

Unit-2

Linear Algebra-2

Orthogonality

The scalar product in \mathbb{R}^n

Let x and y be any vectors in \mathbb{R}^n . If

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then the scalar product of

x and y is

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Ex1 : If

$$x = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \text{ and } y = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \text{ then}$$

$$x^T y = 12 - 6 + 2 = 8$$

Euclidean length in \mathbb{R}^n .

Let $x \in \mathbb{R}^n$ be a vector. If $x = (x_1, x_2, \dots, x_n)^T$, then

Euclidean length is

$$\|x\| = (x^T x)^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.$$

Defn: Let x and y be any vectors in \mathbb{R}^n . Then the distance between x and y is defined to be the number

$$\|x - y\|.$$

Ex2 : If $x = (3, 4)^T$ and $y = (-1, 7)^T$, Then

i) length of x is $\|x\| = \sqrt{3^2 + 4^2} = 5$

ii) distance between x and y is

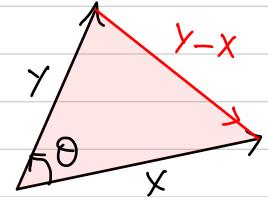
$$\|x - y\| = \sqrt{(3 - (-1))^2 + (4 - 7)^2} = 5$$

Theorem: If x and y are two vectors in \mathbb{R}^2 (or \mathbb{R}^3) and θ is the angle between them, then

$$x^T y = \|x\| \|y\| \cos \theta$$

pf: By the law of cosines, we have

$$\begin{aligned}\|y-x\|^2 &= \|x\|^2 + \|y\|^2 - 2 \|x\| \|y\| \cos \theta \\ \Rightarrow 2 \|x\| \|y\| \cos \theta &= \frac{1}{2} (\|x\|^2 + \|y\|^2 - \|y-x\|^2)\end{aligned}$$



$$\begin{aligned}&= \frac{1}{2} (x^T x + y^T y - (y-x)^T (y-x)) \\ &= \frac{1}{2} (x^T x + y^T y - x^T y + y^T x + x^T y - x^T x) \\ &= \frac{1}{2} (y^T x + x^T y) \\ &= x^T y \quad (\because x^T y = y^T x)\end{aligned}$$

Note: If x and y are vectors in \mathbb{R}^n , then we can specify their directions by forming unit vectors

$$u_1 = \frac{x}{\|x\|}, \quad u_2 = \frac{y}{\|y\|}$$

If θ is the angle b/w x and y , then

$$\cos \theta = \frac{x^T y}{\|x\| \|y\|} = u_1^T u_2$$

Ex 3: Obtain unit vector in the direction of the vector

$$x = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

Soln: Unit vector $u = \frac{x}{\|x\|}$,

where $\|x\| = \sqrt{4+25+16} = \sqrt{45}$

$$\therefore u = \begin{bmatrix} -2/\sqrt{45} \\ 5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}$$

Ex 4: Find the angle between the vectors v and w , where

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$$

Soln: If θ is angle b/w v and w , then

$$\cos \theta = \frac{v^T w}{\|v\| \|w\|}$$

$$\text{where } v^T w = 12 + 3 + 27 = 42$$

$$\|v\| = \sqrt{4+1+9} = \sqrt{14}$$

$$\|w\| = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$$

$$\therefore \cos \theta = \frac{42}{3\sqrt{14}\sqrt{14}} = 1$$

$$\therefore \theta = 0$$

As you can see $w = 3v$, both are in same direction. Thus angle between them is 0 .

(Corollary: [Cauchy-Schwarz inequality])

If x and y are vectors in \mathbb{R}^2 or \mathbb{R}^3 , then

$$|x^T y| \leq \|x\| \|y\|$$

equality holds iff one of the vectors is 0 or one vector is a multiple of the other.

pf: $\because |\cos \theta| \leq 1, \quad \|x^T y\| \leq \|x\| \|y\| \quad (\text{from previous thm})$

Definition: The vectors x and y in \mathbb{R}^2 (or \mathbb{R}^3) are said to be orthogonal if $x^T y = 0$.

Ex 3: Vectors $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$ are orthogonal in \mathbb{R}^2 .

$$\text{Since } x^T y = -12 + 12 = 0$$

Ex 4: The vector 0 is orthogonal to every vector in \mathbb{R}^2 .

Scalar and Vector projections

Let x and y be vectors in \mathbb{R}^2 (or \mathbb{R}^3).

Vector projection of x onto y is

$$P = c y, \quad c \text{ is a scalar.}$$

From fig it is clear that,

y and $x - P$ are orthogonal.

$$\therefore y^T(x - P) = 0$$

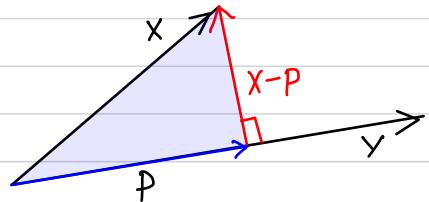
$$\Rightarrow y^T(x - cy) = 0$$

$$\Rightarrow y^T x - cy^T y = 0$$

$$\text{or } c = \frac{y^T x}{y^T y}$$

∴ Vector projection of x onto y is

$$P = y \frac{y^T x}{y^T y}$$



Scalar projection of x onto y is

$$d = c\|y\| = \|y\| \cdot \frac{y^T x}{y^T y} \Rightarrow d = \frac{y^T x}{\|y\|} \quad (\because y^T y = \|y\|^2)$$

Ex5: Determine the point Q on the line $y = \frac{1}{3}x$ that is closest to the point $(1, 4)$.

Soln: The vector $w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a vector in the direction of the line $y = \frac{1}{3}x$.

$$\text{Let } r = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

If Q is the desired point, then

\overrightarrow{OQ} is the vector projection of r onto w .



$$\therefore \overrightarrow{OQ} = w \frac{v^T w}{w^T w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \left(\frac{3+4}{9+1} \right) = \frac{7}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \overrightarrow{OQ} = \begin{bmatrix} 0.21 \\ 0.7 \end{bmatrix}$$

Thus, $Q = (0.21, 0.7)$ is the closest point.

Ex6: For each of the following pairs of vectors x and y ,

Find the vector projection p of x onto y .

a) $x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $x = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Soln: a) Scalar projection of x onto y is

$$\alpha = \frac{x^T y}{\|y\|} = \frac{3+5}{\sqrt{1+1}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

vector proj. of x onto y is

$$p = y \frac{x^T y}{y^T y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{8}{2} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

b) Scalar proj. of x onto y is

$$\alpha = \frac{x^T y}{\|y\|} = \frac{2 - 10 - 4}{\sqrt{6}} = \frac{-12}{\sqrt{6}} = -2\sqrt{6}$$

Vector proj.

$$P = y \frac{x^T y}{y^T y} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \left(\frac{-12}{6} \right) = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

Notation: Vector projection of x onto y is denoted
by $\text{Proj}_y(x)$

Inner product space

It is a generalization of scalar products to other vector spaces.

Defn: Let V be a vector space. The inner product on V is a function, $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfying the foll. conditions:

a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

b) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$

c) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{R}$.

A vector space with inner product is called an inner product space.

Ex 1: The vector space \mathbb{R}^n

i) The standard inner product in \mathbb{R}^n is the scalar product

$$\langle x, y \rangle = x^T y$$

ii) We could also define inner product by

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i w_i, \text{ where vector } w \text{ is a}$$

vector with positive entries, and w_i are referred as weights.

Ex 2: The vector space $\mathbb{R}^{m \times n}$

Let A and B in $\mathbb{R}^{m \times n}$, we define inner product by

$$\langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij} \quad \text{--- ①}$$

Let us verify it.

a) $\langle A, A \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \geq 0$ and $\langle A, A \rangle = 0$ iff $A = 0$

$$b) \langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij} = \sum_{i=1}^m \sum_{j=1}^n b_{ij} a_{ij} = \langle B, A \rangle.$$

$$\begin{aligned} c) \langle \alpha A + \beta B, C \rangle &= \sum_{i=1}^m \sum_{j=1}^n (\alpha a_{ij} + \beta b_{ij}) c_{ij} \\ &= \sum_{i=1}^m \sum_{j=1}^n \alpha (a_{ij} c_{ij}) + \beta (b_{ij} c_{ij}) \\ &= \alpha \sum_{i=1}^m \sum_{j=1}^n a_{ij} c_{ij} + \beta \sum_{i=1}^m \sum_{j=1}^n b_{ij} c_{ij} \\ &= \alpha \langle A, C \rangle + \beta \langle B, C \rangle \end{aligned}$$

Thus, ① is an inner product.

Ex 3: The Vector Space $C[a, b]$

We define inner product by

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx \quad \text{--- } ②$$

Let's verify.

$$a) \langle f, f \rangle = \int_a^b (f(x))^2 dx \geq 0 \quad \text{and } \langle f, f \rangle = 0 \text{ iff } f \equiv 0.$$

$$b) \langle f, g \rangle = \int_a^b f(x) g(x) dx = \int_a^b g(x) f(x) dx$$

$$= \langle g, f \rangle$$

$$c) \langle \alpha f + \beta g, h \rangle = \int_a^b (\alpha f(x) + \beta g(x)) h(x) dx$$

$$= \alpha \int_a^b f(x) h(x) dx + \beta \int_a^b g(x) h(x) dx$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

Hence ② is an inner product.

If $w(x)$ is a positive continuous fn. on $[a, b]$, then

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) dx$$

also defines an inner product on $C[a, b]$. The fn. $w(x)$ is called weight fn.

Ex 4: The Vector space P_n .

Let x_1, x_2, \dots, x_n be distinct real numbers. For each pair of polynomials in P_n , we define an inner product by

$$\langle p, q \rangle = \sum_{i=1}^n p(x_i) q(x_i)$$

a) $\langle p, p \rangle = \sum_{i=1}^n (p(x_i))^2 \geq 0$ and $\langle p, p \rangle = 0$ iff

x_1, x_2, \dots, x_n are roots of $p(x)=0$, but $p(x)$ is a polynomial of degree $< n$, it must be zero polynomial.

b) $\langle p, q \rangle = \sum_{i=1}^n p(x_i) q(x_i) = \sum_{i=1}^n q(x_i) p(x_i)$

$$= \langle q, p \rangle$$

c) $\langle \alpha p + \beta q, r \rangle = \sum_{i=1}^n (\alpha p(x_i) + \beta q(x_i)) r(x_i)$

$$= \alpha \sum_{i=1}^n p(x_i) r(x_i) + \beta \sum_{i=1}^n q(x_i) r(x_i)$$

$$= \alpha \langle p, r \rangle + \beta \langle q, r \rangle$$

Basic properties of inner product

1) If v is a vector in an inner product space V , the length or norm of v is given by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

2) Two vectors u and v in an inner product space V is orthogonal if $\langle u, v \rangle = 0$.

3) Unit vector along a vector u is $\frac{u}{\|u\|}$.

Theorem [The Pythagorean law]:

If u and v are orthogonal vectors in an inner product space

V , then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Pf:

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v \rangle \\&= \langle u+v, u \rangle + \langle u+v, v \rangle \\&= \langle u, u \rangle + \langle v, u \rangle + \langle u, v \rangle + \langle v, v \rangle \\&= \|u\|^2 + 2\langle u, v \rangle + \|v\|^2 \\&= \|u\|^2 + \|v\|^2 \quad (\because \langle u, v \rangle = 0)\end{aligned}$$

Ex1: Consider the vector space $[-1, 1]$ with inner product defined by

$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx.$$

- i) Show that vectors l and x are orthogonal.
- ii) Find length of l and length of x .

Soln: i) We show that $\langle 1, x \rangle = 0$.

By defn.

$$\begin{aligned}\langle 1, x \rangle &= \int_{-1}^1 1 \cdot x \, dx = \frac{x^2}{2} \Big|_{-1}^1 \\ &= \frac{1}{2} - \frac{1}{2} = 0\end{aligned}$$

$\therefore 1$ and x are orthogonal.

$$ii) \|1\|^2 = \langle 1, 1 \rangle = \int_{-1}^1 1 \cdot 1 \, dx = x \Big|_{-1}^1 = 2$$

$$\therefore \|1\| = \sqrt{2}$$

$$\|x\|^2 = \langle x, x \rangle = \int_{-1}^1 x^2 \, dx = \frac{x^3}{3} \Big|_{-1}^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\therefore \|x\| = \sqrt{\frac{2}{3}}$$

Since 1 and x are orthogonal, by pythagorean law

$$\begin{aligned}\|1+x\|^2 &= \|1\|^2 + \|x\|^2 \\ &= 2 + \frac{2}{3} = \frac{8}{3}\end{aligned}$$

Ex2: For the vector space $C[-\pi, \pi]$ define inner product by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx, \quad \frac{1}{\pi} \text{ is weight fn.}$$

i) S.t $\cos x$ and $\sin x$ are orthogonal.

ii) Find length of $\cos x$, $\sin x$ and $\cos x + \sin x$.

Soln: i) We s.t $\langle \cos x, \sin x \rangle = 0$.

By defn,

$$\langle \cos x, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin x \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin 2x \, dx$$

$$= 0 \quad (\because \sin(-2x) = -\sin(2x))$$

ii) $\|\cos x\|^2 = \langle \cos x, \cos x \rangle$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x \, dx \quad (\cos^2 x = \frac{1 + \cos 2x}{2})$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + \cos 2x) \, dx$$

$$= \frac{1}{2\pi} \left[x + \frac{\sin 2x}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} (\pi + \pi) = 1$$

$$\therefore \|\cos x\| = 1$$

$$\|\sin x\|^2 = \langle \sin x, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{1}{\pi} \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = 1$$

$$\therefore \|\sin x\| = 1$$

By Pythagorean law :

$$\|\cos x + \sin x\|^2 = \|\cos x\|^2 + \|\sin x\|^2$$

$$= 1 + 1 = 2$$

$$\therefore \|\cos x + \sin x\| = \sqrt{2}.$$

Ex3: Let A and B vectors in $\mathbb{R}^{3 \times 2}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 3 & 0 \\ -3 & 4 \end{bmatrix}.$$

Inner product is define by-

$$\langle A, B \rangle = \sum_{i=1}^3 \sum_{j=1}^2 a_{ij} b_{ij}$$

i) Find $\langle A, B \rangle$, $\|A\|$ and $\|B\|$

Soln: By defn.

$$\begin{aligned} \langle A, B \rangle &= 1 \cdot -1 + 1 \cdot 1 + 1 \cdot 3 + 2 \cdot 0 + 3 \cdot -3 + 3 \cdot 4 \\ &= -1 + 1 + 3 - 9 + 12 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \|A\|^2 &= \langle A, A \rangle = 1^2 + 1^2 + 1^2 + 2^2 + 3^2 + 3^2 \\ &= 25 \end{aligned}$$

$$\therefore \|A\| = 5$$

$$\begin{aligned} \|B\|^2 &= \langle B, B \rangle = 1^2 + 1^2 + 3^2 + 0^2 + (-3)^2 + 4^2 \\ &= 36 \end{aligned}$$

$$\therefore \|B\| = 6.$$

Note: Norm defined above i.e

$$\|A\| = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \text{ is called Frobenius norm}$$

denoted by $\|\cdot\|_F$

Defn: If u and v are vectors in an inner product space V and $v \neq 0$, then

The scalar projection of u onto v is

$$\alpha = \frac{\langle u, v \rangle}{\|v\|} \quad \text{--- (a)}$$

and the vector projection of u onto v is

$$\text{Proj}_v(u) = v \frac{\langle u, v \rangle}{\langle v, v \rangle} \quad \text{--- (b)}$$

From (a) and (b), $\text{Proj}_v(u) = \alpha \frac{v}{\|v\|}$

Observations

If $v \neq 0$ and $p = \text{Proj}_v(u)$, then

- 1) $u - p$ and p are orthogonal.
- 2) $u = p$ iff u is a scalar multiple of v .

Pf of 1) : Consider

$$\langle u - p, p \rangle$$

$$= \langle u, p \rangle - \langle p, p \rangle$$

$$= \left\langle u, \frac{\alpha v}{\|v\|} \right\rangle - \left\langle \frac{\alpha v}{\|v\|}, \frac{\alpha v}{\|v\|} \right\rangle$$

$$= \frac{\alpha}{\|v\|} \langle u, v \rangle - \frac{\alpha^2}{\|v\|^2} \langle v, v \rangle$$

$$= \alpha^2 - \alpha^2 \quad \left(\because \langle v, v \rangle = \|v\|^2 \right)$$

$$= 0$$

Pf 2) : If $u = \beta v$, then

$$\text{proj}_v(u) = v \frac{\langle \beta v, v \rangle}{\langle v, v \rangle} = \beta v = u$$

Conversely, if $u = \text{proj}_v(u)$, Then

$$u = \alpha \frac{v}{\|v\|} = \beta v, \text{ where } \beta = \frac{v}{\|v\|}$$

Theorem: The Cauchy - Schwarz inequality

If u and v are any two vectors in an inner product

space V , then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

Equality hold iff u and v are linearly dependent.

Pf: If $p = \text{proj}_v(u)$, Then $u-p$ and p are orthogonal.

∴ by Pythagorean law

$$\|u-p\|^2 + \|p\|^2 = \|u\|^2$$

$$\Rightarrow \|p\|^2 = \|u\|^2 - \|u-p\|^2$$

$$\text{But } \|p\| = \frac{\langle u, v \rangle}{\|v\|}$$

$$\Rightarrow \frac{(\langle u, v \rangle)^2}{\|v\|^2} = \|u\|^2 - \|u-p\|^2$$

$$\Rightarrow (\langle u, v \rangle)^2 = \|u\|^2 \|v\|^2 - \|u-p\|^2 \|v\|^2 \quad \textcircled{*}$$
$$\leq \|u\|^2 \|v\|^2$$

Equality holds in $\textcircled{*}$ iff $u=p$ or $v=0$

or simply u and v are dependent.

Orthonormal Sets

Defn: Non-zero vectors v_1, v_2, \dots, v_n in an inner product space V is said to be an orthogonal set if

$$\langle v_i, v_j \rangle = 0 \text{ whenever } i \neq j$$

Ex: The set $\{(1, 1, 1)^T, (2, 1, -3)^T, (4, -5, 1)^T\}$ is an orthogonal set in \mathbb{R}^3 , since

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} = 0$$

Theorem: If $\{v_1, v_2, \dots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space V , then v_1, v_2, \dots, v_n are linearly independent.

Pf : Let v_1, v_2, \dots, v_n be orthogonal set.

$$\text{Let } c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad \text{--- (1)}$$

By taking the inner product of v_j with both sides of (1), we see that $(1 \leq j \leq n)$

$$c_1 \langle v_j, v_1 \rangle + c_2 \langle v_j, v_2 \rangle + \dots + c_n \langle v_j, v_n \rangle = 0$$

$$\Rightarrow c_j \|v_j\|^2 = 0$$

and hence all the scalars c_1, c_2, \dots, c_n must be 0.

Defn: An orthonormal set of vectors is an orthogonal set of unit vectors.

Ex: Let v_1, v_2 and v_3 be orthogonal set in \mathbb{R}^3 , where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}.$$

Form an orthonormal set.

Soln: let $u_1 = \frac{v_1}{\ v_1\ }$ $\ v_1\ = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$	let $u_2 = \frac{v_2}{\ v_2\ }$ $\ v_2\ = \sqrt{4+1+9} = \sqrt{14}$	let $u_3 = \frac{v_3}{\ v_3\ }$ $\ v_3\ = \sqrt{16+25+1} = \sqrt{42}$
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\therefore orthonormal set is

$$u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}, \quad u_3 = \begin{bmatrix} 4/\sqrt{42} \\ -5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$$

Ex: Let $C[-\pi, \pi]$ be an inner product space, where inner product is defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx$$

S.T The set $\{1, \cos x, \cos 2x, \dots, \cos nx\}$ is an orthogonal set.

and form its orthonormal set.

Soln: Consider

$$\langle 1, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx dx = 0$$

$$\langle \cos jx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jx \cdot \cos kx dx \quad (j \neq k)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos(j+k)x + \cos(j-k)x) dx$$

$$= \frac{1}{2\pi} \left[\frac{\sin(j+k)x}{j+k} + \frac{\sin(j-k)x}{j-k} \right]_{-\pi}^{\pi}$$

$$= 0$$

Thus, the set is orthogonal.

$$\text{Consider } \|1\|^2 = \langle 1, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 dx = 2 \Rightarrow \|1\| = \sqrt{2}$$

$$\text{and } \|\cos kx\|^2 = \langle \cos kx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 kx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 + \cos 2x}{2} dx$$

$$= \frac{1}{2\pi} \left[x + \frac{\sin 2x}{2} \right]_{-\pi}^{\pi}$$

$$= 1$$

\therefore The orthonormal set is

$$\left\{ \frac{1}{\sqrt{2}}, \cos x, \cos 2x, \dots, \cos nx \right\}$$

Orthonormal basis

A set $S = \{u_1, u_2, \dots, u_n\}$ is said to be an orthonormal basis of an inner product space V if

- i) $S = \{u_1, u_2, \dots, u_n\}$ is orthonormal set.
- ii) $\text{Span}(S) = V$

Theorem: Let $S = \{u_1, u_2, \dots, u_n\}$ be an orthonormal basis for an inner product space V . For any vector $v \in V$

Suppose $v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$. Then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \vdots \\ \langle v, u_n \rangle \end{bmatrix}$

i.e. Coordinate vector of v w.r.t S is

$$[v]_S = \begin{bmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \vdots \\ \langle v, u_n \rangle \end{bmatrix}.$$

Pf: Let $v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n = \sum_{i=1}^n c_i u_i$

By taking inner product with u_j on both sides,

$$\begin{aligned} \langle v, u_j \rangle &= \left\langle \sum_{i=1}^n c_i u_i, u_j \right\rangle \\ &= \sum_{i=1}^n c_i \langle u_i, u_j \rangle = \sum_{i=1}^n c_i \delta_{ij} = c_j \end{aligned}$$

Thus, $[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \vdots \\ \langle v, u_n \rangle \end{bmatrix}$

Corollary: Let $S = \{u_1, u_2, \dots, u_n\}$ be orthonormal basis for an inner product space V . If $u = \sum_{i=1}^n a_i u_i$ and $v = \sum_{i=1}^n b_i u_i$, then

i) $\langle u, v \rangle = \sum_{i=1}^n a_i b_i \quad (= \langle [u]_S, [v]_S \rangle)$

ii) $\|v\|^2 = \sum_{i=1}^n c_i^2 \quad (= \| [v]_S \|^2) \quad [\text{Parseval's formula}]$

pf : Consider

$$\begin{aligned}
 i) \quad \langle u, v \rangle &= \left\langle \sum_{i=1}^n a_i u_i, v \right\rangle = \sum_{i=1}^n a_i \langle u_i, v \rangle \\
 &= \sum_{i=1}^n a_i \langle v, u_i \rangle \\
 &= \sum_{i=1}^n a_i b_i \quad (\because b_i = \langle v, u_i \rangle)
 \end{aligned}$$

$$ii) \quad \|v\|^2 = \langle v, v \rangle = \sum_{i=1}^n c_i^2$$

Ex: Let $S = \{u_1, u_2, u_3\}$ be orthonormal basis for \mathbb{R}^3 ,

where

$$u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}, \quad u_3 = \begin{bmatrix} 4/\sqrt{42} \\ -5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$$

Let $x = \begin{bmatrix} 2 \\ 4 \\ -7 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ be vectors in \mathbb{R}^3 .

i) Determine coordinate vectors of x and y wrt S .

(That is, $[x]_S$ and $[y]_S$)

ii) Verify that $\langle x, y \rangle = \langle [x]_S, [y]_S \rangle$

iii) Verify that $\|x\| = \|[x]_S\|$.

$$\text{Soln i)} \quad [x]_S = \begin{bmatrix} \langle x, u_1 \rangle \\ \langle x, u_2 \rangle \\ \langle x, u_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{7}{\sqrt{3}} \\ \frac{4}{\sqrt{14}} + \frac{4}{\sqrt{14}} + \frac{21}{\sqrt{14}} \\ \frac{8}{\sqrt{42}} - \frac{20}{\sqrt{42}} - \frac{7}{\sqrt{42}} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ 29/\sqrt{14} \\ -19/\sqrt{42} \end{bmatrix}$$

$$[y]_S = \begin{bmatrix} \langle y, u_1 \rangle \\ \langle y, u_2 \rangle \\ \langle y, u_3 \rangle \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ -11/\sqrt{14} \\ -1/\sqrt{42} \end{bmatrix}$$

$$i) \quad \langle x, y \rangle = -2 + 0 - 21 = -23$$

$$\langle [x]_s, [y]_s \rangle = -\frac{2}{3} - \frac{319}{14} + \frac{19}{42} = \frac{-28 - 1157 + 19}{42} = -23$$

$$ii) \quad \|x\| = \sqrt{4+16+49} = \sqrt{69}$$

$$\|[x]_s\| = \sqrt{\frac{1}{3} + \frac{29^2}{14} + \frac{19^2}{42}} = \sqrt{\frac{14 + 3 \times 29^2 + 19^2}{42}} = \sqrt{69}$$

Orthogonal Matrices

Defn: An $n \times n$ matrix Q is said to be orthogonal matrix if the column vectors of Q form an orthonormal set in \mathbb{R}^n .

Theorem: An $n \times n$ matrix is orthogonal iff $Q^T Q = I$.

Pf: It follows from the defn.

This implies that $Q^T = Q^{-1}$.

Examples: 1) For a fixed θ

$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is orthogonal.

Because $Q^T Q = I$

2) Permutation matrix,

$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is orthogonal.

Since $Q^T Q = I$

3) Let $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is orthogonal too.

Properties of Orthogonal matrix:

If Q is an $n \times n$ orthogonal matrix, then

- The column vectors of Q form an orthonormal basis for \mathbb{R}^n .
- $Q^T Q = I$
- $Q^T = Q^{-1}$
- Inner product are preserved under multiplication by an orthogonal matrix.

$$\text{i.e. } \langle x, y \rangle = \langle Qx, Qy \rangle$$

$$\text{Because, } \langle Qx, Qy \rangle = (Qx)^T Qy$$

$$= (x^T Q^T) Qy$$

$$= x^T (Q^T Q)y$$

$$= x^T I y = x^T y = \langle x, y \rangle$$

- Multiplication by an orthogonal matrix preserves the length of vectors.

$$\text{i.e., } \|Qx\| = \|x\|$$

$$\text{Since from d)} \quad \langle Qx, Qx \rangle = \|Qx\|^2 = \langle x, x \rangle = \|x\|^2.$$

Note: If Q is any $m \times n$ rectangular matrix such that column vectors are orthonormal vectors, then

$$Q^T Q = I, \text{ but } Q Q^T \text{ need not be } I.$$

Ex:

$$\text{Let } Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \quad \text{Clearly column vectors of } Q \text{ are orthonormal vectors.}$$

$$Q^T Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem (Projection onto subspaces with orthonormal bases.)

Let V be a subspace of \mathbb{R}^n and $B = \{u_1, u_2, u_3, \dots, u_k\}$ be orthonormal basis of V .

Then projection of $x \in \mathbb{R}^n$ onto the subspace V is

$$\begin{aligned}\text{Proj}_V(x) &= \text{Proj}_{u_1}(x) + \text{Proj}_{u_2}(x) + \dots + \text{Proj}_{u_k}(x) \\ &= \langle u_1, x \rangle u_1 + \langle u_2, x \rangle u_2 + \dots + \langle u_k, x \rangle u_k\end{aligned}$$

Moreover, if

$$Q = \begin{bmatrix} u_1 & u_2 & \dots & u_k \end{bmatrix}_{n \times k},$$

then

$$\text{Proj}_V(x) = Q Q^T(x)$$

Where, $Q Q^T$ is called the projection matrix onto V with orthonormal basis $\{u_1, u_2, \dots, u_k\}$.

Pf: Let $x \in \mathbb{R}^n$ be any vector. Then $x = v + w$, where $v \in V$ and $w \in V^\perp$. Thus, $v = \text{Proj}_V(x)$ and $w = \text{Proj}_{V^\perp}(x)$.

Since B is orthonormal basis of V

$$v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k \quad \text{for some } c_i \in \mathbb{R} \quad \text{(1)}$$

$$\text{and } x = c_1 u_1 + c_2 u_2 + \dots + c_k u_k + w \quad \text{(2)}$$

$$V^\perp = \{w \in \mathbb{R}^n \mid \langle w, v \rangle = 0 \text{ for every } v \in V\}$$

Called orthogonal complement of V .
 V^\perp is also subspace of \mathbb{R}^n

$$\Rightarrow \langle u_1, x \rangle = c_1 \cancel{\langle u_1, u_1 \rangle} + c_2 \cancel{\langle u_1, u_2 \rangle} + \dots + c_k \cancel{\langle u_1, u_k \rangle} + \cancel{\langle u_1, w \rangle} = c_1$$

$$\text{In general, } \langle u_i, x \rangle = c_i, \quad i = 1, 2, \dots, k$$

Thus,

$$\begin{aligned}v &= \text{Proj}_V(x) = \langle u_1, x \rangle u_1 + \langle u_2, x \rangle u_2 + \dots + \langle u_k, x \rangle u_k \quad (\text{from (1)}) \\ &= \text{Proj}_{u_1}(x) + \text{Proj}_{u_2}(x) + \dots + \text{Proj}_{u_k}(x)\end{aligned}$$

OR

We have

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \langle u_1, x \rangle \\ \langle u_2, x \rangle \\ \vdots \\ \langle u_k, x \rangle \end{bmatrix} \quad \text{--- } (3)$$

From (2),

$$X = \begin{bmatrix} u_1 & u_2 & \dots & u_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} + w$$

Taking Q^T on BS.

$$Q^T X = Q^T Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} + Q^T w \quad \left(\text{Since } Q^T Q = I \right)$$

$$Q^T w = \begin{bmatrix} u_1^T w \\ u_2^T w \\ \vdots \\ u_k^T w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$Q^T X = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \quad \text{---} \quad ④$

$$\therefore \text{proj}_V(x) = \langle u_1, x \rangle u_1 + \langle u_2, x \rangle u_2 + \cdots + \langle u_k, x \rangle u_k$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix} \begin{bmatrix} \langle u_1, x \rangle \\ \langle u_2, x \rangle \\ \vdots \\ \langle u_k, x \rangle \end{bmatrix} = (Q Q^T) x$$

(from ③ and ④)

Ex: Let $V = \text{span}(u_1, u_2)$ be subspace in \mathbb{R}^3 , let $x \in \mathbb{R}^3$

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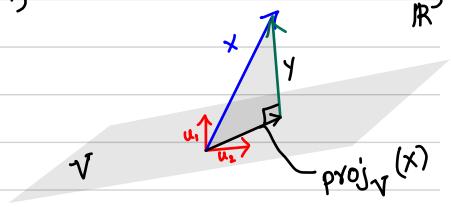
$$U_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Obtain i) Projection of x onto the plane V .

ii) the shortest distance between the point x and the plane V .

Soln: Clearly, $\{u_1, u_2\}$ is an orthonormal basis of V .

Let $Q = \begin{bmatrix} 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \\ 1/3 & 0 \end{bmatrix}$, $Q^T = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$



i) projection of x onto V is

$$\begin{aligned} P_{\text{proj}_V} x &= \text{proj}_{u_1} x + \text{proj}_{u_2} x \\ &= \langle u_1, x \rangle u_1 + \langle u_2, x \rangle u_2 \\ &= 2 \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} + \frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 17/6 \\ -1/6 \\ 2/3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \langle u_1, x \rangle &= \frac{4}{3} - \frac{2}{3} + \frac{4}{3} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \langle u_2, x \rangle &= \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} \end{aligned}$$

Or

$$\text{projection matrix onto } V = Q Q^T = \begin{bmatrix} \frac{17}{18} & -\frac{1}{18} & \frac{2}{9} \\ -\frac{1}{18} & \frac{17}{18} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix}$$

\therefore proj of x onto V is

$$\text{proj}_V(x) = Q Q^T x = \begin{bmatrix} \frac{17}{18} & -\frac{1}{18} & \frac{2}{9} \\ -\frac{1}{18} & \frac{17}{18} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 17/6 \\ -1/6 \\ 2/3 \end{bmatrix}$$

$$\text{ii}) y = x - \text{proj}_V(x) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 51/18 \\ -1/6 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -5/6 \\ -5/6 \\ 10/3 \end{bmatrix}$$

\therefore Shortest distance is

$$\|y\| = \|x - \text{proj}_V(x)\| = \left\| \begin{bmatrix} -5/6 \\ -5/6 \\ 10/3 \end{bmatrix} \right\| = \sqrt{\frac{25}{36} + \frac{25}{36} + \frac{100}{9}} = \frac{5\sqrt{2}}{2}$$

The Gram-Schmidt Orthogonalization Process

We discuss a process of constructing orthonormal basis for an n -dimensional inner product space V .

In particular, we learn to transform an ordinary basis $\{w_1, w_2, w_3, \dots, w_n\}$ into an orthonormal basis $\{u_1, u_2, u_3, \dots, u_n\}$

Construction is such that

$$\text{Span}(w_1, w_2, w_3, \dots, w_n) = \text{Span}(u_1, u_2, u_3, \dots, u_n)$$

$$\text{Let } V_1 = \text{Span}(w_1), \quad V_2 = \text{Span}(w_1, w_2).$$

In general

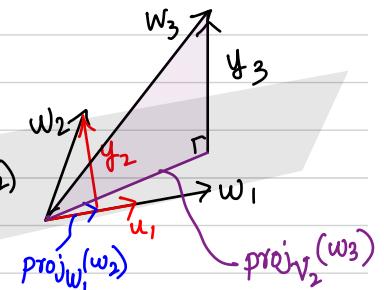
$$V_i = \text{Span}(w_1, w_2, \dots, w_i), \quad i = 1, 2, 3, \dots, n$$

$$\text{and } V_n = V.$$

$$\text{Let } u_1 = \frac{w_1}{\|w_1\|}. \quad \text{Then } V_1 = \text{Span}(u_1) \text{ and } V_2 = \text{Span}(u_1, u_2)$$

$$\text{From fig, } y_2 = w_2 - \text{proj}_{V_1}(w_2)$$

$$\Rightarrow y_2 = w_2 - \langle u_1, w_2 \rangle u_1, \quad u_2 = \frac{y_2}{\|y_2\|} \quad V_2 = \text{Span}(w_1, w_2)$$



$$\therefore V_2 = \text{Span}(u_1, u_2) \text{ and } V_3 = \text{Span}(u_1, u_2, u_3)$$

$$\text{Now, } y_3 = w_3 - \text{proj}_{V_2}(w_3)$$

$$= w_3 - \text{proj}_{u_1}(w_3) - \text{proj}_{u_2}(w_3)$$

$$\Rightarrow y_3 = w_3 - \langle u_1, w_3 \rangle u_1 - \langle u_2, w_3 \rangle u_2, \quad u_3 = \frac{y_3}{\|y_3\|}$$

$$\therefore V_3 = \text{Span}(u_1, u_2, u_3)$$

In general,

$$y_i = w_i - \text{proj}_{V_i}(w_i)$$

$$= w_i - \text{proj}_{U_1}(w_i) - \text{proj}_{U_2}(w_i) - \cdots - \text{proj}_{U_{i-1}}(w_i)$$

$$\Rightarrow y_i = w_i - \langle u_1, w_i \rangle u_1 - \langle u_2, w_i \rangle u_2 - \cdots - \langle u_{i-1}, w_i \rangle u_{i-1}$$

$$\text{and } u_i = \frac{y_i}{\|y_i\|}, \quad i = 1, 2, \dots, n$$

Ex1: Let

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Find orthonormal basis for the column space of A.

Soln: Let

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

w_1, w_2 and w_3 are LI, hence form a basis for 3-dimensional subspace of \mathbb{R}^4 .

We use Gram-Schmidt process to construct orthonormal basis of u_1, u_2, u_3

$$u_1 = \frac{w_1}{\|w_1\|} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\text{Let } y_2 = w_2 - \langle u_1, w_2 \rangle u_1, \quad u_2 = \frac{y_2}{\|y_2\|}$$

$$\text{where } \langle u_1, w_2 \rangle = u_1^T w_2 = -\frac{1}{2} + \frac{4}{2} + \frac{4}{2} - \frac{1}{2} = 3$$

$$\therefore y_2 = w_2 - 3u_1$$

$$= \begin{bmatrix} -1 - 3 \cdot \frac{1}{2} \\ 4 - 3 \cdot \frac{1}{2} \\ 4 - 3 \cdot \frac{1}{2} \\ -1 - 3 \cdot \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix}$$

$$\|y_2\| = \sqrt{\frac{25}{4} + \frac{25}{4} + \frac{25}{4} + \frac{25}{4}} = \sqrt{\frac{100}{4}} = 5$$

$$\therefore u_2 = \frac{y_2}{\|y_2\|}$$

$$\Rightarrow u_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\text{Let } y_3 = w_3 - \langle u_1, w_3 \rangle u_1 - \langle u_2, w_3 \rangle u_2 \quad \text{and } u_3 = \frac{y_3}{\|y_3\|}$$

$$\text{where } \langle u_1, w_3 \rangle = u_1^T w_3 = \frac{4}{2} - \frac{2}{2} + \frac{2}{2} + 0 = 2$$

$$\langle u_2, w_3 \rangle = u_2^T w_3 = -\frac{1}{2} - \frac{2}{2} + \frac{2}{2} + 0 = -2$$

$$\therefore y_3 = w_3 - 2u_1 + 2u_2$$

$$= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\|y_3\| = \sqrt{4+4+4+4} = \sqrt{16} = 4$$

$$\therefore u_3 = \frac{y_3}{\|y_3\|} \Rightarrow u_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Thus orthonormal basis is $u_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, u_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

Gram - Schmidt QR factorization

If A is an $m \times n$ matrix of rank n , then A can be factored into QR ,

where Q is $m \times n$ matrix with orthonormal column vectors

R is $n \times n$ upper triangular matrix (diagonal entries are positive)

Pf: Let

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

Given that $\{a_1, a_2, \dots, a_n\}$ are L.I ($\because \text{rank}(A) = n$)

Let $\{u_1, u_2, \dots, u_n\}$ be orthonormal basis for $C(A)$.

$$\text{Let } \gamma_{11} = \|a_1\|, \quad \gamma_{kk} = \left\| a_k - \sum_{i=1}^{k-1} \langle a_k, u_i \rangle u_i \right\|, \quad k=2,3,\dots,n.$$

Then by Gram - Schmidt process,

$$u_1 = \frac{a_1}{\gamma_{11}}$$

$$\Rightarrow a_1 = \gamma_{11} u_1$$

$$u_2 = \frac{a_2 - \langle u_1, a_2 \rangle u_1}{\gamma_{22}}$$

$$\Rightarrow a_2 = \langle u_1, a_2 \rangle u_1 + \gamma_{22} u_2$$

$$u_3 = \frac{a_3 - \langle u_1, a_3 \rangle u_1 - \langle u_2, a_3 \rangle u_2}{\gamma_{33}}$$

$$\Rightarrow a_3 = \langle u_1, a_3 \rangle u_1 + \langle u_2, a_3 \rangle u_2 + \gamma_{33} u_3$$

$$u_n = \frac{a_n - \langle u_1, a_n \rangle u_1 - \dots - \langle u_{n-1}, a_n \rangle u_{n-1}}{\gamma_{nn}}$$

$$\Rightarrow a_n = \langle u_1, a_n \rangle u_1 + \langle u_2, a_n \rangle u_2 + \dots + \langle u_{n-1}, a_n \rangle u_{n-1} + \gamma_{nn} a_n$$

We have $\gamma_{kk} = \langle u_k, a_k \rangle$

Consider

$$y_{kk} u_k = a_k - \langle u_1, a_k \rangle u_1 - \cdots - \langle u_{k-1}, a_k \rangle u_{k-1}$$

$$\Rightarrow \langle u_k, \gamma_{kk} u_k \rangle = \langle u_k, a_k - \langle u_1, a_k \rangle u_1 - \cdots - \langle u_{k-1}, a_k \rangle u_{k-1} \rangle$$

$$\Rightarrow \gamma_{kk} < u_k, u_k >$$

$$= \langle u_k, a_k \rangle - \langle u_1, a_k \rangle \langle u_k, u_1 \rangle - \dots - \langle u_{k-1}, a_k \rangle \langle u_k, u_{k-1} \rangle$$

But $\langle u_k, u_k \rangle = 1$ and $\langle u_k, u_j \rangle = 0$, $j = \{1, 2, \dots, k-1\}$

$$\therefore \gamma_{KK} = \langle u_K, q_K \rangle$$

Therefore,

$$a_1 = \langle u_1, a_1 \rangle u_1$$

$$a_2 = \langle u_1, a_2 \rangle u_1 + \langle u_2, a_2 \rangle u_2$$

$$a_3 = \langle u_1, a_3 \rangle u_1 + \langle u_2, a_3 \rangle u_2 + \langle u_3, a_3 \rangle u_3$$

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$$a_n = \langle u_1, a_n \rangle u_1 + \langle u_2, a_n \rangle u_2 + \cdots + \langle u_n, a_n \rangle u_n$$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \cdots & \langle u_1, a_n \rangle \\ 0 & \langle u_2, a_2 \rangle & \cdots & \langle u_2, a_n \rangle \\ 0 & 0 & \cdots & \langle u_2, a_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle u_n, a_n \rangle \end{bmatrix}$$

A Q R

$$\Rightarrow A = Q R,$$

OR

From Gram - Schmidt process,

if $\{u_1, u_2, \dots, u_n\}$ are orthonormal set of $\{a_1, a_2, \dots, a_n\}$,

then

$$a_1 \in \text{Span}(u_1) \quad (\because \text{Span}(a_1) = \text{Span}(u_1))$$

$$\Rightarrow a_1 = \langle u_1, a_1 \rangle u_1$$

$$a_2 \in \text{Span}(u_1, u_2) \quad (\because \text{Span}(a_1, a_2) = \text{Span}(u_1, u_2))$$

$$\Rightarrow a_2 = \langle u_1, a_2 \rangle u_1 + \langle u_2, a_2 \rangle u_2$$

$$a_3 \in \text{Span}(u_1, u_2, u_3) \quad (\because \text{Span}(a_1, a_2, a_3) = \text{Span}(u_1, u_2, u_3))$$

$$\Rightarrow a_3 = \langle u_1, a_3 \rangle u_1 + \langle u_2, a_3 \rangle u_2 + \langle u_3, a_3 \rangle u_3$$

:

$$a_n \in \text{Span}(u_1, u_2, \dots, u_n)$$

$$\Rightarrow a_n = \langle u_1, a_n \rangle u_1 + \langle u_2, a_n \rangle u_2 + \dots + \langle u_n, a_n \rangle u_n$$

If we set

$$Q = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}, \quad R = \begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \langle u_1, a_3 \rangle & \dots & \langle u_1, a_n \rangle \\ 0 & \langle u_2, a_2 \rangle & \langle u_2, a_3 \rangle & \dots & \langle u_2, a_n \rangle \\ 0 & 0 & \langle u_3, a_3 \rangle & \dots & \langle u_3, a_n \rangle \\ 0 & 0 & 0 & \dots & \langle u_4, a_n \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \langle u_n, a_n \rangle \end{bmatrix}$$

Then

$$A = Q R$$

Ex2: Obtain QR factorization for

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Soln: Let

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

If u_1, u_2 and u_3 are orthonormal sets, Then from previous

example 1,

$$u_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad u_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad u_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \langle u_1, a_3 \rangle \\ 0 & \langle u_2, a_2 \rangle & \langle u_2, a_3 \rangle \\ 0 & 0 & \langle u_3, a_3 \rangle \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

is required QR factorization of A.

Ex3: Compute the Gram-Schmidt QR factorization of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

Soln: Column vectors of A are linearly independent. Since $\text{rank}(A)=3$ (verify). Hence we can obtain QR factorization.

Let $a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 0 \end{bmatrix}$, $a_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$.

Let $\{u_1, u_2, u_3\}$ be corresponding orthonormal set.

Then

$$u_1 = \frac{a_1}{\|a_1\|}$$

where $\|a_1\| = (1^2 + 2^2 + 2^2 + 4^2)^{1/2} = 5$

$$\Rightarrow u_1 = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix}$$

$$\begin{aligned} y_2 &= a_2 - \text{proj}_{u_1} a_2 \\ &= a_2 - \langle u_1, a_2 \rangle u_1 \end{aligned}$$

$$= \begin{bmatrix} -2 \\ 0 \\ -4 \\ 0 \end{bmatrix} - \left(-\frac{2}{5}, -\frac{8}{5} \right) \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -8/5 \\ 4/5 \\ -16/5 \\ 8/5 \end{bmatrix}$$

$$\|y_2\| = \left(\frac{64+16+256+64}{25} \right)^{1/2} = 4$$

$$\therefore u_2 = \frac{y_2}{\|y_2\|} = \begin{bmatrix} -2/5 \\ 1/5 \\ -4/5 \\ 2/5 \end{bmatrix}$$

$$y_3 = a_3 - \langle u_1, a_3 \rangle u_1 - \langle u_2, a_3 \rangle u_2$$

$$= \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \left(-\frac{1}{5} + \frac{2}{5} + \frac{4}{5} \right) \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{bmatrix} - \left(\frac{2}{5} + \frac{1}{5} - \frac{8}{5} \right) \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{5} \\ \frac{4}{5} \\ \frac{4}{5} \\ -\frac{2}{5} \end{bmatrix}$$

$$\|y_3\| = \sqrt{\frac{64+16+16+4}{25}} = 2$$

$$\therefore u_3 = \frac{y_3}{\|y_3\|} = \begin{bmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

Thus

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \langle u_1, a_3 \rangle \\ 0 & \langle u_2, a_2 \rangle & \langle u_2, a_3 \rangle \\ 0 & 0 & \langle u_3, a_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \text{O } R$$

Recall : If A is an $m \times n$ matrix, then

i) Every vector of $R(A)$ is orthogonal to every vector of $N(A)$.

ii) $N(A) \cap R(A) = \{0\}$

iii) $\dim(R(A)) + \dim(N(A)) = \dim(\mathbb{R}^n)$

iv) Basis of $R(A) \cup$ Basis of $N(A)$ = Basis of \mathbb{R}^n .

Ex 4: Let $a_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

i) Extend this to an orthonormal basis of \mathbb{R}^3

ii) Suppose $\{u_1, u_2, u_3\}$ is orthonormal basis, where $u_1 = \frac{a_1}{\|a_1\|}$

- Find the projection matrix P_1 that projects vectors in \mathbb{R}^3 onto $\text{Span}(u_2, u_3)$.

- Find the projection matrix P_2 that projects vectors in \mathbb{R}^3 on $\text{Span}(u_1)$.

Soln: Let A be a matrix given by

$$A = [3 \ 1 \ -1]$$

Clearly, $R(A) = \text{Span}(a_1) \Rightarrow \text{Basis of } R(A) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

We know that $N(A)$ and $R(A)$ are orthogonal complements.

i.e every vector in $N(A)$ is orthogonal to every vector in $R(A)$

and $N(A) \cap R(A) = \{0\}$

$\therefore \text{Basis of } N(A) \cup \text{Basis of } R(A) = \text{Basis of } \mathbb{R}^3$.

Null Space of A :

Consider $Ax = 0$

$$\Rightarrow [3 \ 1 \ -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow 3x_1 + x_2 - x_3 = 0$$

Let $x_2 = k_1$, $x_3 = k_2$ be free variables

$$\Rightarrow x_1 = \frac{-k_1 + k_2}{3}$$

$$\therefore X = \begin{bmatrix} -\frac{k_1+k_2}{3} \\ k_1 \\ k_2 \end{bmatrix}$$

$$N(A) = \left\{ k_1 \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}, k_1, k_2 \in \mathbb{R} \right\}$$

Basis of $N(A) = \left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

Basis of $\mathbb{R}^3 = \left\{ \underbrace{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}}_{a_1}, \underbrace{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}_{a_2}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{a_3} \right\}$

clearly a_1 is orthogonal to a_2 and a_3 .

Let $u_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}$ (\because we use GS process to orthonormal set of $\{a_2, a_3\}$)

$$u_2 = \frac{a_2}{\|a_2\|} = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix}$$

$$y_3 = a_3 - \text{proj}_{u_2} a_3$$

$$= a_3 - \langle u_2, a_3 \rangle u_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \left(-\frac{1}{\sqrt{10}}\right) \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix} = \begin{bmatrix} 9/10 \\ 3/10 \\ 3 \end{bmatrix}$$

$$u_3 = \frac{y_3}{\|y_3\|}, \|y_3\| = \sqrt{\frac{99}{10}}$$

$$u_3 = \begin{bmatrix} \frac{9}{\sqrt{99}} \cdot \sqrt{10} \\ \frac{3}{\sqrt{99}} \sqrt{10} \\ \frac{30}{\sqrt{99}} \sqrt{10} \end{bmatrix}$$

Orthonormal basis of \mathbb{R}^3 .

$$\left\{ \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix}, \begin{bmatrix} 9/\sqrt{99}\sqrt{10} \\ 3/\sqrt{99}\sqrt{10} \\ 30/\sqrt{99}\sqrt{10} \end{bmatrix} \right\}$$

ii) Let $Q = [u_2 \ u_3]$

Projection matrix onto $\text{Span}(u_2, u_3)$ is

$$Q Q^\top = \begin{bmatrix} -1/\sqrt{10} & 9/\sqrt{99}\sqrt{10} \\ 3/\sqrt{10} & 3/\sqrt{99}\sqrt{10} \\ 0 & 30/\sqrt{99}\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 9/\sqrt{99}\sqrt{10} & 3/\sqrt{99}\sqrt{10} & 30/\sqrt{99}\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{11} & -\frac{3}{11} & \frac{3}{11} \\ -\frac{3}{11} & \frac{10}{11} & \frac{1}{11} \\ \frac{3}{11} & \frac{1}{11} & \frac{10}{11} \end{bmatrix}$$

iii) projection matrix onto $\text{Span}(u_1)$ is $u_1 u_1^\top$

$$= \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix} \begin{bmatrix} 3/\sqrt{11} & \frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} \end{bmatrix} = \begin{bmatrix} \frac{9}{11} & \frac{3}{11} & -\frac{3}{11} \\ \frac{3}{11} & \frac{1}{11} & -\frac{1}{11} \\ -\frac{3}{11} & -\frac{1}{11} & \frac{1}{11} \end{bmatrix}$$

Ex5: Let A be an $m \times n$ matrix. If $Ax=b$ is consistent, the projection of b onto $C(A)$ is _____.

Ans : b itself

Since $Ax=b$ is consistent, b is in span of col. vectors of $A \Rightarrow b \in C(A)$.

$$\therefore \text{proj}_{C(A)} b = b$$

Ex6: Obtain the orthogonal projection of the vector $x = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ onto the column space of the matrix A . Given

that the columns of A are L.I.

$$A = \begin{bmatrix} 1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

Soln: Let col. vectors of A be

$$a_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

First we obtain orthonormal basis of $C(A)$ using Gram-Schmidt process, then we obtain proj. of given vector onto $C(A)$.

Let $\{u_1, u_2, u_3\}$ be orthonormal basis of $C(A)$. Then

$$u_1 = \frac{a_1}{\|a_1\|}$$

$$\Rightarrow u_1 = \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

$$u_2 = \frac{y_2}{\|y_2\|}, \quad \text{where} \quad y_2 = a_2 - \langle u_1, a_2 \rangle u_1$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 2\sqrt{12} \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \end{bmatrix}$$

$$\|y_2\| = \sqrt{(64+4+0+4)^{1/2}} = \sqrt{72}$$

$$\therefore u_2 = \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix}$$

$$y_3 = \frac{y_3}{\|y_3\|}, \quad y_3 = a_3 - \langle u_1, a_3 \rangle u_1 - \langle u_2, a_3 \rangle u_2$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{18}{\sqrt{12}} \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} - \frac{48}{\sqrt{72}} \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix}$$

$$= \begin{bmatrix} -5/6 \\ -1/6 \\ 9/2 \\ -19/6 \end{bmatrix}$$

$$\|y_3\| = \left(\frac{25}{36} + \frac{1}{36} + \frac{27^2}{36} + \frac{19^2}{36} \right)^{1/2}$$

$$= \sqrt{31}$$

$$u_3 = \begin{bmatrix} -5/6\sqrt{31} \\ -1/6\sqrt{31} \\ 9/2\sqrt{31} \\ -19/6\sqrt{31} \end{bmatrix}$$

Orthonormal basis of $C(A)$ is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix}, \begin{bmatrix} -5/6\sqrt{31} \\ -1/6\sqrt{31} \\ 9/2\sqrt{31} \\ -19/6\sqrt{31} \end{bmatrix} \right\}$$

$$\text{proj}_{C(A)} x = \text{proj}_{u_1} x + \text{proj}_{u_2} x + \text{proj}_{u_3} x$$

$$= \langle u_1, x \rangle u_1 + \langle u_2, x \rangle u_2 + \langle u_3, x \rangle u_3$$

$$= \frac{2}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{4}{\sqrt{72}} \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix} - \frac{7}{3\sqrt{31}} \begin{bmatrix} -5/6\sqrt{31} \\ -1/6\sqrt{31} \\ 9/2\sqrt{31} \\ -19/6\sqrt{31} \end{bmatrix}$$

$$= \begin{bmatrix} 104/93 \\ 9/31 \\ -16/93 \\ 17/93 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Let A be a square matrix of order $n \times n$. Then a non-zero vector X is said to be an eigenvector of A if there exist a scalar λ such that

$$AX = \lambda X$$

here λ is called corresponding eigenvalue or characteristic value.

That is,

Suppose $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation, a non-zero vector $X \in \mathbb{R}^n$ is said to be an eigenvector if the image of X under A is a vector lies in the same line where X lies. The scaling factor is called the eigenvalue.

Ex 0: If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $IX = X$ for every $X \in \mathbb{R}^3$

\therefore Every vector is an eigenvector with eigenvalue $\lambda = 1$
It has three LI eigenvectors.

Ex 1: Let P be a projection matrix onto a subspace V . Then eigenvalues of P are

- i) $\lambda_1 = 1$ corresponding eigenvectors are basis of V .
- ii) $\lambda_2 = 0$ corresponding eigenvectors are basis of V^\perp
where $V^\perp = \{v \mid \langle v, w \rangle = 0 \text{ for all } w \in V\}$

a) Let $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

It is a projection matrix onto the line $y=x$.

i) $Px = x$ when x is on the line $y=x$

\therefore eigenvector is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_1 = 1$.

ii) $Px=0$ when x is orthogonal to the line $y=x$

\therefore eigenvector is $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_2 = 0$.

b) $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is a projection matrix onto e_1 -axis.

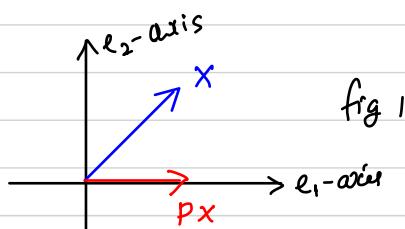


fig 1

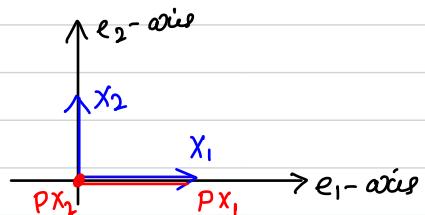


fig 2

From fig 2.

i) $Px_1 = x_1 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda_1 = 1$

ii) $Px_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is another eigenvector with $\lambda_2 = 0$

c) Let $V = \text{span}(u_1, u_2)$ be subspace in \mathbb{R}^3 ,

where $u_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$.

Clealy V is a subspace with orthonormal basis $\{u_1, u_2\}$

If $Q = [u_1 \ u_2]$, then

projection matrix P onto $V = Q Q^T = \begin{bmatrix} \frac{17}{18} & -\frac{1}{18} & \frac{2}{9} \\ -\frac{1}{18} & \frac{17}{18} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix}$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$.

Eigenvectors when $\lambda_1 = 1$ are u_1 and u_2 , and eigenvector vector when $\lambda_2 = 0$ is a vector $u_3 \perp$ to both u_1 and u_2 .

To find u_3 consider

$$A = \begin{bmatrix} u_1^\top \\ u_2^\top \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

we find Null space of A :

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 1 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_2 + x_3$$

$$\Rightarrow 2x_1 + 2x_2 + x_3 = 0 \\ 4x_2 + x_3 = 0$$

$$\text{Let } x_3 = k \Rightarrow x_2 = -\frac{k}{4} \text{ and } x_1 = \frac{k/2 - k}{2} = -\frac{k}{4}$$

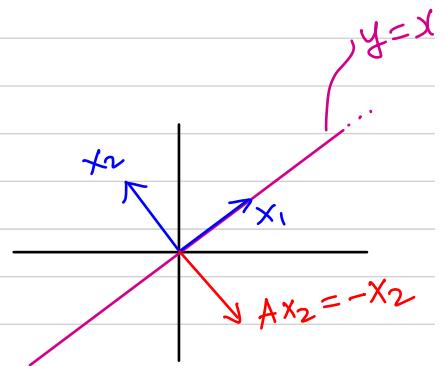
$$x = k \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

$$\therefore \text{eigen vector } u_3 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Note: Clearly, projection matrix is symmetric and if P is a projection matrix of order n , then it has $n+1$ eigenvectors.

Ex2: Reflection matrix about a line $y=x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$



From the fig:

i) $Ax_1 = x_1$, when x_1 is on the line

$y=x$. : eigen vector $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $\lambda_1 = 1$

ii) $Ax_2 = -x_2$, when x_2 is \perp to x_1

: eigen vector $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $\lambda_2 = -1$

Ex3: Rotation matrix by an angle 90° in counter clockwise direction.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ No real eigenvalues and eigen vectors}$$

Method to find eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. If X is an eigenvector and λ is the eigenvalue, then

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow [A - \lambda I] X = 0 \quad \text{--- (1)}$$

It has non-trivial soln iff

$$|A - \lambda I| = 0 \quad \text{--- (2)}$$

(2) is called characteristic equation, solving we get eigenvalues and a corresponding eigenvectors are found using (1).

Property 1 :

If A is an $n \times n$ matrix, then it has at most n linearly independent eigenvectors.

Property 2 :

A symmetric matrix of order n has n linearly independent eigenvectors

Property 3 :

A matrix B is singular iff 0 is an eigenvalue.

If B is singular $\Leftrightarrow |B| = 0 \Leftrightarrow |B - 0I| = 0$
 $\Leftrightarrow 0$ is the eigenvalue.

Note: If A is an $n \times n$ symmetric matrix of rank r , then it has r nonzero eigenvalues and the multiplicity of the eigenvalue 0 is $n-r$.

Ex. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Characteristic eqn :

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\Rightarrow \lambda^2 - \text{trace}(A)\lambda + |A| = 0.$$

Now, let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic eqn :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right) \lambda$$

$$- |A| = 0$$

$$\Rightarrow \lambda^3 - \text{trace}(A)\lambda^2 + (\text{sum of minors of diagonal entries})\lambda - |A| = 0$$

property 4 :

Sum of eigenvalues of $A = \text{trace}(A)$

product of eigenvalues of $A = \det(A)$

property 5 :

Matrices A and A^T have same eigenvalues

Suppose λ is eigenvalue of A , then

$$|A - \lambda I| = 0$$

$$\Rightarrow |(A - \lambda I)^T| = 0 \quad (\because A \text{ matrix and its transpose have same determinant})$$

$$\Rightarrow |A^T - \lambda I| = 0$$

$\Rightarrow \lambda$ is eigenvalue of A^T .

Property 6:

If λ is eigenvalue of A , then eigenvalue of $A + kI$ is $\lambda + k$

Consider $[A + kI]X = Ax + kIX$

$$\begin{aligned} &= Ax + kx \\ &= \lambda x + kx \quad (\because \lambda \text{ is eigenvalue of } A) \\ &= (\lambda + k)x \end{aligned}$$

$\Rightarrow \lambda + k$ is eigenvalue of $A + kI$

Eigenvectors of A and $A + kI$ are same

Property 7:

If A is triangular or diagonal matrix, then its eigenvalues are diagonal entries of A .

For instance, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Char Eqn: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{vmatrix} = 0 \Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$$

\therefore eigenvalues are a_{11}, a_{22} and a_{33} .

property 8

If A is symmetric, then its eigenvalues are real.

property 9.

If A is skew-symmetric, then its eigenvalues are zero or purely imaginary

property 10.

If λ is eigenvalue of A , then λ^k is eigenvalue of A^k , $k \in \mathbb{Z}^+$.

$$\text{Let } Ax = \lambda x$$

$$\Rightarrow A(Ax) = A(\lambda x)$$

$$\Rightarrow (AA)x = \lambda Ax$$

$$\Rightarrow A^2x = \lambda^2 x$$

$\Rightarrow \lambda^2$ is eigenvalue of A^2 . By induction we can easily prove that for any $k \in \mathbb{Z}^+$ λ^k is eigenvalue of A^k .

Property 11

If A is nonsingular and λ is eigenvalue of A , then

$\frac{1}{\lambda}$ is eigenvalue of A^{-1} .

$$\text{Let } Ax = \lambda x$$

$$\Rightarrow (A^{-1}A)x = A^{-1}(\lambda x)$$

$$\Rightarrow x = \lambda(A^{-1}x)$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda}x$$

Property 12:

Eigenvalues of orthogonal matrices are +1 or -1.

If Q is orthogonal matrix, then $Q^{-1} = Q^T$

If λ is eigenvalue of Q , then $\frac{1}{\lambda}$ is eigenvalue of Q^{-1} and λ is eigenvalue of Q^T .

$$\therefore \frac{1}{\lambda} = \lambda \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Property 13:

If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of A , then its k eigenvectors are linearly independent.

Repeated eigenvalues may or may not have linearly independent eigenvectors.

Property 14:

If A is symmetric, then eigenvectors to different eigenvalues are orthogonal.

We can find orthogonal eigenvector for repeated eigenvalues using Gram Schmidt process.

Property 15:

Matrices A and B are said to be similar if there exist non-singular matrix S such that

$$B = S^{-1} A S$$

Two similar matrices have same eigenvalues.

Ex 1: Let $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$. Find eigenvalues and eigenvectors of B.

Soln: To find eigenvalues:

$$\text{Characteristic eqn is } |B - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 15 = 0$$

Solve for λ ,

$$\lambda = \frac{8 \pm \sqrt{64-60}}{2}$$

$$\Rightarrow \lambda = 5, 3$$

Eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$

To find eigenvectors: Consider $[B - \lambda I] x = 0$

i) $\lambda_1 = 5$,

$$[B - 5I] x = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$\text{Let } x_2 = k \Rightarrow x_1 = k \quad (k \neq 0)$$

Eigenvector for $\lambda = 5$ is

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (k = 1)$$

$$\text{Now, } \lambda = 3, \quad [B - 3\lambda] x = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

$$\text{Let } x_2 = k, \quad x_1 = -k$$

Eigenvector for $\lambda = 3$ is

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (k=1)$$

Diagonalisation:

Let A be an $n \times n$ matrix which has n linearly independent eigenvectors. Then we can factor A into

$A = X D X^{-1}$, is called diagonalization.

where columns of X are eigenvectors of A , it is called a diagonalising matrix (or eigenvector matrix).

D is a diagonal matrix, diagonal entries are eigenvalues.

Pf: Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of A and let

x_1, x_2, \dots, x_n be corresponding eigenvectors. Then

$$A x_i = \lambda_i x_i, \quad i = 1, 2, \dots, n$$

$$\Rightarrow A X = A \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}$$

$$= \begin{bmatrix} A x_1 & A x_2 & A x_3 & \cdots & A x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} & & & \\ & x_1 & x_2 & \cdots & x_n \\ & & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= X D$$

$$\Rightarrow A = X D X^{-1}$$

or

$$D = X^{-1} A X$$

Remark 1: If the matrix A has no repeated eigenvalues, then it has n LI eigenvectors. Therefore it is diagonalizable.

Remark 2: Diagonalizing matrix X is not unique. Any eigenvector can be multiplied by a constant, and remains an eigenvector.

Remark 3: Not all matrices possess n linearly independent eigenvectors, so not all matrices are diagonalizable.

Remark 4: If $A = X D X^{-1}$, then $A^k = X D^k X^{-1}$ for any $k \in \mathbb{Z}^+$.

$$A = X D X^{-1}$$

$$\Rightarrow A^2 = (X D X^{-1}) (X D X^{-1})$$

$$\Rightarrow A^2 = X D (X^{-1} X) D X^{-1}$$

$$\Rightarrow A^2 = X D^2 X^{-1} \quad (\because X^{-1} X = I)$$

Ex 2: Let $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$.

i) Diagonalize the matrix A

ii) Find eigenvalues and eigenvectors of $(A^{-1} + I)^3$

Soln: Given $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

Characteristic eqn : $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 3\lambda - 4 = 0$$

Solving, $\lambda = -4, 1$

eigenvalues $\lambda_1 = -4, \lambda_2 = 1$.

To find eigenvectors, consider

$$[A - \lambda I] x = 0$$

$$i) \lambda_1 = -4$$

$$[A + 4I] x = 0$$

$$\Rightarrow \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 = 0$$

$$\text{put } x_2 = k \Rightarrow x_1 = \frac{k}{2}$$

$$\therefore x_1 = \begin{bmatrix} \frac{k}{2} \\ k \end{bmatrix}, k \neq 0$$

Eigenvector for $\lambda_1 = -4$ is $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ($k=2$)

$$ii) \lambda = 1$$

$$[A - I] x = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - 3x_2 = 0$$

$$\text{put } x_2 = k, \Rightarrow x_1 = 3k,$$

$$\therefore x_2 = \begin{bmatrix} 3k \\ k \end{bmatrix}$$

eigenvector for $\lambda_2 = 1$ is $x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ($k = 1$)

Diagonalizing matrix $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

Diagonal matrix $D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$

and $X^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

\therefore Diagonalization : $A = X D X^{-1}$

$$\Rightarrow \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{bmatrix}$$

ii) Eigenvalues of $(A^{-1} + I)^3$ are

$$\lambda_1 = \left(-\frac{1}{4} + 1\right)^3 \quad \text{and} \quad \lambda_2 = (1+1)^3$$

$$\Rightarrow \lambda_1 = \frac{27}{64}, \quad \lambda_2 = 8$$

Corresponding eigenvectors are $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Ex 3: Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

Find eigenvalues and eigenvectors of A. If A has n LI eigenvectors, then diagonalize A.

Soln: To find eigenvalues: Consider

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \left(\begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} \right) \lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda - 0 = 0$$

$$\Rightarrow \lambda(\lambda^2 - 2\lambda + 1) = 0$$

eigenvalues are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1$.

To find eigenvectors:

$$[A - \lambda I]x = 0$$

$$i) \lambda_1 = 0, Ax = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix}} = k$$

$$\Rightarrow x_1 = -k, x_2 = -k, x_3 = -k$$

eigenvector $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (k = -1)$

$$\text{ii) } \lambda = 1$$

$$[A - I] X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - 3x_2 + x_3 = 0$$

Let $x_2 = k_1$ and $x_3 = k_2$ be free variables

$$\Rightarrow x_1 = 3k_1 - k_2$$

$$X = \begin{bmatrix} 3k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $x_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ be eigenvectors

Thus, diagonalising matrix

$$X = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$X^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \quad (\text{use calculator to find})$$

$$\text{Diagonal matrix : } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Diagonalization : } A = X D X^{-1}$$

If A is a symmetric matrix, then we can find orthonormal eigenvectors. Thus diagonalising matrix X is an orthogonal matrix.

$$\text{Diagonalization : } A = X D X^T$$

Since, if X is orthogonal, $X^{-1} = X^T$.

Ex 4: Diagonalize

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Soln: Since A is symmetric, we find orthonormal eigenvectors.

$$\text{Consider : } |A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + \left(\begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix} \right) \lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 21\lambda + 98 = 0$$

$$\Rightarrow \lambda = -2, 7, 7$$

Eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 7$ (repeated)

To find eigenvectors : consider

$$[A - \lambda I] X = 0$$

$$\text{i) } \lambda_1 = -2, \quad [A + 2I] X = 0$$

$$\Rightarrow \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\Rightarrow \frac{x_1}{\begin{vmatrix} 8 & 2 \\ 2 & 6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 8 \\ 4 & 2 \end{vmatrix}} = k$$

$$\Rightarrow x_1 = 36k, \quad x_2 = 18k, \quad x_3 = -36k$$

$$\therefore x_1 = 18k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\text{Let } u_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\text{i) } \lambda = 7, \quad [A - 7I]x = 0$$

$$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - x_2 + 2x_3 = 0$$

$$\text{put } x_2 = k_1, \quad x_3 = k_2 \Rightarrow x_1 = \frac{2k_2 - k_1}{2}$$

$$\therefore x = \begin{bmatrix} \frac{2k_2 - k_1}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $x_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. These two are not orthogonal

We shall use Gram Schmidt method.

$$\text{Let } u_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \quad \text{Let } y_3 = x_3 - \langle u_2, x_3 \rangle u_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{5}} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/\sqrt{5} \\ 1 \end{bmatrix}$$

$$\therefore u_3 = \frac{y_3}{\|y_3\|} = \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

\therefore Diagonalizing matrix is

$$X = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{3} & -1/\sqrt{5} & 4/\sqrt{45} \\ 1/\sqrt{3} & 2/\sqrt{5} & 2/\sqrt{45} \\ -2/\sqrt{3} & 0 & 5/\sqrt{45} \end{bmatrix}$$

Diagonalization : $A = X D X^T$

$$\Rightarrow \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} & -1/\sqrt{5} & 4/\sqrt{45} \\ 1/\sqrt{3} & 2/\sqrt{5} & 2/\sqrt{45} \\ -2/\sqrt{3} & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 2/\sqrt{3} & 1/\sqrt{3} & -2/\sqrt{3} \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 4/\sqrt{45} & 2/\sqrt{45} & 5/\sqrt{45} \end{bmatrix}$$

Singular value decomposition (SVD)

Let A be an $m \times n$ matrix with rank r . Then A can be factored into

$$A = U \Sigma V^T$$

where, U is an $m \times m$ orthogonal matrix.

V is an $n \times n$ orthogonal matrix.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & \sigma_r & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\underbrace{\hspace{1cm}}$ $n-r$ cols

is an $m \times n$ matrix.

here $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r > 0$ are called singular values.

Column vectors of U are called left singular vectors.

Column vectors of V are called right singular vectors.

Remark 1: If A is $m \times n$ matrix of rank r , then

i) $A^T A$ is an $n \times n$ symmetric matrix

ii) $A A^T$ is an $m \times m$ symmetric matrix

iii) $N(A) = N(A^T A)$ and $N(A^T) = N(A A^T)$

iv) $\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(A A^T)$

v) $R(A) = R(A^T A)$ and $R(A^T) = R(A A^T)$

$$i) (A^T A)^T = A^T (A^T)^T = A^T A \Rightarrow A^T A \text{ is symmetric}$$

$$ii) (A A^T)^T = (A^T)^T A^T = A A^T \Rightarrow A A^T \text{ is symmetric}$$

iii) First we shall show that
 $N(A) \subseteq N(A^T A)$

Let $x \in N(A)$ for some $x \in \mathbb{R}^n$

$$\Rightarrow A x = 0$$

$$\Rightarrow A^T (A x) = 0$$

$$\Rightarrow (A^T A) x = 0$$

$$\Rightarrow x \in N(A^T A)$$

$$\therefore N(A) \subseteq N(A^T A) \quad \text{--- (1)}$$

Now, we show that
 $N(A^T A) \subseteq N(A)$

Let $x \in N(A^T A)$ for some $x \in \mathbb{R}^n$

$$\Rightarrow (A^T A) x = 0$$

$$\Rightarrow x^T (A^T A) x = 0$$

$$\Rightarrow (x^T A^T) (A x) = 0$$

$$\Rightarrow (A x)^T (A x) = 0$$

$$\Rightarrow \|A x\|^2 = 0$$

$$\Rightarrow A x = 0$$

$$\Rightarrow x \in N(A)$$

$$\therefore N(A^T A) \subseteq N(A) \quad \text{--- (2)}$$

From (1) and (2), $N(A) = N(A^T A)$

We replace A by A^T , we get $N(A^T) = N(A A^T)$

iv) By rank-nullity Thm : $\text{rank}(A) + \text{nullity}(A) = n$

$$\Rightarrow \text{rank}(A) = n - \text{nullity}(A)$$

Also, $\text{rank}(A^T A) + \text{nullity}(A^T A) = n$.

$$\Rightarrow \text{rank}(A^T A) = n - \text{nullity}(A^T A).$$

But from iii) $\text{nullity}(A) = \text{nullity}(A^T A)$

$$\therefore \text{rank}(A) = \text{rank}(A^T A)$$

Further argument asserts that $\text{rank}(A) = \text{rank}(A A^T)$

v) It easily follows from iii) and iv)

Remark 2: The column vectors of an orthogonal matrix V in SVD are orthonormal eigenvectors of $A^T A$.

Consider $A = U \Sigma V^T$

$$\begin{aligned} \Rightarrow A^T A &= (U \Sigma V^T)^T (U \Sigma V^T) \\ &= V \Sigma^T (V^T U) \Sigma V^T \\ &= V \Sigma^T \Sigma V^T \end{aligned}$$

$\Sigma^T \Sigma$ is a diagonal matrix of order $n \times n$.

i.e $\Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}$

If $V = [v_1 \ v_2 \ \cdots \ v_r \ v_{r+1} \ \cdots \ v_n]$, then

$\sigma_1^2, \sigma_2^2, \dots, \sigma_r^2$ are nonzero eigenvalues of $A^T A$ and the corresponding orthonormal eigenvectors are v_1, v_2, \dots, v_r .

And $v_{r+1}, v_{r+2}, \dots, v_n$ are orthonormal eigenvectors for eigenvalue 0 (multiplicity $n-r$)

Also, note that $A^T A v_j = 0 \quad , \quad j = r+1, \dots, n$
(since eigenvalue is 0)

$\Rightarrow \{v_{r+1}, v_{r+2}, \dots, v_n\}$ is a basis of $N(A^T A)$.

Further, $\{v_1, v_2, \dots, v_r\}$ form basis of $R(A^T A)$

Remark 3: The column vectors of an orthogonal matrix U are orthonormal eigenvectors of $A^T A$.

It easily follows from Remark 2.

Suppose $U = \begin{bmatrix} u_1 & u_2 & \dots & u_r & u_{r+1} & \dots & u_m \end{bmatrix}$

Then u_1, u_2, \dots, u_r are orthonormal eigenvectors for the eigenvalues $\sigma_1^2, \sigma_2^2, \dots, \sigma_r^2$.

Multiplicity of the eigenvalue 0 is $m-r$ and its corresponding orthonormal eigenvectors are

$$u_{r+1}, u_{r+2}, \dots, u_m$$

Note that $A^T A u_j = 0, j = r+1, \dots, m$

$\Rightarrow \{u_{r+1}, u_{r+2}, \dots, u_m\}$ form basis of $N(A^T A)$

And $\{u_1, u_2, \dots, u_r\}$ form basis of $R(A^T A)$

Remark 4: From above three remarks, we see that

U and V give orthonormal bases for all four fundamental subspaces.

First r columns of U : basis of $C(A)$ ($\because R(A^T) = C(A)$)

Last $m-r$ columns of U : basis of $N(A^T)$

First r columns of V : basis of $R(A)$

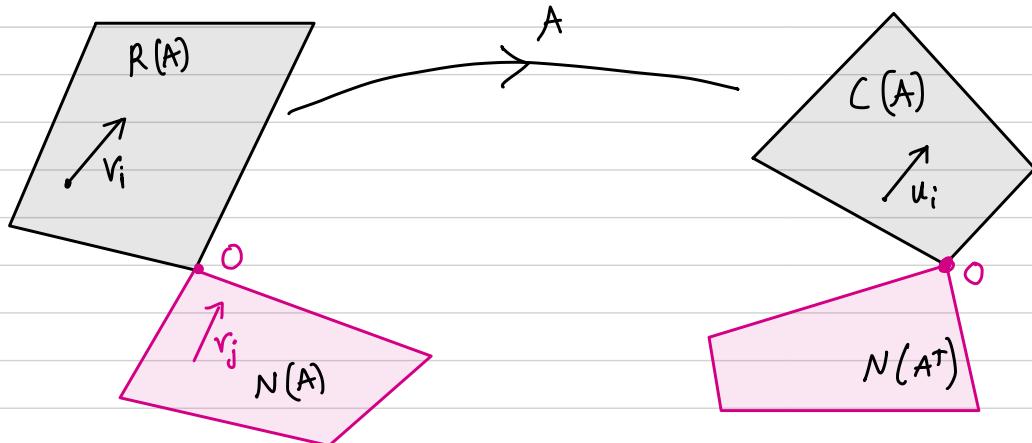
Last $n-r$ columns of V : basis of $N(A)$

SVD chooses those bases in an extremely special value such that

$$A v_i = \sigma_i u_i ; \quad i = 1, 2, \dots, r$$

and

$$A v_j = 0 ; \quad j = r+1, r+2, \dots, n$$



It comes directly from $AV = U\Sigma$, looked at a column at a time.

$$\begin{aligned} \text{i.e } AV &= A \begin{bmatrix} v_1 & v_2 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix} \\ &= \begin{bmatrix} u_1 & u_2 & \cdots & u_r & u_{r+1} & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & \sigma_r & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_r & Av_{r+1} & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_r u_r & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\Rightarrow Av_i = \sigma_i u_i \quad i = 1, 2, \dots, r$$

$$Av_i = 0 \quad i = r+1, \dots, n$$

$$\text{Also, } A = U \Sigma V^T$$

$$\Rightarrow A^T = (U \Sigma V^T)^T$$

$$\Rightarrow A^T = V \Sigma^T U^T$$

$$\Rightarrow A^T U = V \Sigma^T$$

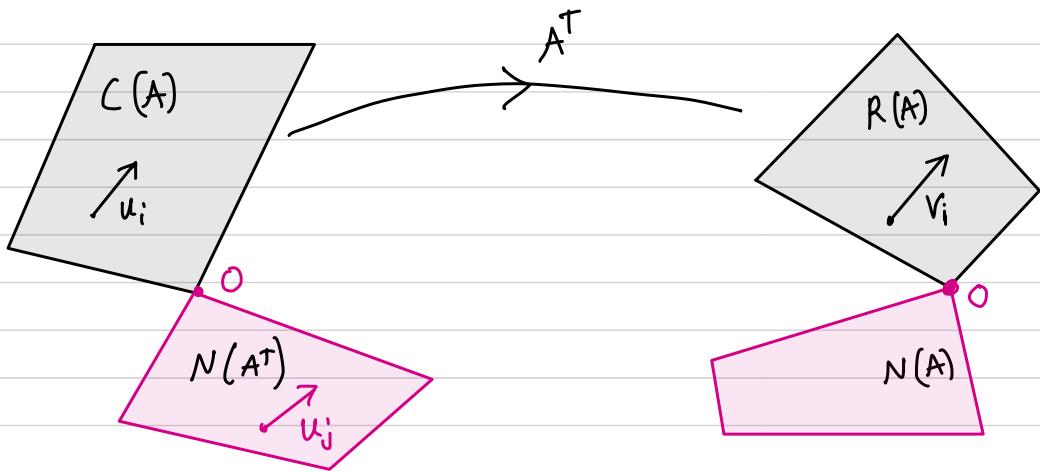
As above if we look at a column at a time, we see

that

$$A^T U_i = \sigma_i v_i ; \quad j=1, 2, \dots, r$$

and

$$A^T U_j = 0 ; \quad j=r+1, \dots, m$$



Method to find SVD of an $m \times n$ matrix A of rank r

$$\text{SVD: } A = U \Sigma V^T$$

If $m \geq n$

- i) Consider $A^T A$ and find its eigenvalues and eigenvectors

If v_1, v_2, \dots, v_r are orthonormal eigenvectors corresponding to the non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$,

and v_{r+1}, \dots, v_n are orthonormal eigenvectors corresponding to the eigenvalue 0,

then

$$V = \begin{bmatrix} v_1 & v_2 & v_3 & \dots & v_r & v_{r+1} & \dots & v_n \end{bmatrix}_{n \times n}$$

Singular values

$$\sigma_i = \sqrt{\lambda_i} \quad i=1, 2, \dots, r$$

- ii) To find V consider

$$A v_i = \sigma_i u_i, \quad i=1, 2, \dots, r$$

$$\Rightarrow u_i = \frac{A v_i}{\sigma_i}, \quad i=1, 2, \dots, r$$

Further, if $u_{r+1}, u_{r+2}, \dots, u_m$

are orthonormal basis of $N(A^T)$,

we know how to find it.

Thus,

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_r & u_{r+1} & \dots & u_m \end{bmatrix}_{m \times m}$$

If $n \geq m$

- i) Consider $A A^T$ and find its eigenvalues and eigenvectors

If u_1, u_2, \dots, u_r are orthonormal eigenvectors corresponding to the non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$

and u_{r+1}, \dots, u_m are orthonormal eigenvectors corresponding to the eigenvalue 0,

then

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_r & u_{r+1} & \dots & u_m \end{bmatrix}_{m \times n}$$

Singular values

$$\sigma_i = \sqrt{\lambda_i} \quad i=1, 2, \dots, r$$

- ii) To find V consider

$$A^T u_i = \sigma_i v_i, \quad i=1, 2, \dots, r$$

$$\Rightarrow v_i = \frac{A^T u_i}{\sigma_i}, \quad i=1, 2, \dots, r$$

Further, if $v_{r+1}, v_{r+2}, \dots, v_n$

are orthonormal basis of $N(A)$

we know how to find it.

Thus,

$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_r & v_{r+1} & \dots & v_n \end{bmatrix}_{n \times n}$$

OR

We find orthonormal eigenvectors
of $A A^T$ to find U

If u_1, u_2, \dots, u_r are orthonormal
eigenvectors of $\lambda_1, \lambda_2, \dots, \lambda_r$

and $u_{r+1}, u_{r+2}, \dots, u_m$ are
orthonormal eigenvectors of the
eigenvalue 0.

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_r & u_{r+1} & \dots & u_m \end{bmatrix}$$

OR

We find orthonormal eigenvectors
of $A^T A$ to find V

If v_1, v_2, \dots, v_r are orthonormal
eigenvectors of $\lambda_1, \lambda_2, \dots, \lambda_r$

and v_{r+1}, \dots, v_m are
orthonormal eigenvectors of the
eigenvalue 0.

$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_r & v_{r+1} & \dots & v_m \end{bmatrix}$$

Ex 1 : Determine the singular value decomposition of

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

OR

Find U, Σ and V such that $A = U \Sigma V^T$.

Soln : Rank of the matrix A is 2, it has two singular
values σ_1 and σ_2 .

Step 1 : Consider $A^T A$

$$A^T A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$$

Step 2 : Find eigenvalues and eigenvectors of $A^T A$.

Consider, characteristic eqn

$$|A^T A - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - 50\lambda + 225 = 0$$

$$\Rightarrow \lambda = 5, 45$$

\therefore eigenvalues are $\lambda_1 = 45, \lambda_2 = 5$

Eigenvector for $\lambda_1 = 45$ is

$$[A^T A - 45I] x = 0$$

$$\Rightarrow \begin{bmatrix} -20 & 20 \\ 20 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -20x_1 + 20x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

Let $x_1 = x_2 = 1$

\therefore Eigenvector for $\lambda_1 = 45$ is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvector for $\lambda_2 = 5$,

$$[A^T A - 5I] x = 0$$

$$\Rightarrow \begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 20x_1 + 20x_2 = 0$$

$$\Rightarrow x_1 + x_2 = 0$$

Let $x_2 = 1 \Rightarrow x_1 = -1$

\therefore Eigenvector for $\lambda_2 = 5$ is $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Step 3: Singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{45}$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{5}$$

Right singular vectors v_1 and v_2 are unit vectors along x_1 and x_2 , resp.

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

Step 4: To find left singular vectors:

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{45}} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

Thus,

$$\begin{aligned} A &= U \Sigma V^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Ex 2: Obtain singular value decomposition for

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Soln: Rank of the matrix is 1. \therefore we get only one non-zero singular value.

$$\text{Let } A_{3 \times 2} = U_{3 \times 3} \Sigma_{3 \times 2} V_{2 \times 2}^T$$

Step 1: First let us find $V = [v_1 \ v_2]$.

$$\text{Consider } A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Step 2: Find eigenvalues and eigenvectors of $A^T A$:

Characteristic Eqn: $|A^T A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - 4\lambda = 0 \Rightarrow \lambda = 0, 4$$

\therefore eigenvalues are $\lambda_1 = 4, \lambda_2 = 0$

Eigenvector for $\lambda_1 = 4$:

$$[A^T A - 4I]x = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2$$

Let $x_2 = 1$ and $x_1 = 1$.

Eigenvector for $\lambda_1 = 4$ is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvector for $\lambda_2 = 0$:

$$[A^T A - 0I]x = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + 2x_2 = 0 \Rightarrow x_1 = -x_2$$

Let $x_1 = 1$ and $x_2 = -1$

Eigenvector for $\lambda_2 = 0$ is $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Step 3: Singular values are $\sigma_1 = \sqrt{\lambda_1} = 2$

$$\sigma_2 = \sqrt{\lambda_2} = 0$$

Right singular vectors v_1 and v_2 are unit vectors along x_1 and x_2 , resp.

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Step 4: To find left singular vectors u_1, u_2 and u_3 .

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

u_2 and u_3 form an orthonormal basis for $N(A^T)$.

Let us compute basis for $N(A^T)$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Consider $A^T x = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + 0x_3 = 0$$

put $x_2 = k_1$ and $x_3 = k_2$ (free variables)

$$\Rightarrow x_1 = -k_1$$

$$\therefore \text{Left Null space } N(A^T) = \begin{bmatrix} -k_1 \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis of } N(A^\top) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

These vectors are already orthogonal, it is not necessary to use the Gram-Schmidt process.

Orthonormal basis are

$$u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$U = [u_1 \ u_2 \ u_3]$$

$$\Rightarrow U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$A = U \Sigma V^T$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Note: Instead of step 4 to find $U = [u_1 \ u_2 \ u_3]$.

We can consider the matrix AA^\top and find its orthonormal eigenvectors. u_1 is the eigenvector for $\lambda_1 = 4$ and, u_2 and u_3 are eigenvectors for $\lambda_2 = \lambda_3 = 0$

Ex 3 : Obtain SVD of

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Soln: Given 2×3 matrix.

$$\text{Let } A_{2 \times 3} = U_{2 \times 2} \Sigma_{2 \times 3} V_{3 \times 3}^T$$

Consider

$$AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Eigenvectors of AA^T are column vectors of U .

To find eigenvalues :

$$\text{char. Eqn } |AA^T - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda = 3, 1$$

\therefore eigenvalues are $\lambda_1 = 3, \lambda_2 = 1$

\therefore Singular values are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$

To find eigenvectors:

$$[AA^T - \lambda I] X = 0$$

$$i) \lambda_1 = 3$$

$$[AA^T - 3I] X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 - x_2 = 0$$

$$\text{Let } x_2 = k$$

$$x_1 = -k$$

$$\therefore x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, k=1$$

$$u_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$ii) \lambda_2 = 1$$

$$[AA^T - I] x = 0$$

$$\Rightarrow u_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

\therefore Matrix

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find $V = [v_1 \ v_2 \ v_3]$. Consider

$$A^T u_j = \sigma_j v_j, \quad j = 1, 2$$

v_3 is normal basis of $N(A)$

$$\therefore v_1 = \frac{1}{\sigma_1} A^T u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$v_2 = \frac{1}{\sigma_2} A^T u_2 = \frac{1}{1} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Consider

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$-x_2 + x_3 = 0$$

$$x_3 = k_1$$

$$\Rightarrow x_2 = k_1$$

$$x_1 = k_1$$

$$\therefore x = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Thus, } V_3 = \frac{x}{\|x\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

OR

To find V consider $A^T A$ and find its orthonormal eigenvectors

$$A^T A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$ and $\lambda_3 = 0$

eigenvector when $\lambda_1 = 3$

$$[A^T A - 3I]X = 0$$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -1 & -1 \\ -1 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -1 \\ 0 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}} = k$$

$$\Rightarrow x_1 = k, x_2 = -2k, x_3 = k$$

$$\Rightarrow x_1 = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad v_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

likewise we find eigenvectors when $\lambda_2 = 1$ and $\lambda_3 = 0$, we get

v_2 and v_3 .

$$\therefore V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \Rightarrow V = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Hence,

$$A = U \Sigma V^T$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Ex 4: Obtain SVD for

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{Ans: } U = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \end{bmatrix}$$