgan's)
$\Leftrightarrow \neg(\neg p \vee \neg q)$ (Modus Tollens)
$\Leftrightarrow p \wedge q$ (De Morgan's)
$\Leftrightarrow p$ (Conjuctive simplification)
$\therefore$ the given argument is valid.

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(xvi) $(\neg p \vee q) \rightarrow r$
$r \rightarrow (s \vee t)$
$\neg s \wedge \neg u$
$\neg u \rightarrow \neg t$
$\therefore p$

Sol. $[(\neg p \vee q) \rightarrow r] \wedge [r \rightarrow (s \vee t)] \wedge (\neg s \wedge \neg u) \wedge (\neg u \rightarrow \neg t)$
$\Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge (\neg s \wedge \neg u) \wedge (\neg u \rightarrow \neg t)$ (Syllogism)
$\Leftrightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge (\neg s) \wedge [\neg u \wedge (\neg u \rightarrow \neg t)]$ (associative)
$\Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge (\neg s) \wedge (\neg t)$ (Modus Ponens)
$\Leftrightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge [\neg (s \vee t)]$ (De Morgan's)
$\Rightarrow \neg(\neg p \vee q)$ (Modus Tollens)
$\Leftrightarrow p \wedge \neg q$ (De Morgan's)
$\Rightarrow p$ (conjunctive simplification)
$\therefore$ The argument is valid.

(xvii) $\{p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)\} \rightarrow (s \vee t)$

Sol^n Given
p
$p \rightarrow q$
$s \vee r$
$r \rightarrow \neg q$
$\therefore s \vee t$

Sol. $p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)$
$\Leftrightarrow p \wedge (p \rightarrow q) \wedge (\neg s \rightarrow r) \wedge (r \rightarrow \neg q)$ (conditional)
$\Rightarrow p \wedge (p \rightarrow q) \wedge (\neg s \rightarrow \neg q)$ (Syllogism)
$\Leftrightarrow p \wedge (p \rightarrow q) \wedge (q \rightarrow s)$ (Contrapositive)
$\Rightarrow p \wedge (p \rightarrow s)$ (Syllogism)
$\Rightarrow s$ (Modus Ponens)
$\Rightarrow s \vee t$ (Disjunctive Amplification)
$\therefore$ The argument is valid

(xviii)

(xix) Either I will go to the party or I will stay home
If I go to the party, I will have fun.
If I stay home, I will study.
∴ Either I will have fun or I will study

Sol.

$$
\begin{array}{lll}
p \vee q & (p \vee q) \wedge (p \to r) \wedge (q \to s) & \\
p \to r & \Leftrightarrow (\neg p \to q) \wedge (p \to r) \wedge (q \to s) & \text{(conditional)} \\
\underline{q \to s} & \Leftrightarrow (\neg p \to q) \wedge (q \to s) \wedge (p \to r) & \text{(Commutative)} \\
\therefore r \vee s & \Rightarrow (\neg p \to s) \wedge (p \to r) & \text{(Syllogism)} \\
 & \Leftrightarrow (\neg s \to p) \wedge (p \to r) & \text{(Contrapositive)} \\
 & \Rightarrow \neg s \to r & \text{(Syllogism)} \\
 & \Leftrightarrow s \vee r & \text{(Conditional)} \\
 & \Leftrightarrow r \vee s & \text{(Commutative)}
\end{array}
$$

∴ The argument is valid.

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Open Statements

Consider the declarative sentences such as (i) $x+3=6$, (ii) $x^2<10$, (iii) x divides 4, (iv) $x=\sqrt{2}$

These sentences are not propositions unless the symbol x is specified. Sentences of this kind are called open statements or open sentences, and the unspecified symbols, such as x in the sentences given above, are called free variables.

The sentence (i) becomes a proposition if x is replaced by any element of $\mathbb{R}$ (the set of real numbers). This sentences becomes a true proposition, if x is replaced by 3, and becomes a false proposition, if x is replaced by any other real number (other than 3). Here, we say $\mathbb{R}$ is a universe (or universe of disclosure) for the variable x in the sentence (i). Similarly $\mathbb{R}$ is a universe for x in sentences (ii), (iii) and (iv).

Open statements containing a variable x are denoted by $p(x)$, $q(x)$ etc. If U is the universe for the variable x in an open statement $p(x)$ and if $a \in U$, then the proposition got by replacing x by a in $p(x)$ is denoted by $p(a)$.

In the open statement: $p(x): x+3=6$, $p(3)$ is true, whereas $p(2)$ is false.

Hence an open statement $p(x)$ becomes a proposition only when x is replaced by a chosen element of the universe. The truth or falsity of the proposition $p(a)$ depends upon the element a of the universe that is chosen to replace x.

Compound open statements can be formed using the logical connectives. (i) $\neg p(x)$ is the negation of $p(x)$, (ii) $p(x) \wedge q(x)$ is conjunction (iii) $p(x) \vee q(x)$ is disjunction, (iv) $p(x) \rightarrow q(x)$ is conditional, (v) $p(x) \leftrightarrow q(x)$ is biconditional of $p(x)$ and $q(x)$.

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* Suppose the universe consists of all integers. Consider the following open statements:

$p(x): x \le 3$, $q(x): x+1$ is odd, $r(x): x > 0$.

Write the truth values of the following:

(i) $p(2)$, (ii) $\neg q(4)$, (iii) $p(-1) \wedge q(1)$, (iv) $\neg p(3) \vee r(0)$,
(v) $p(0) \rightarrow q(0)$, (vi) $p(1) \leftrightarrow \neg q(2)$, (vii) $p(4) \vee (q(1) \wedge r(2))$,
(viii) $p(2) \wedge (q(0) \vee \neg r(2))$

Sol

(i) $p(2): 2 \le 3$, ~~is~~ true.

(ii) ~~$\neg$~~$q(4)$: $4+1$ is odd ~~is~~ true $\therefore \neg q(4)$ is false.

(iii) $p(-1): -1 \le 3$ is true, $q(1): 1+1$ is odd is false
$\therefore p(-1) \wedge q(1)$ is false.

(iv) $p(3): 3 \le 3$ is true $\therefore \neg p(3)$ is false
$r(0): 0 > 0$ is false
$\therefore \neg p(3) \vee r(0)$ is false

(v) $p(0): 0 \le 3$ is true, $q(0): 0+1$ is odd is true
$\therefore p(0) \rightarrow q(0)$ is true

(vi) $p(1): 1 \le 3$ is true,
$q(2): 2+1$ is odd is true, $\therefore \neg q(2)$ is false
$p(1) \leftrightarrow \neg q(2)$ is false

(vii) $p(4): 4 \le 3$ is false, $q(1): 1+1$ is odd is false
$r(2): 2 > 0$ is true.
$\therefore q(1) \wedge r(2)$ is false.
$\therefore p(4) \vee (q(1) \wedge r(2))$ is false

(viii) $p(2): 2 \le 3$ is true, $q(0): 0+1$ is odd is true
$r(2): 2 > 0$ is true, $\neg r(2)$ is false
$q(0) \vee \neg r(2)$ is true.
$\therefore p(2) \wedge (q(0) \vee \neg r(2))$ is true.

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Quantifiers

Consider the following propositions:

1. All squares are rectangles.
2. For every integer x, x^2 is a non-negative integer
3. Some determinants are equal to zero.
4. There exists a real number whose square is equal to itself.

In these propositions, the words "all", "every", "some", "there exists" are associated with the idea of a quantity. Such words are called quantifiers.

The above propositions can be rewritten in alternative form as:

1. For all $x \in S$, x is a rectangle.
or $\forall x \in S, p(x)$,
where $\forall$ denotes the phrase "for all"
and $p(x)$ stands for the open statement "x is a rectangle".

Similarly

2. $\forall x \in Z, q(x)$
where $\forall$ is "for every"
$q(x)$ is x^2 is a non-negative integer

Here the symbol $\forall$ is called a universal quantifier

3. can be rewritten as for some D, x is equal to zero or symbolically $\exists x \in D, p(x)$, where $\exists$ denotes "for some" and $p(x)$ is "x is equal to zero".

Similary 4 is $\exists x \in R, q(x)$, where $\exists$ denotes "there exists" and $q(x)$ is "x is a real number whose square is equal to itself".

Here the symbol $\exists$ is called the existence quantifiers

A proposition involving the universal or the existential quantifier is called a quantified statement and the variable present in the quantified statement is called a bounded variable

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For the universe of all integers, let
$p(x): x > 0$, $q(x): x$ is even, $r(x): x$ is a perfect square, $s(x): x$ is divisible by 3, $t(x): x$ is divisible by 7.
Write down the following quantified statements in symbolic form.

(i) At least one integer is even.
(ii) There exists a positive integer that is even.
(iii) Some even integers are divisible by 3.
(iv) Every integer is either even or odd.
(v) If x is even and a perfect square, then x is not divisible by 3.
(vi) If x is odd or is not divisible by 7, then x is divisible by 3.

Sol^n

(i) $\exists x, q(x)$
(ii) $\exists x, [p(x) \wedge q(x)]$
(iii) $\exists x, [q(x) \wedge s(x)]$
(iv) $\forall x, [q(x) \vee \neg q(x)]$
(v) $\forall x, [\{q(x) \wedge r(x)\} \rightarrow \neg s(x)]$
(vi) $\forall x, [\{\neg q(x) \vee \neg t(x)\} \rightarrow s(x)]$

Truth value of a quantified Statement

Rule 1: The statement "$\forall x \in S, p(x)$" is true only when $p(x)$ is true for each $x \in S$.
Accordingly, to infer that a proposition of the form "$\forall x \in S, p(x)$" is false, it is enough to exhibit one element a of S such that $p(a)$ is false.

Rule 2: The statement "$\exists x \in S, p(x)$" is false only when $p(x)$ is false for every $x \in S$.
Accordingly, to infer that a "$\exists x \in S, p(x)$" is true it is enough to exhibit one element $a \in S$ such that $p(a)$ is true.

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Two Rules of Inference

Rule 3: If an open statement $p(x)$ is known to be true for all x in a universe S and if $a \in S$, then $p(a)$ is true. (This is known as the Rule of Universal Specification).

Rule 4: If an open statement $p(x)$ is proved to be true for any (arbitrary) x chosen from a set S, then the quantified statement, $\forall x \in S, p(x)$ is true. (This is known as the Rule of Universal Generalization)

Logical Equivalence

Two quantified statements are said to be logically equivalent whenever they have the same truth values in all possible situations.

(i) $\forall x, [p(x) \wedge q(x)] \iff (\forall x, p(x)) \wedge (\forall x, q(x))$

(ii) $\exists x, [p(x) \vee q(x)] \iff (\exists x, p(x)) \vee (\exists x, q(x))$

(iii) $\exists x, [p(x) \rightarrow q(x)] \iff \exists x, [\neg p(x) \vee q(x)]$

Rule for Negation of a Quantified statement

Rule 5: To construct the negation of a quantified statement, change the quantifier from universal to existential and vice-versa, and also replace the open statement by its negation.

ie, $\neg(\forall x, p(x)) \equiv \exists x, (\neg p(x))$

$$\neg(\exists x, p(x)) \equiv \forall x, (\neg p(x))$$

$\rightarrow$

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examples

1. "All equilateral triangles are isoceles" can be read as "$\forall x \in T, p(x)$" where T is the set of equilateral triangles, $p(x)$ is "x is isoceles".
The negation is "$\exists x \in T, \neg p(x)$".
In words it reads "For some equilateral triangle, x is not isoceles" OR, equivalently
"Some equilateral triangles are not isoceles."

2. "Some integers are even", can be read as "$\exists x \in Z, p(x)$", where Z is the set of integers, $p(x)$ is "x is even".
The negation is "$\forall x \in Z, \neg p(x)$".
In words it reads "For every integer x, x is not even", or equivalently
"For no integer x, x is even" OR
"All integers are not even".

3. "No ~~even~~ integer is divisible by 7"
which can be read as: "For any even integer x, x is not divisible by 7". ~~Symb~~
Symbolically, $\forall x \in E, \neg p(x)$
where E is set of all even integers
$p(x)$ is "x is divisible by 7"
The negation is $\exists x \in E, p(x)$,
In words it is "There exists an even integer divisible by 7" OR
"Some even integer is divisible by 7".

* Consider the open statements $p(x), q(x), r(x), s(x), t(x)$ given by $p(x): x > 0$, $q(x)$: x is even, $r(x)$: x is a perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7. Express each of the following symbolic statements in words and indicate its truth value. (i) $\forall x, [r(x) \rightarrow p(x)]$ (ii) $\exists x, [s(x) \wedge \neg q(x)]$ (iii) $\forall x, [\neg r(x)]$, (iv) $\forall x, [r(x) \vee t(x)]$.

Sol. (i) For ~~some~~ any integer x, if x is a perfect square, then $x > 0$. – false, (for $x = 0$).

(ii) For some integer x, x is divisible by 3 and x is not even. – true (for $x = 9$).

(iii) For any integer x, x is not a perfect square. – false. (for $x = 25$)

(iv) For any integer x, x is a perfect square or x is divisible by 7 – false (for $x = 8$)

* Consider the following open statements with the set of all real numbers as the universe.
$p(x): x \geq 0$, $q(x): x^2 \geq 0$, $r(x): x^2 - 3x - 4 = 0$, $s(x): x^2 - 3 > 0$.
Determine the truth values of the following statements (i) $\exists x, p(x) \wedge q(x)$ (ii) $\forall x, r(x) \vee s(x)$, (iii) $\forall x, q(x) \rightarrow s(x)$, (iv) ~~$\forall x, r(x) \rightarrow p(x)$~~

Sol. (i) There exists a real number x for which ~~both~~ x is positive and the square of x is positive. (~~eg~~ $x = 1$) – true, truth value is 1

(ii) For every real number x, $x^2 - 3x - 4 = 0$ or $x^2 - 3 > 0$ ($x^2 - 3x - 4 = 0 \Rightarrow x = 4, -1$, & $x^2 > 3$) – false (not true for $x = -1$), truth value is 0

(iii) For every $x \in R$, $q(x)$ is true, $s(x)$ is false $\therefore$ $\forall x\ q(x) \rightarrow s(x)$ is false. the truth value is 0.

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* Negate and simplify each of the following:

(i) $\exists x, [p(x) \vee q(x)]$, (ii) $\forall x, [p(x) \wedge \neg q(x)]$

(iii) $\forall x, [p(x) \to q(x)]$, (iv) $\exists x, [[p(x) \vee q(x)] \to r(x)]$

Solⁿ

(i) $\neg\{\exists x, [p(x) \vee q(x)]\} \equiv \forall x, \neg[p(x) \vee q(x)]$

$\equiv \forall x, [\neg p(x) \wedge \neg q(x)]$

(ii) $\neg\{\forall x, [p(x) \wedge \neg q(x)]\} \equiv \exists x, [\neg p(x) \vee q(x)]$

(iii) $\neg\{\forall x, [p(x) \to q(x)]\} \equiv \neg\{\forall x, [\neg p(x) \vee q(x)]\}$

$\equiv \exists x, [p(x) \wedge \neg q(x)]$

(iv) $\neg\{\exists x, [[p(x) \vee q(x)] \to r(x)]\} \equiv \neg\{\exists x, [(\neg p(x) \wedge \neg q(x)) \vee r(x)]\}$

$\equiv \forall x, [[p(x) \vee q(x)] \wedge \neg r(x)]$

* Let the set Z of all integers be the universe. Obtain the negation of the quantified statement $\exists x \in Z, [p(x) \wedge q(x)]$ for $p(x): 2x+1=5$ and $q(x): x^2=9$, and express it in words.

Solⁿ $\neg\{\exists x \in Z, [p(x) \wedge q(x)]\} \equiv \forall x \in Z, [\neg p(x) \vee \neg q(x)]$

In words: For all integers, $2x+1 \neq 5$ or $x^2 \neq 9$.

* Let Z be the universe. Given $p(x)$: x is odd and $q(x)$: x^2-1 is even, express the conditional "For any x, if x is odd, then x^2-1 is even", in symbolic form and negate it.

Solⁿ In symbolic form: $\forall x \in Z, [p(x) \to q(x)]$

Negation is $\neg\{\forall x \in Z, [p(x) \to q(x)]\} \equiv \neg\{\forall x, [\neg p(x) \vee q(x)]\}$

$\equiv \exists x, [p(x) \wedge \neg q(x)]$

In words: For some integer x, x is odd and x^2-1 is not even.

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* Write the following propositions in symbolic form and find its negation: "All integers are rational numbers and some rational numbers are not integers."

Sol: let $p(x)$: x is a rational number.
$q(x)$: x is an integer.
Z = set of all integer, Q = set of all rational numbers.

Then in symbolic form: $[\forall x \in Z, p(x)] \wedge [\exists x \in Q \; \neg q(x)]$

negation is:

$$\neg\{[\forall x \in Z, p(x)] \wedge [\exists x \in Q \; \neg q(x)]\}$$
$$\equiv [\exists x \in Z, \neg p(x)] \vee [\forall x \in Q, q(x)]$$

In words:
"Some integers are not rational numbers or every rational number is an integer".

* Write the following proposition in symbolic form, and find its negation: "If all triangles are right-angled, then no triangle is equiangular".

Sol: let T = set of all triangles.
$p(x)$: x is right angled, $q(x)$: x is equiangular.

In symbolic form: $[\forall x \in T, p(x)] \rightarrow [\forall x \in T, \neg q(x)]$

Negation is:

$$\neg\{[\forall x \in T, p(x)] \rightarrow [\forall x \in T, \neg q(x)]\}$$
$$\equiv [\forall x \in T, p(x)] \vee [\exists x \in T, q(x)]$$

In words:
"All triangles are right-angled and some triangles are equiangular".

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* Consider the following open statements with the set of all real numbers as the universe. $p(x): |x| > 3$, $q(x): x > 3$. Find the truth value of the statement, $\forall x, [p(x) \rightarrow q(x)]$.
Also write the converse, inverse and the contrapositive of this statement and find their truth values.

Sol Given $p(x): |x| > 3$, $q(x): x > 3$

$p(-4) \equiv |-4| = 4 > 3$ is: ~~p(-4)~~ ~~is~~ true.

$q(-4) \equiv -4 > 3$ is false.

$\therefore$ $p(x) \rightarrow q(x)$ is false for $x = -4$.

$\therefore$ $\forall x [p(x) \rightarrow q(x)]$ is false (for $x = -4$ its false)

(i) Converse is: $\forall x, [q(x) \rightarrow p(x)]$

In words: "For every real number x, if $x > 3$ then $|x| > 3$"

This is a true statement.

(ii) Inverse is: $\forall x, [\neg p(x) \rightarrow \neg q(x)]$

In words: "For every real number x, if $|x| \leq 3$, then $x \leq 3$.

This is a true statement.

(iii) Contrapositive is: $\forall x, [\neg q(x) \rightarrow \neg p(x)]$

In words: "For every real number x, if ~~|x|~~ $x \leq 3$, then $|x| \leq 3$"

This is a false statement.

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Logical implication involving Quantifiers

A quantified statement P is said to logically imply a quantified statement Q if Q is true whenever P is true. Then we write $P \Rightarrow Q$.

Given a set of quantified statements $P_1, P_2, \ldots, P_n$ and Q, we say that Q is a valid conclusion from the premises $P_1, P_2, \ldots, P_n$ or $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ is a valid argument if Q is true whenever each of $P_1, P_2, \ldots, P_n$ is true, or equivalently if $P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n \Rightarrow Q$.

The validity of an argument involving quantified statements is analysed on the basis of the laws of logic and the rules of inference.

* Prove the following:

(i) $\forall x, p(x) \Rightarrow \exists x, p(x)$, (ii) $\forall x, [p(x) \vee q(x)] \Rightarrow \forall x, p(x) \vee \exists x, q(x)$

Sol[n] (i) $\forall x, p(x) \Rightarrow p(x)$ is true for every $x \in U$
$\Rightarrow p(a)$ is true for $x = a \in U$
$\Rightarrow p(x)$ is true for some $x \in U$
$\Rightarrow \exists x, p(x)$.

(ii) $\forall x, [p(x) \vee q(x)] \rightarrow p(x) \vee q(x)$ is true for every $x \in U$.
$\rightarrow$ [$p(x)$ is true for every $x \in U$],
or [$q(x)$ is true for every $x \in U$]
$\Rightarrow \forall x, p(x) \vee q(a)$ is true for $a \in U$
$\Rightarrow \forall x, p(x) \vee \exists x, q(x)$

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* Prove that the statement $\exists x, q(x)$ follows logically from the premises $\forall x, p(x) \to q(x)$ and $\exists x, p(x)$.

Sol[n]:
$$[\exists x, p(x)] \wedge [\forall x, p(x) \to q(x)]$$
$$\Rightarrow p(a) \wedge [p(a) \to q(a)]$$
$$\Rightarrow q(a) \quad \text{(Modus Ponens)}$$
$$\Rightarrow \exists x, q(x).$$

* Prove that the following argument is valid:

$$\forall x, [p(x) \to q(x)]$$
$$\underline{\forall x, [q(x) \to r(x)]}$$
$$\therefore \forall x, [p(x) \to r(x)].$$

Sol[n]:
$$[\forall x, [p(x) \to q(x)]] \wedge [\forall x, [q(x) \to r(x)]]$$
$$\Rightarrow [p(a) \to q(a)] \wedge [q(a) \to r(a)]$$
$$\Rightarrow p(a) \to r(a) \quad \text{[Syllogism]}$$
$$\Rightarrow \forall x, [p(x) \to r(x)], \text{ by the Rule of Universal Generalization.}$$

* Establish the validity of the following argument:

$$\forall x, [p(x) \vee q(x)]$$
$$\underline{\forall x, [[\neg p(x) \wedge q(x)] \to r(x)]}$$
$$\therefore \forall x, [\neg r(x) \to p(x)]$$

Sol[n]:
$$\{\forall x, [p(x) \vee q(x)]\} \wedge \{\forall x [[\neg p(x) \wedge q(x)] \to r(x)]\}$$
$$\Leftrightarrow \{\forall x, [p(x) \vee q(x)]\} \wedge \{\forall x [\neg r(x) \to \neg[\neg p(x) \wedge q(x)]]\} \quad \text{(Contrapositive)}$$
$$\Leftrightarrow \{\forall x, [p(x) \vee q(x)]\} \wedge \{\forall x [\neg r(x) \to [p(x) \vee \neg q(x)]]\} \quad \text{(De Morgan's)}$$
$$\Rightarrow \{\forall x, [p(x) \vee q(x)]\} \wedge \forall x\, [p(x) \vee \neg q(x)] \quad \text{(Modus Ponens) (assuming } \neg r(x) \text{ is true)}$$
$$\Leftrightarrow \forall (x), [p(x) \vee [q(x) \wedge \neg q(x)]] \quad \text{(Distributive).}$$
$$\Leftrightarrow \forall (x), [p(x) \vee F_0] \quad \text{(Inverse)}$$
$$\Leftrightarrow \forall (x), p(x) \quad \text{(Identity)}$$

Thus when $\neg r(x)$ is true, the given premises implies $p(x)$

$\therefore \neg r(x) \to p(x)$ is true

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* Establish the validity of the following argument:

$\forall x, [p(x) \to [q(x) \wedge r(x)]]$

$\forall x, [p(x) \wedge s(x)]$

$\therefore \forall x, [r(x) \wedge s(x)]$.

Sol: Take any a from the universe.

Then $[p(a) \to [q(a) \wedge r(a)]] \wedge [p(a) \wedge s(a)]$

$\Leftrightarrow p(a) \wedge [p(a) \to [q(a) \wedge r(a)]] \wedge s(a)$ (Commutative)

$\Rightarrow [q(a) \wedge r(a)] \wedge s(a)$ (Modus Ponens)

$\Rightarrow q(a) \wedge [r(a) \wedge s(a)]$ (Associative)

$\Rightarrow r(a) \wedge s(a)$ (Conjunctive simplification)

$\Rightarrow \forall x, [r(x) \wedge s(x)]$ (Universal Generalization)

This proves that the given argument is valid.

* Find whether the following argument is valid:

No engineering student of First or Second semester studies Logic.

Anil is an engineering student who studies Logic.

$\therefore$ Anil is not in Second Semester.

Sol: $p(x)$: x is in First semester, $q(x)$: x is in Second Semester

$r(x)$: x studies logic, a: Anil.

The given argument reads:

$\forall x, [[p(x) \vee q(x)] \to \neg r(x)]$

$r(a)$

$\therefore \neg q(a)$

$\forall x [[p(x) \vee q(x)] \to [\neg r(x)]] \wedge r(a)$

$\Rightarrow [[p(a) \vee q(a)] \to \neg r(a)] \wedge r(a)$

$\Rightarrow r(a) \wedge [[p(a) \vee q(a)] \to \neg r(a)]$ (commutative)

$\Rightarrow r(a) \wedge [r(a) \to \neg [p(a) \vee q(a)]]$ (Contrapositive)

$\Rightarrow \neg [p(a) \vee q(a)]$ (Modus Ponens)

$\Rightarrow \neg p(a) \wedge \neg q(a)$ (De Morgan's)

$\Rightarrow \neg q(a)$ (Conjuctive simplification)

$\therefore$ the given argument is valid.

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* Prove the following argument is valid:

$\forall x, [p(x) \vee q(x)]$
$\exists x, \neg p(x)$
$\forall x, [\neg q(x) \vee r(x)]$
$\forall x, [s(x) \rightarrow \neg r(x)]$
$\therefore \exists x, \neg s(x)$

Solⁿ $[\forall x, [p(x) \vee q(x)]] \wedge [\exists x, \neg p(x)] \wedge [\forall x, [\neg q(x) \vee r(x)]] \wedge [\forall x, [s(x) \rightarrow \neg r(x)]]$

$\Rightarrow [p(a) \vee q(a)] \wedge [\neg p(a)] \wedge [\neg q(a) \vee r(a)] \wedge [s(a) \rightarrow \neg r(a)]$

$\Rightarrow q(a) \wedge [\neg q(a) \vee r(a)] \wedge [s(a) \rightarrow \neg r(a)]$ (Disjunctive Syllogism)

$\Rightarrow r(a) \wedge [s(a) \rightarrow \neg r(a)]$ (Disjunctive Syllogism)

$\Rightarrow \neg s(a)$ (Modus Tollens)

$\Rightarrow \exists x, \neg s(x)$

This proves that the given argument is valid.

* Find whether the following argument is valid:

If a triangle has two equal sides, then it is isoceles.
If a triangle is isoceles, then it has two equal angles.
A certain triangle ABC does not have two equal angles.
$\therefore$ The triangle ABC does not have two equal sides.

Solⁿ let the universe be the set of all triangles,
let $p(x)$: x has equal sides, $q(x)$: x is isoceles,
$r(x)$: x has two equal angles. C: triangle ABC

The given argument is: $\forall x, [p(x) \rightarrow q(x)]$
$\forall x, [q(x) \rightarrow r(x)]$
$\neg r(C)$
$\therefore \neg p(C)$

$[\forall x, [p(x) \rightarrow q(x)]] \wedge [\forall x, [q(x) \rightarrow r(x)]] \wedge [\neg r(C)]$

$\Rightarrow [p(C) \rightarrow q(C)] \wedge [q(C) \rightarrow r(C)] \wedge [\neg r(C)]$ [Universal Specification]

$\Rightarrow [p(C) \rightarrow r(C)] \wedge [\neg r(C)]$ [Modus Ponens]

$\Rightarrow \neg p(C)$ (Modus Tollens).

This proves that the given argument is valid.

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* Find whether the following is a valid argument for which the universe is the set of all students.

No Engineering student is bad in studies
Anil is not bad in studies
∴ Anil is an Engineering student.

Sol: let $p(x)$: x is an engineering student,
$q(x)$: x is bad in studies

Then the argument is
$$\begin{array}{l} \forall x\,[p(x) \rightarrow \neg q(x)] \\ \neg q(a) \\ \hline \therefore\ p(a) \end{array}$$

$\forall x\,[p(x) \rightarrow \neg q(x)] \wedge \neg q(a)$
$\Rightarrow [p(a) \rightarrow \neg q(a)] \wedge \neg q(a)$
$\not\Rightarrow p(a)$

because $p(a)$ can be false when both $p(a) \rightarrow \neg q(a)$ and $\neg q(a)$ are true.

As such the given argument is not valid.

* Prove that $\exists x, [p(x) \wedge q(x)] \Rightarrow \exists x, p(x) \wedge \exists x\, q(x)$. Is the converse true?

Sol^n let S denote the universe.

$\exists x, [p(x) \wedge q(x)] \Rightarrow p(a) \wedge q(a)$, for some $a \in S$.
$\Rightarrow p(a)$ for some $a \in S$ and $q(a)$ for some $a \in S$.
$\Rightarrow \exists x, p(x) \wedge \exists x\, q(x)$

∴ the required implification follows.

Next consider $\exists x, p(x) \wedge \exists x\, q(x)$

$\exists x, p(x) \Rightarrow p(a)$ for some $a \in S$
and $\exists x, q(x) \Rightarrow q(b)$ for some $b \in S$

∴ $\exists x, p(x) \wedge \exists x\, q(x) \Rightarrow p(a) \wedge q(b)$
$\not\Rightarrow p(a) \wedge q(a)$ [∵ b need not be same as a]

Thus $\exists x, [p(x) \wedge q(x)]$ need not be true when $\exists x, p(x) \wedge \exists x\, q(x)$ is true.

i.e, $\exists x, p(x) \wedge \exists x, q(x) \not\Rightarrow \exists x, [p(x) \wedge q(x)]$

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Methods of Proof and Methods of Disproof

The propositions that commonly appear in mathematical discussions are conditionals of the form $p \to q$, where p and q are single or compound propositions which may involve quantifiers as well. Given such a conditional, the process of establishing that the conditional is true by using the rules/laws of logic and other known facts constitutes a proof of the conditional.

The process of establishing that a proposition is false constitutes a disproof.

Direct Proof:

The direct method of proving a conditional $p \to q$ has the following lines of argument:

1. Hypothesis: First assume that p is true.
2. Analysis: Starting with the hypothesis and employing the rules/laws of logic and other known facts, infer that q is true.
3. Conclusion: $p \to q$ is true.

* Give a direct proof of the statement: "The square of an odd integer is an odd integer".

Sol^n The conditional to be proved is:

"If n is an odd integer, then n^2 is an odd integer."

Assume that n is an odd integer (hypothesis)

Then $n = 2k+1$ for some integer k.

Consequently $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

Observe that RHS is not divisible by 2

Therefore, n^2 is not divisible by 2.

ie, n^2 is an odd integer (Conclusion)

The given statement is thus proved by direct proof.

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* Prove that for all integers k and l, if k and l are both odd, then $k+l$ is even and kl is odd.

Sol^n Take any two integers k and l, and assume that both of these are odd (hypothesis)
Then $k=2m+1$, $l=2n+1$ for some integers m and n.
$\therefore$ $k+l=(2m+1)+(2n+1) = 2(m+n+1)$
and $kl = (2m+1)(2n+1) = 4mn + 2(m+n)+1$.
Observe that $k+l$ is divisible by 2
and kl is not divisible by 2
$\therefore$ $k+l$ is an even integer. and kl is an odd integer.

Indirect Proof

We know that the conditional $p \to q$ and its contrapositive $\neg q \to \neg p$ are logically equivalent. In some situations, given a conditional $p \to q$, a direct proof of the contrapositive $\neg q \to \neg p$ is easier. On the basis of this proof, we infer that the conditional $p \to q$ is true. This method of proving a conditional is called an indirect method of proof.

* Let n be an integer. Prove that if n^2 is odd, then n is odd.

Sol^n Here, the conditional to be proved is $p \to q$, where p: n^2 is odd, q: n is odd.
Consider: $\neg q \to \neg p$: If n is even then n^2 is even
assume $\neg q$ is true, ie, n is even,
then $n = 2k$, where k is an integer.
Consequently $n^2 = (2k)^2 = 4k^2$ which is divisible by 2.
$\therefore$ n^2 is even. ie, $\neg p$ is true. This prove $\neg q \to \neg p$ is true.
$\therefore$ $\neg q \to \neg p$ is true serves as an indirect proof of $p \to q$.

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* Give an indirect proof of the statement:
"The product of two even integers is an even integer."

Sol^n The given statement is equivalent to
"If a and b are even integers, then ab is an even integer",
i.e. $p \to q$, where p: a and b are even integers.
q: ab is an even Integer.

The contrapositive $\neg q \to \neg p$ is:
If ab is an odd integer, then a and b are odd integers.

assume that $\neg q$ is true.
i.e. ab is an odd integer.
$\Rightarrow$ ab is not divisible by 2
$\Rightarrow$ a is not divisible by 2 and b is not divisible by 2
$\Rightarrow$ a and b are odd integers.
$\Rightarrow$ $\neg p$ is true.
$\therefore$ $\neg q \to \neg p$ is true $\Rightarrow p \to q$ is true.

* Provide an indirect proof of the following statement
"For all positive real numbers x and y, if the product xy exceeds 25, then $x>5$ or $y>5$.

Sol^n The given statement reads: $p \to (q \vee r)$.
where $p: xy > 25$, $q: x>5$, $r: y>5$
The contrapositive is: $(\neg q \wedge \neg r) \to \neg p$.
let $(\neg q \wedge \neg r)$ be true, i.e. $x \leq 5$ and $y \leq 5$.
then this implies $xy \leq 25$.
$\therefore$ $\neg p$ is true.
$\therefore$ $(\neg q \wedge \neg r) \to \neg p$ is true.
$\Rightarrow$ $p \to (q \vee r)$ is true.

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Proof by contradiction

The indirect method of proof is equivalent to what is known as the Proof by contradiction. The lines of argument in this method of proof of the statement $p \to q$ are as follows:

1. Hypothesis: Assume that $p \to q$ is false.
   ie, assume that p is true and q is false.
2. Analysis: Starting with the hypothesis that q is false and employing the rules of logic and other know facts, infer that p is false. This contradicts the assumption that p is true.
3. Conclusion: Because of the contradiction arrived in the analysis, we infer that $p \to q$ is true.

* Provide a proof by contradiction of the following statement: For every integer n, if n^2 is odd, then n is odd.

Soln: Let n be an Integer. Then the given statement reads $p \to q$, where $p: n^2$ is odd, and $q: n$ is odd.

Assume that $p \to q$ is false, that is, assume that p is true and q is false.

Now, q is false means: n is even, so that $n = 2k$ for some integer k. This yields $n^2 = (2k)^2 = 4k^2$, from which it is evident that n^2 is even; ie, p is false. This contradicts the assumption that p is true. In view of this contradiction, we infer that the given conditional $p \to q$ is true.

* Prove that if m is even integer, then $m+7$ is an odd integer.

Soln: Given $p \to q$, where $p: m$ is even, $q: m+7$ is odd.

Assume $p \to q$ is false, ie, p is true and q is false.

q is false $\Rightarrow m+7 = 2k \Rightarrow m = 2k - 7 \Rightarrow m = 2k - 8 + 1 \Rightarrow m = 2(k-4)+1$ which is odd.

This contradics m is even, ie. contradicts p is true.

$\therefore$ the assumption $p \to q$ is false is wrong, hence $p \to q$ is true.

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* Prove that there is no rational number whose square is 2.

Sol^n Let Q denote the set of all rational numbers.
Then we need to prove: $\forall x \in Q, p \to q$,
where p: x is rational number and q: $x^2 \neq 2$
Assume $p \to q$ is false, ie, p is true and q is false
ie, x is a rational number and $x^2 = 2$.
since x is rational, $x = \frac{a}{b}$ where $a, b (\neq 0)$ are integers, which have no common factors.
Since $x^2 = 2 \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a$ is even
$\Rightarrow a = 2n$ for some integer n.
$\Rightarrow 2b^2 = (2n)^2 \Rightarrow b^2 = 2n^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even
Since a and b are both even, hence have common factor 2.
This is a contradiction to the assumption that a & b have no common factors.
Hence our assumption is wrong & $p \to q$ is true.

* Give a proof by contradiction for the following statement: "If n is an odd integer, then $n+9$ is an even integer".

Sol^n $p \to q$; p: n is an odd integer
q: $n+9$ is an even integer.
assume $p \to q$ is false ie, p is true and q is false
q is false $\Rightarrow$ $n+9$ is an odd integer
ie, $n+9 = 2k+1 \Rightarrow n = 2k-8 \Rightarrow n = 2(k-4)$
which shows n is even, This contradicts that n is odd. Hence the assumption $p \to q$ is false is wrong. Hence the given statement is true.

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Proof by Exhaustion

The quantified statement "$\forall x \in S, p(x)$" is true if $p(x)$ is true for every (each) x in S. If S consists of only a limited number of elements, we can prove that the statement "$\forall x \in S, p(x)$" is true by considering $p(a)$ for each a in S and verifying that $p(a)$ is true (in each case). Such a method of proof is called the method of exhaustion.

* Prove that every even integer n with $2 \le n \le 26$ can be written as a sum of atmost three perfect squares.

* Let $S = \{2, 4, 6, \ldots, 24, 26\}$.

We observe that $2 = 1^2 + 1^2$, $4 = 2^2$, $6 = 2^2 + 1^2 + 1^2$, $8 = 2^2 + 2^2$, $10 = 3^2 + 1^2$, $12 = 2^2 + 2^2 + 2^2$, $14 = 3^2 + 2^2 + 1^2$, $16 = 4^2$, $18 = 4^2 + 1^2 + 1^2$, $20 = 4^2 + 2^2$, $22 = 3^2 + 3^2 + 2^2$, $24 = 4^2 + 2^2 + 2^2$, $26 = 5^2 + 1^2$.

The above facts verify that each x is S is a sum of at most three perfect squares.

Proof by Existence

The quantified statement "$\exists x \in S, p(x)$" is true if any one element $a \in S$ such that $p(a)$ is true is exhibited. Hence the best way of providing a proposition of the form "$\exists x \in S, p(x)$" is to exhibit the existence of one $a \in S$ such that $p(a)$ is true. This method of proof is called Proof of existence.

* Prove that there exists a real number x such that $x^3 + 2x^2 - 5x - 6 = 0$

Sol: For $x = -1$, $x^3 + 2x^2 - 5x - 6 = -1 + 2 + 5 - 6 = 0$.

Hence the proof.

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Disproof by Contradiction

Suppose we wish to disprove a conditional $p \to q$. For this purpose, we start with the hypothesis that p is true and q is true, and end up with a contradiction. In view of the contradiction, we conclude that the conditional $p \to q$ is false. This method of disproving $p \to q$ is called Disproof ~~of~~ by Contradiction.

* Disprove the statement: "The sum of two odd integers is an odd integer".

Sol^n Here the proposition to be disproved is $p \to q$, where p: a and b are odd integers and q: $a+b$ is odd integer.

Assume that p is true and q is true.

Then $a = 2k_1 + 1$, $b = 2k_2 + 1$ — (1) and $a+b = 2k_3 + 1$ — (2)

From (1) $a+b = 2k_1 + 1 + 2k_2 + 1 = 2(k_1 + k_2 + 1)$

$\Rightarrow a+b$ is even integer.

This contradicts the assumption (2)

In view of this contradiction, we infer that $p \to q$ is false. This disproves the given statement.

Disproof by Counterexample

The proposition of the form "$\forall x \in S, p(x)$" is false if any one element $a \in S$ such that $p(a)$ is false is exhibited. Hence the best way to disproving a proposition involving the universal quantifier is to exhibit just one case where the proposition is false. This method of disproof is called Disproof by counterexample.

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* Disprove the proposition: The product of any odd integers is a perfect square.

Sol^n Note that $m=3$ and $n=5$ are odd integers, but $mn=15$ is not a perfect square.
Thus the given proposition is disproved, with $m=3, n=5$ serving as counter examples

* Disprove the proposition: If m and n are positive integers which are perfect squares, then $m+n$ is a perfect square.

Sol^n Note that $m=9$ and $n=4$ are perfect squares, but $m+n=14$ is not a perfect square.
Therefore the given statement is not true. It is disproved through the counterexample $m=9$, $n=4$.

* Disprove that the sum of squares of any four non-zero integers is an even integer.

Sol^n Here, the proposition is:
For any four non-zero integers a, b, c, d,
$a^2+b^2+c^2+d^2$ is an even integer.
Note that for $a=1, b=1, c=1, d=2$, the proposition is false.
Thus the given proposition is not a true proposition. The proposition is disproved through the counter example $a=1, b=1, c=1$ and $d=2$.

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