DEPARTMENT OF MATHEMATICS

Discrete Mathematical Structures and Combinatorics (CS241AT)
Practice Problems
Fundamentals of logic

Set 2

1. Let $P(x)$ denote the statement "$x \leq 4$". What are these truth values?
   a) $P(0)$ b) $P(4)$ c) $P(6)$
   Ans: a) T b) T c) F

2. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.
   a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x P(x)$ d) $\forall x P(x)$

3. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.
   a) $\forall x(C(x) \rightarrow F(x))$ b) $\forall x(C(x) \wedge F(x))$ c) $\exists x(C(x) \rightarrow F(x))$ d) $\exists x(C(x) \wedge F(x))$

4. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.
   a) There is a student at your school who can speak Russian and who knows C++.
   b) There is a student at your school who can speak Russian but who doesn't know C++.
   c) Every student at your school either can speak Russian or knows C++.
   d) No student at your school can speak Russian or knows C++.
   Ans: a) $\exists x\ (P(x) \wedge Q(x))$ b) $\exists x\ (P(x) \wedge \neg Q(x))$ c) $\forall x\ (P(x) \vee Q(x))$ d) $\forall x\ \neg(P(x) \vee Q(x))$

5. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?
   a) $P(-1)$ b) $\exists x P(x)$ c) $\forall x P(x)$
   Ans: a) F b) T c) F

6. Determine the truth value of each of these statements if the domain consists of all integers.
   a) $\forall n(n + 1 > n)$ b) $\exists n(2n = 3n)$
   Ans: a) T b) T

7. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.
   a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$ e) $\neg \exists x P(x)$ f) $\neg \forall x P(x)$
   Ans: a) $P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$ b) $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$
   c) $\neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$ d) $\neg P(0) \wedge \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$
   e) $\neg(P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4))$ f) $\neg(P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4))$

8. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.
   a) Someone in your class can speak Hindi.
   b) Everyone in your class is friendly.
   c) There is a person in your class who was not born in California.
   Ans: Let $H(x)$: x can speak Hindi, $C(x)$: x is in your class, $F(x)$: x is friendly, $B(x)$: x was born in California

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	First way	Second way
a)	$\exists x H(x)$	$\exists x(C(x) \wedge H(x))$
b)	$\forall x F(x)$	$\forall x(C(x) \rightarrow F(x))$
c)	$\exists x \neg B(x)$	$\exists x(C(x) \wedge \neg B(x))$

9. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.
   a) No one is perfect. b) Not everyone is perfect.
   c) All your friends are perfect. d) At least one of your friends is perfect
   Ans: Let $P(x)$: x is perfect, $F(x)$: x is your friend. Domain be all people.
   a) $\forall x \neg P(x)$ b) $\neg \forall x P(x)$ c) $\forall x(F(x) \rightarrow P(x))$ d) $\exists x(F(x) \wedge P(x))$

10. Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")
    a) Some old dogs can learn new tricks. b) No rabbit knows calculus.
    Ans: a) $T(x)$: x can learn new tricks. Domain be old dogs. Original statement is $\exists x T(x)$. Negation is $\forall x \neg T(x)$. In English this reads "All old dogs can't learn new tricks".
    b) Let $C(x)$: x knows calculus. Domain be rabbits. Original statement is $\neg \exists x C(x)$. Negation is $\exists x C(x)$. In English this reads "There is a rabbit that knows calculus".

11. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
    a) $\forall x(x^2 \geq x)$ b) $\forall x(x > 0 \vee x < 0)$
    Ans: a) T b) F as $x = 0$ is counterexample

12. Express the negation of each of these statements in terms of quantifiers without using the negation symbol.
    a) $\forall x(x > 1)$ b) $\exists x(x \geq 4)$ c) $\forall x((x < -1) \vee (x > 2))$
    Ans: a) $\exists x(x \leq 1)$ b) $\forall x(x < 4)$ c) $\exists x((x \geq -1) \wedge (x \leq 2))$

13. Translate these system specifications into English, where the predicate $S(x, y)$ is "x is in state y" and where the domain for x and y consists of all systems and all possible states, respectively.
    a) $\exists x S(x, open)$ b) $\forall x(S(x, malfunctioning) \vee S(x, diagnostic))$
    Ans: a) "There is at least one system that is in the open state"
    b) "All systems are either malfunctioning or in diagnostic state" OR "For every system, it is either malfunctioning or in a diagnostic state"

14. Express each of these system specifications using predicates, quantifiers, and logical connectives.
    a) At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.
    b) Whenever there is an active alert, all queued messages are transmitted.
    Ans: a) Let $F(x, y)$:Disk x has more than y kilobytes of free space. $S(x)$: Mail message x can be saved.
    $(\exists x F(x, 10)) \rightarrow \exists x S(x)$
    b) Let $A(x)$: Alert x is active. $Q(x)$: Message x is queued. $T(x)$:Message x is transmitted.
    $(\exists x A(x)) \rightarrow \forall x\big(Q(x) \rightarrow T(x)\big)$.

15. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.
16. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\, P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

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17. What are the truth values of these statements?
   a) $\exists! xP(x) \rightarrow \exists xP(x)$ b) $\forall xP(x) \rightarrow \exists! xP(x)$
   Ans: a) T b) F

18. Translate these statements into English, where the domain for each variable consists of all real numbers.
   a) $\forall x\exists y(x < y)$ b) $\forall x\forall y(((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$

19. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.
   a) $\exists x\exists yQ(x, y)$ b) $\exists x\forall yQ(x, y)$ c) $\forall x\exists yQ(x, y)$ d) $\exists y\forall xQ(x, y)$

20. Let $S(x)$ be the predicate "x is a student," $F(x)$ the predicate"x is a faculty member," and $A(x, y)$ the predicate "x has asked y a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.
   a) Lois has asked Professor Michaels a question.
   b) Every student has asked Professor Gross a question.
   c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
   Ans: a) A(Lois, Professor Michaels)
   b) $\forall x(S(x) \rightarrow A(x,$ Professor Gross))
   c) $\forall x(F(x) \rightarrow (A(x,$ Professor Miller)$\vee A($Professor Miller,x)))

21. Use quantifiers and predicates with more than one variable to express these statements.
   a) Every computer science student needs a course in discrete mathematics.
   b) Every student in this class has taken at least one computer science course.
   c) There is a student in this class who has been in every room of at least one building on campus.
   Ans: a) $\forall xN(x, \text{discrete mathematics})$ where $N(x, y)$: x needs a course in y. Domain for x is C.S. students and domain for y is academic subjects.
   b) $\forall x\exists yP(x, y)$ where $P(x, y)$: $x\ has\ taken\ y$; x ranges over students in class and y ranges over C.S. courses.
   c) $\exists x\exists y\forall z(P(z, y) \rightarrow Q(x, z))$ where $P(z, y)$: $z\ is\ in\ y$ and $Q(x, z)$: $x\ has\ been\ in\ z$; x ranges over students in class. y ranges over buildings on campus and z ranges over rooms.

22. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
   a) $\exists x\forall y(xy = y)$ b) $\forall x\forall y(((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$
   Ans: a) There exists a real number x such that for every real number y, the product xy equals y $i.e.$ there is a multiplicative identity for the real numbers.
   b) The product of 2 negative real numbers is always a positive real number.

23. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
   a) $\forall n\exists m(n^2 < m)$ b) $\exists n\forall m(n < m^2)$ c) $\forall n\exists m(n + m = 0)$
   d) $\exists n\exists m(n + m = 4 \wedge n - m = 1)$
   Ans: a) T b) T c) T d) F

24. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
   a) $\forall x\exists y\forall zT(x, y, z)$ b) $\forall x\exists yP(x, y) \vee \forall x\exists yQ(x, y)$
   Ans: a) $\exists x\forall y\exists z \neg T(x, y, z)$ b) $\exists x\forall y \neg P(x, y) \wedge \exists x\forall y \neg Q(x, y)$

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25. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
    a) $\forall x \forall y(x^2 = y^2 \rightarrow x = y)$ b) $\forall x \exists y(y^2 = x)$
    Ans: a) $x = 2, y = -2$ b) $x = -4$

26. For each of these arguments determine whether the argument is correct or incorrect and explain why.
    a) "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."

    b) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

27. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.
28. Use a direct proof to show that every odd integer is the difference of two squares. [Hint: Find the difference of the squares of $k + 1$ and k where k is a positive integer.]
29. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind of proof did you use?
30. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.
31. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
    a) a proof by contraposition. b) a proof by contradiction.

32. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.
33. Prove that if x is irrational, then $\frac{1}{x}$ is irrational.
34. Prove that if x is an irrational number and $x > 0$, then $\sqrt{x}$ is also irrational.
35. Show that at least ten of any 64 days chosen must fall on the same day of the week.
36. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
37. Show that these statements about the real number x are equivalent: (i) x is irrational, (ii) $3x + 2$ is irrational, (iii) $\frac{x}{2}$ is irrational.
38. Prove that if n is an integer, these four statements are equivalent: (i) n is even, (ii) $n + 1$ is odd, (iii) $3n + 1$ is odd, (iv) $3n$ is even.