Graph Theory

A major publishing company has ten editors (referred to by $1, 2, \ldots, 10$) in the scientific, technical and computing areas. These ten editors have a standard meeting time during the first Friday of every month and have divided themselves into seven committees to meet later in the day to discuss specific topics of interest to the company namely (i) advertising (ii) securing reviewers, (iii) contacting new potential authors, (iv) finances (v) used copies and new editions, (vi) competing textbooks and (vii) textbook representatives.

The ten editors have decided on the seven committees: $C_1 = \{1, 2, 3\}$, $C_2 = \{1, 3, 4, 5\}$, $C_3 = \{2, 5, 6, 7\}$, $C_4 = \{4, 7, 8, 9\}$, $C_5 = \{2, 6, 7\}$, $C_6 = \{8, 9, 10\}$, $C_7 = \{1, 3, 9, 10\}$. They have set aside three time periods for the seven committees to meet on those Fridays when all ten editors are present. Some pairs of committees cannot meet during the same period because one or two of the editors are on both committees. This situation can be modeled visually as shown in the below figure.

In this ~~graph~~ figure there are several small dots, representing the seven committees and a straight line segment is drawn between two circles if the committees they represent have at least one committee member in common. In other words, a straight line segment between two committees tells us that these two committees should be scheduled to meet at the same time. This gives us a picture or a model of the committees and the overlapping nature of their membership.

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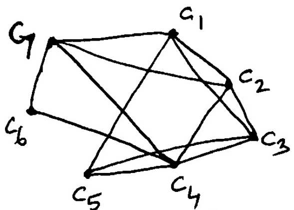


The above figure is called a graph.

Graph:

A graph is a figure consisting of finite nonempty set V of objects called vertices and a set E of 2-element subsets of V called edges. The sets V and E are the vertex set and edge set of G, respectively. So, a graph G is a pair, $G = \{V, E\}$.

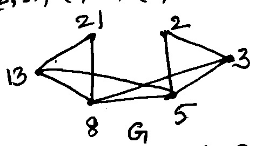
example

Consider the set $S = \{2, 3, 5, 8, 13, 21\}$. Let the edge set be those pair of distinct numbers of S such that the sum or difference also belongs to S. Construct the graph with S and $E = (x, y)$ s.t $|x-y| \in S$.

Sol^n vertex set $= \{2, 3, 5, 8, 13, 21\}$

edge set is $\{(2,3), (2,5), (3,5), (3,8), (5,8); (5,13), (8,13), (8,21), (13,21)\}$

∴ The graph is:



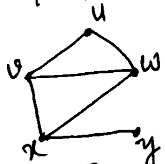
Order(n) and size(m) of a graph.

The number of vertices in the graph G is called the order of G, and the number of edges in the graph G is called the size of G.

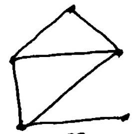
The above graph G has order 6 and size 8.

* A graph with exactly one vertex is called a trivial graph, implying that the order of a nontrivial graph is at least 2.

A graph G with $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, uw, vw, vx, wx, xy\}$ is shown below.



(a) Labelled graph



(b) unlabelled graph.

Here (a) is drawn with labelling and (b) is drawn without labelling. Hence the graph (a) is a labelled graph and (b) is an unlabelled graph.

<u>Terminologies</u>

* The two vertices u and v are end vertices of the edge (u, v).
* Edges that have the same end vertices are parallel edges.
* An edge of the form (v, v) is a loop.
* A graph is simple graph if it has no parallel edges or loops.
* A graph with no edges is empty graph.
* A graph with no vertices (and hence no edges) is a null graph.
* A graph with only one vertex is trivial graph.
* Two vertices u and v are adjacent vertices if they are connected by an edge, in other words (u, v) is an edge. Also u and v are referred to as neighbours of each other.
* Edges are said to be adjacent edges if they share a common end vertex (or they are incident with a common vertex).

Subgraphs

A graph H is called a <u>subgraph</u> of a graph G, written $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We also say that G contains H as a subgraph.

If $H \subset G$ and either $V(H)$ is proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a <u>proper subgraph</u> of G.

If a subgraph of a graph G has the same vertex set a G, then it is a <u>spanning subgraph</u> of G.

A subgraph F of a graph G is called an <u>induced subgraph</u> of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well.

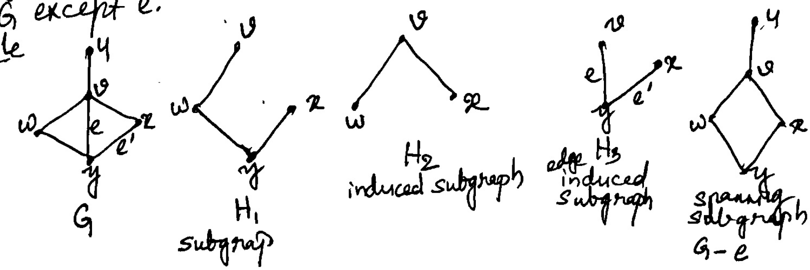
If S is a nonempty set of vertices of a graph G, then the subgraph of G induced by S is the induced subgraph with vertex set S. The induced subgraph is denoted by $\langle S \rangle$.

For a nonempty set X of edges, the subgraph induced by X has edge set X and consisting of all vertices that are incident with atleast one edge in X. This subgraph is called an <u>edge-induced subgraph</u> of G. It is denoted by $\langle X \rangle$.

Any proper subgraph of a graph G can be obtained by removing vertices and edges from G.

For an edge e of G, we write $G-e$ for the spanning subgraph of G whose edge set consists of all edges of G except e.

<u>example</u>



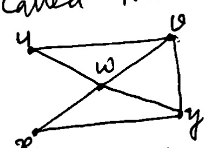
Walk, Trail, Path, Circuit, Cycle.

A u-v walk W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as $W: u=v_0, v_1, \ldots, v_k=v$, where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i=0,1,\ldots k-1$. Each vertex v_i, $(0 \leq i \leq k)$ and each edge $v_i v_{i+1}$ $(0 \leq i \leq k-1)$ is said to lie on or belong to W.

If $u=v$, then the walk W is closed, ~~while~~ while if $u \neq v$, then W is open.

As we move from one vertex of W to the next, we are encountering or traversing edges of G, possibly traversing some edges of G more than once.

The number of edges encountered in a walk (including multiple occurrences of an edge) is called the length of the walk.



W_1: x, y, w, y, v, w is a walk of length 5 also an open walk.

W_2: w x y w is a closed walk of length 3

A walk of length 0 is a trivial walk. eg W_3: v is a trivial walk.

A u-v trail in a graph G is a u-v walk in which no edge is traversed more than once.

A u-v path is a u-v walk in which no vertex is repeated. If no vertex in a walk is repeated, then no edge is repeated either. (Hence every path is a trail).

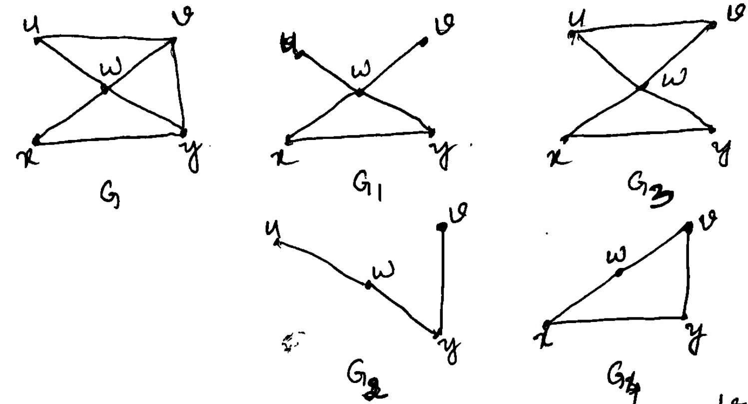
A circuit is a closed trail.

A cycle is a closed path.

A k-cycle is a cycle of length k.

A cycle of odd length is called an odd cycle.

A cycle of even length is called an even cycle.



Here the subgraphs G_1, G_2, G_3, G_4 are a trail, path, circuit and cycle respectively.

Connected and Disconnected Graphs.

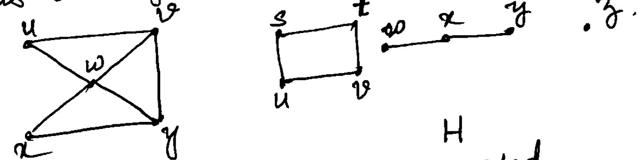
If G contains a u-v path, then u and v are said to be connected, and u is connected to v.

A graph G is connected if every two vertices of G are connected, that is, if G contains a u-v path for every pair u, v of distinct vertices of G.

A graph G that is not connected is called disconnected.

A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a component of G.

A graph G is then connected if and only if it has exactly one component.



G connected.

H disconnected.

[$= H_1 \cup H_2 \cup H_3$, where H_1, H_2, H_3 are called components of H]

<u>Remark</u>: If $x_1, x_2, \ldots, x_k \geq 1$ are real numbers and $x = x_1 + x_2 + \ldots + x_k$, then $\sum_{i=1}^{k} x_i^2 \leq x^2 - (k-1)(2x-k)$.

<u>Proof</u>: For each i let $x_i = y_i + 1$, where $y_i \geq 0$.

Let $y = y_1 + y_2 + \ldots + y_k$

Then $x^2 - (k-1)(2x-k)$

$$= (x_1 + x_2 + \ldots + x_k)^2 - (k-1)(2(x_1 + x_2 + \ldots + x_k)) - k$$

$$= (y_1 + 1 + y_2 + 1 + \ldots + y_k + 1)^2 - (k-1)(2(y_1 + 1 + y_2 + 1 + \ldots y_k + 1)) - k$$

$$= (y+k)^2 - (k-1)(2y + 2k - k)$$

$$= y^2 + 2yk + k^2 - 2yk - k^2 + 2y + k$$

$$= y^2 + 2y + k$$

$$= \left(\sum_{i=1}^{k} y_i\right)^2 + 2\sum_{i=1}^{k} y_i + k$$

$$\geq \sum_{i=1}^{k} y_i^2 + 2\sum_{i=1}^{k} y_i + k$$

$$= \sum_{i=1}^{k} (y_i^2 + 2y_i + 1)$$

$$= \sum_{i=1}^{k} (y_i + 1)^2$$

$$= \sum_{i=1}^{k} x_i^2$$

Hence $\sum_{i=1}^{k} x_i^2 \leq x^2 - (k+1)(2x+k)$.

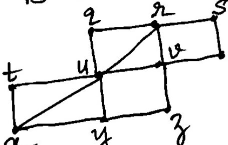
Theorem:
For a simple graph G with n vertices and k components $m \leq \frac{(n-k)(n-k+1)}{2}$.

Proof: Let $H_1, \ldots, H_k$ be the components of G.
For $1 \leq i \leq k$, denote by n_i the number of vertices of H_i.
Note that $n = n_1 + n_2 + \cdots + n_k$.
Each H_i is a simple graph.
Each H_i can have at most $\frac{n_i(n_i-1)}{2}$ edges.
Hence, the maximum number of edges G has is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$
$$\leq \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - \frac{1}{2} \times n$$
$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$
$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$
$$= \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$
$$= \frac{(n-k)(n-k+1)}{2}$$

Distance between two vertices

Let G be a connected graph of order n and let u and v be two vertices of G. The *distance* between u and v is the smallest length of any u-v path in G and is denoted by $d(u,v)$.
Hence if $d(u,v)=k$, then there exists a u-v path $P: u=v_0, v_1, \ldots, v_k=v$ of length k in G, but no u-v path of smaller length exists in G.
A u-v path of length $d(u,v)$ is called a *u-v geodesic*.
The greatest distance between any two vertices of a connected graph G are called the diameter of G and is denoted by $\mathrm{diam}(G)$.

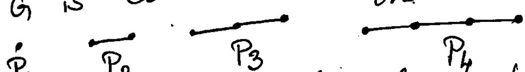


Here $d(q,z)=3$ $d(u,v)=1$
$d(t,s)=3$ $d(u,x)=1$
$d(x,v)=2$ $d(w,x)=3$
$d(y,r)=2$ $d(x,y)=1$

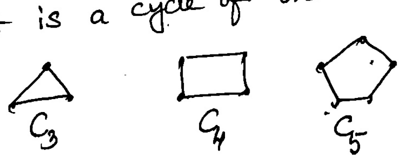
the diameter is 3.

Common classes of graphs.

If a graph G of order n can be labeled (or relabeled) as $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$, then G is called a path. A graph that is a path of order n is denoted as P_n.

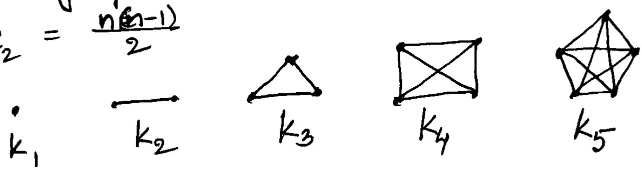


If a graph G of order $n \geq 3$ can be labeled (or relabeled) as $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$, then G is called a cycle. A graph that is a cycle of order $n \geq 3$ is denoted as C_n.

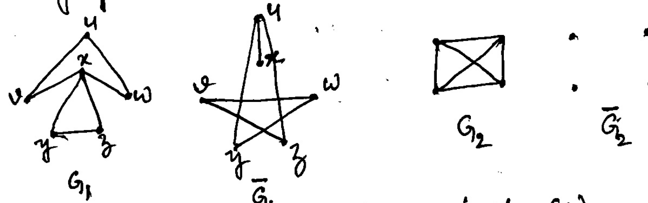


A graph G is <u>complete</u> if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n.
Since every two distinct vertices of K_n are joined by an edge, K_n has the maximum possible size for a graph with n vertices and is equal to

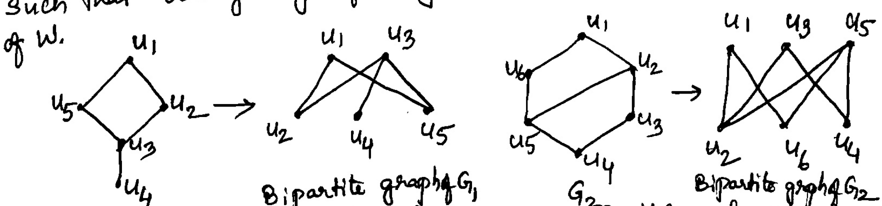
$$^{n}C_2 = \frac{n(n-1)}{2}$$



The <u>complement</u> $\bar{G}$ of a graph G is that graph whose vertex set is V(G) such that for each pair u,v of vertices of G, uv is an edge of $\bar{G}$ if and only if uv is not an edge of G.
If G is a graph of order n and size m, then $\bar{G}$ is a graph of order n and size $^{n}C_2 - m$. ~~The~~
The graph $\bar{K}_n$, then has n vertices and no edges.



A graph G is a <u>bipartite graph</u> if V(G) can be partitioned into two subsets U and W called partite sets, such that every edge of G joins a vertex of U and a vertex of W.

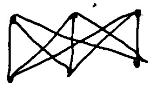


Bipartite graph G_1
partite sets $U = \{u_1, u_3\}$
$W = \{u_2, u_4, u_5\}$

Bipartite graph G_2
Partite sets $U = \{u_1, u_3, u_5\}$
$W = \{u_2, u_4, u_6\}$

If for every $u \in U$ and every $w \in W$, the edge (u,w) is an edge of the graph G, where U and W are partite sets, then G is called complete bipartite graph and it is denoted by $K_{p,q}$. where p is the number of vertices in U and q is the number of vertices in W.

examples



$K_{3,3}$

$K_{2,3}$

* Number of vertices in $K_{p,q}$ is $n = p+q$.
* Number of edges in $K_{p,q}$ is $m = pq$

<u>Theorem:</u>
If a nontrivial, connected graph G is bipartite, then G contains no odd cycles.

<u>Proof:</u>
Suppose G is a nontrivial, connected bipartite graph. containing ~~and~~ odd cycle $v_1, v_2, \ldots, v_{2k+1}, v_1$, for some integer k. ~~&~~

Let V_o and V_e be the partite sets.

Let $v_1 \in V_o$.

Since v_1v_2 is an edge, $v_2 \in V_e$.

Since v_2v_3 is an edge, $v_3 \in V_o$.

Continuing this reasoning $2k$ times we conclude that $v_{2k+1} \in V_o$.

Since $v_{2k+1}v_1$ is an edge, $v_1 \in V_e$.

Now we have $v_1 \in V_o$ and $v_1 \in V_e$, a contradiction

Hence our assumption is wrong.

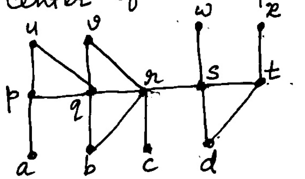
$\therefore$ If a nontrivial, connected graph G is bipartite then G contains no odd cycles.

[The converse: If G has no odd cycles then G is bipartite is also true
* We are not proving that here *]

Eccentricity of a vertex

For a vertex v in a connected graph G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The radius of a graph is the minimum eccentricity among all vertices. The diameter of a graph is the maximum eccentricity among all vertices. The center of G is a vertex having minimum eccentricity.

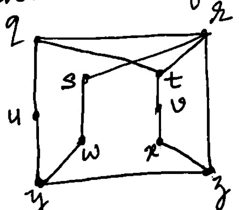
* Find the eccentricity of each vertex, radius, diameter and center of the following graph.



Soln $e(u)=5$, $e(v)=5$, $e(w)=5$, $e(x)=6$, $e(p)=5$, $e(q)=4$, $e(r)=3$, $e(s)=4$, $e(t)=5$, $e(a)=6$, $e(b)=4$, $e(c)=4$, $e(d)=5$.

radius $=3$, diameter $=6$, center is 3.

* Find the eccentricity of each vertex, radius, diameter and center of the following graph



Soln $e(q)=3$, $e(r)=2$, $e(s)=3$, $e(t)=3$, $e(u)=3$, $e(v)=4$, $e(w)=4$, $e(x)=3$, $e(y)=3$, $e(z)=2$

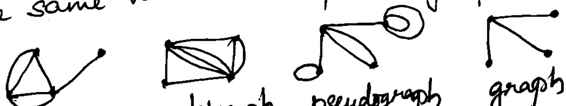
radius $=2$, diameter $=4$, centers are r and z.

Multigraphs and Digraphs

In a graph two vertices are either adjacent or they are not, that is, two vertices are joined by one edge or no edge.

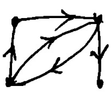
A multigraph M consists of a finite nonempty set V of vertices and a set E of edges, where every two vertices of M are joined by a finite number of edges (possibly zero). If two or more edges join the same pair of (distinct) vertices, then these edges are called parallel edges.

In a pseudograph, not only are parallel edges permitted but an edge is also permitted to join a vertex to itself. Such an edge is called a loop. If a loop e joins a vertex v to itself, then e is said to be a loop at v. There can be any finite number of loops at the same vertex in a pseudograph.



multigraph multigraph pseudograph graph

A digraph (or directed graph) D is a finite nonempty set V of objects called vertices together with a set E of ordered pairs of distinct vertices. The elements of E are called directed edges or arcs. If (u,v) is a directed edge, then we indicate this in a diagram representing D by drawing a directed line segment or curve from u to v. Then u is said to be adjacent to v and v is adjacent from u. Arcs (u,v) and (v,u) may both be present in some directed graph.



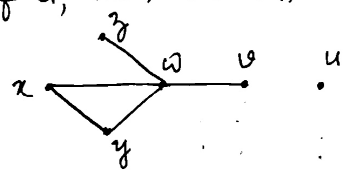
Degree of a vertex

The *degree* of a vertex v in a graph G is the number of edges incident with v and is denoted by $\deg v$. Also, $\deg v$ is the number of vertices adjacent to v. Two adjacent vertices are referred to as neighbours of each other. The set $N(v)$ of neighbours of a vertex v is called the neighbourhood of v. Thus $\deg v = |N(v)|$.

A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is called a *pendant vertex* (end-vertex).

The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$, the maximum degree of G is denoted by $\Delta(G)$.

So if G is a graph of order n and v is any vertex of G, then $0 \le \delta(G) \le \deg v \le \Delta(G) \le n-1$.



Here $n=6$, $m=5$

u is an isolated vertex.

$\deg(u)=0$, $\deg(v)=1$, $\deg(w)=4$

$\deg(x)=2$, $\deg(y=2$, $\deg(z)=1$.

v and z are pendant vertices

$\delta(G)=0$, & $\Delta(G)=4$

The first theorem of graph theory

If G is a graph of size m, then $\sum_{v \in V(G)} \deg v = 2m$.

Proof:

When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its incident vertices.

Note: Suppose that G is a bipartite graph of size m with partite sets $U=\{u_1,u_2,\dots u_s\}$ and $W=\{w_1,w_2\dots w_t\}$. Since every edge of G joins a vertex of U and a vertex of W it follows that adding degrees of the vertices in U (or W) gives the number of edges in G, i.e, $\sum_{i=1}^{s} \deg u_i = \sum_{i=1}^{t} \deg w_i = m$

Note: The vertex of even degree is called an even vertex, while a vertex of odd degree is an odd vertex

Corollary
Every graph has an even number of odd vertices.

Proof: Let G be a graph of size m.
Divide $V(G)$ into two subsets V_1 and V_2, where V_1 consists of odd vertices of G and V_2 consists of the even vertices of G.
By First Theorem of graph theory,

$$\sum_{v \in V(G)} \deg(v) = \sum_{v \in V_1} \deg v + \sum_{v \in V_2} \deg v = 2m$$

The sum $\sum_{v \in V_2} \deg v$ is even since it is a sum of even integers

$2m$ is even.

$$\therefore \sum_{v \in V_1} \deg v = 2m - \sum_{v \in V_2} \deg v,$$

$$\Rightarrow \sum_{v \in V_1} \deg v = \text{even}.$$

Since each of the members $\deg v \in V_1$ is odd, the number of odd vertices of G is even.

Corollary
If G is a graph of order n with $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

example
A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5?

Soln: Let x be the number of vertices of degree 3, and y of degree 5
Then $x+y+6=14 \Rightarrow y=8-x$ $\quad x\times 3+6\times 4+(8-x)\times 5=2\times 27 \Rightarrow x=5$
i.e. 5 vertices are of degree 3 and 3 vertices are of degree 5. & $y=3$

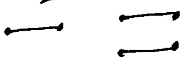
Regular graphs

We know that $0 \le \delta(G) \le \Delta(G) \le n-1$ for every graph G of order n. If $\delta(G) = \Delta(G)$ then the vertices of G have the same degree and G is called regular.

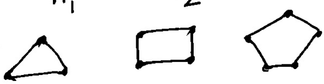
If $\deg v = r$ for every vertex v of G, where $0 \le r \le n-1$, then G is r-regular or regular of degree r.

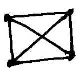
0-regular 

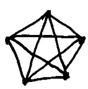
G_1 G_2 G_3

1-regular 

H_1 H_2

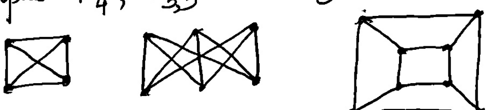
2-regular 

3-regular 

4-regular 

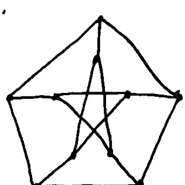
A 3-regular graph is also referred to as a cubic graph.

The graphs K_4, $K_{3,3}$ and Q_3 are cubic graphs



K_4 $K_{3,3}$ Q_3

The best known cubic graph is the Petersen graph, shown below.

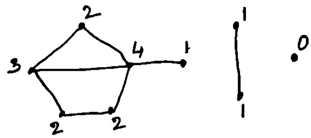


Theorem: Let r and n be integers with $0 \le r \le n-1$. There exists an r-regular graph of order n if and only if at least one of r and n is even.

Degree Sequences

It is typical for the vertices of a graph to have a variety of degrees. If the degrees of the vertices of a graph G are listed in a sequence s, then s is called a degree sequence of G.

For example, $s: 4,3,2,2,2,1,1,1,0$; $s': 0,1,1,1,2,2,2,3,4$; $s'': 4,3,2,1,2,2,1,1,0$. all of the sequences are degree sequences of the graph G of the below figure, each of whose vertices is labeled by its degree



The sequence s is non-increasing, s' is non-decreasing and s'' is neither.

Suppose that we are given a finite sequence s of nonnegative integers. This finite sequence of nonnegative integers is called graphical if it is a degree sequence of some graph.

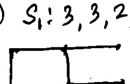
examples

Which of the following sequences are graphical.

(i) S_1: 3,3,2,2,1,1, (ii) S_2: 6,5,5,4,3,3,3,2,2, (iii) S_3: 7,6,4,4,3,3,3,

(iv) S_4: 3,3,3,1.

Sol (i) S_1: 3,3,2,2,1,1, $\sum \deg v_i = 12$ (even) ✓, 4 vertices of odd degree. $n=6$, maximum deg ≤ 5 ✓ $\therefore$ graphical



(ii) S_2: 6,5,5,4,3,3,3,2,2, $\sum \deg v_i = 33$ not even $\therefore$ not graphical

(iii) S_3: 7,6,4,4,3,3,3, $\sum \deg v_i = 30$ (even) ✓ 4 vertices of odd degree. $n=7$, maximum deg $\not\le 6$ $\therefore$ not graphical.

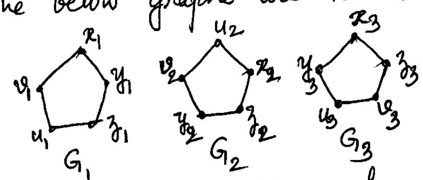
(iv) S_4: 3,3,3,1, $\sum \deg v_i = 10$ (even) ✓ 4 vertices of odd degree ✓ $n=4$, max deg ≤ 3 ✓.

But we can not draw a graph with this sequence. Only 1 vertex can be of deg 3, other two have to be of deg 2. $\therefore$ not graphical

Isomorphic Graphs

Two (labeled) graphs G_1 and G_2 are isomorphic (have the same structure), if there exists a one-to-one correspondence from $V(G_1)$ to $V(G_2)$ such that if $uv \in E(G_1)$ then $f(u)f(v) \in E(G_2)$ and if $uv \notin E(G_2)$ then $f(u)f(v) \notin E(G_2)$. In this case, f is called an isomorphism from G_1 to G_2.
Thus, if G_1 and G_2 are isomorphic graphs, then we say that G_1 is isomorphic to G_2 and we write $G_1 \cong G_2$.
If G_1 and G_2 are unlabeled then they are isomorphic if under any labeling of their vertices, they are isomorphic as labeled graphs.
If two graphs G_1 and G_2 are not isomorphic, then they are called non-isomorphic graphs and we write $G_1 \ncong G_2$.

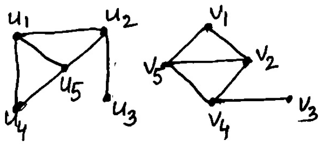
The below graphs are isomorphic.



The necessary conditions for two graphs to be isomorphic are:

1. Both graphs must have same number of vertices.
2. Both graphs must have same number of edges.
3. Both graphs must have equal number of vertices with the same degree.
4. Both the graphs must have the same degree sequence and same cycle vector $(c_1, c_2, \dots, c_n)$, where c_i is the number of cycles of length i.

* Show that the below graphs are isomorphic.



Sol^n The order of both graphs is 5

The size of both graphs is 6

The degree sequence of both graphs is: 1, 2, 3, 3, 3

$deg(u_3) = deg(v_3) = 1 \therefore$ u_3 can be mapped with v_3

$deg(u_4) = deg(v_1) = 2 \therefore$ u_4 can be mapped with v_1

u_3 is adjacent to u_2 & v_3 is adjacent to v_4

$\therefore$ u_2 can be mapped with v_4

u_2 is also adjacent to u_1 and u_5.

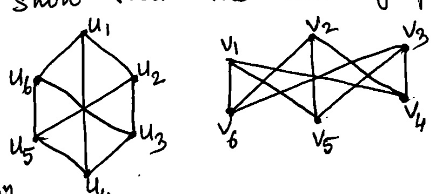
v_4 is also adjacent to v_2 and v_5

u_1 can be mapped with v_2 & u_5 can be mapped to v_5

$\therefore$ The one-to-one correspondence of the two graphs

is: $u_1 \sim v_2$, $u_2 \sim v_4$, $u_3 \sim v_3$, $u_4 \sim v_1$, $u_5 \sim v_5$

* Show that the below graphs are isomorphic.



Sol^n $n=6$, $m=9$, degree sequence is: 3, 3, 3, 3, 3, 3.

(i) $u_1 < \begin{matrix} u_2 \\ u_4 \\ u_6 \end{matrix}$ & $v_1 < \begin{matrix} v_4 \\ v_5 \\ v_6 \end{matrix}$ $\therefore u_1 \sim v_1$, $u_2 \sim v_4$, $u_4 \sim v_5$, $u_6 \sim v_6$

(ii) $u_2 < \begin{matrix} u_1 \\ u_3 \\ u_5 \end{matrix}$ & $v_4 < v_1$ & $v_4 < \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}$ $\therefore u_3 \sim v_2$, $u_5 \sim v_3$

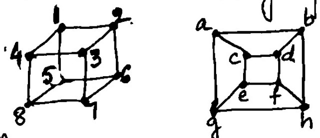
from graph   from mapping   from graph

(iii) $u_3 < \begin{matrix} u_2 \\ u_4 \\ u_6 \end{matrix}$ & $v_2 < \begin{matrix} v_4 \\ v_5 \\ v_6 \end{matrix}$ = $v_2 < \begin{matrix} v_4 \\ v_5 \\ v_6 \end{matrix}$ $\therefore$ verified. Hence the one-to-one correspondence is

from graph   from mapping = from graph

$u_1 \sim v_1$, $u_2 \sim v_4$, $u_3 \sim v_2$, $u_4 \sim v_5$, $u_5 \sim v_3$, $u_6 \sim v_6$

* Show that the below graphs are isomorphic.



Sol^n $n=8$, $m=12$, degree sequence is 3, 3, 3, 3, 3, 3, 3, 3

(i)

$1 \left< \begin{matrix} 2 \\ 4 \\ 5 \end{matrix} \right.$ $\quad a \left< \begin{matrix} b \\ c \\ g \end{matrix} \right.$ $\therefore$ $1 \sim a$, $2 \sim b$, $4 \sim c$, $5 \sim g$

(ii)

$2 \left< \begin{matrix} 1 \\ 3 \\ 6 \end{matrix} \right.$ $\quad b \left< \begin{matrix} a \\ \\ \end{matrix} \right.$ & $b \left< \begin{matrix} a \\ d \\ h \end{matrix} \right.$ $\therefore$ $3 \sim d$, $6 \sim h$

from graph   from mapping   from graph

observe 3 is adjacent to 4
& d is adjacent to c

(iii)

$3 \left< \begin{matrix} 2 \\ 4 \\ 7 \end{matrix} \right.$ $\quad d \left< \begin{matrix} b \\ c \\ \end{matrix} \right.$ & $d \left< \begin{matrix} b \\ c \\ f \end{matrix} \right.$ $\therefore$ $7 \sim f$

from graph   from mapping   from graph

(iv)

$4 \left< \begin{matrix} 1 \\ 3 \\ 8 \end{matrix} \right.$ $\quad c \left< \begin{matrix} a \\ d \\ \end{matrix} \right.$ & $c \left< \begin{matrix} a \\ d \\ e \end{matrix} \right.$ $\therefore$ $8 \sim e$

from graph   from mapping   from graph

(v)

$5 \left< \begin{matrix} 1 \\ 6 \\ 8 \end{matrix} \right.$ $\quad g \left< \begin{matrix} a \\ h \\ e \end{matrix} \right.$ $=$ $g \left< \begin{matrix} a \\ h \\ e \end{matrix} \right.$ $\therefore$ verified.

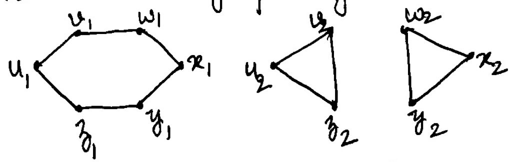
from graph   from mapping = from graph

Hence the one-to-one correspondence is:

$1 \sim a$, $2 \sim b$, $3 \sim d$, $4 \sim c$, $5 \sim g$, $6 \sim h$, $7 \sim f$, $8 \sim e$.

Hence the given graphs are isomorphic.

* Are the two graphs given below isomorphic?



G_1 G_2

$\underline{\text{sol}^n}$ $n=6$, $m=6$, degree sequence is $2,2,2,2,2,2$

$u_1 \left\langle \begin{matrix} v_1 \\ z_1 \end{matrix} \right.$ from graph $u_2 \left\langle \begin{matrix} v_2 \\ z_2 \end{matrix} \right.$ from graph $\therefore u_1 \sim u_2,\ v_1 \sim v_2,\ z_1 \sim z_2$

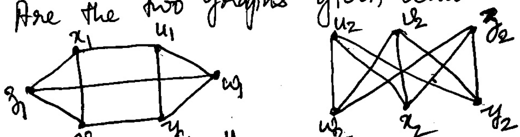
$v_1 \left\langle \begin{matrix} u_1 \\ w_1 \end{matrix} \right.$ from graph $v_2 \left\langle \begin{matrix} u_2 \\ \ \end{matrix} \right.$ from mapping $v_2 \left\langle \begin{matrix} u_2 \\ z_2 \end{matrix} \right.$ $\therefore w_1 \sim z_2$

but already $z_1 \sim z_2$

$\therefore$ one-to-one correspondence is not possible

$\therefore$ the graphs G_1 and G_2 are not isomorphic.

* Are the two graphs given below isomorphic?



H_1 $\underline{H_2}$

$\underline{\text{sol}^n}$ $n=6$, $m=6$, degree sequence is $2,2,2,2,2,2$

$z_1 \left\langle \begin{matrix} x_1 \\ w_1 \\ v_1 \end{matrix} \right.$ from graph $u_2 \left\langle \begin{matrix} w_2 \\ x_2 \\ y_2 \end{matrix} \right.$ from graph. $\therefore z_1 \sim u_2,\ x_1 \sim w_2,\ w_1 \sim x_2,\ v_1 \sim y_2$

$x_1 \left\langle \begin{matrix} z_1 \\ u_1 \\ v_1 \end{matrix} \right.$ from graph $w_2 \left\langle \begin{matrix} u_2 \\ y_2 \end{matrix} \right.$ from mapping & $w_2 \left\langle \begin{matrix} u_2 \\ v_2 \\ z_2 \end{matrix} \right.$ $\because x_1 \sim w_2$ & $v_1 \sim y_2$ & $x_1 v_1$ is an edge but $w_2 y_2$ is not an edge

$\therefore$ one-to-one correspondence is not possible

H_1 & H_2 are complements of G_1 & G_2 respectively. $\therefore$ the graphs are not isomorphic

Note! Two graphs G and H are isomorphic if and only if their complements are isomorphic.

Matrix representation of a graph.

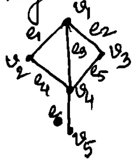
A graph G can be defined by two sets, namely its vertex set $V(G)$ and edge set $E(G)$ or by a diagram. A graph can also be described by a matrix and for some purposes this is especially useful.

Let G be a graph of order n and size m, where $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$.

The adjacency matrix of G is the $n \times n$ matrix $A = [a_{ij}]$, where
$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

The incidence matrix of G is the $n \times m$ matrix $B = [b_{ij}]$, where
$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

* Find the adjacency matrix and incidence matrix of the graph



sol^n

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

adjacency matrix $\qquad$ incidence matrix

* Find the adjacency matrix of K_4.

sol^n

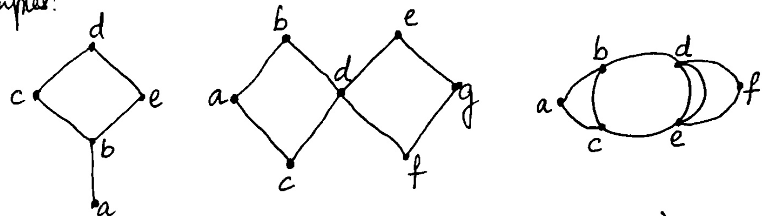
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

* Find the incidence matrix of $K_{3,2}$

sol^n

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

A circuit C in a graph G is called an <u>Eulerian circuit</u> if C contains every edge of G. Since no edge is repeated in a circuit, every edge appears exactly once in an Eulerian circuit.
A connected graph that contains an Eulerian circuit is called an <u>Eulerian graph</u>.
In a connected graph G, an open trail that contains every edge of G is an <u>Eulerian trail</u>.
examples:



(a) (b) (c)

(a) has an Euler trail but no Euler circuit.
(b) has both Euler circuit and Euler trail.
(c) has an Euler trail but no Euler circuit.

<u>Theorem:</u> A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

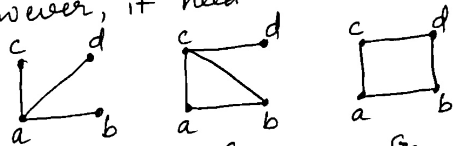
<u>Corollary:</u> A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

has both Euler circuit and Euler trail

has an Euler trail but no Euler circuit

A cycle in a graph G that contains every vertex
of G is called a Hamiltonian cycle of G. Thus a
Hamiltonian cycle of G is a spanning cycle of G.
A Hamiltonian graph is a graph that contains a
Hamiltonian cycle. The graph C_n $(n \geq 3)$ is Hamiltonian.
Also, for $n \geq 3$, the complete graph K_n is a Hamiltonian
graph.

A path in a graph G that contains every vertex
of G is called a Hamiltonian path in G. If a
graph contains a Hamiltonian cycle, then it contains
a Hamiltonian path. In fact, removing any edge
from a Hamiltonian cycle produces a Hamiltonian
path. If a graph contains a Hamiltonian path,
however, it need not contain a Hamiltonian cycle



G_1 G_2 G_3

The graph G_1 has no hamiltonian path (and no hamiltonian cycle)
The graph G_2 has hamiltonian path but no hamiltonian cycle.
The graph G_3 has both hamiltonian path and hamiltonian cycle

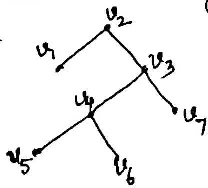
Theorem: Let G be a graph of order $n \geq 3$. If $\deg u + \deg v \geq n$
for each pair u, v of non adjacent vertices of G, then
G is Hamiltonian (Converse need not be true).

Corollary: Let G be a graph of order $n \geq 3$. If
$\deg v \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian.
(Converse need not be true).

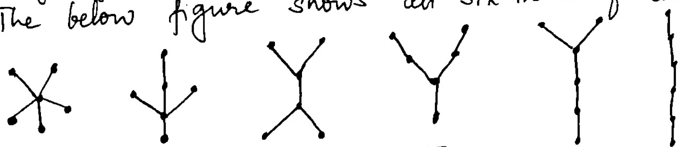
Tree.

A graph G is called acyclic if it has no cycles.
A tree is an acylic connected graph.

example



When dealing with trees, we often use T rather than G to denote a tree.
Indeed, we could define a tree as a connected graph, every edge of which is a bridge.
The below figure shows all six trees of order 6.

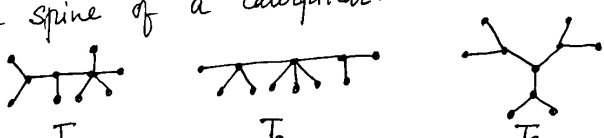


T_1 T_2 T_3 T_4 T_5 T_6.

The tree $T_1 = K_{1,5}$ is a star and $T_6 = P_6$ is a path.
The number of end vertices in the trees of the above figure ranges from 2 to 5. A tree containing exactly two vertices that are not end vertices is called a double star. The trees T_2 and T_3 in the above figure are double star.

double stars.

A caterpillar is a tree of order 3 or more, the removal of whose end-vertices produces a path called the spine of a caterpillar.

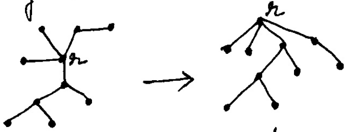


T_1 T_2 T_3

T_1 and T_2 are caterpillars but T_3 is not a caterpillar.

Sometimes it is convenient to select a vertex of a tree T under discussion and designate this vertex as the root of T. The tree T then becomes a rooted tree. Often the rooted tree T is drawn with the root r at the top and the other vertices of T drawn below, in levels according to their distance from r.

example



Acyclic graphs are also referred to as forests. Therefore, each component of a forest is a tree. One fact that distinguishes trees from forests is that a ~~trees~~ tree is required to be connected, while a forest is not required to be connected. Since a tree is connected, every two vertices in a tree are connected by a path.

Theorem:
A graph G is a tree if and only if every two vertices of G are connected by a unique path.

Proof:
First, let G be a tree. Then G is connected by definition. Thus every two vertices of G are connected by a path. Assume, to the contrary, that there are two vertices of G that are connected by two distinct paths. Then a cycle is produced from some or all of the edges of these two paths. This is a contradiction.
For the converse, suppose that every two distinct vertices of G are connected by a unique path. Certainly then, G is connected.
Assume, to the contrary, that G has a cycle C.
Let u and v be two distinct vertices of C.
Then C determines two distinct u-v paths, producing a contradiction. Thus G is acyclic and so G is a tree.

Theorem:
Every tree of order n has size $n-1$.

Proof

There is only one tree of order 1, namely K_1, which has size 0. Thus the result is true for $n=1$.
Assume for a positive integer k that the size of every tree of order k is $k-1$.
Let T be a tree of order $k+1$.
Every nontrivial tree T contains at least two end vertices.
Let v be one of them. Then $T' = T-v$ is a tree of order k.
By the induction hypothesis, the size of T' is $m=k-1$.
Since T has exactly one more edge than T', the size of T is $m+1 = (k-1)+1 = (k+1)-1$, as desired.

Problem: The degrees of the vertices of a certain tree T of order 13 are 1, 2 and 5. If T has exactly three vertices of degree 2, how many end-vertices does it have?

Ans: let x be the number of vertices of degree 1,
y be the number of vertices of degree 5,

then $x+3+y=13 \Rightarrow y=10-x$

Also $x\times 1 + 3\times 2 + (10-x)\times 5 = 2\times(13-1) \Rightarrow x=8$.

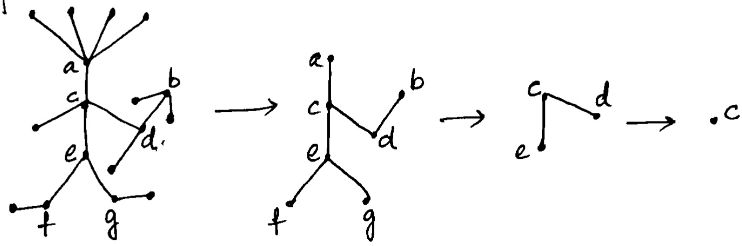
Hence the tree has 8 end vertices.

Corollary
Every forest of order n with k components has size $n-k$.

Eccentricity
For a vertex v in a connected graph G, the eccentricity $e(v)$ ~~of~~ v is the distance between v and a vertex farthest from v in G. The center of G is a vertex having minimum eccentricity.

The centre of a tree can be obtained by removing the leaves of the tree continuously, until it reduces to a single edge or a single vertex.

example



Theorem:
There are one or more centers in every tree; in the later case they are adjacent.

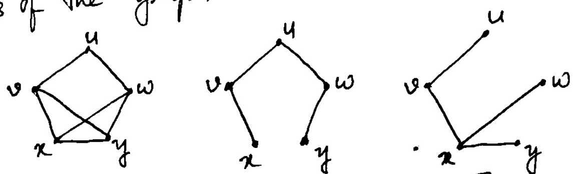
Theorem:
Let T be a tree on more than one vertex.

(i) If a longest path of T has an even length, then T has exactly one center, which is the mid-vertex of each longest path.

(ii) If a longest path of T has an odd length, then T has exactly two adjacent centers, which are the two mid most vertices of each longest path.

Spanning Trees of a graph.

Suppose G is a connected graph. We can produce trees T that are subgraphs of the given connected graph G such that $V(T) = V(G)$. Also it is a spanning subgraph of G. In the below figure T_1 and T_2 represent spanning trees of the graph G.



G $\qquad$ T_1 $\qquad$ T_2

Let G be a connected graph each of whose edge is assigned a number (called the cost or weight of the edge). We denote the weight of an edge e of G by $w(e)$. Such a graph is called a weighted graph. For each subgraph H of G, the weight $w(H)$ of H is defined as the sum of the weights of its edges, that is $w(H) = \sum_{e \in E(H)} w(e)$

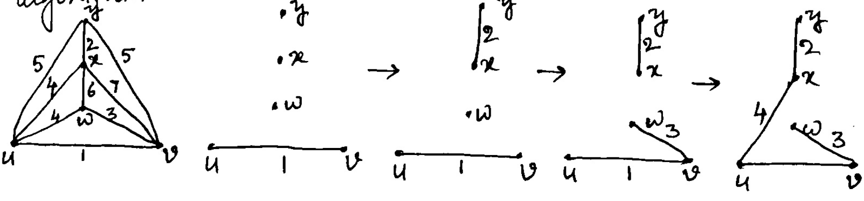
We seek a spanning tree of G whose weight is minimum among all spanning trees of G. Such a spanning tree is called a minimum spanning tree. The problem of finding a minimum spanning tree in a connected weighted graph is called the minimum spanning tree problem.

Kruskal's algorithm

For a connected weighted graph G, a spanning tree T of G is constructed as follows:

For the first edge e_1 of T, we select any edge of G of minimum weight and for the second edge e_2 of T, we select any remaining edge of G of minimum weight. For the third edge e_3 of T, we choose any remaining edge of G of minimum weight that does not produce a cycle with the previously selected edges. We continue in this manner until a spanning tree is produced.

The below figure shows on how a spanning tree of a connected weighted graph is constructed using Kruskal's algorithm.



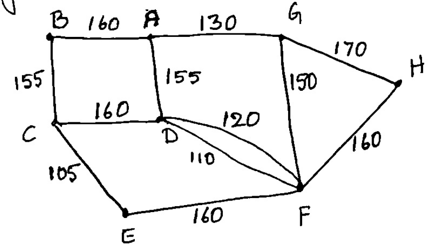
* Eight cities A, B, C, D, E, F, G and H are are required to be connected by a new railway network. The possible tracks and the cost involved to lay them (in crores of rupees) are summarized in the following table:

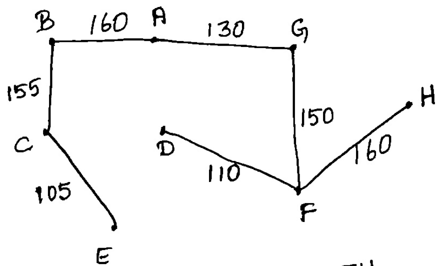
Between	Cost	Between	Cost	Between	Cost	Between	Cost
A and B	160	B and C	155	D and F	110	F and H	160
A and D	155	C and D	160	E and F	160	G and H	170
A and G	130	C and E	105	F and G	150	D and F	120

(i) Draw the weighted graph which represents the new railway network.

(ii) Further determine a railway network of minimal cost that connects all these cities using Kruskal's algorithm. Also mention the minimum cost.

sol





CE + DF + AG + FG + BC + AB + FH

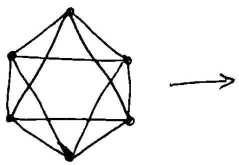
= 105 + 110 + 130 + 150 + 155 + 160 + 160

= 970

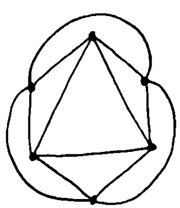
<u>Planar Graphs</u>

A graph G is called a planar graph if G can be drawn in the plane so that no two of its edges cross each other. A graph that is not planar is called nonplanar. A graph G is called a plane graph if it is drawn in the plane so that no two edges of G cross.

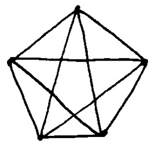
Eg



planar

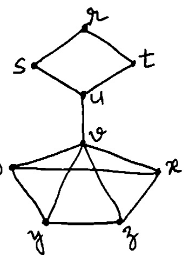


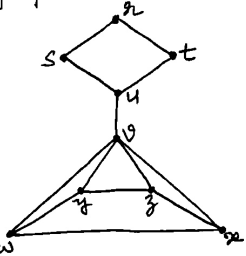
planar, plane.



non-planar

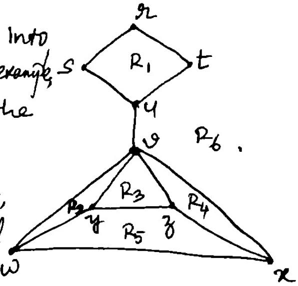
Consider the first graph shown in the below figure.



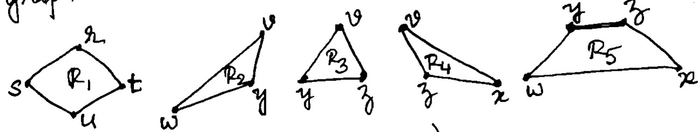


The graph is connected. But it is also planar, as we can see from the second graph, where it is drawn as a plane graph.

A plane graph divides the plane into connected pieces called regions. For example, in the case of the plane graph of the above figure, there are six regions. This graph is redrawn on the right, where the six regions are denoted by $R_1, R_2, R_3, R_4, R_5, R_6$



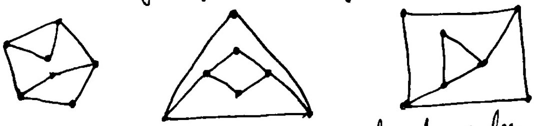
In every plane graph, there is always one region that is unbounded. This is the exterior region. For the previous graph, R_6 is the exterior region. The subgraph of a plane graph whose vertices and edges are incident with a given region R is the boundary of R. The boundaries of the six regions of the above graph are also shown in the below figure.



<u>Theorem</u>: (The Euler identity)
If G is a connected plane graph of order n, size m and having r regions, then $n-m+r=2$.

<u>Proof</u>
First, if G is a tree of order n, then $m=n-1$ and $r=1$; so $n-m+r=2$.
Therefore, we need only be concerned with connected graphs that are not trees.
Assume, to the contrary, that the theorem does not hold. Then there exists a connected plane graph G of smallest size for which the Euler identity does not hold. Suppose that G has order n, size m and r regions.
So $n-m+r \neq 2$.
Since G is not a tree, there is an edge e that is not a bridge.
Thus $G-e$ is a connected plane graph of order n and size $m-1$ having $r-1$ regions.
Because the size of $G-e$ is less than m, the Euler identity holds for $G-e$.
So $n-(m-1)+(r-1)=2$, but then $n-m+r=2$, which is a contradiction. Hence our assumption is incorrect.
Therefore $n-m+r=2$ holds.

The below figure shows a planar graph G and several ways of drawing G as a plane graph.



However, since G has a fixed order $n=7$ and fixed size $m=9$ and the Euler Identity holds ($n-m+r=7-9+r=2$), each drawing of G as a plane graph always produces the same number of regions, namely $r=4$.

<u>Theorem</u>:
If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n-6$.

The above theorem provides a necessary condition for a graph to be planar and so provides a sufficient condition for a graph to be nonplanar. In particular, the contrapositive of the above Theorem gives us the following.

If G is a graph of order $n \geq 3$ and size m such that $m > 3n-6$, then G is nonplanar.

Also,

<u>Theorem</u>:
If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 2n-4$ if G has no 3-cycles.

The contropositive,
If G is a graph with no triangles and $m > 2n-4$ then G is a nonplanar graph.

Corollary:
Every planar graph contains a vertex of degree 5 or less.

Proof:
Suppose that G is a graph, every vertex of which has degree 6 or more.
Let G have order n and size m.
~~Ce~~ Certainly, $n \geq 7$,
Then $2m = \sum_{v \in V(G)} \deg v \geq 6n$.

Thus $m \geq 3n > 3n-6$.

Hence G is nonplanar.
Therefore, if G is a planar graph then G contains a vertex of degree 5 or less.

Corollary
The complete graph K_5 is nonplanar.

Proof:
The graph K_5 has order $n=5$ and size $m=10$.
Since $m=10>9=3n-6$, it follows that K_5 is nonplanar.

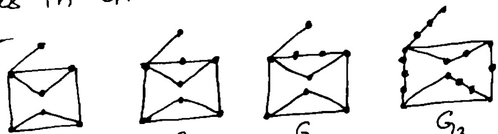
Corollary
The complete bipartite graph $K_{3,3}$ is nonplanar.

Proof
The graph $K_{3,3}$ has order $n=6$ and size $m=9$.
Since $m=9>8=2n-4$, it follows that $K_{3,3}$ is nonplanar

Note: There exists a graph of order $n \geq 3$ and size $m > 3n-6$ that contains neither K_5 nor $K_{3,3}$ as a subgraph.

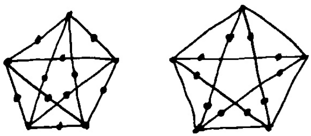
Let G be a graph and $e = uv$ an edge of G. A subdivision of e is the replacement of the edge e by a simple path $u_0, u_1, \ldots, u_k$, where $u_0 = u$ and $u_k = v$ are the only vertices of the path in $V(G)$.
We say that G' is a subdivision of G, if G' is obtained from G by a sequence of subdivisions of edges in G.

example

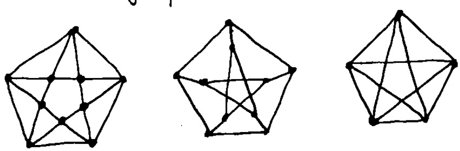


G G_1 G_2 G_3

A graph G is said to be homeomorphic to G', if G' can be obtained from G by insertion or deletion of vertices of degree 2 between the edges of G.
The below graphs are homeomorphic to each other.



The below graphs are non-homeomorphic to eachother.



<u>Kuratowski's Theorem:</u>
A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

OR

A graph G is planar if and only if G does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.

Detection of planarity of a graph.

If a given graph G is planar or non planar is an important problem. We must have some simple and efficient criterion. We follow the following single steps to detect the planarity of a graph.

a. Since a disconnected graph is planar if and only if each of its component is planar, consider only one component at a time. Therefore, for the given arbitrary graph G, determine the ~~components~~ such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$, where each G_i is a connected graph. Then check ~~the~~ each component G_i for planarity.

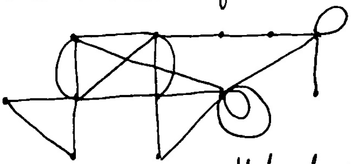
1. Since parallel edges do not affect planarity, eliminate the edges in parallel by removing all but one edge between every pair of vertices.
2. Since addition or removal of a selfloop does not affect planarity, remove all self-loops.
3. Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Hence eliminate all edges in series.

Repeating the above steps yields one of the following graphs:

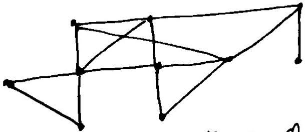
(i) A single edge.
(ii) A complete graph on four vertices.
(iii) $m > 3n-6$
(iv) A K_5 or $K_{3,3}$ subgraph.

If the graph reduces to (i) or (ii), the given graph is planar.
If the graph reduces to (iii) or (iv), the given graph is nonplanar.

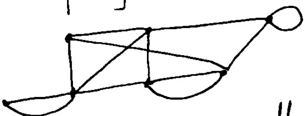
* Check the planarity of the following graph by the method of elementary deduction!



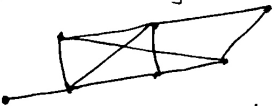
(i) removing parallel edges and self loops, eliminating edges in series



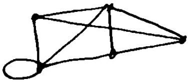
(ii) collapsing vertices of degree 1 and degree 2.



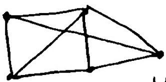
(iii) eliminating parallel edges and self loop



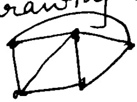
(iv) collapsing vertex of degree 1 and eliminating ~~vertex~~ edges in series



(v) removing the self loop

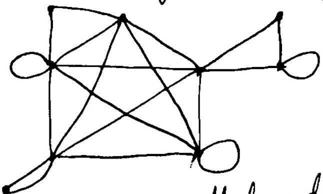


(vi) redrawing the graph

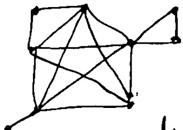


The graph is planar

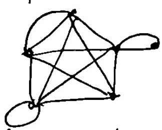
* Check the planarity of the following graph by the method of elementary deduction.



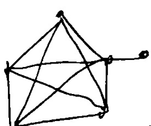
(i) eliminate parallel edges and self loops



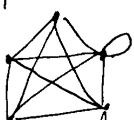
(ii) collapse vertices of degree 1 and degree 2



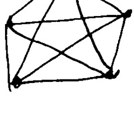
(iii) eliminate self loop and parallel edges



(iv) collapse vertex of degree 1



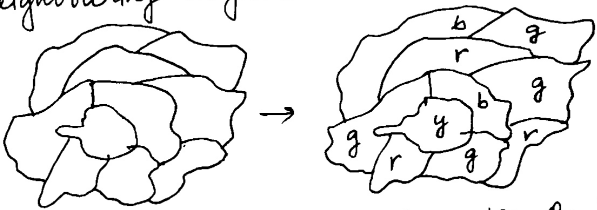
(v) eliminate self loop



The graph is K_5, which is non-planar

Vertex Colouring

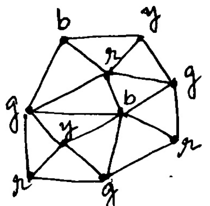
Consider the problem of colouring the regions on a map with different colors, such that no two neighbouring regions have the same color.



It can be seen that each region of the map can be assigned one of four given colours such that neighbouring regions are colored differently. Indeed, one such coloring is shown in the figure, where r, b, g and y denoted, red, blue, green and yellow respectively.

With each map, there is associated a graph G, called the dual of the map, whose vertices are the regions of the map and such that two vertices of G are adjacent if the corresponding regions are neighbouring regions.

Coloring the regions of a map suggests coloring the vertices of its dual. Indeed, it suggests coloring the vertices of any graph. By a proper coloring (or, more simply, a coloring) of a graph G, we mean an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, such that adjacent vertices are colored differently.

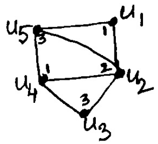


The smallest number of colors in any coloring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$. If it is possible to color (the vertices of) G from a set of k colors, then G is said to be k-colorable. A coloring that uses k colors is called a k-coloring. If $\chi(G) = k$, then G is said to be k-chromatic and every k-coloring of G is a minimum coloring of G.

The following observations can be made w.r.t the coloring of a graph.

1. A graph is 1-chromatic if and only if it is totally disconnected.
2. A graph having at least one edge is at least 2-chromatic (bichromatic)
3. A graph G having n vertices has $\chi(G) \le n$.
4. If H is a subgraph of a graph G, then $\chi(H) \le \chi(G)$.
5. $\chi(K_n) = n$ and $\chi(\overline{K_n}) = 1$.
6. $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$.
7. If $G_1, G_2, \ldots, G_r$ are the components of a disconnected graph G, then $\chi(G) = \max\{\chi(G_1), \chi(G_2), \ldots \chi(G_r)\}$.
8. $\chi(K_{m,n}) = 2$.

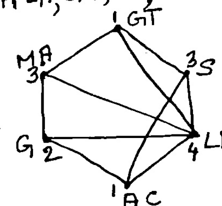
A proper coloring of a graph naturally induces a partition of the vertices into different subsets. For example, the coloring in the following graph produces the partitioning, called the chromatic partitioning $V_1 = \{u_1, u_4\}$, $V_2 = \{u_2\}$, $V_3 = \{u_3, u_5\}$.



Problem.

The mathematics department of a certain college plans to schedule the classes Graph theory (GT), Statistics (S), Linear Algebra (LA), Advanced Calculus (AC), Geometry (G) and Modern Algebra (MA). Ten students have indicated the course they plan to take. With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having a student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period. Below is mentioned the student preferences.

A: LA, S ; B: MA, LA, G ; C: MA, G, LA ; D: G, LA, AC ; E: AC, LA, S ; F: G, AC ; G: GT, MA, LA ; H: LA, GT, S ; I: AC, S, LA ; J: GT, S.

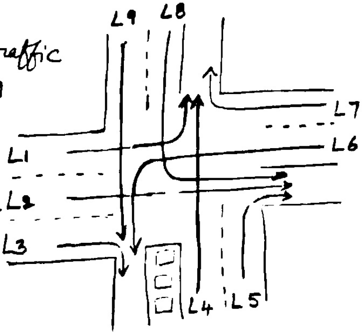


Soln The above situation can be represented by the following graph, where the vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking both classes. The graph can be colored by 4 colors as shown. Hence $\chi(H) = 4$. This also tells us one way to schedule these six classes during four time periods.

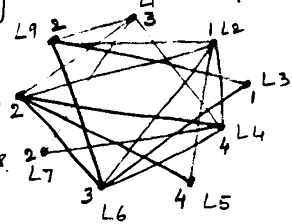
Period 1: GT, AC ; Period 2: G ; Period 3: S, MA ; Period 4: LA

Problem

The following figure shows the traffic lanes L1, L2, L3, L4, L5, L6, L7, L8, L9 at the intersection of two busy streets. A traffic light is located at this intersection. During a certain phase of the traffic light, those cars in lane for which the light is green may proceed safely through the * intersection. What is the minimum number of phases needed for the traffic light so that all cars may proceed through the intersection?



Sol^n Construct a graph G to model the situation, where, $V(G)=\{L1, L2, L3, L4, L5, L6, L7, L8, L9\}$ and two vertices (lanes) are joined by a edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is a possibility of an accident.



Answering this question require determining the chromatic number of the graph. Notice that $\{L2, L4, L6, L8\} \cong K_4$. Since there exists a 4-coloring of G, as indicated in the graph, $\chi(G)=4$.

So a minimum of 4 phases is needed for the traffic light so that all cars may proceed through the intersection.

__Theorem__ (Greedy Algorithm): For every graph G, $\chi(G) \le 1+\Delta(G)$.

__Brook's Theorem__: For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \le \Delta(G)$.

__Five Color Theorem__: For every planar graph G, $\chi(G) \le 5$

__The Four Color Theorem__: The chromatic number of every planar graph is at most 4.

Coloring Enumeration

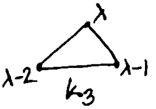
Let G be a graph and $\lambda \in N$. Define the number $P(G;\lambda)$ to be the number of proper λ-vertex colorings $c: V(G) \to \{1,2,3,\ldots,\lambda\}$. This property of a graph can be expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G.

i.e, Let G be a labeled graph. A coloring of G from λ colors is a coloring of G which uses λ or fewer colors. Two colorings of G from λ colors will be considered different as atleast one of the labeled vertex is assigned different colors.

<u>Note</u>: (i) For each $\lambda < \chi(G)$, we have $P(G:\lambda)=0$

(ii) For each $\lambda \geq \chi(G)$, we have $P(G:\lambda)>0$

(iii) Indeed the smallest λ for which $P(G:\lambda)>0$ is the chromatic number of G.



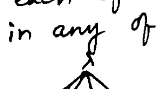
There are λ ways of coloring any given vertex of K_3. For a second vertex, any of $\lambda-1$ colors may be used, while there are $\lambda-2$ ways of coloring the remaining vertex.

Thus $P(K_3:\lambda) = \lambda(\lambda-1)(\lambda-2)$.

This can be generalized to any complete graph.

$$P(K_n;\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1).$$

The corresponding polynomial of the totally disconnected graph (null graph) $\overline{K_n}$ is particularly easy to find, since each of its n vertices may be colored independently in any of λ ways. Thus $P(\overline{K_n}:\lambda) = \lambda^n$.

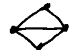


The central vertex of complete bipartite graph $K_{1,4}$ can be colored in λ ways, while each end vertex may be colored in any $\lambda-1$ ways.

Thus $P(K_{1,4}:\lambda) = \lambda(\lambda-1)^4$.

* A graph G with n vertices is a tree if and only if $P(G:\lambda) = \lambda(\lambda-1)^{n-1}$

Theorem: Let u and v be two non adjacent vertices in a graph G. ... $G+e$ be a graph obtained by adding an edge between u and v. Let $G.e$ be a single graph obtained from G by fusing the vertices u and v together and replacing sets of parallel edges with single edge.
Then $P(G;\lambda) = P(G+e:\lambda) + P(G.e:\lambda)$

Problem: Find the chromatic polynomial of the graph 

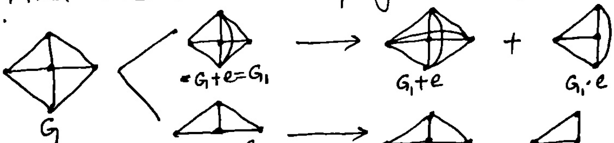
Sol^n

$G \qquad G+e \qquad G.e$

$$\therefore P(G:\lambda) = P(G+e;\lambda) + P(G.e;\lambda) = P(K_4:\lambda) + P(K_3:\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)(\lambda-3+1)$$
$$= \lambda(\lambda-1)(\lambda-2)^2$$

Problem: Find the chromatic polynomial of the graph 

Sol^n.



$G \qquad G+e=G_1 \qquad G_1+e \qquad G_1.e$

$G.e=G_2 \qquad G_2+e \qquad G_2.e$

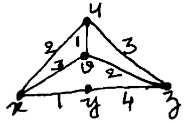
$$\therefore P(G:\lambda) = P(G_1+e:\lambda) + P(G_1.e:\lambda) + P(G_2+e:\lambda) + P(G_2.e:\lambda)$$
$$= P(K_5:\lambda) + P(K_4:\lambda) + P(K_4:\lambda) + P(K_3:\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda-4\lambda+12+2\lambda-6+1)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

Edge Coloring

A k-edge-coloring of G is a labeling $f: E(G) \to S$, where $|S| = k$. The labels are colors; the edges of one color from a color class. A k-edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is k-edge-colorable if it has a proper k-edge-coloring. The edge-chromatic number (chromatic index) $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.

<u>example</u>

The edge chromatic number of the following graph is four



For a graph G and any vertex $u \in V(G)$, all edges with u as an end vertex are adjacent and hence must receive different colors in a proper edge coloring of G. Hence, we note the obvious lower bound for the edge chromatic number of G $\chi'(G) \geq \Delta(G)$; the maximum degree of G.

Edge chromatic number of some basic graphs:

$\chi'(K_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$, $\chi'(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$, $\chi'(P_n) = 2$

<u>Theorem</u>: For a bipartite graph G, we have $\chi'(G) = \Delta(G)$.

<u>Theorem</u>: For the complete bipartite graph $K_{m,n}$, we have $\chi'(K_{m,n}) = \max\{m, n\}$.

<u>Theorem</u>: If G is a simple graph, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

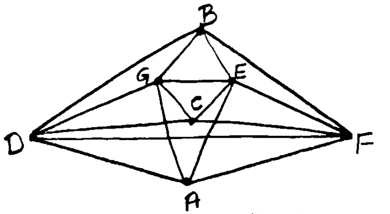
<u>Vizing's Theorem</u>: For a non empty graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = 1 + \Delta(G)$.

<u>Theorem</u>: Let G be a graph of odd order n and size m. If $m > \frac{(n-1)\Delta(G)}{2}$, then $\chi'(G) = 1 + \Delta(G)$.

Problem:

Alvin (A) has invited three married couples to his summer house for a week: Bob (B) and Carrie (C) Hanson, David (D) and Edith (E) Irwin and Frank (F) and Gena (G) Jackson. Since all six guests enjoy playing tennis, he decides to set up some tennis matches. Each of his six guests will play a tennis match against every other guest except his/her spouse. In addition, Alvin will play a match against each of David, Edith, Frank and Gena. If no one is to play two matches on the same day, what is a schedule of matches over the smallest number of days?

Sol^n First, we construct a graph H whose vertices are the people at Alvin's summer house, so V(H)=A,B,C,D,E,F,G, where two vertices of H are adjacent if the two vertices (people) are to play a tennis match. To answer the question, we determine the chromatic index of H.



First, observe that $\Delta(H)=5$.
Hence $\chi'(H)=5$ or $\chi'(H)=6$. Also, the order of H is $n=7$ and its size is $m=16$. Since $m=16>15=\frac{(7-1)\times 5}{2}=\frac{(n-1)\Delta(H)}{2}$, it follows that $\chi'(H)=6$. And the following figure shows the 6-edge coloring of H, which provides a schedule of matches that take place over the smallest number of days (namely 6).

Day 1: B-G, C-E, D-F
Day 2: B-D, G-E, A-F
Day 3: B-F, A-E, C-G
Day 4: B-E, C-D, A-G
Day 5: D-G, E-F
Day 6: A-D, C-F