Fundamental Principles of Counting and Combinatorics. (1)

The word discrete refers to quantities that are individual, separate or distinct.

Discrete mathematics typically uses only the whole numbers on a number line.

Combinatorics, the study of arrangements of objects is an important part of discrete mathematics. Any type of application in the sciences that involves choices and possibilities often uses the concepts of combinatorics.

Combinatorial chemistry explores the results when a series of different chemical groups are added to the same basic chemical structure to investigate the qualities of the resulting compound.

In addition, combinatorics is very important to the study of probability. In order to calculate the probability of an event, it is often necessary to calculate how many different ways something can happen.

Enumeration or counting, is an obvious process one needs to learn when first studying arithmetic. Enumeration has applications in such areas as coding theory, probability and statistics (in mathematics) and in the analysis of algorithms (in computer science).

Enumeration, the counting of objects with certain properties, is an important part of combinatorics. We must count objects to solve many different types of problems.

For instance, counting is used to determine the complexity of algorithms. Counting is also required to determine whether there are enough telephone numbers or internet protocol addresses to meet demand. Furthermore, counting techniques are used extensively when probabilities of events are computed.

The binary codes that computers use are generally controlled and kept (mostly) error free through the use of discrete mathematics. Computer security for the simplest (checking your on-line bank balance) and the most complex (high level classified data) digital information is handled through encryption that relies on the concepts of discrete mathematics.

The basics of counting

Suppose that a password on a computer system consists of six, seven or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must ~~consist~~ contain at least one digit. How many such passwords are there?

Counting problems arise throughout mathematics and computer science.

For example, we must count the successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events. We need to count the number of operations used by an algorithm to study its time complexity.

We will introduce the basic techniques of counting in this section. These methods serve as the foundation for all most all counting techniques.

Fundamental Principle of Counting

The fundamental counting principle is a rule to count all the possible ways for an event to happen or the total number of possible outcomes in a situation.

The product rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are m ways to do the first task and for each of these ways of doing the first task, there are n ways to do the second task, then there are $m \times n$ ways to do the procedure. At times this rule is also referred to as the principle of choice.

Eg There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

Soln The procedure of choosing a port consists of two tasks, first picking a computer and then picking a port on this computer. Therefore the total number of computer ports is $32 \times 24 = 768$ ports.

Eg A new company with just two employees, A and B rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Soln The procedure of assigning offices to these two employees consists of assigning an office to A, which can be done in 12 ways and then assigning an office to B different from the office assigned to A, which can be done in 11 ways. Hence the office to A and B can be assigned in $12 \times 11 = 132$ ways.

The Sum rule

If a task can be done either in one of m ways or in one of n ways, where none of the set of m ways is the same as any of the set of n ways, then there are m+n ways to do the task.

eg: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for their representative if there are 37 members of the mathematics faculty and 83 mathematics majors and ~~more~~ no one is both a faculty member and a student?

sol: A faculty can be chosen in 37 ways and a student can be chosen in 83 ways. Choosing a faculty is never the same as choosing a student, as no one is both a faculty and a student. Therefore there are $37+83=120$ possible ways to pick this representative.

eg: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects respectively. No project is on more than one list. How many possible projects are there to choose from?

sol: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, ~~by~~ there are $23+15+19=57$ ways to choose a project.

At times it is necessary to combine several different counting principles in the solution of one problem. &

eg* In a certain version of the programming language BASIC, a variable name consists of a single letter (A,B,C, . . .) or a single letter followed by a single digit. Find the total number of variable names possible.

Sol: Since the computer does not distinguish between capitals and lowercase letters, a and A are considered the same variable names, as are A7 and a7.
By the rule of product there are $26 \times 10 = 260$ variable names consisting of a letter followed by a digit; and because there are 26 variable names consisting of a single letter, by the rule of sum there are $260 + 26 = 28\cancel{8}6$ variable names in this programming language.

eg* The menu at ~~a~~ restaurant is limited: six kinds of rice, eigh kinds of sandwiches, and five beverages (2 hot beverages - hot coffee & hot tea, 3 cold beverages - iced tea, cola and orange juice). X went to the shop to get a lunch - either a rice and a hot beverage or a sandwich and a cold beverage. In how many ways can X purchae the lunch?

Sol: X can get a rice and hot beverage in 6×2 ways, and can get a sandwich and cold beverage in 8×3 ways. Thus, the total number of ways, X can get the lunch is: $6 \times 2 + 8 \times 3 = 36$ ways.

Permutations

A permutation is a bijection from $\{1, 2, 3, \dots n\}$ (n distinct objects) to itself.

Equivalently, permutation is a linear ordering (ordered arrangement) of n distinct objects.

eg let $A = \{1, 2, 3\}$

The possible permutations are;

123, 132, 231, 213, 312, 321

There are 6 permutations of $\{1, 2, 3\}$.

Theorem:

The number of permutations of $\{1, 2, 3 \dots n\}$ is $n!$.

Proof: For the 1^{st} position we have n choices,

for the 2^{nd} position we have $(n-1)$ choices.

⋮

for the n^{th} position we have 1 choice.

∴ the total number of choices is $n \times (n-1) \times (n-2) \dots \times 2 \times 1 = n!$.

Note $0! = 1$

k-permutation $(0 \le k \le n)$

A k-permutation is a linear ordering (ordered arrangement) on a k-element subset of $\{1, 2, \dots, n\}$

eg $n = 3$, $A = \{1, 2, 3\}$

1-permutation: 1, 2, 3 → total 3

2-permutations: 12, 13, 21, 23, 31, 32 → total 6

Theorem:

The number of k-permutations of $\{1, 2, \ldots, n\}$ is $n(n-1)(n-2)\ldots(n-(k-1)) = \frac{n!}{(n-k)!}$

Proof: There are n- possibilities for the 1^{st} position

$(n-1)$-possibilities for the 2^{nd} position

$\vdots$

$(n-(k-1))$- possibilities for the k^{th} position.

$\therefore$ Total number of k-permutations is

$$n(n-1)\ldots(n-(k-1)) = \frac{n(n-1)\ldots(n-(k-1))(n-k)\ldots 2.1}{(n-k)\ldots 2.1}$$

$$= \frac{n!}{(n-k)!}$$

The k-permutations of n objects is denoted as ^nP_k or $P(n,k)$.

Another interpretation:

let $X = \{1, 2, \ldots k\}$

$Y = \{1, 2 \ldots n\}$ $k \leq n$).

The number of injective maps from X to Y is ^nP_k

Definition: Let A be a set of alphabets ($A=\{a,b,c\ldots\}$)
A word is defined as a finite sequence of elements of A.
$A=\{a,b,c,d\}$
word of length 3: aaa, aab, abc, bcd, bdb, ...

Theorem: The number of words of length k ~~from a~~ that can be formed from a set consisting ~~alphabet~~ of n alphabets is equal to n^k.

Proof: $A=\{a_1, a_2 \ldots a_n\}$

words of length k : | 1 | 2 | | | ... | k |

The 1^{st} position can be filled by n ways
Similarly the other positions can also be filled in n ways.
Thus the total number of words of length k is

$$\underbrace{n\times n\times\cdots\times n}_{k \text{ times}} = n^k$$

* In other words, the number of linear ordering (Ordered arrangement) of k elements of n distinct objects with repetitions is n^k.
It is also called k-permutations with repetition.

Another interpretation

Let $X=\{1,2,\ldots k\}$, $Y=\{a_1,a_2\ldots a_n\}$
The number of maps from X to Y is n^k.
Let $f: X\to Y$, f corresponds to the word $(f(1), f(2)\ldots f(k))$ of length k in n letters where n is $|Y|$.

eg. How many different bit strings of length seven are there?

Sol

Sol^n Here $A = \{0, 1\}$.

The number of words ~~that can be formed~~ of length 7 ~~is~~ that can be formed from 2 objects is $2^7 =$

or $\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7.}$

0 or 1 → there are 2 possibilites for 1st position and similarly for other positions.

Thus the total is $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 =$

eg. How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

Case 1! $\boxed{1}\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }$

fixing the 1st position by $\boxed{1}$, all other positions can be filled in 2^7 ways.

Case 2! $\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{0}\boxed{0}$

fixing the last 2 positions by $\boxed{0}\boxed{0}$, all other positions can be filled in 2^6 ways.

Case 3! $\boxed{1}\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{0}\boxed{0}$

fixing the 1st position by $\boxed{1}$, and last two positions by $\boxed{0}\boxed{0}$, the remaining positions can be filled in 2^5 ways.

But case 3 occurs in both case 1 & case 2.

∴ total bit strings of length seven is

$$2^7 + 2^6 - 2^5 = 160.$$

* The principle used in this problem is called the inclusion-exclusion principle

eg. How many ways are there to select a first prize winner, a second prize winner and a third prize winner from 100 different people who have entered a contest?

sol: The number of ways to pick the three prize winners is the number of ordered selection of 3 elements from a set of 100 elements.

$\therefore$ possible ways is $^{100}P_3 = 100 \times 99 \times 98$

eg: How many permutations of the letters ABCDEFGH contain the string ABC?

sol: Let ABC be one character.

Then the characters used in the permutation are

ABC, D, E, F, G, H $\longrightarrow$ 6 characters.

total permutations is $6!$

eg: Three cards are chosen one after the other from a 52-card deck. Find the number of ways this can be done (a) with replacement, (b) without replacement

sol: with replacement: $52 \times 52 \times 52 = 52^3 =$

without replacement: $52 \times 51 \times 50 = {}^{52}P_3 =$

Permutations with indistinguishable objects.

The number of different permutations of n objects, where, there are n_1 indistinguishable objects of type 1
n_2 indistinguishable objects of type 2,
⋮
n_k indistinguishable objects of type k,
where $n = n_1 + n_2 + \cdots + n_k$ is $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$

eg Find the number of possible arrangements of the word MASSASAUGA. In how many of these arrangements A's are together?

Sol: In MASSASAUGA, there are 1-M, 4 A's, 3S's, 1U, 1G.
The number of possible arrangements of these letters is
$\dfrac{10!}{4!\,3!} =$

The number of possible arrangements in which A's are together is
$\dfrac{7!}{3!} = 840$.

eg How many arrangements are there of the letters in SOCIOLOGICAL? (a) In how many of these arrangements A & G are adjacent? (b) In how many of these arrangements all vowels are adjacent?

Sol: In SOCIOLOGICAL, there are 1S, 3O's, 2C's, 2I's, 2L's, 1G, 1A.
number of arrangement of these letters is $\dfrac{12!}{3! \times 2! \times 2! \times 2!} =$

(a) number of arrangements in which A & G are together is
$\dfrac{11!}{3! \times 2! \times 2! \times 2!} \times 2 =$ [∵ AG is a single character and can be arranged in two ways]

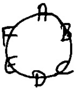
(b) the vowels A, I, O are together
$= \dfrac{7!}{2! \times 2!} \times \dfrac{6!}{2! \times 3!} =$ [∵ vowels are kept together and these vowels can be arranged in $\dfrac{6!}{2! \times 3!}$ ways].

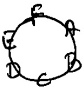
eg Each user on a computer system has a password, which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Sol There are 26 uppercase letters and 10 digits.
The number of passwords of length six that can be formed using $26+10=36$ characters is 36^6.
The number of passwords of length six that can be formed using 26 letters is 26^6.
$\therefore$ Possible passwords of length six with atleast one digit are $36^6 - 26^6 =$
Similarly possible passwords of length 7 and 8 with atleast one digit are $36^7 - 26^7 =$
and $36^8 - 26^8 =$ respectively.
$\therefore$ Total possible passwords of length 6, 7 or 8 is $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8 =$.

Circular permutations

Q If six people, designated as A, B, C, D, E, F are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

Sol^n Consider the circular arrangement 

which is equivalent to 

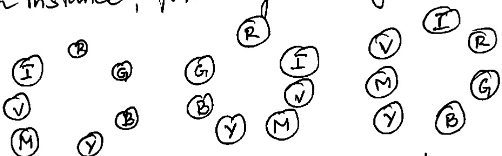
. The linear arrangement corresponding to these circular arrangements are ABCDEF and FABCDE. In addition to these ~~two~~ four other linear arrangements EFABCD, DEFABC, CDEFAB and BCDEFA correspond to the same circular arrangements mentioned above.

So, as each circular arrangement corresponds to six linear arrangements, we have 6 × (number of circular arrangements of A, B, C, D, E, F) = (number of linear arrangements of A, B, C, D, E, F) = 6!.

∴ There are $\frac{6!}{6} = 5! = 120$ arrangements of A, B, C, D, E, F around the circular table.

* In general, if n people are seated about a round table, then $\frac{n!}{n} = (n-1)!$ arrangements are possible.

eg: How many necklaces can be formed of 7 distinct beads?
Note: necklace can be rotated or/and flipped over;
for instance, ~~all beads~~ following arrangements of beads are same.



N_1 N_2 N_3

N_2 is obtained by flipping N_1.
N_3 is obtained by rotating N_1.
Circular arrangements of beads can be done in $(7-1)! = 6!$ ways.
Two circular arrangements are same, if one is obtained from the other by flipping.
$\therefore$ total number of necklaces formed $= \frac{6!}{2} = 720$ ~~ways~~

Combinations

Combinations of elements of a set is an unordered selection of objects.

eg let $S = \{1, 2, 3, 4\}$

Suppose we want to select 2 objects from this set. Then the possible combinations ~~are 12, 23, 34, 41, 13, 24~~

$$12, 13, 14, 23, 24, 34$$

Suppose we want to select 3 objects from this set. Then the possible combinations are:

$$123, 124, 134, 234$$

k-combination

An k-combination of elements of a set is an unordered selection of k elements from the set. Thus, an k-combination is simply a subset of the set with k elements.

The number of k-combinations of a set with n elements is denoted as ^nC_k or $C(n,k)$ or $\binom{n}{k}$, read as n choose k. & $\binom{n}{k}$ is also called the binomial coefficient.

Theorem: The number of k-combinations of a set with n elements, where n is a nonnegative integer and k is an integer with $0 \le k \le n$ equals $\frac{n!}{k!\,(n-k)!}$

Proof: $P(n,k)$ is the ordered arrangement of k-elements from the set of n-elements.

$C(n,k)$ is the unordered arrangement of k-elements from the set of n-elements.

$\therefore P(n,k) = C(n,k) \times k!$ [since ordering of k elements in each k-combinations can be done in k! ways].

$$\Rightarrow C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

* How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

$\underline{\text{sol}^n}$ Five cards from a standard deck of 52 cards can be selected in $^{52}C_5$ ways.

47 cards can be selected in $^{52}C_{47}$ ways.

$\underline{\text{Note}}$: Once k ~~cards~~ objects are selected from n objects, n-k objects will be automatically selected on the other side.

$\therefore \quad {}^nC_k = {}^nC_{n-k}$

* How many bit strings of length 8 contain exactly 4 1s?

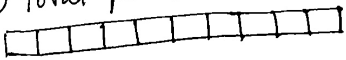
$\underline{\text{sol}^n}$ Number of bit strings of length 8 containing exactly 4 1s = $\underline{\underline{\text{no}}}$ of possible selection of 4 positions from 8 positions $= {}^8C_4 = \frac{8!}{4!\,4!} =$

sol^n This can also be analysed as number of possible arrangements of 8 objects where there are
4 undistinguishable objects of type 1 (4 1s)
4 undistinguishable objects of type 2 (4 0s)
$= \frac{8!}{4!\,4!} =$

$\underline{\text{Note}}$ Number of k-combinations from a set of n objects
= $\underline{\text{no}}$ of permutations of n objects where there are k undistinguishable objects of type 1 and k undistinguishable objects of type 2.

* A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes (a) are there in total?, (b) contain exactly two heads?, (c) contain at most three tails?, (d) contain the same number of heads and tails?

Sol^n (a) total possible outcomes = total no of ways we can fill the 10 positions by head or tail



$= 2^{10}$

(b) total possible outcomes containing exactly two heads $\Big\}$ = no of ways of selecting 2 positions out of 10 positions

$= {}^{10}C_2 = \dfrac{10!}{2!\,8!} = 45.$

(c) total possible outcomes containing atmost 3 tails $\Big\}$ = possible outcomes containing 3 tails + possible outcomes containing 2 tails + possible outcomes containing 1 tail + possible outcomes containing 0 tails

$= {}^{10}C_3 + {}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0 =$

(d) total possible outcomes containing same number of heads and tails $\Big\}$ = ${}^{10}C_5 =$

* In how many of the possible arrangements of the word TALLAHASSEE have no adjacent A's?

Sol^n The word TALLAHASSEE has 1T, 3A's, 2L's, 1H, 2S's, 2E's.

_ T _ L _ L _ H _ S _ S _ E _ E _

A's can take only the blank (_) positions.

Since there are 9 blank positions, and 3 A's, the A's can be ~~filled~~ selected in $^9C_3$ ways =

And the other letters TLLHSSEE can be ~~filled~~ arranged in $\frac{8!}{2!2!2!}$ = ways.

By product rule, required possible arrangements are $^9C_3 \times \frac{8!}{2!2!2!}$ =

* Suppose that a person X draws five cards from a standard deck of 52 cards. In how many ways can his/her selection result in a ~~card~~ with no clubs?

Sol^n There are 13 clubs.

possible ways of selecting 5 cards from 52 cards with no clubs is same as number of ways of selecting 5 cards from (52-13) = 39 cards, which is $^{39}C_5$ = ways.

* In how many ways can 12 different books be distributed among four children so that
(a) each child gets three books
(b) the two oldest children get four books each and the two youngest get two book each?

Sol: There are 12 books and 4 children (say c_1, c_2, c_3, c_4).

(a) 3 books can be given to c_1 in ${}^{12}C_3 =$ ways

After this event, only 9 books remain.

Then 3 books can be given to c_2 in ${}^{9}C_3 =$ ways.

later 3 books can be given in ${}^{6}C_3 =$ and ${}^{3}C_3 =$ ways respectively.

Thus, total number of ways 12 books can be distributed among 4 children such that each get 3 books are ${}^{12}C_3 \times {}^{9}C_3 \times {}^{6}C_3 \times {}^{3}C_3 =$

(b) Let c_1 and c_2 be the two oldest, and c_3 and c_4 be the two youngest children.

We can distribute 4 books each among the oldest and 2 books each among the youngest in

${}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_2 \times {}^{2}C_2 =$ ways.

* Suppose that there are 9 faculty members in the Mathematics department and 11 in the Computer Science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the Mathematics department and four from the Computer Science department?

Sol: The number of ways to select 3 faculty from Mathematics department is ${}^{9}C_{3}$

And for each Mathematics faculty selected, the number of ways to select 4 faculty from Computer Science department is ${}^{11}C_{4}$.

∴ By product rule the possible ways to select 3 from Mathematics department and 4 from Computer Science department is

$${}^{9}C_{3} \times {}^{11}C_{4} = 27,720.$$

The Binomial Theorem

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A binomial expression is simply the sum of two terms, such as $a+b$.

Consider the expansion of $(x+y)^3$, which can be written as $(x+y)^3 = (x+y)(x+y)(x+y)$.

when we expand the above expansion, all products of a term in first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3, x^2y, xy^2 and y^3 arise.

To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in one way

$\therefore$ The coefficient of x^3 term is 1.

To obtain a term of the form x^2y, an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence the number of such terms is the number of 2-combinations of 3 objects, namely $\binom{3}{2}$.

Similarly, the number of terms of the form xy^2 is the number of 1-combinations of 3 objects, namely $\binom{3}{1}$.

The only way to obtain the y^3 term is to choose y in each of the sums, that can be done in one-way.

$$\therefore (x+y)^3 = x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

$$= \sum_{k=0}^{3} {}^{n}C_k\, x^{3-k} y^k$$

The Binomial Theorem

Let x and y be variables, and let n be a non-negative integer. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n} y^n.$$

Proof:

$$(x+y)^n = \underbrace{(x+y)(x+y)\ldots(x+y)}_{n \text{ times}}.$$

The terms in the product when it is expanded are of the form $x^{n-k}y^k$ for $k=0,1,2,\ldots n$. To count the number of terms of the form $x^{n-k}y^k$, It is necessary note that to obtain such a term it is necessary to choose $(n-k)$ xs from the n sums (so that the other k terms in the product are ys). Therefore the coefficient of $x^{n-k}y^k$ is $\binom{n}{n-k}$, which is equal to $\binom{n}{k}$.

$$\therefore (x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \cdots \binom{n}{k} x^{n-k} y^k + \cdots + y^n$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

This triangle is known as Pascal's triangle, named after the French mathematician Blaise Pascal.

Pascal's identity show that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

Properties of binomial coefficient

1. $\binom{n}{0} = \binom{n}{n} = 1$

2. $\binom{n}{1} = \binom{n}{n-1} = n$

3. $\binom{n}{k} = \binom{n}{n-k}$

4. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Proof: $\binom{n}{k}$ = no of k-element subsets of $\{1, 2, \dots n\}$

= {k-element subsets containing 1}

+ {k-element subsets not containing 1}

= $K1 + K2$

Since $k1$ already contains 1, we have to choose $(k-1)$ elements from $(n-1)$ elements, which can be done in $\binom{n-1}{k-1}$ ways.

Since $k2$ doesnot contain 1, we have to choose k elements from $(n-1)$ elements, which can be done in $\binom{n-1}{k}$ ways.

$\therefore \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

From Pascal's triangle and induction it is easy to see that

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

The Pascal's triangle

1. 1

2. $\binom{1}{0}_1 \quad \binom{1}{1}_1$

3. $\binom{2}{0}_1 \quad \binom{2}{1}_2 \quad \binom{2}{2}_1$

4. $\binom{3}{0}_1 \quad \binom{3}{1}_3 \quad \binom{3}{2}_3 \quad \binom{3}{3}_1$

5. $\binom{4}{0}_1 \quad \binom{4}{1}_4 \quad \binom{4}{2}_6 \quad \binom{4}{3}_4 \quad \binom{4}{4}_1$

5. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$ OR $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

Proof: A set with n elements has a total of 2^n different subsets.
Each subset has zero elements, one element, ... or n elements in it.
There are $\binom{n}{0}$ subsets with zero elements,
$\binom{n}{1}$ subsets with one element.
$\vdots$
$\binom{n}{n}$ subsets with n elements.
$\therefore \sum_{k=0}^{n}\binom{n}{k}$ counts the total number of subsets of a set with n elements.
By equating the two formulas, we have for the number of subsets of a set with n elements, we see that $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

~~The Binomial Theorem~~
Also, substituting $a=1$, $b=1$ in the Binomial Theorem, we have $(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k$ or $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

6. $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^n\binom{n}{n}=0$.
substituting $a=1$, $b=-1$ in the Binomial Theorem, we have
$(1-1)^n=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}(-1)+\binom{n}{2}1^{n-2}(-1)^2+\cdots+\binom{n}{n}(-1)^n$
ie, $0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-+\cdots+(-1)^n\binom{n}{n}$ is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$

Multinomial Theorem
For positive integer n, t, the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_t)^n$ is
$\frac{n!}{n_1!\,n_2!\cdots n_t!}$, $0\le n_i\le n$, $i=1,2,\ldots,t$, and $n_1+n_2+\cdots+n_t=n$

* Obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$.

Sol^n: coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is $\binom{n}{k}$.

let $x = 2a$, $y = -3b$

then coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{2}$

and $\binom{7}{2}x^5y^2 = \binom{7}{2}(2a)^5(-3b)^2$

$= \binom{7}{2}2^5 \times (-3)^2 a^5b^2$

$\therefore$ The coefficient of a^5b^2 in the expansion of $(2a-3b)^7$ is $\binom{7}{2}2^5 \times (-3)^2 =$

* Obtain the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

Sol^n let $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, $x_5 = 5$ in the expansion

coefficient of $x_1^2 x_2^3 x_3^2 x_4^5 x_5^4$ of $(x_1+x_2+x_3+x_4+x_5)^{16}$ is $\frac{16!}{2!\,3!\,2!\,5!\,4!}$

and $\frac{16!}{2!\,3!\,2!\,5!\,4!}x_1^2x_2^3x_3^2x_4^5x_5^4$

$= \frac{16!}{2!3!2!5!4!}a^2(2b)^3(-3c)^2(2d)^5 5^4$.

$= \frac{16!}{2!3!2!5!4!} \times 2^3(-3)^2 2^5 5^4 a^2b^3c^2d^5$

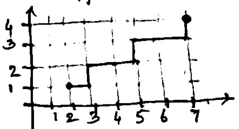
$\therefore$ Coefficient of $a^2b^3c^2d^5$ is $\frac{16!}{2!3!2!5!4!} \times 2^3 \times 3^2 \times 2^5 \times 5^4$

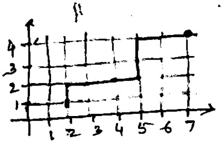
$=$

Cab Driver Problem

* Determine the number of (staircase) paths in the xy plane from (2,1) to (7,4) where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U).

Sol The bold lines in the below figure show two of these paths:





individual steps RURRURRU URRRUURR

The paths consist of 5 Rs for moves to the right and 3 Us for moves upward.

In general, the overall trip from (2,1) to (7,4) requires $(7-2)=5$ horizontal moves to the right and $(4-1)=3$ vertical moves upward. Consequently, each path corresponds with a list of 5 R's and 3 U's and the solution for the number of paths emerges as the number of arrangements of the 5 R's and 3 U's, which is $\frac{8!}{5!\,3!} = 56$.

* Suppose, the overall trip from point A to point B requires m horizontal moves to the right and n vertical moves upward. Then there are

$$\binom{m+n}{m} = \binom{m+n}{n}$$ ways to go from A to B.

Combinations with repetition

Consider a fruit bowl containing apples, oranges and pears. ~~If~~ How many ways are there to select four pieces of fruits. if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters and there are at least four pieces of each type of fruit in the bowl?

Sol^n. Below is the list of all possible ways to select the fruit.

(4 a), (4 o), (4 p)
(3a, 1o), (3a, 1p), (3o, 1a), (3o, 1p), (3p, 1a), (3p, 1o)
(2a, 2o), (2a, 2p), (2o, 2p),
(2a, 1o, 1p), (2o, 1a, 1p), (2p, 1a, 1o)

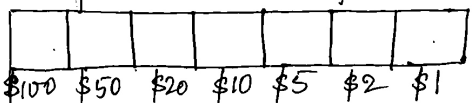
There are 15 ways to select the fruits.
The solution is the number of 4-combinations with repetition allowed from a three-element set.

* How many ways are there to select 5 bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

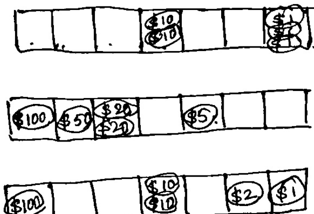
Sol^n This problem involves counting 5-combinations with repetition allowed from a set with seven elements.

→

The below figure illustrates a cash ~~box~~ ~~to~~ with seven compartments, one to hold each type of bill.
The compartments are separated by six dividers.



The below figure illustrates three different ways to select five bills, and the corresponding figure illustrates the six dividers represented by bars and the five bills by stars



$| \; | \; | * * | \; | \; | * * *$

$* | * | * * | \; | * | \; |$

$* | \; | \; | * * | \; | * | *$

The number of ways to select five bills correspond to the number of ways to arrange six bars and five stars in a row with a total of 11 positions.
Consequently, the number of ways to select the five bills is the number of ways to select the positions of the five stars from the 11 positions.
This corresponds to the number of unordered selection of 5 objects from a set of 11 objects, which can be done in ${}^{11}C_5$ ways $= 462$ ways.

<u>Theorem</u>: There are $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ k-combinations from a set with n elements when repetition of elements is allowed.

<u>Note</u>: The above theorem can also be used to find the number of solutions of certain linear equations where the variables are integers subject to constraints.

* Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

sol^n The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. This equals $\binom{4+6-1}{6} = \binom{9}{6} = \frac{9!}{6!\,3!} = 84$.

* How many solutions does the equation $x_1+x_2+x_3=4$ have, where x_1, x_2 and x_3 are nonnegative integers?

sol^n To count the number of solutions, we note that a solution corresponds to a way of selecting 4 items from a set with three elements so that x_1 items of type one, x_2 items of type two and x_3 items of type three are chosen. Hence the number of solutions is equal to the number of 4-combinations with repetition allowed from a set with three elements.

ie, there are $\binom{3+4-1}{4} = \binom{6}{4} = \frac{6!}{4!\,2!} = \cancel{10}\ 15$

Note: The number of integer solutions of the equation $x_1+x_2+\dots+x_n=k$, $x_i \geq 0$, $1 \leq i \leq n$ is $\binom{n+k-1}{k}$

* Determine all integer solutions to the equation $x_1+x_2+x_3+x_4=7$, where $x_i \geq 0$, for all $1 \leq i \leq 4$.

sol^n There are $\binom{4+7-1}{7} = \binom{10}{7} = \frac{10!}{7!\,3!} =$

$x_1=2,\ x_2=4,\ x_3=0,\ x_4=1$

$x_1=0,\ x_2=4,\ x_3=2,\ x_4=1$

no of ways we can put 7 balls in 4 boxes.

* How many integer solutions are there to the inequality $x_1+x_2+x_3+x_4+x_5+x_6 < 10$, $x_i \geq 0$, $1 \leq i \leq 6$

Sol^n. Note that there is a correspondence between the non negative integer solution of

$$x_1+x_2+x_3+x_4+x_5+x_6 < 10 \quad -(1)$$

and the integer solution of

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 10, \quad x_i \geq 0, \; 1 \leq i \leq 6, \; x_7 > 0 \quad -(2)$$

The number of solutions of (2) is same as the number of nonnegative integer solutions of

$$x_1+x_2+x_3+x_4+x_5+x_6+y+1=10, \quad \text{where } y+1 = x_7 \text{ or } y = x_7 - 1 \geq 0$$

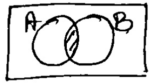
$$\Rightarrow x_1+x_2+x_3+x_4+x_5+x_6+y = 9.$$

$\therefore$ number of non-negative solutions is $\binom{7+9-1}{9} = \binom{15}{9}$

$$= \frac{15!}{9!\,6!} =$$

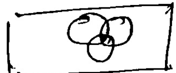
Principle of Inclusion - exclusion

1. Let A and B be two finite sets.
Number of elements in $A\cup B$ i.e $|A\cup B|$ is



$$|A\cup B| = |A| + |B| - |A\cap B|$$

2. Let A, B and C be three finite sets.



Then $|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$.

Theorem:
Let $A_1, A_2, \ldots, A_n$ be finite subsets of U.
Then
$$|A_1\cup A_2\cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| - \sum_{1\le i<j\le n} |A_i\cap A_j| + \sum_{1\le i<j<k\le n} |A_i\cap A_j\cap A_k| - + \cdots + (-1)^{n-1}|A_1\cap A_2\cap \cdots \cap A_n|$$

Proof
Let $x\in A_1\cup A_2\cup \cdots \cup A_n$ [x is counted once on the LHS].
Suppose $x\in A_1\cap A_2\cap\cdots\cap A_p$ and $x\notin A_{p+1}, A_{p+2}, \ldots A_n$
This element is counted $\binom{p}{1}$ times by $\sum|A_i|$
counted $\binom{p}{2}$ times by $\sum|A_i\cap A_j|$
or in general, it is counted $\binom{p}{m}$ times by the summation involving m of the sets A_i
i.e, x contributes to the RHS with multiplicity

$$\binom{p}{1} - \binom{p}{2} + \binom{p}{3} - + \cdots + (-1)^{p-1}\binom{p}{p}$$

$$= 1 - 1 + \binom{p}{1} - \binom{p}{2} + \binom{p}{3} - + \cdots + (-1)^{p-1}\binom{p}{p}$$

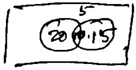
$$= 1 - \left[1 - \binom{p}{1} + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^{p}\binom{p}{p}\right]$$

$$= 1 - 0 = 1$$. This proves the theorem.

* In a class of 50 college freshmen, 30 are studying C++, 25 are studying Java, and 10 are studying both languages. How many freshmen are (i) not studying C++, (ii) studying either computer language (iii) studying C++ but not Java?

Sol^n Let set U = ~~total number~~ set of students in the class, set C = those studying C++, set J = those studying Java

$\therefore |C| = 30, \quad |J| = 25, \quad |C \cap J| = 10, \quad |U| = 50$



$\therefore |C \cup J| = |C| + |J| - |C \cap J| = 30 + 25 - 10 = 45$

(i) Those not studying C++ $= |U - C| = 20$

(ii) those studying either C++ or Java $= |C \cup J| = 45$.

(iii) those studying C++ but not Java $= |C| - |C \cap J| = 20$.

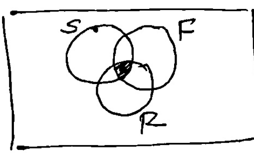
* A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French and Russian, how many students have taken a course in all three languages?

Sol^n $|S| = 1232, \ |F| = 879, \ |R| = 114, \ |S \cap F| = 103, \ |S \cap R| = 23,$

$|F \cap R| = 14, \ |S \cup F \cup R| = 2092, \quad |S \cap F \cap R| = ?$

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$

$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$



$\Rightarrow |S \cap F \cap R| = 7$

* In a survey of 120 people, it was found that 65 read Newsweek magazine, 45 read Times magazine, 42 read Fortune magazine, 20 read both Newsweek and Times, 25 read both Newsweek and Fortune, 15 read both Times and Fortune, 8 read all three magazines. Find (i) the number of people who read at least one of the three magazines, (ii) ~~the~~ number of people who read exactly one magazine.

Soln. $|U| = 120,\ |N| = 65,\ |T| = 45,\ |F| = 42$

$|N \cap T| = 20,\ |N \cap F| = 25,\ |T \cap F| = 15$

$|N \cap T \cap F| = 8$

(i) $|N \cup T \cup F| = |N| + |T| + |F| - |N \cap T| - |N \cap F| - |T \cap F| + |N \cap T \cap F|$

$= 65 + 45 + 42 - 20 - 25 - 15 + 8$

$= 100$

(ii) only $|N|$ + only $|T|$ + only $|F|$

$= (|N| - |N \cap T| - |N \cap F| + |N \cap T \cap F|) + (|T| - |N \cap T| - |T \cap F| + |N \cap T \cap F|)$

$+ (|F| - |N \cap F| - |T \cap F| + |N \cap T \cap F|)$

$= 28 + 18 + 10 = 56$

* A computer company receives 350 applications from computer graduates for a job planning a line of new webservers. Suppose that 220 of these applicants majored in computer science, 147 majored in bussiness and 51 majored both in computer science and in bussiness. How many of these applicants majored neither in computer science nor in bussiness?

Soln $|U| = 350,\ |C| = 220,\ |B| = 147,\ |C \cap B| = 51$

$|C \cup B| = |C| + |B| - |C \cap B| = 220 + 147 - 51 = 316$

those majored neither in CS ~~not~~ nor in B $=$ ~~tot~~ $|U| - |C \cup B| = 350 - 316 = 34.$

Derangements: Nothing is in its Right place

A derangement is a permutation of objects with no fixed points (i.e, never maps an object to itself)

example

Let us consider 4 objects $\{1,2,3,4\}$

The possible permutations are $4! = 24$.

Here we want to know in how many ways we can arrange the numbers 1, 2, 3, 4, so that 1 is not in first place (its natural position), 2 is not in second place, 3 is not in third place, 4 is not in fourth place. These arrangements are called the derangements of 1, 2, 3, 4.

The possible derangements are:
2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.
There are 9 derangements.

Among the 24, $24-9=15$ permutations are not derangements and they are:
1234, 1243, 1324, 1342, 1423, 1432, 3214, 3241, 4213, 4231, 2134, 2431, 4132, 2314, 3124.

The principle of inclusion and exclusion provides the key to calculating the number of derangements. For each $1 \le i \le 4$, an arrangement of 1, 2, 3, 4 is said to satisfy condition c_i, if integer i is in the i^{th} place. We obtain the number of derangements, denoted by d_4 as follows:

$$d_4 = N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$$

$$= 4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0!$$

$$= 4!\left[1 - \binom{4}{1}\frac{3!}{4!} + \binom{4}{2}\frac{2!}{4!} - \binom{4}{3}\frac{1!}{4!} + \binom{4}{4}\frac{0!}{4!}\right]$$

$$= 4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

Theorem:

The number of derangements of a set with n elements is $d_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$

Proof:

We compute the number of permutations with fixed points.

Let $A_i = \{$permutations mapping i to $i\}$, where $i = 1, 2, \ldots, n$.

$A_i \cap A_j = \{$permutations mapping i to i and j to $j\}$.

$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = \{$permutations mapping $i_1 \to i_1, i_2 \to i_2, \ldots, i_k \to i_k\}$

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$

$\vdots$

$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$

$\therefore$ The number of derangements of a set with n elements is $d_n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$

$$= n! - \left\{ \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cdots \cap A_n| \right\}$$

$$= n! - \left[\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n}{n}0!\right]$$

$$= n! - \left[n! - \frac{n!}{2!} + \frac{n!}{3!} - + \cdots + (-1)^{n-1}\frac{n!}{n!}\right]$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - + \cdots + (-1)^n \frac{n!}{n!}$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}\right]$$

$$= n! \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!} \approx \left\{ n!\, e^{-1} = \frac{n!}{e} \right\} \left[\because e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots\right]$$

* While at the race track, Mr X bets on each of the ten horses in a race to come in according to how they are favoured. In how many ways can they reach the finish line so that he loses all of his bets?

Sol^n Here, we have to find the number of ways of arranging the horses 1, 2, 3, .., 9, 10 so that 1 is not in its favoured place, 2 is not in its favoured place. ... and 10 is not in its favoured place.
Thus, the required number of ways is the number of derangements of 10 objects,

ie. $d_{10} = 10!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\right)$

$= 10! \times e^{-1} = \frac{10!}{e} = 1334960916.$

* How many derangements of {1, 2, 3, 4, 5, 6} begin with the integers 1, 2, 3 in some order.

Sol^n Integers 1, 2 and 3 will get one of the positions in the first 3 places and that can be done in

$$\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) \times 3!$$

Integers 2, 4 and 6 will get one of the positions in the last 3 places and that can be done in

$$\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) \times 3!$$

$\therefore$ total number of derangements is $\left(\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) \times 3!\right)^2$

$= 4.$

which are 231564, 231645, 312564, 312645

* How many derangements of $\{1,2,3,4,5,6\}$ end with the integers 1, 2, 3 in some order?

Sol^n Integers 1, 2 and 3 occupiye one of the last 3 places and integers 2, 4 and 6 occupy one of the first 3 places in $3! \times 3! = 36$ ways.

* There are 10 couples, each with a single child. The 10 children get lost in a dark cave. Each couple goes into the cave and pulls out one child. How many ways are there in which at least one child gets paired with their parents? What is the probability of that occuring?

Sol^n. The number of ways that no child gets paired is $d_{10} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}$

$$= \frac{10!}{e}$$

Thus, the number of ways that at least one child gets paired is $10! - \frac{10!}{e} = 10!\left(1-\frac{1}{e}\right) = 2293839$

The probability of that occuring is $\frac{10!\left(1-\frac{1}{e}\right)}{10!}$

$$= 0.6321$$

* There are eight letters to eight different people to be placed in eight different addressed envelopes. Find the number of ways of doing this so that at least one letter gets to the right person.

Sol^n The number of ways of placing 8 letters in 8 envelopes is $8!$

The number of ways of placing 8 letters in 8 envelopes such that no letter is in the right envelope is d_8.

$\therefore$ the number of ways of placing 8 letters in 8 envelopes such that at least one letter is in the right envelope is $8! - d_8$

$$= 25486$$

* Thirty students take a quiz. Then for the purpose of grading, the teacher asks the students to exchange papers so that no one is grading his own paper. In how many ways can this be done?

Sol. This is the case of 30 derangements.

$$\text{ie, } d_{30} = \frac{30!}{e}$$

* Each of the n students is given a book. The books are to be returned and redistributed to the same student. In how many ways can the two distributions be made so that no student will get the same book in both the distributions.

Sol^n A student can be given a book in $n!$ ways. Upon return and redistribution, the student would not get the same book can be done in d_n ways.

$\therefore$ totally the first distribution and redistribution $\Big\} = n! \times d_n = n! \times n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$

$$= (n!)^2 \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

Recursive Definitions, Recurrence Relations.

Sequences is a real / complex valued function on the positive integers. Sequences are generally defined by specifying their general terms. Alternatively, a sequence may be defined by indicating a relation connecting its general term with one or more of the preceding terms. In other words, a sequence $\{a_n\}$ may be defined by indicating a relation connecting its general term a_n with $a_{n-1}, a_{n-2}, \ldots$ etc. Such a relation is called a recurrence relation for the sequence.

The process of determining a_n from a recurrence relation is called solving of the relation.

A value a_n that satisfies a recurrence relation is called its general solution. ~~If~~

If the values of some particular terms of the sequence are specified, then by making use of these values in the general solution we obtain particular solution that uniquely determines the sequence.

<u>example</u>

Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$.

The list of the terms of the sequence are $a_1, a_2 \ldots$ which are $1, \frac{1}{2}, \frac{1}{3}, \ldots$

Fibonacci's problem about rabbits.

notation B B - baby rabbits.
A A - adult rabbits.

Assume that after one month baby rabbits will be fully grown up and every other month adult rabbits give birth to baby rabbits.

Month	Rabbits	No of rabbit pairs
1	B B	1
2	A A	1
3	A A B B	2
4	A A B B A A	3
5	A A B B A A A A B B	5

How many pairs of rabbits will you have after a year?

Sol^n Let F_n be the number of pairs of rabbits during n^{th} month.

Then $\underbrace{F_n = F_{n-1} + F_{n-2}}_{\text{recurrence relation for } F_n}$, $\underbrace{F_1 = 1, \; F_2 = 1}_{\text{initial conditions}}$

List of terms of the sequence F_n is

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $144 = F_{12}$

This sequence is called Fibonacci sequence

Q Find the number of bit strings of length n that do not have two consecutive 0's.

Sol^n. Let a_n be the no of bit strings of length n that don't have consecutive 0's = no of such bit strings ~~of length with~~ ending with 0.

+ no of such bit strings ending with 1.

= the bit strings of length $(n-1)$ with no consecutive 0's

+ the bit strings of length $(n-2)$ with no consecutive 0's

$\therefore a_n = a_{n-1} + a_{n-2}$

$n=4$

0101	1010	1110
0110	1011	1111
0111	1101	

First-order Recurrence Relations

Consider a recurrence relation of the form

$$a_n = c a_{n-1} + f(n), \quad \text{for } n \geq 1, \quad \text{---} (1)$$

where c is a known constant and $f(n)$ is a known function.

Such a relation is called a linear recurrence relation of first-order with constant coefficient. If $f(n) \equiv 0$, the relation is called homogeneous, otherwise, it is called non-homogeneous.

The relation (1) can be solved in a trivial way.

The relation (1) can be rewritten as

$$a_{n+1} = c a_n + f(n+1), \quad \text{for } n \geq 0 \quad \text{---} (2)$$

For $n = 0, 1, 2, \ldots$ the relation (2) yields, respectively

$$a_1 = c a_0 + f(1)$$

$$a_2 = c a_1 + f(2)$$

$$= c^2 a_0 + c f(1) + f(2)$$

$$a_3 = c a_2 + f(3)$$

$$a_3 = c^3 a_0 + c^2 f(1) + c f(2) + f(3)$$

By induction

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c f(n-1) + f(n)$$

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k), \quad \text{for } n \geq 1 \quad \text{---} (3)$$

This is the general solution of the recurrence relation (2) which is equivalent to the relation (1).

If $f(n) = 0$, then (3) becomes $a_n = c^n a_0$, for $n \geq 1$ --- (4)

The equations (3) and (4) yield particular solutions if a_0 is specified. The specified value of a_0 is the initial condition

* Solve the recurrence relation $a_{n+1} = 4a_n$, $n \geq 0$, given $a_0 = 3$.

<u>sol^n</u> $a_{n+1} = 4a_n$, $n \geq 0$ is homogeneous.

The general solution is $a_n = 4^n a_0$, for $n \geq 1$.

substituting $a_0 = 3$, then $\underline{a_n = 3.4^n}$, for $n \geq 1$ is the particular solution of the given relation.

* Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$, given that $a_2 = 98$.

<u>sol^n</u> $a_n = 7a_{n-1}$ is homogeneous.

$\therefore$ G.S. is $a_n = 7^n a_0$ for $n \geq 1$

$\Rightarrow$ $a_2 = 7^2 a_0 \Rightarrow 98 = 49 a_0 \Rightarrow a_0 = 2$.

$\therefore$ P.S is $\underline{a_n = 2 \cdot 7^n}$, for $n \geq 1$

* Solve the recurrence relation $a_n = n a_{n-1}$ for $n \geq 1$, given that $a_0 = 1$.

<u>sol^n</u> $a_n = n a_{n-1}$, for $n \geq 1$.

$\Rightarrow$ $a_1 = 1 \cdot a_0$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 \cdot a_0 = 2!\, a_0$

$a_3 = 3 a_2 = 3 \cdot 2 \cdot 1 \cdot a_0 = 3!\, a_0$.

$\vdots$

$\Rightarrow$ $a_n = (n!)\, a_0$, for $n \geq 1$.

using $a_0 = 1 \Rightarrow a_n = 1 \cdot n!$ ~~or~~

or $a_n = n!$ is the required solution

* If a_n is a solution of the recurrence relation $a_{n+1} = k a_n$ for $n \geq 0$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$, what is k?

Sol^n: Given $a_{n+1} = k a_n$, for $n \geq 0$

$\Rightarrow$ G.S is $a_n = k^n a_0$, for $n \geq 1$.

$a_3 = \frac{153}{49} \Rightarrow \frac{153}{49} = k^3 a_0$

$a_5 = \frac{1377}{2401} \Rightarrow \frac{1377}{2401} = k^5 a_0$

$\Rightarrow \frac{153/49}{1377/2401} = \frac{k^3 a_0}{k^5 a_0}$

$\Rightarrow k^2 = \frac{9}{49} \Rightarrow k = \pm \frac{3}{7}$.

* Find a_{12} if $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ for $n \geq 0$, given that $a_0 = 2$.

Sol^n: let $b_n = a_n^2$, then the recurrence relation is $b_{n+1} = 5b_n$,

The G.S is $b_n = 5^n b_0$

$a_0 = 2 \Rightarrow b_0 = a_0^2 = 4 \Rightarrow b_n = 4 \cdot 5^n$

$\Rightarrow a_n^2 = 4 \cdot 5^n$,

$\Rightarrow a_n = 2 \cdot (\sqrt{5})^n$ ($\because a_n > 0$ for $n \geq 0$)

$\therefore a_{12} = 2(\sqrt{5})^{12} = 2 \cdot 5^6 = 31250$.

* Find a recurrence relation and the initial condition for the sequence 2, 10, 50, 250, ... and hence find the general term of the sequence.

Sol^n: $a_0 = 2$.

$a_1 = 10 = 5 \cdot 2 = 5a_0$.

$a_2 = 50 = 5 \cdot 10 = 5 \cdot a_1$

$a_3 = 250 = 5 \cdot 50 = 5a_2$

$\therefore$ the recurrence relation is $a_n = 5a_{n-1}$ for $n \geq 1$ with $a_0 = 2$.

$\Rightarrow$ the general term is $a_n = 5^n a_0 = 2 \cdot 5^n$

* Solve the recurrence relation $a_n - 3a_{n-1} = 5\times 7^n, n\geq 1$, given that $a_0 = 2$.

Solⁿ. Given $a_n = 3a_{n-1} + 5\times 7^n, n\geq 1$

which can be written as $a_{n+1} = 3a_n + 5\times 7^{n+1}$, for $n\geq 0$

$= 3a_n + f(n+1)$, where $f(n) = 5\times 7^n$.

The general solution is:

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k)$$

substituting $a_0 = 2$

$$\Rightarrow a_n = 2.3^n + \sum_{k=1}^{n} 3^{n-k}.5.7^k$$

$$= 2.3^n + \sum_{k=1}^{n} 3^n.5.\left(\frac{7}{3}\right)^k$$

[For a G.P $S_n = \frac{a(r^n-1)}{(r-1)}$]

$a = \frac{7}{3}, r = \frac{7}{3}$

$$= 2.3^n + 3^n.5.\frac{\frac{7}{3}\left[\left(\frac{7}{3}\right)^n - 1\right]}{\left[\frac{7}{3} - 1\right]}$$

$$= 2.3^n + 5.7.3^{n-1}.\frac{7^n - 3^n}{3^n}.\frac{3}{4}$$

$$= 2.3^n + \frac{1}{4}(5.7.(7^n - 3^n))$$

$$= \frac{5}{4}7^{n+1} - \frac{1}{4}3^{n+3}$$

* Solve the recurrence relation $a_n - 3a_{n-1} = 5\times 3^n, n\geq 1$ given $a_0 = 2$.

Solⁿ $a_{n+1} = 3a_n + 5.3^{n+1}, n\geq 0.$

$= 3a_n + f(n+1)$, where $f(n) = 5.3^n$.

G.S is. $a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k}(5.3^k)$

substituting $a_0 = 2$: $a_n = 2.3^n + 3^n.5\sum_{k=1}^{n} 1$

$$= 2.3^n + 5.3^n.n$$

$$= (2+5n)3^n$$

* Solve the recurrence relation
$a_n = 2a_{n/2} + (n-1)$ for $n = 2^k$, $k \geq 1$, given $a_1 = 0$

sol^n we have $a_n - 2a_{n/2} = n-1 \Rightarrow a_n - 2a_{n/2} = (n-1)$

$a_{n/2} - 2a_{n/4} = \left(\frac{n}{2} - 1\right) \Rightarrow 2a_{n/2} - 2^2 a_{n/4} = (n-2)$

$a_{n/4} - 2a_{n/8} = \left(\frac{n}{4} - 1\right) \Rightarrow 2^2 a_{n/4} - 2^3 a_{n/8} = (n - 2^2)$

$\vdots$

$a_{n/(2^{k-2})} - 2a_{n/(2^{k-1})} = \left(\frac{n}{2^{k-2}} - 1\right) \Rightarrow 2^{k-2} a_{n/(2^{k-2})} - 2^{k-1} a_{n/(2^{k-1})} = (n - 2^{k-2})$

$a_{n/(2^{k-1})} - 2a_{n/(2^k)} = \left(\frac{n}{2^{k-1}} - 1\right) \Rightarrow 2^{k-1} a_{n/(2^{k-1})} - 2^k a_{n/(2^k)} = (n - 2^{k-1})$

adding all these equation, we have

$a_n - 2^k a_{n/(2^k)} = (n-1) + (n-2) + (n-2^2) + \dots + (n - 2^{k-1})$
$= kn - (1 + 2 + 2^2 + \dots + 2^{k-1})$

$\because n = 2^k$, $a_{n/2^k} = a_1 = 0$, and the above equation implies

$a_n - 0 = kn - (1 + 2 + 2^2 + \dots + 2^{k-1})$ G.P $a = 1, r = 2$

$= kn - \frac{1.(2^k - 1)}{(2-1)}$

$= kn - (n-1)$

$= 1 + (k-1)n$ $2^k = n \Rightarrow \log_2 2^k = \log_2 n \Rightarrow k = \log_2 n$

$a_n = 1 + (\log_2 n - 1)n$

* Find the recurrence relation and the initial condition for the sequence 0, 2, 6, 12, 20, 30, 42, ..., and hence find the general term of the sequence.

sol^n $a_0 = 0$,

$a_1 = 2 \Rightarrow a_1 - a_0 = 2 = 2.1$

$a_2 = 6 \Rightarrow a_2 - a_1 = 4 = 2.2$

$a_3 = 12 \Rightarrow a_3 - a_2 = 6 = 2.3$

$a_4 = 20 \Rightarrow a_4 - a_3 = 8 = 2.4$

$a_5 = 30 \Rightarrow a_5 - a_4 = 10 = 2.5$

$\therefore a_n - a_{n-1} = 2n$

or $a_n = a_{n-1} + 2n$, for $n \geq 1$

$\therefore a_n - a_{n-1} = 2n$

$a_{n-1} - a_{n-2} = 2(n-1)$

$\vdots$

$a_3 - a_2 = 2 \times 3$

$a_2 - a_1 = 2 \times 2$

$a_1 - a_0 = 2 \times 1$

adding all the equations

$\Rightarrow a_n - a_0 = 2[n + (n-1) + (n-2) + \dots + 3 + 2 + 1]$

$a_n - a_0 = 2 \times \frac{n(n+1)}{2}$

$\Rightarrow a_n = n(n+1)$

$a_n = n^2 + n$

* The number of virus affected files in a system is 1000 (to start with) and this increases 250% every two hours. Use a recurrence relation to determine the number of virus affected files in the system after one day.

Sol: The number of virus affected files initially is 1000.

$\therefore a_0 = 1000$.

Let a_n denote the number of virus affected files after $2n$ hours. and the number increases by 250% every 2 hours.

Then a_{n+1} denotes the number of virus affected files after $2(n+1)$ / $(2n+2)$ hours

$\therefore a_{n+1} = a_n + a_n \times \frac{250}{100}$

$a_{n+1} = 3.5\, a_n$, such that $a_0 = 1000$.

$\therefore$ the G.S is $a_n = (3.5)^n a_0 = 1000\,(3.5)^n$.

$\therefore$ The number of virus affected files after one day (or 24 hours) or $2n$ hours (where $n=12$)

is a_n or $a_{12} = 1000\,(3.5)^{12} = 337922 0508$.

* A person invests Rs10,000 at 10.5% interest (per year) compounded monthly. Find and solve the recurrence relation for the value of the investment at the end of n months. What is the value of the investment at the end of the first year? How long will it take to double the investment?

Sol: $\because$ annual interest is 10.5%, monthly interest is $\frac{10.5\%}{12} = 0.875\%$
$= 0.00875$

initial investment $a_0 = 10{,}000$,

value of investment after n months is a_n.

$\therefore a_1 = a_0 + (0.00875)a_0 = 1.00875\, a_0$

$a_2 = a_1 + (0.00875)a_1 = (1.00875)\, a_1$

$\therefore a_n = (1.00875)a_{n-1}$

$\therefore$ solution is $a_n = (1.00875)^n a_0$

$a_n = 10{,}000\,(1.00875)^n$

investment after 1 year is $a_{12} = 10{,}000\,(1.00875)^{12}$

$\simeq 11{,}102$

To double ie, $a_n = 2a_0$

$\Rightarrow (1.00875)^n \cancel{a_0} = 2\cancel{a_0}$

$\Rightarrow n = \frac{\log 2}{\log(1.00875)} \simeq 79.6$

about 80 months.

* A bank pays a certain percentage of annual interest on deposits, compounding the interest once in 3 months. If a deposit doubles in 6 years and 6 months, what is the annual percentage of interest paid by the bank?

Sol: Let the annual rate of interest be $x\%$.

Then the interest every 3 months (quarterly) $= \left(\frac{x}{4}\right)\% = \frac{x}{400}$

Let a_0 denote the deposit made,

a_n denote the value of the deposit at the end of n^{th} quarter.

Then $a_{n+1} = a_n + \left(\frac{x}{400}\right) a_n = \left(1+\frac{x}{400}\right) a_n$ for $n \geq 0$

Then the G.S is $a_n = \left(1+\frac{x}{400}\right)^n a_0$, for $n \geq 1$

Given $a_n = 2a_0$ in $n = 26$ quarters [i.e, $6\times4+2=26$]

i.e, $\left(1+\frac{x}{400}\right)^{26} a_0 = 2a_0 \Rightarrow 26\log\left(1+\frac{x}{400}\right) = \log 2$

$\Rightarrow \log\left(1+\frac{x}{100}\right) = 0.0267$

$\Rightarrow 1+\frac{x}{400} \doteq e^{0.0267}$.

$\Rightarrow 1+\frac{x}{400} = 1.027$

$\Rightarrow x = 10.8$

$\therefore$ the annual rate of interest paid by the bank is 10.8% (compounded once in every 3 months).

Suppose that there are $n \geq 2$ persons at a party and that each of these person shakes hands (exactly once) with all of the other persons present. Using a recurrence relation, find the number of hand shakes.

Sol^n. Let a_{n-2} denote the number of hand shakes among the $n \geq 2$ persons present. If $n=2$, then $a_0 = 1$.
If a new person joins the party, he will shake hands with each of the n persons already present.
$\therefore$ The number of hank shakes increases by n when the number of persons changes to $n+1$ from n.

$$\therefore \quad a_{(n+1)-2} = a_{n-2} + n \quad \text{for } n \geq 2.$$

$$\text{or } \quad a_{n-1} = a_{n-2} + n$$

Let $m = n-1$ then $a_m = a_{m-1} + (m+1)$, for $m \geq 1$

~~or $a_{m+1} = a_m + (m+2)$ for $m \geq 0$~~

where $c = 1, \; f(m) = m+1$

$\therefore$ G.S is
$$a_m = 1^n . a_0 + \sum_{k=1}^{m} 1^{m-k} f(k)$$
$$= a_0 + \sum_{k=1}^{m} (k+1).$$
$$= 1 + [2 + 3 + \cdots + m+1]$$
$$= 1 + 2 + 3 + \cdots + (m+1)$$
$$a_m = \frac{(m+1)(m+2)}{2} \quad \text{for } m \geq 0$$

$$\text{or } \quad a_{n-1} = \frac{(n-1+1)(n-1+2)}{2}$$
$$a_{n-1} = \frac{n(n+1)}{2}$$
$$a_{n-2} = \frac{(n-1)(n-1+1)}{2}$$
$$a_{n-2} = \frac{(n-1)n}{2}, \quad \text{for } n \geq 2.$$

This is the number of handshakes in the party when $n \geq 2$ persons are present.

Second-order Homogeneous Recurrence Relation

Consider the recurrence relation of the form

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0 \quad \text{for } n \geq 2, \quad (1)$$

where c_n, c_{n-1}, c_{n-2} are real constants with $c_n \neq 0$.
A relation of this type is called a second-order linear homogeneous recurrence relation with constant coefficients.

We seek a solution of (1) in the form $a_n = ck^n$, where $c \neq 0$, $k \neq 0$. Substituting $a_n = ck^n$ is (1) implies

$$c_n c k^n + c_{n-1} c k^{n-1} + c_{n-2} c k^{n-2} = 0$$

$$\Rightarrow c_n k^2 + c_{n-1} k + c_{n-2} = 0. \quad -(2)$$

Thus $a_n = ck^n$ is a solution of (1) if k satisfies the quadratic equation (2). This equation (2) is called the auxillary equation or characteristic equation for the relation (1).

There are 3 cases;

(1) The roots of (2) are real and distinct (say k_1, k_2)
Then $a_n = Ak_1^n + Bk_2^n$ where A, B are real constants, is the general solution of relation (1).

(2) The roots of (2) are real and equal (say k, k).
Then $a_n = (A + Bn)k^n$, where A, B are real constants is the general solution of relation (1).

(3) The roots of (2) are complex (say $k_1 = p + iq$, $k_2 = p - iq$).
Then $a_n = r^n(A\cos n\theta + B\sin n\theta)$, where A, B are complex constants, $r = |k_1| = |k_2| = \sqrt{p^2 + q^2}$ and $\theta = \tan^{-1}(q/p)$ is the general solution of relation (1).

The constants A and B can be evaluated if a_n is specified for two particular values of n.

* Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, with $a_0 = 2$, $a_1 = 7$.

Sol^n. Given $a_n - a_{n-1} - 2a_{n-2} = 0$

The characteristic equation is

$$k^2 - k - 2 = 0 \Rightarrow (k-2)(k+1) = 0$$
$$\Rightarrow k = 2, -1$$

$\therefore$ G.S is $a_n = A(2)^n + B(-1)^n$

$a_n = A\,2^n \cancel{+}B$

$a_0 = 2 \Rightarrow \quad 2 = A + B$

$a_1 = 7 \Rightarrow \quad 7 = 2A \cancel{+} B$

$\Rightarrow A = 3, \; B = -1$

$\therefore$ The solution is $\underline{a_n = 3.2^n \cancel{+} (-1)^n}$

* Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$, given $a_1 = 5$ and $a_2 = 3$

Sol^n Given $a_n - 3a_{n-1} + 2a_{n-2} = 0$.

The characteristic equation is:

$$k^2 - 3k + 2 = 0 \Rightarrow (k-2)(k-1) = 0 \Rightarrow k = 2, 1$$

$\therefore$ G.S is $a_n = A(2)^n + B(1)^n$

$a_n = A.2^n + B$

$a_1 = 5 \Rightarrow \quad 5 = 2A + B$

$a_2 = 3 \Rightarrow \quad 3 = 4A + B$

$\Rightarrow A = -1, \; B = 7$.

$\therefore$ The solution is $\underline{a_n = -(2)^n + 7}$

* Solve the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 0$, for $n \geq 2$, given that $a_0 = 5$, $a_1 = 12$

Sol^n Given $a_n - 6a_{n-1} + 9a_{n-2} = 0$

The characteristic equation is:

$k^2 - 6k + 9 = 0 \Rightarrow (k-3)^2 = 0 \Rightarrow k = 3, 3$

$\therefore$ G.S is $a_n = (A + Bn)\, 3^n$

$a_0 = 5 \Rightarrow \quad 5 = A$

$a_1 = 12 \Rightarrow \quad 12 = (A + B) \times 3 \Rightarrow B = -1$

$\therefore$ the solution is $a_n = (5 - n)\, 3^n$.

* Solve the recurrence relation (Fibonacci relation):

$F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, $F_0 = 0$, $F_1 = 1$.

$F_n = F_{n-1} + F_{n-2}$, or $F_n - F_{n-1} - F_{n-2} = 0$ for $n \geq 2$

The characteristic equation is:

$k^2 - k - 1 = 0 \Rightarrow k = \frac{1 \pm \sqrt{5}}{2}$, ie, $k_1 = \frac{1+\sqrt{5}}{2}$, $k_2 = \frac{1-\sqrt{5}}{2}$

$\therefore$ The G.S is $F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$F_0 = 0 \Rightarrow \quad 0 = A + B \Rightarrow B = -A$

$F_1 = 1 \Rightarrow \quad 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$\Rightarrow 1 = A\left[\frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}\right] \Rightarrow A = \frac{1}{\sqrt{5}}, \; B = -\frac{1}{\sqrt{5}}$

$\therefore$ the solution is $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

* Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$, for $n \geq 2$, given that $a_0 = 1$, $a_1 = 2$

Sol^n Given $a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is $k^2 - 2k + 2 = 0$

$\Rightarrow k = \frac{2 \pm 2i}{2} = 1 \pm i \qquad r = \sqrt{1^2 + 1^2} = \sqrt{2}, \; \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\therefore$ G.S is $a_n = (\sqrt{2})^n \left[A \cos\frac{n\pi}{4} + B \sin\frac{n\pi}{4}\right]$

$a_0 = 1 \Rightarrow \quad 1 = A \times 1 \Rightarrow A = 1$

$a_1 = 2 \Rightarrow \quad 2 = \sqrt{2}\left[A \times \frac{1}{\sqrt{2}} + B \times \frac{1}{\sqrt{2}}\right] \Rightarrow 2 = A + B$

$\Rightarrow B = 1$

$\therefore a_n = (\sqrt{2})^n \left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]$

Second order non-homogeneous recurrence relation

Consider the recurrence relation of the form

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = f(n), \text{ for } n \geq 2 \quad \text{--- (1)}$$

where c_n, c_{n-1}, c_{n-2} are real constants with $c_n \neq 0$.

A relation of this type is called a second-order linear non homogeneous recurrence relation with constant coefficients.

We seek a solution of (1) in the form

$$a_n = a_n^{(h)} + a_n^{(p)},$$

where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation, and $a_n^{(p)}$ is a particular solution of the non-homogeneous recurrence relation.

Let $f(n) = (b_0 + b_1 n + \cdots + b_k n^k) r^n$.

There arises two cases.

(i) when r is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, then the particular solution is of the form $(p_0 + p_1 n + \cdots + p_k n^k) r^n$

(ii) when r is a root of the characteristic equation with multiplicity m, then the particular solution is of the form $n^m (p_0 + p_1 n + \cdots + p_k n^k) r^n$.

* Find all solutions of the recurrence relation

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$

Solⁿ: Let a_n be the solution of the given recurrence relation. Then $a_n = a_n^{(h)} + a_n^{(q)}$

To find $a_n^{(h)}$

Consider the associated homogeneous recurrence relation

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

The characteristic equation is: $k^2 - 5k + 6 = 0 \Rightarrow k = 2, 3$

$\therefore \quad a_n^{(h)} = A2^n + B3^n.$

To find $a_n^{(q)}$

Here $f(n) = 7^n$, $\therefore$ $a_n^{(q)}$ is of the form $a_n^{(q)} = p_0 7^n$

Then $a_n^{(q)} = 5a_{n-1}^{(q)} - 6a_{n-2}^{(q)} + 7^n.$

$\Rightarrow \quad p_0 7^n = 5p_0 7^{n-1} - 6p_0 7^{n-2} + 7^n.$

$\Rightarrow \quad p_0 \cdot 7^2 = 5p_0 \times 7 - 6p_0 + 7^2$

$\Rightarrow \quad (49 - 35 + 6)p_0 = 49$

$\Rightarrow \quad p_0 = \frac{49}{20}.$

$\therefore$ The required solution is $\underline{a_n = A2^n + B3^n + \frac{49}{20}7^n}$

* Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$, with $a_0=0$ and $a_1=1$

Sol^n: Let a_n be the solution of the given recurrence relation

Then $a_n = a_n^{(p)} + a_n^{(q)}$

To find $a_n^{(p)}$

Consider the homogeneous relation $a_n - 4a_{n-1} + 4a_{n-2} = 0$

The characteristic equation is $k^2 - 4k + 4 = 0 \Rightarrow k = 2, 2$

$\therefore \boxed{a_n^{(p)} = (A+Bn)2^n}$

To find $a_n^{(q)}$

Here $f(n) = (1+n)2^n$ $\therefore$ $a_n^{(q)}$ is of the form $a_n^{(q)} = n^2(p_0 + p_1 n)2^n$

Then r.h.s $n^2(p_0+p_1 n)2^n = 4(n-1)^2(p_0+p_1(n-1))2^{n-1} - 4(n-2)^2(p_0+p_1(n-2))2^{n-2} + (n+1)2^n$

$\Rightarrow n^2(p_0+p_1 n)2^n = 4(n^2-2n+1)(p_0+p_1 n - p_1)2^{n-1} - 4(n^2-4n+4)(p_0+p_1 n - 2p_1)2^{n-2} + n2^n + 2^n$

$\div 2^{n-2}$

$\Rightarrow \cancel{4n^2p_0} + \cancel{4n^3p_1} = \cancel{8n^2p_0} + \cancel{8n^3p_1} - \cancel{8n^2p_1} \cancel{\mp 16np_0} - \cancel{16n^2p_1} + \cancel{16np_1}$

$+ 8p_0 + 8np_1 - 8p_1 - \cancel{4n^2p_0} - \cancel{4n^3p_1} + \cancel{8n^2p_1}$

$+ \cancel{16np_0} + \cancel{16n^2p_1} - 32np_1 + 4n + 4$

$\Rightarrow$ $- 16p_0 - \cancel{16np_1} + 32p_1$

$0 = -24np_1 + 4n - 8p_0 + 24p_1 + 4$

$\Rightarrow -24p_1 + 4 = 0$ or $-8p_0 + 24p_1 + 4 = 0$

$\Rightarrow p_1 = 1/6$, $p_0 = 1$

$\boxed{\therefore a_n^{(q)} = n^2\left(1+\frac{n}{6}\right)2^n}$

Then the G.S is $a_n = (A+Bn)2^n + n^2\left(1+\frac{n}{6}\right)2^n$.

with $a_0 = 0 \Rightarrow \boxed{0 = A}$, $a_1 = 1 \Rightarrow 1 = 2B + \frac{7}{3} \Rightarrow \boxed{B = -2/3}$

$\therefore \boxed{a_n = -\frac{2}{3}n\cdot 2^n + n^2\left(1+\frac{n}{6}\right)2^n}$

Generating Functions

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series.

Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations.

Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function.

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series $G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$

example

The generation function for the sequence $\{a_k\}$ with (i) $a_k = 3$ is $\sum_{k=0}^{\infty} 3x^k$,

(ii) $a_k = k+1$ is $\sum_{k=0}^{\infty} (k+1)x^k$

(iii) $a_k = 2^k$ is $\sum_{k=0}^{\infty} 2^k x^k$

* What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?

sol^n G.F is $G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \dfrac{x^6 - 1}{(x-1)}$

$\left\{ \text{G.S: } a = 1, r = x, \; S_6 = \dfrac{1(x^6 - 1)}{x-1} = \dfrac{(x^6-1)}{(x-1)} \right\}$

* What is the generating function for $a_k = C(m,k)$? $k = 0, 1, 2 \ldots m$.

sol^n $a_k = C(m,k) = {}^mC_k$.

G.F is $G(x) = {}^mC_0 \cdot x^0 + {}^mC_1 x^1 + {}^mC_2 x^2 + \cdots + {}^mC_m x^m$

$= 1 + {}^mC_1 x + {}^mC_2 x^2 + \cdots + x^m$

$G(x) = (1+x)^m$

* What is the generating function for the sequence $1, 1, 1, 1, \ldots$?

sol^n G.F is $G(x) = 1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$

$\left\{ \text{G.S, } a = 1, r = x, \; S_\infty = \dfrac{1}{1-x} \right\}$

* What is the generating function for the sequence $1, a, a^2, a^3, \ldots$?

sol^n G.F. is $G(x) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$

$= \dfrac{1}{1-ax}$

* Find the generating function for the finite sequence $1, 4, 16, 64, 256$.

$G(x) = 1 + 4x + 16x^2 + 64x^3 + 256x^4$ $\quad a = 1, r = 4x$

$= \dfrac{1((4x)^5 - 1)}{(4x-1)} = \dfrac{1024x^5 - 1}{(4x-1)}$

* Find a closed form for the ~~generating~~ function of the sequence $0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, \ldots$

Sol^n $G(x) = 0 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + 0 + 0 + 0 + \ldots$

$= 2x[1 + x + x^2 + x^3 + x^4 + x^5] + 0 \quad (a = 1, r = x)$

$= 2x\left[1 \cdot \frac{(x^6 - 1)}{(x - 1)}\right] = \frac{2x(1 - x^6)}{(1 - x)}$

* Find a closed form for the generating function of the sequence $0, 0, 0, 1, 1, 1, 1, 1, 1, \ldots$

Sol^n $G(x) = 0 + 0 \cdot x + 0 \cdot x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \ldots$

$= x^3[1 + x + x^2 + x^3 + x^4 + x^5 + \ldots] \quad \left(\begin{matrix} a = 1 \\ r = x \end{matrix}\right)$

$= x^3 \cdot \frac{1}{1 - x} = \frac{x^3}{1 - x}$

* Find a closed form for the generating function of the sequence $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$

Sol^n $G(x) = 0 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 1 \cdot x^4 + 0 \cdot x^5 + 0 \cdot x^6 + 1 \cdot x^7 + 0 \cdot x^8 + 0 \cdot x^9 + 1 \cdot x^{10} + \ldots$

$= x + x^4 + x^7 + x^{10} + \ldots \quad (a = x, r = x^3)$

$= \frac{x}{1 - x^3}$

* Find a closed form for the generating function of the sequence $1, 1, 0, 1, 1, 1, 1, 1, 1, 1, \ldots$

Sol^n $G(x) = 1 + 1 \cdot x + 0 \cdot x^2 + 1 \cdot x^3 + 1 \cdot x^4 + 1 \cdot x^5 + 1 \cdot x^6 + 1 \cdot x^7 + 1 \cdot x^8 + 1 \cdot x^9 + \ldots$

$= [1 + x + x^2 + x^3 + x^4 + x^5 + \ldots] - x^2$

$= \frac{1}{1 - x} - x^2$

Theorem:
Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$.

Then $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

and $f(x) g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$.

example

* Let $f(x) = \frac{1}{(1-x)^2}$. Find the coefficients $a_0, a_1, a_2 \ldots$ in the expansion of $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

sol^n: $f(x) = \frac{1}{(1-x)^2} = \frac{1}{(1-x)} \cdot \frac{1}{(1-x)}$ $\because \frac{1}{1-x} = 1 + x + x^2 + \cdots$

then $\frac{1}{(1-x)^2} = \frac{1}{(1-x)} \cdot \frac{1}{(1-x)} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1) x^k$.

$= 1 + 2x + 3x^2 + \cdots$

$\therefore a_0, a_1, a_2 \ldots$ are $1, 2, 3, \ldots$

* **Useful generating functions**

a_k	$G(x)$
$\binom{n}{k}$	$\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n = (1+x)^n$
1	$1 + x + x^2 + \cdots = \frac{1}{1-x}$
λ^k	$1 + \lambda x + \lambda^2 x^2 + \cdots = \frac{1}{1-\lambda x}$
$k+1$	$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$
$\binom{n+k-1}{k}$	$\binom{n-1}{0} + \binom{n}{1} x + \binom{n+1}{2} x^2 + \cdots = \frac{1}{(1-x)^n}$
$\frac{1}{k!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$
$\frac{(-1)^{k+1}}{k}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots = \log(1+x)$

* Solve the recurrence relation $a_n = 3a_{n-1}$, $n \geq 1$ and given $a_0 = 2$. using generating function

Sol^n The generating function is given by

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

Consider the r.r $a_k = 3a_{k-1}$, $k \geq 1$

multiplying both sides by $x^k \Rightarrow$

$$a_k x^k = 3a_{k-1} x^k$$

$\Rightarrow$ $k=1$ $a_1 x = 3a_0 x$

$k=2$ $a_2 x^2 = 3a_1 x^2$

$\vdots$

$$\Rightarrow \quad a_1 x + a_2 x^2 + \cdots = 3(a_0 x + a_1 x^2 + \cdots)$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} a_k x^k = 3\sum_{k=1}^{\infty} a_{k-1} x^k$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k x^k - a_0 = 3\sum_{k=0}^{\infty} a_k x^{k+1}$$

$$\sum_{k=0}^{\infty} a_k x^k - a_0 = 3x\sum_{k=0} a_k x^k$$

$$\Rightarrow \quad G(x) - a_0 = 3x\,G(x)$$

$$\Rightarrow \quad G(x)(1-3x) = a_0$$

$$\Rightarrow \quad G(x) = \frac{a_0}{1-3x}$$

$$= \frac{2}{1-3x}$$

$$= 2\sum_{k=0}^{\infty} (3x)^k$$

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2.3^k x^k$$

$$\Rightarrow \quad a_k = 2.3^k$$

$\therefore$ The solution of the given r.r is $\boxed{a_n = 2.3^n}$, $k \geq 0$

* Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$, $a_0 = 1$, using generating function.

Sol^n Consider $a_n = 8a_{n-1} + 10^{n-1}$, $n \geq 1$

multiplying both sides by $x^n \Rightarrow$

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 8a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - a_0 = \sum_{n=0}^{\infty} 8a_n x^{n+1} + \sum_{n=0}^{\infty} 10^n x^{n+1}$$

$$\Rightarrow G(x) - a_0 = 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n$$

$$\Rightarrow G(x) - a_0 = 8x\,G(x) + x\left[1 + 10x + 10^2x^2 + \cdots\right]$$

(G.S $a = 1, r = 10x$) $S_\infty = \frac{a}{1-r}$

$$\Rightarrow G(x) - 1 = 8x\,G(x) + x \cdot \frac{1}{(1-10x)}$$

$$\Rightarrow (1-8x)\,G(x) = 1 + \frac{x}{1-10x}$$

$$\Rightarrow G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

$$= \frac{A}{1-8x} + \frac{B}{1-10x}$$

$$= \frac{1}{2}\cdot\frac{1}{(1-8x)} + \frac{1}{2}\cdot\frac{1}{(1-10x)}$$

$$\frac{1-9x}{(1-8x)(1-10x)} = \frac{A}{1-8x} + \frac{B}{1-10x}$$

$x = \frac{1}{8} \Rightarrow A = \frac{-1/8}{-2/8} = \frac{1}{2}$

$x = \frac{1}{10} \Rightarrow B = \frac{1/10}{2/10} = \frac{1}{2}$

$$G(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(8x)^n + \sum_{n=0}^{\infty}(10x)^n\right]$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[\frac{1}{2}\cdot 8^n + \frac{1}{2}\cdot 10^n\right]x^n$$

$$\Rightarrow \underline{a_n = \frac{1}{2}(8^n + 10^n)}$$ is the solution of the given ~~generati~~ recurrence relation.