

Ordered n-tuple.

An ordered pair of two elements consists of two elements say 'a' and 'b', in a given fixed order and denoted by (a, b) . Here 'a' is called the first element and 'b' is called a second element.

Two ordered pairs (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$. i.e., $(a, b) = (c, d) \Leftrightarrow a = c, b = d$

Similarly an ordered triplet is defined and is denoted as (a, b, c) .

In general an ordered n-tuple is defined and denoted by $(x_1, x_2, x_3, \dots, x_n)$.

Cartesian Product

Let A and B be any two sets. The cartesian product of the sets A and B, denoted by $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

i.e., $A \times B = \{(a, b) \mid a \in A, b \in B\}$

In particular, $A \times A = \{(a, a) \mid a \in A, a \in A\}$

Note

1. In general $A \times B \neq B \times A$

2. If the set A has n elements and the set B has m elements, then the set $A \times B$ has mn elements

3. If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$

Note

1. The product set $\mathbb{R} \times \mathbb{R}$, where \mathbb{R} is the set of all real numbers is called Euclidean plane and this set is also denoted by \mathbb{R}^2 .

2 The product set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, where \mathbb{R} is the set of all real numbers is called Euclidean space of three dimension and is also denoted by \mathbb{R}^3 .

example

If $A = \{1, 2\}$ and $B = \{3, 1, 4\}$,

$$\text{then } A \times B = \{(1, 3), (1, 1), (1, 4), (2, 3), (2, 1), (2, 4)\}$$

$$B \times A = \{(3, 1), (3, 2), (1, 1), (1, 2), (4, 1), (4, 2)\}$$

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$B \times B = \{(3, 3), (3, 1), (3, 4), (1, 3), (1, 1), (1, 4), (4, 3), (4, 1), (4, 4)\}$$

Results related to Cartesian products

$$1. A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$2. A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$3. \text{ If } B \subset A \text{ then } B \times B = (B \times A) \cap (A \times B)$$

4. Let A and B be two sets. Show that the

The sets $A \times B$ and $B \times A$ have an element in common if and only if the set A and B have an element in common.

example

$$\text{If } A = \{1, 2, 3\}, B = \{4, 3\}, C = \{1, 2\}$$

$$\text{then } A \cap B = \{3\}, A \cap C = \{1, 2\}, B \cap C = \emptyset$$

$$A \cup B = \{1, 2, 3, 4\}, A \cup C = \{1, 2, 3\}, B \cup C = \{1, 2, 3, 4\}$$

$$A \times B = \{(1, 4), (1, 3), (2, 4), (2, 3), (3, 4), (3, 3)\}$$

$$A \times C = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$$

$$A \times (B \cap C) = \emptyset = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$C \times A, C \times C = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$(C \times A) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$

$$A \times (C \times A) \cap (A \times C) = \{(1, 1), (1, 2), (2, 1), (2, 2)\} = C \times C$$

Relations

Let A and B be two sets. A relation R from A to B is a subset of $A \times B$.

If A is any set then a relation R on A is a subset of $A \times A$.

If R is a relation from A to B and $(a, b) \in R$, then we say "a is related to b under the relation R " or we say "a is R -related to b" and we denote it by aRb .

If $(a, b) \notin R$, then we write $a \not R b$ i.e., 'a is not related to b under the relation R '.

examples:

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 5, 7\}$

then $R = \{(a, b) \mid a < b\}$ is a subset of $A \times B$,
 $\alpha, R = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 5), (3, 7)\}$
 is a relation from A to B . $(4, 5), (4, 7)\}$

2. Let A be the set of all integers and B be the set of all non negative integers.

Then $R = \{(a, a^2) \mid a \in A\}$ is a relation from A to B

Domain and Range of a relation

Let $R \subseteq A \times B$ be a relation from A to B .

Then the set $\{a \mid (a, b) \in R\}$ is called the domain of the relation R . The set $\{b \mid (a, b) \in R\}$ is called the range of the relation R .

example: Consider the relation R defined on the set N of natural numbers by $R = \{(a, b) \mid a, b \in N \text{ & } a+b=7\}$.
 Then $R = \{(1, 6), (2, 5), (3, 4)\}$

Thus Domain = $\{1, 2, 3\}$, Range = $\{5, 3, 1\}$.

Types of Relations

1. Identity relation

The relation denoted by I_A on a set A is called the identity relation on A , if

$$I_A = \{ (x, y) \mid x = y \} = \{ (x, x) \mid x \in A \}$$

2. Universal relation

The relation $R = A \times A$ on A is called universal relation on A .

3. Void relation

Since the null set \emptyset is also a subset of $A \times A$, the null set \emptyset is also a relation of the set A and is called the void relation or null relation.

4. Inverse relation

Let R be a relation from the set A into set B . The relation denoted by R^{-1} from B to A and defined by $R^{-1} = \{ (b, a) \mid (a, b) \in R \}$ is called the inverse relation of R .

5. Reflexive relation

A relation R on a set A is said to be reflexive, if $\forall x \in A, (x, x) \in R$.

6. Irreflexive relation

A relation R on a set A is said to be irreflexive if for every $x \in A, (x, x) \notin R$.

7. Symmetric relation.

A relation R defined on a set A is called symmetric relation, if $(x, y) \in R \Rightarrow (y, x) \in R$.

8. Anti-symmetric relation

A relation R on a set A is called anti-symmetric if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$

9. Transitive relation

A relation R on a set A is said to be transitive if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in A$.

10. Equivalence relation

A relation R defined on a set A is called an equivalence relation, if it is reflexive, symmetric and transitive.

* Let $A = \{1, 2, 3\}$. Find the nature of the relations on A given below:

- (i) $R_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$
- (ii) $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$
- (iii) $R_3 = \{(1, 1), (2, 2), (3, 3)\}$
- (iv) $R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$
- (v) $R_5 = \{(1, 2), (3, 2)\}$
- (vi) $R_6 = \{(2, 3), (3, 1), (2, 1)\}$

Soln

R_1 - symmetric and irreflexive

R_2 - reflexive and transitive

R_3 - reflexive and symmetric

R_4 - reflexive and symmetric

R_5 - irreflexive

R_6 - irreflexive, transitive.

* Give an example of a relation defined on a suitable set which is:

- i) reflexive, symmetric and transitive.
- ii) reflexive, symmetric but not transitive
- iii) reflexive, transitive but not symmetric
- iv) symmetric, transitive but not reflexive.
- v) reflexive but not symmetric and not transitive
- vi) symmetric but not reflexive and not transitive
- vii) transitive but not reflexive and not symmetric
- viii) not reflexive, not symmetric and not transitive
- ix) reflexive, anti-symmetric and transitive.

~~Q1~~ Consider $A = \{a, b, c\}$

i) Consider $R_1 = \{(a, a), (b, b), (c, c)\}$.

ii) $R_2 = \{(a, a)\}$

iii) $R_3 = \{(a, a), (b, b), (c, c), (a, b)\}$

iv) $R_4 = \{(c, c)\}$

v) $R_5 = \{(a, a), (b, b), (c, c), (a, b), (c, a)\}$.

vi) $R_6 = \{(a, a), (c, c), (a, b), (b, a)\}$

vii) $R_7 = \{(a, a), (b, b), (a, b)\}$

viii) $R_8 = \{(a, b), (b, c)\}$

ix) $R_9 = \{(a, a), (b, b), (c, c)\}$

* Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on A defined by $(a, b) \in R$ if and only if a is a multiple of b . Write R as a set of ordered pairs.

~~Q1~~ $R = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$

* A relation R on the set of all integers \mathbb{Z} is defined recursively by i) $(0, 0) \in R$ and ii) if $(s, t) \in R$, then $(s+1, t+7) \in R$. Find R as a set of ordered pairs.

~~Q1~~ $R = \{(0, 0), (1, 7), (2, 14), (3, 21), (4, 28), \dots\}$

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* Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. On this set define the relation R by $(x, y) \in R$ if and only if $x-y$ is a multiple of 5. Verify that R is an equivalence relation.

Sol) For any $x \in A$, $x-x=0$ which is a multiple of 5.
 $\therefore (x, x) \in R$, hence R is reflexive.

For any $x, y \in A$, if $(x, y) \in R$ then $x-y=5k$ for some integer k .

Consequently $y-x=5(-k)$

$\therefore (y, x) \in R$, hence R is symmetric.

For any $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$,
then $x-y=5k_1$ and $y-z=5k_2$ for some integers k_1 and k_2 .

$$\begin{aligned} \text{Consequently } x-z &= (x-y)+(y-z) \\ &= 5k_1 + 5k_2 \\ &= 5(k_1 + k_2). \end{aligned}$$

$\therefore (x, z) \in R$, hence R is transitive.

Therefore R is an equivalence relation.

C * Let S be the set of all non-zero integers and $A=S \times S$. On A define the relation R by $(a, b)R(c, d)$ if and only if $ad=bc$. Show that R is an equivalence relation.

Sol) $(a, a)R(a, a)$ as $aa=aa$. $\therefore R$ is reflexive

Suppose $(a, b)R(c, d)$,

$$\text{Then } ad=bc \Rightarrow cb=da \Rightarrow (c, d)R(b, a)$$

$\therefore R$ is symmetric.

Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$.

Then $ad=bc$ and $cf=df$

$$\Rightarrow d=\frac{bc}{a}, \text{ then } cf=\frac{bc}{a}e \Rightarrow af=be \Rightarrow (a, b)R(e, f)$$

$\therefore R$ is transitive

Therefore R is an equivalence relation.

* For a fixed integer $n \geq 1$, prove that the relation "congruent modulo n " is an equivalence relation on the set of all integers, \mathbb{Z} .

Sol) "a is congruent to b modulo n" written as
 $a \equiv b \pmod{n} \Rightarrow n \mid a-b$ or $a-b = kn$, $k \in \mathbb{Z}$

As $n \mid a-a \Rightarrow a \equiv a \pmod{n} \Leftrightarrow aRa$, hence R is
 Let aRb ,

then $a \equiv b \pmod{n} \Rightarrow n \mid a-b$ or $a-b = kn$
 $\Rightarrow b-a = (-k)n \Rightarrow b \equiv a \pmod{n}$

Let aRb and bRc
 then $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

$$\Rightarrow n \mid a-b \text{ and } n \mid b-c$$

$$\Rightarrow a-b = k_1n \text{ and } b-c = k_2n$$

$$\Rightarrow a-b+b-c = k_1n + k_2n$$

$$\Rightarrow a-c = (k_1+k_2)n$$

$$\Rightarrow n \mid a-c$$

$$\Rightarrow a \equiv c \pmod{n}$$

$\Leftrightarrow aRc$, hence R is transitive.

Therefore R ["congruent modulo n"] is an equivalence relation.

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* Let R_1 and R_2 be relations on a set A. Prove that (i) If R_1 and R_2 are reflexive, so are $R_1 \cap R_2$ and $R_1 \cup R_2$.

(ii) If R_1 and R_2 are symmetric, so are $R_1 \cap R_2$ and $R_1 \cup R_2$.

(iii) If R_1 and R_2 are antisymmetric, so is $R_1 \cap R_2$.

(iv) If R_1 and R_2 are transitive, so is $R_1 \cap R_2$.

Soln (i) Suppose R_1 and R_2 are reflexive

$$\Rightarrow (a, a) \in R_1, (a, a) \in R_2 \forall a \in A.$$

consequently $(a, a) \in R_1 \cap R_2$ and $(a, a) \in R_1 \cup R_2$
 $\therefore R_1 \cap R_2$ and $R_1 \cup R_2$ are reflexive.

(ii) Suppose R_1 and R_2 are symmetric.

Take any $(a, b) \in R_1 \cap R_2$

then $(a, b) \in R_1$ and $(a, b) \in R_2$

$$\therefore (b, a) \in R_1 \text{ and } (b, a) \in R_2 [\because R_1, R_2 \text{ are symmetric}]$$

consequently $(b, a) \in R_1 \cap R_2$

$$\therefore R_1 \cap R_2 \text{ is symmetric.}$$

Now let $(a, b) \in R_1 \cup R_2$

then $(a, b) \in R_1$ or $(a, b) \in R_2$

$$\therefore (b, a) \in R_1 \text{ or } (b, a) \in R_2 [\because R_1, R_2 \text{ are symmetric}]$$

Consequently $(b, a) \in R_1 \cup R_2$

$$\therefore R_1 \cup R_2 \text{ is symmetric.}$$

(iii) Suppose R_1 and R_2 are antisymmetric.

Take any $(a, b), (b, a) \in R_1 \cap R_2$

Then $(a, b), (b, a) \in R_1$ and $(a, b), (b, a) \in R_2$

$\Rightarrow a=b$ in R_1 and $a=b$ in R_2

$$\therefore a=b \text{ in } (R_1 \cap R_2)$$

$$\therefore R_1 \cap R_2 \text{ is antisymmetric.}$$

(iv) Suppose R_1 and R_2 are transitive.

Take $(a, b), (b, c) \in R_1 \cap R_2$

$\Rightarrow (a, b), (b, c) \in R_1$ and $(a, b), (b, c) \in R_2$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2$ ($\because R_1, R_2$ are transitive)

$\Rightarrow (a, c) \in R_1 \cap R_2$. $\therefore R_1 \cap R_2$ is transitive.

* Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$
be a relation on A. Verify that R is an equivalence relation.

So i) $(1,1), (2,2), (3,3), (4,4) \in R$,

$\rightarrow (a,a) \in R \forall a \in A$. $\therefore R$ is reflexive.

ii) $(1,2), (2,1), (3,4), (4,3) \in R$

$\Rightarrow (a,b) \in R, (b,a) \in R \text{ for } a, b \in A$

$\therefore R$ is symmetric

iii) $(1,2), (2,1), (1,1) \in R, (2,1), (1,2), (2,2) \in R,$
 $(4,3), (3,4), (4,4) \in R, (3,4), (4,3), (3,3) \in R$

Thus if $(a,b), (b,c) \in R$ then $(a,c) \in R$ for $a, b, c \in A$
 $\therefore R$ is transitive.

Thus R is an equivalence relation.

* Let $A = \{1, 2, 3, 4\}$

and $R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3), (1,3), (4,1), (4,4)\}$
be a relation on A. Is R an equivalence relation?

i) $(a,a) \in R \forall a \in A$. $\therefore R$ is reflexive.

ii) $(4,1) \in R$ but $(1,4) \notin R$. $\therefore R$ is not symmetric.

Hence R is not an equivalence relation.

If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1, 2\}$, $A_2 = \{2, 3, 4\}$

and $A_3 = \{5\}$ define the relation R on A by xRy

if and only if x and y are in the same set A_i .

Is R an equivalence relation?

Since x and x belong to the same A_i , xRx $\therefore R$ is reflexive.

If $x, y \in A_i$ then $y, z \in A_i \therefore xRy$ and $yRz \therefore R$ is symmetric.

$(1,2) \in R$ ($\because 1$ and $2 \in A_1$) and $(2,3) \in R$ ($\because 2$ and $3 \in A_2$)

but $(1,3) \notin R$, as 1 and 3 are not in the same A_i .

$\therefore R$ is not transitive.

Hence R is not an equivalence relation.

Union and Intersection of relations

Let R_1 and R_2 be relations from set A to set B.
 The union of R_1 and R_2 i.e., $R_1 \cup R_2$ is defined by
 $(a, b) \in R_1 \cup R_2$ if and only if $(a, b) \in R_1$ or $(a, b) \in R_2$
 The intersection of R_1 and R_2 i.e., $R_1 \cap R_2$ is defined by
 $(a, b) \in R_1 \cap R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \in R_2$.

Complement of a relation

Given a relation R from A to B the complement of R , i.e., \bar{R} is defined with the property that
 $(a, b) \in \bar{R}$ if and only if $(a, b) \notin R$. $(a, b) \in A \times B, (a, b) \notin R$

Converse of a relation

Let R be a relation from a set A to set B,
 the converse of R , i.e., R^C is defined with the property that $(a, b) \in R^C$ if and only if $(b, a) \in R$.

Q. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$

$R_1 = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$, $R_2 = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$,
 find \bar{R}_1 , R_2^C , $R_1 \cup R_2$, $R_1 \cap R_2$.

Sol.

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$$

then $\bar{R}_1 = \{(a, 2), (a, 3), (b, 2), (b, 3), (c, 1)\}$

$$R_2^C = \{(1, a), (2, a), (1, b), (2, b)\}$$

$$R_1 \cup R_2 = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 2), (c, 3)\}$$

$$R_1 \cap R_2 = \{(a, 1), (b, 1)\}$$

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Equivalence classes

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements of A which are related to a by R is called the equivalence class of a with respect to R . The equivalence class is denoted by $R(a)$ or $[a]$ or \bar{a} .

Thus $\bar{a} = [a] = R(a) = \{x \in A \mid (x, a) \in R\}$.

example let $A = \{1, 2, 3\}$

and the equivalence relation $R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$.
The elements x of A for which $(x, 1) \in R$ are $x=1, 3$.
 $\therefore \{1, 3\}$ is the equivalence class of 1. i.e., $[1] = \{1, 3\}$.
Similarly $[2] = \{2\}$, $[3] = \{1, 3\}$.

example

Consider the relation R on \mathbb{Z} defined by xRy if and only if $x-y$ is even, where R is an equivalence relation.

For any $a \in \mathbb{Z}$, the equivalence class of a (w.r.t. R) is

$$[a] = \{x \in \mathbb{Z} \mid xRa\} = \{x \in \mathbb{Z} \mid (x-a) \text{ is even}\}.$$

$$= \{x \in \mathbb{Z} \mid (x-a) = 2k, k \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid x = a+2k, k = 0, \pm 1, \pm 2, \dots\}$$

$$\therefore [0] = \{x \in \mathbb{Z} \mid x = 2k, k = \dots, -2, -1, 0, 1, 2, \dots\}$$

$$\therefore [0] = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

$$\text{Similarly } [1] = \{x \in \mathbb{Z} \mid x = 1+2k, k = \dots, -2, -1, 0, 1, 2, \dots\}$$

$$\therefore [1] = \{\dots, -3, -1, 1, 3, \dots\}$$

$$\text{and } [2] = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

Properties of equivalence classes

1. Let R be an equivalence relation on a set A and $a \in A$, then $a \in [a]$

2. Let R be an equivalence relation on a set A , and let $a, b \in A$, Then aRb if and only if $[a] = [b]$

3. Let R be an equivalence relation on a set A , and let $a, b \in A$. If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$

Partition of a set.

Let A be a nonempty set. Suppose there exist nonempty subsets A_1, A_2, \dots, A_k of A such that the following two conditions hold.

(i) A is the union of A_1, A_2, \dots, A_k

$$\text{i.e., } A = A_1 \cup A_2 \cup \dots \cup A_k$$

(ii) Any two of the subsets A_1, A_2, \dots, A_k are disjoint.

$$\text{i.e., } A_i \cap A_j = \emptyset \text{ for } i \neq j.$$

Then the set $P = \{A_1, A_2, \dots, A_k\}$ is called a partition of A . Also A_1, A_2, \dots, A_k is called the blocks or cells of the partition.

examples

* Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its following subsets $A_1 = \{1, 3, 5, 7\}, A_2 = \{2, 4\}, A_3 = \{6, 8\}$. Observe that $A = A_1 \cup A_2 \cup A_3$. Also any two of the subsets A_1, A_2, A_3 are disjoint. Therefore, $P = \{A_1, A_2, A_3\}$ is a partition of A , with A_1, A_2, A_3 as the blocks(cells) of the partition.

* Suppose $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its subsets are given as $A_1 = \{1, 3, 5\}, A_2 = \{2, 4\}, A_3 = \{6, 8\}$.

Then $P = \{A_1, A_2, A_3\}$ is not a partition of the set A because although the subsets A_1, A_2, A_3 are disjoint, A is not the union of the subsets.

* Suppose $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its subsets are given as $A_1 = \{1, 3, 5, 7\}, A_2 = \{2, 4\}, A_3 = \{5, 6, 8\}$. Then $P = \{A_1, A_2, A_3\}$ is not a partition of the set A because although A is the union of A_1, A_2, A_3 , the sets A_1 and A_3 are not disjoint.

Note If A is a non-empty set, then:

- Any equivalence relation R on A induces a partition of A .
- Any partition of A gives rise to an equivalence relation R on A .

* For the equivalence relation

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$$

defined on the set $A = \{1, 2, 3, 4\}$, determine the partition induced.

Soln The equivalence classes of the elements of A w.r.t R are: $[1] = \{1, 2\}$, $[2] = \{1, 2\}$, $[3] = \{3, 4\}$, $[4] = \{3, 4\}$.
 Of these equivalence classes $[1]$ and $[3]$ are distinct. These two distinct equivalence classes constitute the partition $P = \{[1], [3]\}$.
 i.e., $P = \{\{1, 2\}, \{3, 4\}\}$

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

* For the set A and the relation R on A defined by $(x, y) \in R$ if and only if $x - y$ is a multiple of 5, find the partition of A induced by R .

Soln The equivalence classes of R are

$$[1] = \{1, 6, 11\} = [6] = [11]$$

$$[2] = \{2, 7, 12\} = [7] = [12]$$

$$[3] = \{3, 8\} = [8]$$

$$[4] = \{4, 9\} = [9]$$

$$[5] = \{5, 10\} = [10]$$

Therefore the partition of A induced by R is

$$P = \{\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8\}, \{4, 9\}, \{5, 10\}\}$$

* Let $A = \{a, b, c, d, e\}$. Consider the partition $P = \{\{a, b\}, \{c, d\}, \{e\}\}$ of A . Find the equivalence relation inducing this partition.

Since a, b belong to same block, we have the ordered pairs $(a, a), (a, b), (b, a), (b, b)$

Since c, d belong to same block, we have the ordered pairs $(c, c), (c, d), (d, c), (d, d)$

For the block $\{e\}$ we have (e, e)

Hence the equivalence relation inducing this partition is

$$P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (c, d), (d, c)\}$$

* On the set \mathbb{Z} of all integers, a relation R is defined by aRb if and only if $a^2 = b^2$. Verify that R is an equivalence relation. Determine the partition induced by this relation.

Q1) For any $a \in \mathbb{Z}$, $a^2 = a^2$, i.e., $(a, a) \in R$. Hence R is reflexive.

$$\text{Q2) let } (a, b) \in R \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow (b, a) \in R.$$

Hence R is symmetric.

$$\text{Q3) let } (a, b) \in R \text{ and } (b, c) \in R \Rightarrow a^2 = b^2 \text{ and } b^2 = c^2.$$

$$\Rightarrow a^2 = c^2 \Rightarrow (a, c) \in R. \text{ Hence } R \text{ is transitive.}$$

∴ Hence R is an equivalence relation.

$$\begin{aligned} \text{For any } a \in \mathbb{Z}, [a] &= \{x \in \mathbb{Z} \mid (x, a) \in R\} \\ &= \{x \in \mathbb{Z} \mid x^2 = a^2\} \\ &= \{x \in \mathbb{Z} \mid x = \pm a\} \end{aligned}$$

$$\text{Thus } [0] = \{0\}.$$

$$[n] = \{n, -n\} \text{ for } n \in \mathbb{Z}^+$$

$$\text{also } [-n] = \{-n, n\} \text{ for } n \in \mathbb{Z}^+$$

∴ The distinct partitions are $[0]$ and $[n]$.

$$\text{Hence } P = \{[0], [n]\} \text{ where } n \in \mathbb{Z}^+.$$

zero-one matrices

Consider the sets $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ of orders m and n respectively. Then $A \times B$ consists of all ordered pairs of the form (a_i, b_j) , $1 \leq i \leq m$, $1 \leq j \leq n$, which are mn in number. Let R be a relation from A to B so that R is a subset of $A \times B$.

Let $m_{ij} = (a_i, b_j)$ and assign the values 1 or 0 to m_{ij} according to the following rule:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

The mn matrix formed by these m_{ij} 's is called the matrix of the relation R , or the relation matrix for R and is denoted by M_R or $M(R)$. Since M_R contains only 0 and 1 as its elements, it is also called the zero-one matrix for R .

example

Consider the sets $A = \{0, 1, 2\}$ and $B = \{p, q\}$ and the relation R from A to B defined by $R = \{(0, p), (1, q), (2, p)\}$, then the matrix relation of R is

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{or} \quad M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Consider the set $A = \{1, 2, 3, 4\}$ and the relation R defined by $R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$ then the matrix of the relation R is

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Digraph of a relation

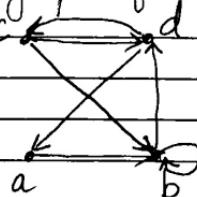
Let R be a relation on a finite set A . Then R can be represented pictorially as described. Draw a small circle or a bullet for each element of A and label the circle (bullet) with the corresponding element of A . These circles (bullets) are called vertices or nodes. Draw an arrow, called an edge, from a vertex x to a vertex y if and only if $(x, y) \in R$. The resulting pictorial representation of R is called a directed graph or digraph of R .

If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the origin or the source for that edge, and a vertex where an edge ends is called the terminus for that edge. A vertex which is neither a source nor a terminus of any edge is called an isolated vertex. An edge for which the source and terminus are the same vertex is called a loop. The number of edges (arrows) terminating at a vertex is called the in-degree of the vertex and the number of edges (arrows) leaving a vertex is called the out-degree of that vertex.

Q) Consider $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, b), (b, b), (b, d), (c, b), (c, d), (d, a), (d, c)\}$$

defined on A . The digraph of this relation is as shown below:



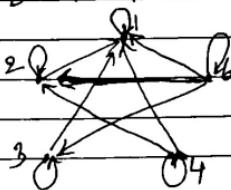
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- * Let $A = \{1, 2, 3, 4, 6\}$ and R be a relation on A defined by aRb if and only if a is a multiple of b . Represent the relation R as a matrix, draw its digraph and determine the in-degrees and out-degrees of the vertices in the digraph.

Sol: $R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4), (6,1), (6,2), (6,3), (6,6)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$



vertex	indegree	outdegree
1	5	1
2	3	2
3	2	2
4	1	3
6	1	4

- * Determine the relation R from a set A to a set B as described by the following matrix. ~~Also write the digraph~~

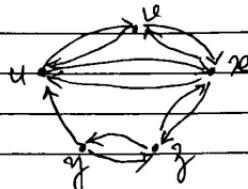
$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Sol: Since the order of M_R is 4×3 , $|A| = 4$ and $|B| = 3$.
 let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3\}$
 Then $R = \{(a_1, b_1), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_1)\}$

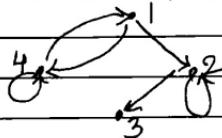
* Let $S A = \{u, v, x, y, z\}$ and R be a relation on A whose matrix is as given below. Determine R and also draw the associated digraph.

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Sol: $R = \{(u, v), (u, x), (v, u), (v, x), (x, u), (x, v), (x, z), (y, u), (y, z), (z, x), (z, y)\}$



* Find the relation represented by the digraph given below. Also write its matrix



$R = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Composition of relations

Consider a relation R_1 from a set A to a set B and a relation R_2 from the set B to a set C. With these relations, we can define a new relation called the product or the composition of R_1 and R_2 , denoted by $R_1 \circ R_2$.

Note: Composition of relations is not commutative but associative.

example

Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$. Also let R_1 be a relation from A to B and R_2, R_3 be relations from B to C, defined by,

$$R_1 = \{(1, w), (2, x), (3, y), (3, z)\}$$

$$R_2 = \{(w, 5), (x, 6)\}, R_3 = \{(w, 5), (w, 6)\}$$

Then $R_1 \circ R_2 = \{(1, 6), (2, 6)\}$ and $R_1 \circ R_3 = \emptyset$

$$\because (1, w) \in R_1, (w, 6) \in R_2 \Rightarrow (1, 6) \in R_1 \circ R_2$$

$$(2, x) \in R_1, (x, 6) \in R_2 \Rightarrow (2, 6) \in R_1 \circ R_2$$

Since the second element of an order pair in R_1 is not the first element of an order pair in R_2 , the an order pair in R_1 can't compose with an order pair in R_3 . Hence $R_1 \circ R_3 = \emptyset$.

Let $R = \{(1, 2), (3, 4), (2, 2)\}$ and $R_2 = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$ be relations on the set $A = \{1, 2, 3, 4, 5\}$.

Find $R_1 \circ (R_2 \circ R_3)$, $R_1 \circ (R_3 \circ R_2)$, $R_2 \circ (R_1 \circ R_3)$, $R_2 \circ (R_3 \circ R_1)$

Soln $R_2 \circ R_3 = \{(1, 5), (3, 2), (2, 5)\}$

$$R_3 \circ R_1 = \{(4, 2), (3, 2), (1, 4)\}$$

$$R_1 \circ (R_2 \circ R_3) = \{(1, 5), (2, 5)\}$$

$$R_1 \circ (R_3 \circ R_2) = \{(3, 2)\}$$

$$R_2 \circ (R_1 \circ R_3) = \{(4, 5), (3, 5), (1, 2)\}$$

$$R_2 \circ (R_3 \circ R_1) = \{(3, 4), (1, 2)\}$$

* For the relation R_1 and R_2 where
 $R_1 = \{(1, 2), (2, 1), (3, 1), (3, 2)\}$ and $R_2 = \{(1, 3), (2, 3), (2, 1)\}$, defined
on $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, find M_{R_1} , M_{R_2} ,
 $M_{R_1 R_2}$. Also verify that $M_{R_1 R_2} = M_{R_1} \cdot M_{R_2}$.

Soln

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{R_1 R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 R_2 = \{(1, 3), (2, 3), (2, 1)\}$

$$M_{R_1} \cdot M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_{R_1 R_2} \quad \text{Hence verified.}$$

* Let $A = \{a, b, c\}$ and R_1, R_2 be relations on A ,
whose matrices are given below. Find the composite
relations R_1^2, R_2^2 and their matrices.

$$M(R_1) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M(R_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Soln

$$R_1 = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, b)\}$$

$$R_2 = \{(a, a), (b, b), (b, c), (c, a), (c, c)\}$$

$$R_1^2 = R_1 \circ R_1 = \{(a, a), (a, c), (a, b), (b, a), (b, b), (b, c), (c, a), (c, b)\}$$

$$R_2^2 = R_2 \circ R_2 = \{(a, a), (b, b), (b, c), (c, a), (c, c)\}$$

$$M(R_1)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad M(R_2^2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Partial Orders

A relation R on set A is said to be a partial ordering relation or a partial order on A if (i) R is reflexive, (ii) R is antisymmetric, and (iii) R is transitive on A .

A set A with a partial order R defined on it is called a partially ordered set or an ordered set or a poset, and is denoted by the pair (A, R) .

- 1. The relation "less than or equal to", on the set \mathbb{Z} of all integers is a partial order on \mathbb{Z} . (\mathbb{Z}, \leq) is a poset.
- 2. The "divisibility relation" on the set \mathbb{Z}^+ defined by a divides b for all $a, b \in \mathbb{Z}^+$ is a partial order on \mathbb{Z}^+ .
- 3. The "subset relation" defined on the power set of a set S is a partial order on S . $(P(S), \subseteq)$ is a poset.
- 4. The relation "is less than" and "is greater than" are not partial orders on \mathbb{Z} , because, these are not reflexive.
- 5. The relation "congruent modulo n " defined on the set of all integers \mathbb{Z} is not a partial order, because this relation is not antisymmetric.

Total order

Let R be a partial order on a set A . Then R is called a total order on A if for all $x, y \in A$, either xRy , or yRx . In this case, the poset (A, R) is called a totally ordered set.

- 1. The partial order relation "less than or equal to" is a total order on the set \mathbb{R} , because for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$. Thus (\mathbb{R}, \leq) is a totally ordered set.
- 2. Consider the divisibility relation on the set $A = \{1, 2, 4, 6, 8\}$. R is a partial order on A , but it is not a total order on A , since neither 4 divides 6 nor 6 divides 4.

Hasse Diagrams

Since a partial order is a relation on a set, we can think of the digraph of a partial order if the set is finite. The following rules are applied while drawing the digraph of a partial order.

- (i) Since a partial order is reflexive, at every vertex in the digraph of a partial order there would be a cycle of length 1. In view of this, while drawing the digraph of a partial order, we need not exhibit such cycles explicitly.
- (ii) If, in the digraph of a partial order, there is an edge from a vertex a to a vertex b and there is an edge from the vertex b to a vertex c , then there should be an edge from a to c (because of transitivity). As such we need not exhibit an edge from a to c explicitly.
- (iii) To simplify the format of the digraph of a partial order, we represent the vertices by dots (bullets) and draw the digraph in such a way that all edges point upward. With this convention, we need not put arrows in the edges.

The digraph of a partial order drawn by adopting the conventions indicated in the above points is called a poset diagram or the Hasse Diagram for the partial order.

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* Let $A = \{1, 2, 3, 4\}$

and $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$

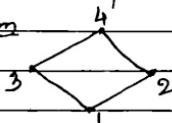
Verify that R is a partial order on A . Also write the Hasse diagram for R .

Soln) $(x, x) \in R \forall x \in A \therefore R$ is reflexive.

R does not contain ordered pairs of the form (y, x) and (x, y) with $y \neq x$. $\therefore R$ is antisymmetric.

Also $(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R \therefore R$ is transitive.
Hence R is a partial order on A .

Hasse Diagram



* Let $A = \{1, 2, 3, 4, 6, 12\}$. On A , define the relation R by aRb if and only if a divides b . Prove that R is a partial order on A . Draw the Hasse Diagram from this relation.

Soln) $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,12), (4,4), (4,12), (6,6), (12,12)\}$

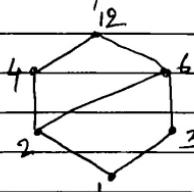
$\therefore (a, a) \in R \forall a \in A$, R is reflexive.

R does not contain ordered pairs of the form (a, b) and (b, a) with $b \neq a$, R is antisymmetric.

\therefore the elements of the forms $(a, c) \in R$, whenever (a, b) and $(b, c) \in R$, R is transitive.

Hence R is a partial order on A .

Hasse Diagram



- * Draw the Hasse diagram representing the positive divisors of 36, and R is the divisibility relation.

$$S = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

R will have the ordered pairs such that:

1 is related to 1, 2, 3, 4, 6, 9, 12, 18, 36;

2 is related to 2, 4, 6, 12, 18, 36;

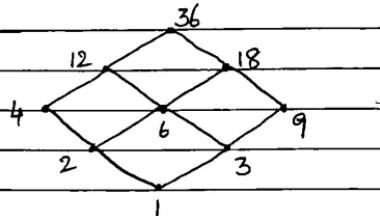
3 is related to 3, 6, 9, 12, 18, 36;

4 is related to 4, 12, 36; 6 is related to 6, 12, 18, 36;

9 is related to 9, 18, 36; 12 is related to 12, 36;

18 is related to 18, 36; 36 is related to 36.

Hasse Diagram



- * Let $S = \{1, 2, 3\}$ and $P(S)$ be the power set of S . On $P(S)$, define the relation R by $X R Y$ if and only if $X \subseteq Y$. Show that the relation is a partial order on $P(S)$. Draw its Hasse Diagram.

$\because A \subseteq A$, $(A, A) \in R \therefore R$ is reflexive.

$\because A \subseteq B \text{ and } B \subseteq A \Rightarrow A = B$, $(A, B) \in R$ and $(B, A) \in R \Rightarrow B = A$.

$\therefore R$ is antisymmetric.

$\because A \subseteq B \text{ and } B \subseteq C \Rightarrow A \subseteq C$, $(A, B) \text{ and } (B, C) \in R \Rightarrow (A, C) \in R$

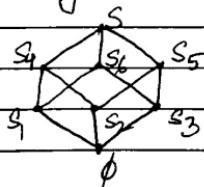
$\therefore R$ is transitive.

Hence $P(S) R$ is a partial order on $P(S)$

Hasse Diagram $P(S) = \{\emptyset, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$

where $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$

$S_4 = \{1, 2\}$, $S_5 = \{2, 3\}$, $S_6 = \{3, 1\}$

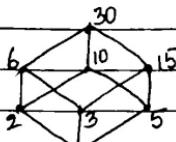


- * In the following cases, consider the partial order of divisibility on the set A. Draw the Hasse Diagram for the poset and determine whether the poset is totally ordered or not.

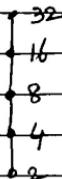
(i) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$, (ii) $A = \{2, 4, 8, 16, 32\}$

Sol

(i)



(ii)



(i) is not a totally ordered set as $(2, 3) \notin R \wedge x, y \in A$
example neither $(2, 3)$, nor $(3, 2) \in R$.

(ii) is a totally ordered set as $(a, b) \in R \forall a, b \in A$.

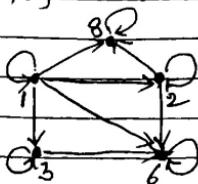
- * Prove that the set of all positive integers is not totally ordered by the relation of divisibility.

For a set A to be totally ordered by a partial order R, we should have aRb or bRa , for every $a, b \in A$.

If R is the divisibility relation on \mathbb{Z}^+ , aRb or bRa need not hold for every $a, b \in \mathbb{Z}^+$.

For example, if $a=2$ and $b=3$, then
a does not divide b and b does not divide a.
Therefore, \mathbb{Z}^+ is not totally ordered by the relation of divisibility.

- * The digraph for a relation on the set $A = \{1, 2, 3, 6, 8\}$ is as shown below:



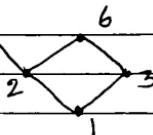
Sol.

Verify that (A, R) is a poset and write its Hasse Diagram

$$R = \{(1,1), (1,2), (1,3), (1,6), (1,8), (2,2), (2,6), (2,8), (3,3), (3,6), (6,6), (8,8)\}$$

~~R is reflexive, anti-symmetric and transitive,~~
Hence R is a partial order on A. i.e., (A, R) is a poset.

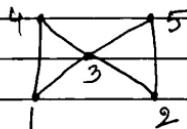
Hasse Diagram



- * Draw the Hasse Diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$ whose matrix is as given below:

1	0	1	1	1
0	1	1	1	1
0	0	1	1	1
0	0	0	1	0
0	0	0	0	1

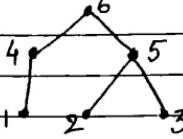
Q) $R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$



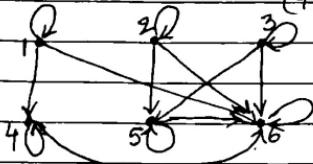
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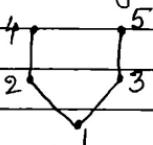
- * The Hasse diagram of a partial order R on the set $A = \{1, 2, 3, 4, 5, 6\}$ is as given below. Write R as a subset of $A \times A$. Construct its digraph.



Solⁿ $R = \{(1, 1), (1, 4), (1, 6), (2, 2), (2, 5), (2, 6), (3, 3), (3, 5), (3, 6), (4, 4), (4, 6), (5, 5), (5, 6), (6, 6)\}$



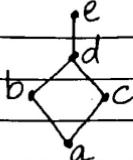
- * Determine the matrix of the partial order whose Hasse diagram is given below:



Solⁿ $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (3, 5), (4, 4), (5, 5)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

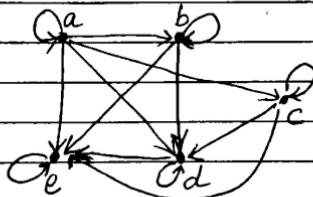
- * For $A = \{a, b, c, d, e\}$, the Hasse diagram for the poset (A, R) is as shown below:



- (Q) Determine the relation matrix for R
 (i) Construct the digraph of R .

$$R = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,d), (b,e), (c,c), (c,d), (c,e), (d,d), (d,e), (e,e)\}$$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Extremal elements in Posets

Consider a poset (A, R) .

1. An element $a \in A$ is called a maximal element of A if there exists no element $x \neq a$ in A such that aRx . In other words, $a \in A$ is a maximal element of A if whenever there is $x \in A$ such that aRx then $x=a$.

This means that a is a maximal element of A if and only if in the Hasse diagram of R no edge starts at a .

2. An element $a \in A$ is called a minimal element of A if there exists no element $x \neq a$ in A such that xRa . In other words, a is a minimal element of A if whenever there is $x \in A$ such that xRa , then $x=a$.

This means that a is a minimal element of A if and only if in the Hasse diagram of R no edge terminates at a .

3. An element $a \in A$ is called a greatest element of A if xRa for all $x \in A$.
4. An element $a \in A$ is called a least element of A if aRx for all $x \in A$.
5. An element $a \in A$ is called an upper bound of a subset B of A if xRa for all $x \in B$.
6. An element $a \in A$ is called a lower bound of a subset B of A if aRx for all $x \in B$.
7. An element $a \in A$ is called the least upper bound (LUB) of a subset B of A if (i) a is an upper bound of B (ii) If a' is an upper bound of B then $a'Ra'$. $\sup_{B \subseteq A}$ is also called supremum.
8. An element $a \in A$ is called the greatest lower bound (GLB) of a subset B of A if (i) a is a lower bound of B (ii) If a' is a lower bound of B then $a'Ra'$. $\inf_{B \subseteq A}$ is also called infimum.

The following theorems contain some results on extremal elements:

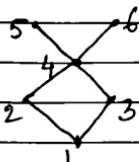
Theorem 1: If (A, R) is a poset and A is finite, then A has at least one maximal element and at least one minimal element.

Theorem 2: Every poset has at most one greatest element and at most one least element.

Theorem 3: If (A, R) is a poset and $B \subseteq A$, then B has at most one LUB and at most one GLB.

examples

(i)



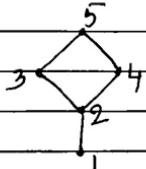
5, 6 are maximal elements

1 is the minimal element

1 is the least element

there is no greatest element

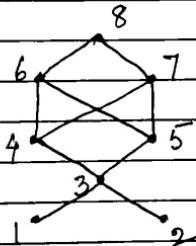
(ii)



5 is the maximal as well as the greatest element.

1 is the minimal as well as the least element.

(iii)



let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$B_1 = \{1, 2\}$, $B_2 = \{3, 4, 5\}$

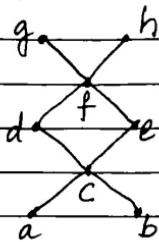
(i) Since $1R3, 2R3$, 3 is an upper bound of B_1 . Similarly 4, 5, 6, 7, 8 are also upper bounds of B_1 .

(ii) Since 3 is the least among

all the upper bounds of B_1 , $LUB(B_1) = 3$.

- (iii) Since there is no element x such that $xR1, xR2$, B_1 , and no lower bounds, and hence no greatest lower bound.
- (iv) 6, 7, 8 are upper bounds of B_2 and $6 \not R 7$, hence no least upper bound.
- (v) 1, 2, 3 are lower bounds of B_2 and 3 is the greatest lower bound of B_2 .

- * Consider the Hasse diagram of a poset (A, R) given below:



- If $B = \{c, d, e\}$, find (i) all upper bounds of B ,
 (ii) all lower bounds of B , (iii) the least upper bound of B ,
 (iv) the greatest lower bound of B .

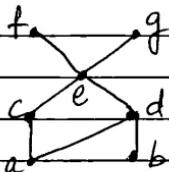
Sol: Upper bounds of B are f, g, h

Lower bound of B are a, b, c

The least upper bound of B is f

The greatest lower bound of B is c .

- * Consider the poset whose Hasse diagram is shown below. Find LUB and GLB of $B = \{c, d, e\}$



- Sol: All upper bounds of B are e, f, g , hence $LUB = e$
 All lower bounds of B are a , hence $GLB = a$

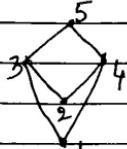
- * Let \mathbb{R} be the set of all real numbers with \leq as the partial order. Also, let B be the open interval $(1, 2)$. Find (i) all upper bounds of B , (ii) all lower bounds of B , (iii) the LUB of B , (iv) the GLB of B

- Sol: (i) All upper bounds of B = every real number > 2
 (ii) All lower bounds of B = every real number ≤ 1 .
 (iii) $LUB(B) = 2$, (iv) $GLB(B) = 1$

Lattice

Let (A, R) be a poset. This poset is called a lattice, if for all $x, y \in A$, the element $\text{LUB}\{x, y\}$ and $\text{GLB}\{x, y\}$ exist in A .

- Consider the set N of all natural numbers, and let R be the partial order "less than or equal to". Then for any $x, y \in N$, we note that $\text{LUB}\{x, y\} = \max\{x, y\}$ and $\text{GLB}\{x, y\} = \min\{x, y\}$ and both of these belong to N . Therefore, the poset (N, \leq) is a lattice.
- Consider the poset whose Hasse diagram is shown below.



since the $\text{GLB}\{3, 4\}$ does not exist, the above poset is not a lattice.

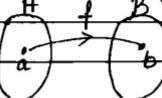
Functions

Let A and B be two non-empty sets. Then a function (or mapping) f from A to B is a relation from A to B such that for each a in A there is a unique b in B such that $(a, b) \in f$.

Then we write $b = f(a)$. Here b is called the image of a , and a is called a preimage of b , under f . The element a is also called an argument of the function f , and $b = f(a)$ is then called the value of the function f for the argument a .

Pictorially f can be represented as follows:

The function f from A to B
 $A \rightarrow B$ is denoted by $f: A \rightarrow B$



For the function $f: A \rightarrow B$, A is called the domain of f and B is called the codomain of f . The subset of B consisting of the images of all elements of A under f is called the range of f and is denoted by $f(A)$.

The following observations are immediate consequences of the definition of a function $f: A \rightarrow B$ and other associated definitions given above:

1. Every a in A belongs to some pair $(a, b) \in f$, and if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$. That is every element of A has an unique image.
2. An element $b \in B$ need not have a preimage in A .
3. If an element $b \in B$ has a preimage $a \in A$ under f , the preimage need not be unique.
4. The statements $(a, b) \in f$, $a \in b$ and $b = f(a)$ are equivalent.

5. If g is a function from A to B , then $f = g$ if and only if $f(a) = g(a)$ for every $a \in A$.
6. The range of $f: A \rightarrow B$ is given by $f(A) = \{f(x) | x \in A\}$ and $f(A)$ is a subset of B .
7. For $f: A \rightarrow B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by $f(A_1) = \{f(x) | x \in A_1\}$, then $f(A_1) \subseteq f(A)$.
8. For $f: A \rightarrow B$, if $b \in B$ and $f^{-1}(b)$ is defined by $f^{-1}(b) = \{x \in A | f(x) = b\}$, then $f^{-1}(b) \subseteq A$.
9. For $f: A \rightarrow B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by $f^{-1}(B_1) = \{x \in A | f(x) \in B_1\}$, then $f^{-1}(B_1) \subseteq A$.
- * Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ and R be a relation from A to B defined by,
 $R = \{(1, -1), (1, 0), (2, -1), (3, 0)\}$. Is R a function from A to B ?
- Solⁿ Since $1 \in A$ is related to two different element $-1, 0 \in B$, R is not a function.
- * Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$, and S be a relation from A to B defined by $S = \{(1, -1), (2, -1), (3, 0)\}$. Is S a function?
- Solⁿ Since each element of A is having a unique image in B , S is a function.
- * Let $A = \{0, \pm 1, \pm 2, 3\}$. Consider the function $f: A \rightarrow \mathbb{R}$, where \mathbb{R} is the set of all real numbers, defined by $f(x) = x^3 - 2x^2 + 3x + 1$, for $x \in A$. Find the range of f .
- Solⁿ $f(0) = 1, f(1) = 3, f(-1) = -5, f(2) = 7, f(-2) = -21, f(3) = 19$
Hence range of f i.e., $f(A) = \{1, 3, -5, 7, -21, 19\}$

* Let a function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Determine the images of the following subsets of \mathbb{R} .

(i) $A_1 = \{2, 3\}$, (ii) $A_2 = \{-2, 0, 3\}$, (iii) $A_3 = (0, 1)$, (iv) $A_4 = [-6, 3]$

Sol. (i) $f(A_1) = \{5, 10\}$

(ii) $f(A_2) = \{5, 1, 10\}$

(iii) $A_3 = \{x \in \mathbb{R} \mid 0 < x < 1\}$

then $f(A_3) = \{f(x) \mid 0 < x < 1\} = \{x^2 + 1 \mid 0 < x < 1\} = (1, 2)$

(iv) $f(A_4) = \{f(x) \mid -6 \leq x \leq 3\} = \{x^2 + 1 \mid -6 \leq x \leq 3\}$

$f(-6) = 37, f(-5) = 26, \dots, f(0) = 1, \dots, f(2) = 5, f(3) = 10\} = [1, 37]$

* Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x - 5, & x > 0 \\ -3x + 1, & x \leq 0 \end{cases}$$

(i) Determine, $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-6)$, ~~$f^{-1}(4)$~~

(iii) What are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?

Sol. (i) $f(0) = -3 \times 0 + 1 = 1, f(-1) = -3 \times (-1) + 1 = 4,$

$$f(5/3) = 3 \times \frac{5}{3} - 5 = 0, f(-5/3) = -3 \times \left(-\frac{5}{3}\right) + 1 = 6$$

(ii) let $f^{-1}(0) = x \Rightarrow f(x) = 0 \Rightarrow \begin{cases} 3x - 5 = 0 \\ -3x + 1 = 0 \end{cases} \Rightarrow x = 5/3 (> 0)$

$$\Rightarrow -3x + 1 = 0 \Rightarrow x = 1/3 (\neq 0) \times$$

$$\therefore f^{-1}(0) = \{5/3\}$$

$f^{-1}(1) = x \Rightarrow f(x) = 1 \Rightarrow \begin{cases} 3x - 5 = 1 \\ -3x + 1 = 1 \end{cases} \Rightarrow x = 2 (> 0) \therefore f^{-1}(1) = \{2\}$

$f^{-1}(3) = x \Rightarrow f(x) = 3 \Rightarrow \begin{cases} 3x - 5 = 3 \\ -3x + 1 = 3 \end{cases} \Rightarrow x = 8/3 (> 0) \therefore f^{-1}(3) = \{8/3, -2/3\}$

$f^{-1}(-1) = x \Rightarrow f(x) = -1 \Rightarrow \begin{cases} 3x - 5 = -1 \\ -3x + 1 = -1 \end{cases} \Rightarrow x = 4/3 (> 0) \therefore f^{-1}(-1) = \{4/3\}$

$f^{-1}(-6) = x \Rightarrow f(x) = -6 \Rightarrow \begin{cases} 3x - 5 = -6 \\ -3x + 1 = -6 \end{cases} \Rightarrow x = -1/3 (x \neq 0) \therefore f^{-1}(-6) = \{-1/3\}$

$$\therefore f^{-1}(-6) = \{-1/3\}$$

$$(iii) f^{-1}([-5, 5]) = \{x \in \mathbb{R} \mid f(x) \in [-5, 5]\}$$

$$= \{x \in \mathbb{R} \mid -5 \leq f(x) \leq 5\}$$

$$-5 \leq 3x - 5 \leq 5 \Rightarrow 0 \leq 3x \leq 10 \Rightarrow 0 \leq x \leq \frac{10}{3} \quad (x > 0)$$

$$-5 \leq -3x + 1 \leq 5 \Rightarrow -6 \leq -3x \leq 4 \Rightarrow -4 \leq x \leq \frac{2}{3} \quad (x < 0)$$

$$\therefore f^{-1}([-5, 5]) = \{x \in \mathbb{R} \mid -4 \leq x \leq \frac{10}{3}\} \rightarrow -\frac{4}{3} \leq x \leq \frac{10}{3} \quad (\because x \text{ should be } \leq 0)$$

$$= [-4/3, 10/3]$$

$$f^{-1}([-6, 5]) = \{x \in \mathbb{R} \mid f(x) \in [-6, 5]\}$$

$$= \{x \in \mathbb{R} \mid -6 \leq f(x) \leq 5\}$$

$$-6 \leq 3x - 5 \leq 5 \Rightarrow -1 \leq 3x \leq 10 \Rightarrow -\frac{1}{3} \leq x \leq \frac{10}{3}$$

$$\Rightarrow 0 < x \leq \frac{10}{3} \quad (\because x \text{ should be } \geq 0)$$

$$-6 \leq -3x + 1 \leq 5 \Rightarrow -7 \leq -3x \leq 4 \Rightarrow -\frac{4}{3} \leq x \leq \frac{7}{3}$$

$$\Rightarrow -\frac{4}{3} \leq x \leq \frac{7}{3} \quad (\because x \text{ should be } \leq 0)$$

$$\therefore f^{-1}([-6, 5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \leq x \leq \frac{7}{3}\}$$

$$= [-4/3, 10/3]$$

- (a) Let A and B be finite sets with $|A|=m$ and $|B|=n$.
 Find how many functions are possible from A to B?
 (b) If there are 2187 functions from A to B and $|B|=3$, what is $|A|$?

(a) Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$
 Then a typical function $f: A \rightarrow B$ is of the form
 $f = \{(a_1, x), (a_2, x), \dots, (a_m, x)\}$ where x can be
 any of the b_j from the set B.

Therefore the total possible number of choices
 ordered pairs would be $n \times n \times \dots \times n$ (n times)

Thus, there are $n^m = |B|^{|A|}$ possible functions from A to B

- (b) given $n^m = 2187$ and $n=3$.

$$\Rightarrow 3^m = 2187 \Rightarrow m = \frac{\ln 2187}{\ln 3} \Rightarrow m = 7$$

$$\therefore |A| = 7$$

Types of functions

Identity function

A function $f: A \rightarrow A$ such that $f(a) = a$ for every $a \in A$ is called the identity function on A . The identity function defined on a set A is usually denoted as I_A .

Constant function

A function $f: A \rightarrow B$ such that $f(a) = c$ for every $a \in A$, where c is a fixed element of B , is called a constant function.

Onto function

A function $f: A \rightarrow B$ is said to be an onto function if for every b of B there is an element a of A such that $f(a) = b$.

An onto function is also called a surjective function.

One-to-One function

A function $f: A \rightarrow B$ is said to be a one-to-one function if different elements of A have different images in B under f . That is if whenever $a_1, a_2 \in A$ with $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$; or equivalently that whenever $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$, then $a_1 = a_2$. A one-to-one function is also called an injective function.

One-to-One correspondence

A function which is both one-to-one and onto is called a one-to-one correspondence or a bijective function. If $f: A \rightarrow B$ is such a function, then every element of A has a unique image in B and every element in B has a unique preimage in A .

* Find the nature of the following functions defined on $A = \{1, 2, 3\}$

$$(i) f = \{(1, 1), (2, 2), (3, 3)\} \quad (ii) g = \{(1, 2), (2, 2), (3, 2)\}$$

$$(iii) h = \{(1, 2), (2, 2), (3, 1)\} \quad (iv) p = \{(1, 2), (2, 3), (3, 1)\}$$

Sol. (i) $\forall a \in A, (a, a) \in f$, hence f is a identity function.

(ii) $\forall a \in A, (a, 2) \in f$ hence g is a constant function.

(iii) Since distinct elements of A don't have distinct images h is not one-to-one.

Since 3 does not have a preimage, h is not onto.

(iv) Since distinct elements of A have distinct images and all elements have preimages, p is both one-to-one and onto.

* The functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are defined by $f(x) = 3x + 7$ for all $x \in \mathbb{R}$ and $g(x) = x(x^2 - 1)$ for all $x \in \mathbb{R}$. Verify that f is one-to-one but g is not.

Sol. Consider $f(x_1) = f(x_2)$

$$\Rightarrow 3x_1 + 7 = 3x_2 + 7$$

$$\Rightarrow x_1 = x_2$$

Hence f is one-to-one.

Consider $g(0) = 0$ and $g(1) = 0$

Since both 0 and 1 are mapped to the same element 0, g is not one-to-one.

* Let $A = \mathbb{R}$ and $B = \{x | x \text{ is real and } x \geq 0\}$. Is the function $f: A \rightarrow B$ defined by $f(a) = a^2$ an onto function?

~~a one-to-one function~~

Sol. Let $f(a) = b \Rightarrow a^2 = b \Rightarrow a = \pm\sqrt{b}$,

i.e. $\pm\sqrt{b} \in \mathbb{R}$ s.t $f(\sqrt{b}) = b$ and $f(-\sqrt{b}) = b$

Hence f is onto.

Topic _____

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- * Let $A = \mathbb{R}$ and $B = \{z \mid z \text{ is real and } z \geq 0\}$.
 Is the function $f: A \rightarrow B$ defined by $f(a) = a^2$ a one-to-one function?

Sol) Consider $f(a_1) = f(a_2)$
 $\Rightarrow a_1^2 = a_2^2$

$$\Rightarrow a_1 = \pm a_2.$$

Since a_1 is not only equal to a_2 , says that both $+a_1$ and $-a_1$ are mapped to a_2^2 .
 hence f is not one-one.

- * Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(a) = a+1$ for $a \in \mathbb{Z}$.
 Find whether f is one-to-one or onto.

Sol) Consider $f(a_1) = f(a_2)$
 $\Rightarrow a_1 + 1 = a_2 + 1$

$$\Rightarrow a_1 = a_2$$

$\therefore f$ is one-to-one

Let $f(a) = b \Rightarrow a+1 = b \Rightarrow a = b-1$

i.e., $\exists b-1 \in \mathbb{Z}$ s.t $f(b-1) = b$.

Hence f is onto.

Composition of functions

Consider three non-empty sets A, B, C and the functions $f: A \rightarrow B$ and $g: B \rightarrow C$. The composition (product) of these two functions is defined as the function $gof: A \rightarrow C$ with $(gof)(a) = g(f(a))$ for all $a \in A$.

- * Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{x, y, z\}$ with $f: A \rightarrow B$ and $g: B \rightarrow C$ given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, z)\}$. Find gof .

$$gof(1) = g(f(1)) = g(a) = x.$$

$$gof(2) = g(f(2)) = g(a) = x.$$

$$gof(3) = g(f(3)) = g(b) = y$$

$$gof(4) = g(f(4)) = g(c) = z$$

$$\text{Thus } gof = \{(1, x), (2, x), (3, y), (4, z)\}$$

- * Consider the functions f and g defined by $f(x) = x^3$ and $g(x) = x^2 + 1$, $\forall x \in \mathbb{R}$. Find gof , fog , f^2 and g^2 .

$$gof(x) = g(f(x)) = g(x^3) = x^6 + 1$$

$$fog(x) = f(g(x)) = f(x^2 + 1) = (x^2 + 1)^3$$

$$f^2(x) = (f \circ f)(x) = f(f(x)) = f(x^3) = x^9$$

$$g^2(x) = (g \circ g)(x) = g(g(x)) = g(x^2 + 1) = (x^2 + 1)^2 + 1$$

- * Let f and g be functions from \mathbb{R} to \mathbb{R} defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$. If gof

If $(gof)(x) = 9x^2 - 9x + 3$, determine a and b .

$$(gof)(x) = g(f(x)) = g(ax + b) = 1 - (ax + b) + (ax + b)^2 \quad (1)$$

$$\text{given } (gof)(x) = 9x^2 - 9x + 3 \quad (2)$$

$$\text{Equating (1) + (2)} \rightarrow 1 - ax - b + ax^2 + 2abx + b^2 = 9x^2 - 9x + 3$$

$$\Rightarrow 1 - b + b^2 - 3 \Rightarrow b^2 - b - 2 = 0 \Rightarrow b = -1 \text{ or } b = 2, \quad 2ab - a = -9$$

$$\Rightarrow a = 3, b = -1 \quad \text{or} \quad a = -3, b = 2$$

Thus $a = 3, b = -1$ and $a = -3, b = 2$

Some important results on composition of functions

Theorem: Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be any two functions. Then the following are true:

- (i) If f and g are one-to-one, so is gof .
- (ii) If gof is one-to-one, then f is one-to-one.
- (iii) If f and g are onto, so is gof .
- (iv) If gof is onto, then g is onto.

Theorem: Let $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$ be three functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$

* Let f, g, h be functions from \mathbb{Z} to \mathbb{Z} defined by $f(x) = x - 1$, $g(x) = 3x$. $h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$

Determine $(f \circ (g \circ h))(x)$ and $((f \circ g) \circ h)(x)$ and verify that $f \circ (g \circ h) = (f \circ g) \circ h$

(1) $(g \circ h)(x) = g(h(x)) = 3h(x)$
 $(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(3h(x)) = 3h(x) - 1$
 $= \begin{cases} -1, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases}$

$$(f \circ g)(x) = f(g(x)) = f(3x) = 3x - 1 \quad (1)$$

$$(f \circ g) \circ h(x) = f(g(h(x))) \quad (f \circ g)(h(x)) = 3h(x) - 1$$

$$= \begin{cases} -1, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases} \quad (2)$$

From (1) and (2) it follows that
 $f \circ (g \circ h) = (f \circ g) \circ h$.

Invertible functions

A function $f: A \rightarrow B$ is said to be invertible if there exists a function $g: B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$, where I_A is the identity function on A and I_B is the identity function on B .

* Let $A = \{1, 2, 3, 4\}$ and f and g be functions from A to A given by,

$$f = \{(1, 4), (2, 1), (3, 2), (4, 3)\} \text{ and } g = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$$

Prove that f and g are inverses of each other.

$$\text{S1 } (gof)(1) = g(f(1)) = g(4) = 1 = I_A(1)$$

$$(gof)(2) = g(f(2)) = g(1) = 2 = I_A(2)$$

$$(gof)(3) = g(f(3)) = g(2) = 3 = I_A(3)$$

$$(gof)(4) = g(f(4)) = g(3) = 4 = I_A(4)$$

$$(fog)(1) = f(g(1)) = f(2) = 1 = I_A(1)$$

$$(fog)(2) = f(g(2)) = f(3) = 2 = I_A(2)$$

$$(fog)(3) = f(g(3)) = f(4) = 3 = I_A(3)$$

$$(fog)(4) = f(g(4)) = f(1) = 4 = I_A(4)$$

Thus, for all $x \in A$, we have $(gof)(x) = I_A(x)$ and $(fog)(x) = I_A(x)$. Therefore, g is an inverse of f and f is an inverse of g .

* Consider the function $f: R \rightarrow R$ defined by

$f(x) = 2x + 5$. Let a function $g: R \rightarrow R$ be defined by $g(x) = \frac{1}{2}(x - 5)$. Prove that g is an inverse of f .

$$\text{S1 } \text{Consider } (gof)(x) = g(f(x)) = g(2x+5) = \frac{1}{2}(2x+5) - 5 \\ = x = I_R(x)$$

$$(fog)(x) = f(g(x)) = f\left[\frac{1}{2}(x-5)\right] = 2x + 5 - 5 \\ = x = I_R(x)$$

These show that g is an inverse of f .

It also follows that f is an inverse of g .

Some results on one-to-one, onto and invertible functions:

1. If a function $f: A \rightarrow B$ is invertible then it has a unique inverse. Further, if $f(a) = b$, then $f^{-1}(b) = a$.
2. A function $f: A \rightarrow B$ is invertible if and only if it is one-to-one and onto.
3. Let A and B be finite sets with $|A| = |B|$ and f be a function from A to B . Then the following statements are equivalent:
 - f is one-to-one,
 - f is onto,
 - f is invertible.
4. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are invertible functions, then $g \circ f: A \rightarrow C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
5. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, $\forall x \in \mathbb{R}$. Is f invertible?
 Solⁿ For any $a \in \mathbb{R}$, we have $f(a) = a^2$ and $f(-a) = a^2$. Thus, both a and $-a$ have the same image a^2 under f . Therefore, f is not one-to-one.
 Consequently, f is not invertible.

- * Let $A = \{x \mid x \text{ is real and } x \geq -1\}$, and $B = \{x \mid x \text{ is real and } x \geq 0\}$. Consider the function $f: A \rightarrow B$ defined by $f(x) = \sqrt{x+1}$ for all $x \in A$. Show that f is invertible and determine f^{-1} .
- Solⁿ $f(a_1) = f(a_2) \Rightarrow \sqrt{a_1+1} = \sqrt{a_2+1} \Rightarrow a_1+1 = a_2+1 \Rightarrow a_1 = a_2$
 $f(a) = b \Rightarrow \sqrt{a+1} = b \Rightarrow a+1 = b^2 \Rightarrow a = b^2 - 1$, $\therefore f$ is one-to-one
 i.e. $\forall b^2 - 1 \in A$ such that $f(b^2 - 1) = b$ $\therefore f$ is onto
 Hence f is invertible.
 As $f(b^2 - 1) = b \Rightarrow f^{-1}(b) = b^2 - 1$ is the inverse of b .

- * Find the inverse (if any) of the function $f(x) = e^x$ defined from $\mathbb{R} \rightarrow \mathbb{R}^+$.
- Soln: $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$. $\therefore f$ is one to one.
- But $f(x) = y \Rightarrow e^x = y \Rightarrow x = \ln y$
- $\therefore \exists \ln y \in \mathbb{R}$ s.t. $f(\ln y) = y$. $\therefore f$ is onto.
- $\therefore f$ is invertible.
- As $f(\ln y) = y \Rightarrow f(y) = \ln y$ is the inverse of y .

* Let $A = B = \mathbb{R}$, the set of all real numbers and the functions $f: A \rightarrow B$ and $g: B \rightarrow A$ be defined by $f(x) = 2x^3 - 1$, $\forall x \in A$; $g(y) = \left(\frac{y+1}{2}\right)^{1/3}$, $\forall y \in B$. Show that each of f and g is the inverse of the other.

Soln: Consider $(g \circ f)(x) = g(f(x))$

$$\begin{aligned} &= g(2x^3 - 1) \\ &= \left(\frac{(2x^3 - 1) + 1}{2}\right)^{1/3} \\ &= \left(\frac{2x^3}{2}\right)^{1/3} \\ &= x \end{aligned}$$

Thus $(g \circ f)(x) = I_A$, hence g is the inverse of f .

Consider $(f \circ g)(y) = f(g(y))$

$$\begin{aligned} &= f\left(\left(\frac{y+1}{2}\right)^{1/3}\right) \\ &= 2\left(\left(\frac{y+1}{2}\right)^{1/3}\right)^3 - 1 \\ &= y \end{aligned}$$

Thus $(f \circ g)(y) = I_B$, hence f is the inverse of g .



* Let $A = B = C = \mathbb{R}$ and $f: A \rightarrow B$ and $g: B \rightarrow C$ be defined by $f(a) = 2a+1$, $g(b) = \frac{1}{3}b$, $\forall a \in A$, $\forall b \in B$. Compute gof and show that gof is invertible. What is $(gof)^{-1}$?

$$\text{Soln. } (gof)(a) = g(f(a)) = g(2a+1) = \frac{1}{3}(2a+1)$$

$$f(a_1) = f(a_2) \Rightarrow 2a_1+1 = 2a_2+1 \Rightarrow a_1 = a_2 \therefore f \text{ is one to one}$$

$$f(a) = b \Rightarrow 2a+1 = b \Rightarrow a = \frac{1}{2}(b-1)$$

$$\therefore \exists \frac{1}{2}(b-1) \in A \text{ s.t. } f\left(\frac{1}{2}(b-1)\right) = b. \therefore f \text{ is onto}$$

$$\text{Hence } f \text{ is invertible. and } f^{-1}(b) = \frac{1}{2}(b-1)$$

$$g(b_1) = g(b_2) \Rightarrow \frac{1}{3}b_1 = \frac{1}{3}b_2 \Rightarrow b_1 = b_2 \therefore g \text{ is one to one}$$

$$g(b) = c \Rightarrow \frac{1}{3}b = c \Rightarrow b = 3c$$

$$\therefore \exists 3c \in B \text{ s.t. } g(3c) = c. \therefore g \text{ is onto.}$$

$$\text{Hence } g \text{ is invertible. and } g^{-1}(c) = 3c$$

Since f and g are invertible gof is invertible.
and $(gof)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$

$$= f^{-1}(g^{-1}(c))$$

$$= f^{-1}[3c]$$

$$= \frac{1}{2}(3c-1)$$

Properties of functions

1. Let A and B be finite sets and f be a function from A to B . Then the following are true.
- (i) If f is one-to-one, then $|A| \leq |B|$.
 - (ii) If f is onto, then $|B| \leq |A|$.
 - (iii) If f is a one-to-one correspondence, then $|A|=|B|$.
 - (iv) If $|A| > |B|$, then at least two different elements of A have the same image under f .

Property (iv) can be interpreted as follows:
If every element of a set A with $|A|=n$ is assigned to a unique element of a set B with $|B|=m$ and $n > m$, then at least two different elements of A are assigned to the same element of B . In other words, if n objects (pigeons, say) are assigned to m places (pigeon holes), and if $n > m$, then at least two objects (pigeons) are assigned to the same place.
This is known as the "Pigeonhole Principle".

2. Suppose A and B are finite sets having the same number of elements, and f is a function from A to B . Then f is one-to-one if and only if f is onto.
3. A function f from a finite set A to a finite set B with $|A|=|B|$ is bijective if and only if f is one-to-one or onto.

Number of one-to-one and onto functions.

If A and B are finite sets with $|A|=m$ and $|B|=n$, where $m \geq n$, then:

(i) The number of one-to-one functions possible from A to B is given by the formula

$$P(n, m) = \frac{n!}{(n-m)!}$$

(ii) The number of onto functions possible from A to B is given by the formula

$$P(n, m) = \sum_{k=0}^n (-1)^k \binom{n}{n-k} (n-k)^m$$

* Let A and B be finite sets. If there are 60 one-to-one functions from A to B and $|A|=3$, what is $|B|$?

Solⁿ Given $m=3$, $P(n, m)=60$

$$\Rightarrow \frac{n!}{(n-3)!} = 60$$

$$\Rightarrow n(n-1)(n-2) = 60 \quad \cancel{n^3 - 3n^2 + 2n - 60 = 0}$$

$$\Rightarrow n=5$$

$$\cancel{n=5}$$

$$\text{i.e., } |B|=5$$

* Let $A=\{1, 2, 3, 4, 5, 6, 7\}$ and $B=\{w, x, y, z\}$. Find the number of onto functions from A to B

Solⁿ Given $m=7$, $n=4$.

$$P(A, 7) = \sum_{k=0}^7 (-1)^k \binom{7}{4-k} (4-k)^7 = 7^7 - 4 \times 3^7 + 6 \times 2^7 - 4 = 8400.$$

* Let $A=\{1, 2, 3, 4\}$ and $B=\{1, 2, 3, 4, 5, 6\}$

(i) How many functions are there from A to B? How many of these are one-to-one? How many are onto?

(ii) How many functions are there from B to A? How many of these are one-to-one? How many are onto?

(i) Possible functions $6^4 = 1296$

one-to-one functions $\frac{6!}{2!} = 360$

(ii) Possible functions $4^6 = 4096$

one-to-one functions $4! = 24$

onto functions $\frac{6!}{(4-4)!} = 1560$

Stirling number of the second kind.
 For $m \geq n$ there are $\sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m$

ways to distribute m distinct objects into n numbered (but otherwise identical) containers with no container left empty. Removing the numbers on the containers, so that they are now identical in appearance, we find that one distribution into these n (nonempty) identical containers corresponds with $n!$ such distributions into the numbered containers. So the number of ways in which it is possible to distribute the m distinct objects into n identical containers, with no container left empty, is $\frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m$.

This will be denoted by $S(m, n)$ and is called a Stirling number of the second kind. We note that for $|A|=m \geq n=|B|$, there are $n! \cdot S(m, n)$ onto functions from A to B .

- * A research chemist who has five laboratory assistants is engaged in a research project that calls for nine compounds that must be synthesized. In how many ways can the chemist assign these syntheses to the five assistants so that each is working on at least one synthesis.

(a) $A =$ the 9 syntheses, $B =$ the 5 assistants.

We need to find the number of onto functions from A to B , which is $\sum_{k=0}^5 (-1)^k \binom{5}{5-k} (5-k)^9$ \Leftarrow
 $= 5^9 - 5 \cdot 4^9 + 10 \cdot 3^9 - 10 \cdot 2^9 + 5 \cdot 1^9 = 834120$

Growth of functions

Computer programmers are often concerned with two questions:

- (i) How much time does an algorithm need to complete?
- (ii) How much memory does an algorithm need for its computation?

Big O notation is a mathematical notation used in computer science to describe the upper bound of an algorithm's runtime complexity, or how its performance scales as the input size grows. It essentially provides a way to analyze the efficiency of an algorithm, focusing on its worst-case scenario.

The Big O notation is denoted as $O(f(n))$, where $f(n)$ is a function that represents the number of operations that an algorithm performs to solve a problem of size n .

Some common Big O notations include

- * $O(1)$: The runtime is independent of input size.
- * $O(\log n)$: The runtime grows logarithmically with input size.
- * $O(n)$: The runtime grows proportionally to input size.
- * $O(n^2)$: The runtime grows with the square of input size.
- * $O(2^n)$: The runtime grows exponentially with the input size.

The size of the input complexities ordered from smallest to largest:

$O(1)$, $O(\log n)$, $O(\sqrt{n})$, $O(n)$, $O(n \log n)$, $O(n^2)$, $O(n^3)$, $O(b^n)$ ($b > 1$), $O(n!)$

The asymptotic behaviour for large n should be determined by the most dominant term in the function for large n .

e.g. $f(x) = x^3 + 2x^2 - 2x$ for large x , is dominated by the term x^3 .

$$= O(f(x)) = x^3$$

Properties of Big O notation

Suppose $f(x)$ is $O(F(x))$ and $g(x)$ is $O(G(x))$.
 Then 1. $c \cdot f(x)$ is $O(F(x))$
 2. $f(x) + g(x)$ is $O(\max(F(x), G(x)))$
 3. $f(x) \cdot g(x)$ is $O(F(x) \cdot G(x))$.

Find the Big O of each of the following:

1. $f(x) = 2x^2 + 4x$,

Big O of $f(x)$ is $O(\max(x^2, x)) = O(x^2)$

2. $h(x) = (x^3 + 1) \cdot (x^2 - x)$

Big O of $h(x)$ is $O(x^3 \cdot x^2) = O(x^5)$

3. $g(x) = x^2(2^x + x^2)$

Big O of $g(x)$ is $O(x^2 \cdot 2^x)$

4. $p(x) = 5x! + 4x^3 \log x$.

Big O of $p(x)$ is $O(\max(x!, x^3 \log x))$
 $= O(x!)$

5. $q(x) = (3x^3 + x) 2^x + (x + x!) x^4$

Big O of $q(x)$ is $O(\max(x^3 \cdot 2^x, x^4 \cdot x!))$
 $= O(x^4 x!)$