# ASSIGNMENT

Piyush Soni [SBU ID # 113273375]

Abhishek Bhattad [SBUID# 113277131]

Aditya Nandan Bhide [SBU ID# 113216565]

Niket Shah [SBU ID# 113274949]

Q. 1

a] Number of ways in which LAC can win 3 ~~any~~ of the 4 matches is $^4C_3$

$M_i$ = Match $i$

| $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|
| LAC win | LAC win | LAC win | LAC lose |
| 0.5 | 0.5 | 0.5 | 0.5 |

Now probability for LAC to win 3 matches with above combination is $(0.5)^4$.

Now number of ways that LAC wins is $^4C_3$

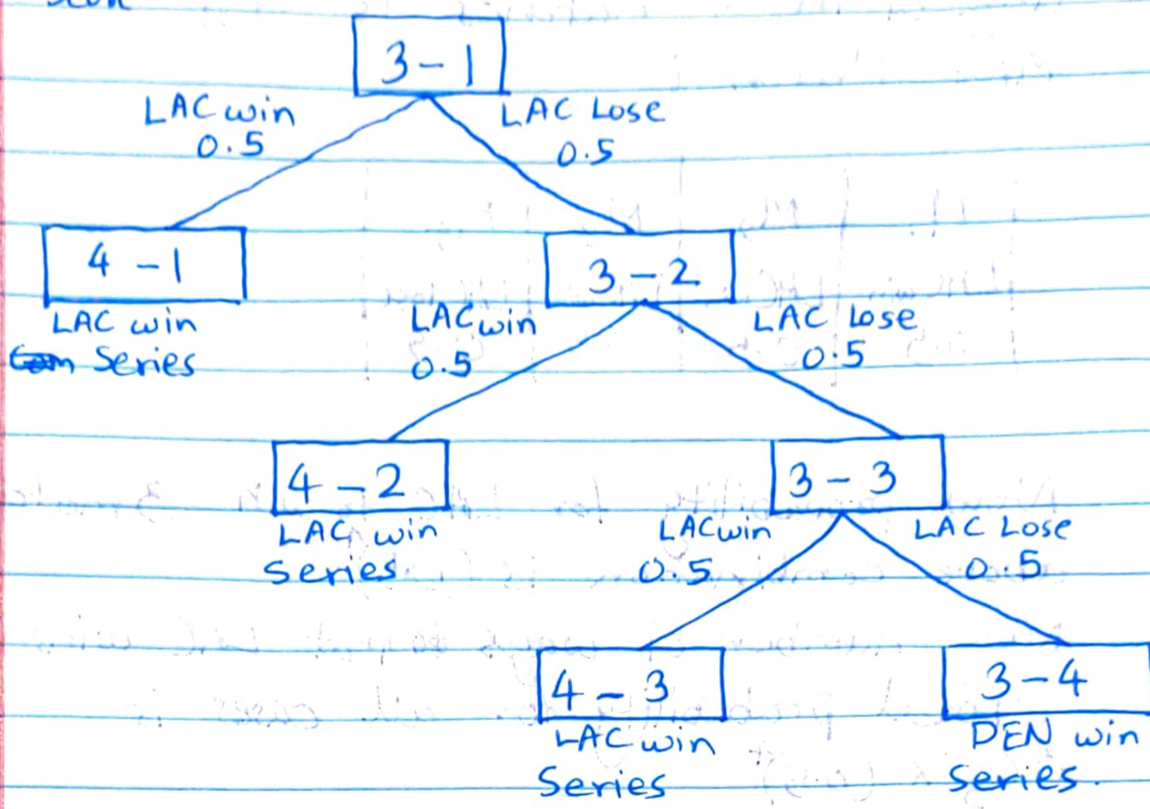
$\therefore$ Total probability for all cases is

$= {}^4C_3 \times (0.5)^4$

$$= \frac{4!}{3!\,1!} \times (0.5)^4 = \frac{4 \times 3!}{3! \times 1} \times \left(\frac{1}{2}\right)^4$$

$$= 4 \times \frac{1}{16} = \frac{1}{4} = \boxed{0.25}$$

Q1 b] After first 4 matches the score between LAC - DEN is 3-1. Node defines current score.



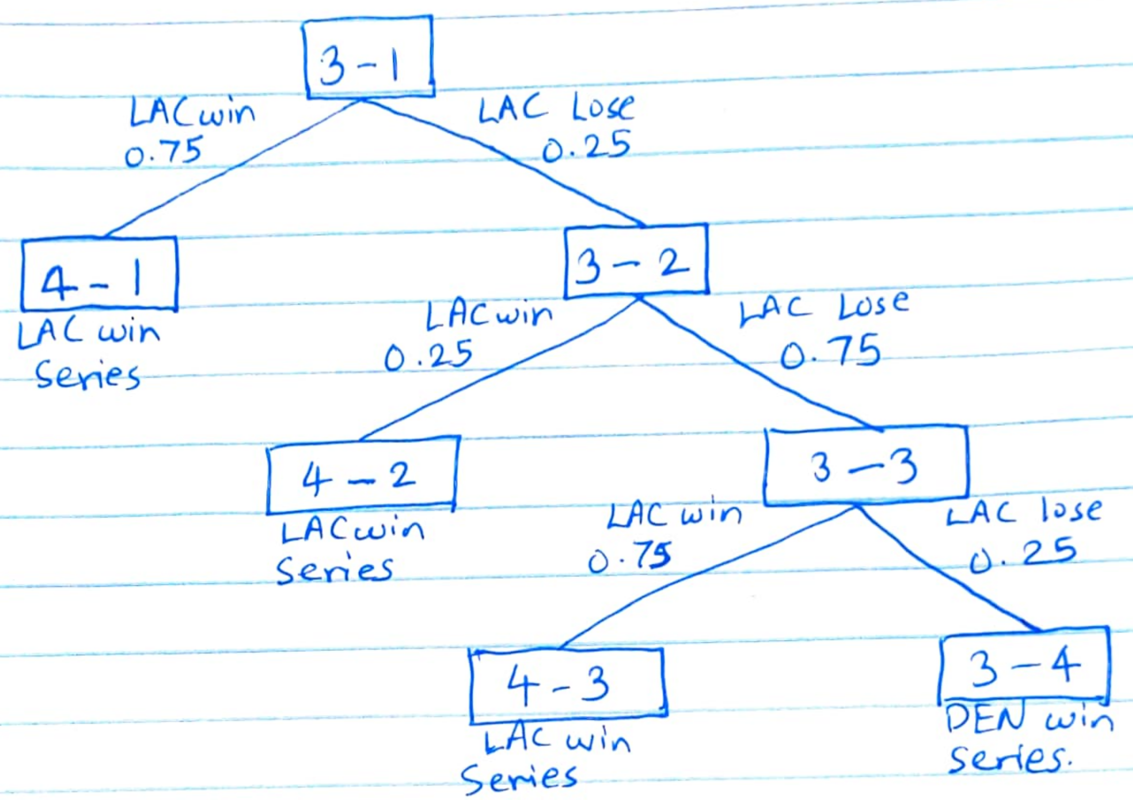

Q1 c] In a decision tree probability of a Leaf node is multiplication of probability on the path to the root node. From decision tree in question b. DEN win series is a leaf node.

$\therefore P(\text{DEN win series}) = 0.5 \times 0.5 \times 0.5$

$= 0.125$

Q1.

d] After first 4 matches the score between LAC-DEN is 3-1. Node defines current score.




e] From decision tree in question d. DEN win series is a leaf node.

$$P(\text{DEN win series}) = 0.25 \times 0.75 \times 0.25$$

$$= \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}$$

$$= \frac{3}{64} = 0.046875$$

Q1 F] As observed from results from program, as N increases, the calculated probability (by code) converges towards theoretical probability values. i.e.

for $n = 3$ $(N = 10^3)$

Part a probability = 0.256

Part c probability = 0.1171875

Part e probability = 0.04803

for $n = 5$ $(N = 10^5)$

a prob. = 0.25123

c prob. = 0.12538

e prob. = 0.04512

For $n = 7$ $(N = 10^7)$

a prob. = 0.2500

c prob. = 0.12501

e prob. = 0.046768

for $n = 4$ $(N = 10^4)$

a prob. = 0.2512

c prob. = 0.12858

e prob. = 0.0514

For $n = 6$ $(N = 10^6)$

a prob = 0.2500

c prob. = 0.12487

e prob. = 0.04683

2) let $E_i$ be the event in which the 'ith' iphone is picked in the 'ith' step, hence not discarded.

$\therefore$ $Pr(E_1 \cup E_2 \cup E_3 \cup \dots E_n)$ denotes the probability that atleast one iphone is not discarded

$\therefore$ Using the Principle of Inclusion-Exclusion:

$$Pr\left(\bigcup_{i=1}^{n} E_i\right) = \sum_i Pr(E_i) - \sum_{i<j} Pr(E_i \cap E_j) + \sum_{i<j<k} Pr(E_1 \cap E_2 \cap E_3) - \dots + (-1)^{n+1} Pr(E_1 \cap \dots \cap E_n) \quad -1$$

$$\sum_{i=1}^{n} Pr(E_i) = \frac{(n-1)!}{n!} \times {}^nC_1 = \frac{(n-1)!}{n!} \times \frac{n!}{(n-1)!\,1!} = \frac{1}{1!}$$

$$\sum_{i<j}^{n} Pr(E_i \cap E_j) = {}^nC_2 \cdot \frac{(n-2)!}{n!} = \frac{\cancel{n!}}{2!\cancel{(n-2)}!} \times \frac{\cancel{(n-2)}!}{\cancel{n!}} = \frac{1}{2!}$$

$\vdots$

$$\sum Pr(E_1 \cap E_2 \cap \dots \cap E_n) = {}^nC_n \cdot \frac{(n-n)!}{n!} = 1/n!$$

$\therefore$ Substituting in eq 1:

$$Pr\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n+1}\frac{1}{n!} \quad -2$$

* We know that:

$$e^{-1} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots \frac{1}{n!} \quad -3$$

for $n \to \infty$

Probability ~~the~~ to have atleast one undiscarded phone :

$$\boxed{\Pr\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}}$$

for $n \to \infty$

$$\Pr\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots \infty$$

$$= 1 - 1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots$$

$$= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \cdots\right)$$

from eq$^n$ 3 —

$$\boxed{\Pr\left(\bigcup_{i=1}^{\infty} E_i\right) = 1 - e^{-1}}$$

ANS 3]

Given :-

Let,

$P_r(A)$ : be the probability that the given ring is the $\boxed{\text{one ring}}$

$\therefore P_r(A) = \frac{1}{10000}$ — (i)

$P_r(B)$ : be the probability that the owner has $\boxed{\text{above}}$ average lifespan.

$P_r(W)$ : be the probability that $\boxed{\text{writing}}$ will appear on the one ring.

$P_r(N)$ : be the probability that the given ring is NOT the one ring

$\therefore P_r(N) = 1 - P_r(A) = 1 - \frac{1}{10000} = \frac{9999}{10000}$ — (ii)

From the problem statement it is given,

$P_r(B/A) = 0.95$ — (iii)

$P_r(X/N) = 0.75$ — (iv)

$P_r(X)$ : be the probability that the owner has $\boxed{\text{below}}$ average lifespan

BAYES' THEOREM:-

$$P_r(E_i/A) = \frac{P_r(A/E_i) \times P_r(E_i)}{\sum_{j=1}^{n} P_r(A/E_j) P(E_j)} \quad \text{--- (A)}$$

a) $P_r(A/B) = ?$

$P_r$ (Ring is the One Ring given Billo is above Average)

Using Bayes' theorem, from (A),

$$P_r(A/B) = \frac{P_r(B/A) \times P_r(A)}{P_r(B/A) \times P_r(A) + P_r(B/N) \times P_r(N)}$$

Total number of events we have,

| RING | NOT THE RING |
|---|---|
| A | N |

$$= \frac{0.95 \times \frac{1}{10000}}{0.95 \times \frac{1}{10000} + (1 - P_r(X/N)) \times P_r(N)}$$

from Eq (iv).

$$= \frac{0.95 \times \frac{1}{10000}}{0.95 \times \frac{1}{10000} + (1 - 0.75) \times \frac{9999}{10000}}$$

$$= \frac{0.95}{0.95 + 0.25 \times 9999}$$

$$= \boxed{0.000379936}$$

i.e. $\boxed{P_r(A/B) = 0.00038}$

b) Let,

$Pr(W)$: be the probability that writing appears on ring

Given,

$$Pr(W/A) = 0.9 \quad -(V)$$

Pr(writing will appear given it's the one ring) = 0.9

$$Pr(W/N) = 0.05 \quad -(vi).$$

To calculate,

Pr(ring is one ring given writing appears and Billo is above average) = ?

$$Pr(A/W \cap B) = ?$$

Using Bayes' theorem, from (A)

$$Pr(A/W \cap B) = \frac{Pr((W \cap B)/A)\, Pr(A)}{Pr(W \cap B/A)\, Pr(A) + Pr(W \cap B/N)\, Pr(N)}.$$

It is given that tests are independent conditioned on one ring and not the one ring.

$\therefore$ By Conditional Independence:

$$Pr(A \cap B/C) = Pr(A/C) \times P(B/C) \quad -(B).$$

$$\therefore P_r(A/W\cap B) = \frac{P_r(W/A) \times P_r(B/A) \times P_r(A)}{P_r(W/A) \times P_r(B/A) \times P_r(A) + P_r(W/N) \times P_r(B/N) \times P_r(N)}$$

$$= \frac{\overset{(v)}{0.9} \times \overset{(iii)}{0.95} \times \overset{(i)}{\frac{1}{10000}}}{\underset{(v)}{0.9} \times \underset{(iii)}{0.95} \times \underset{(i)}{\frac{1}{10000}} + \underset{(vi)}{0.05} \times \underset{(iv)}{(1 - P_r(X/N))} \times \underset{(ii)}{\frac{9999}{10000}}}$$

$$= \frac{0.9 \times 0.95}{0.9 \times 0.95 + 0.05 \times 0.25 \times 9999}$$

$$= \frac{0.855}{0.855 + 124.9875}$$

$$= \frac{0.855}{125.8425}$$

$$\boxed{= 0.006794207}$$

$$\boxed{\text{ie } P_r(A/W\cap B) = 0.0068}$$

4. Given :

X is non-negative, integer valued R.V.

To Prove

$$E[X] = \sum_{x=0}^{\infty} Pr[X > x]$$

let $p(x) = Pr(X = x)$

$$E[X] = \sum_{x=0}^{\infty} x . p(x) . \cancel{dx}$$

$$= \sum_{x=0}^{\infty} x . Pr(X = x) \cancel{dx}$$

$$= 0 . P(X=0) + 1 . P(X=1) + 2 P(X=2) + \ldots\ldots$$

$$= \left(P(X=1) + P(X=2) + \ldots \infty\right) + \left(P(X=2) + P(X=3) + \ldots \infty\right) + \left(P(X=3) + P(X=4) + \ldots \infty\right) \ldots$$

$$= Pr[X > 0] + Pr[X > 1] + Pr[X > 2] + \ldots\ldots$$

$$\therefore \boxed{E[X] = \sum_{x=0}^{\infty} Pr[X > x]}$$

Hence Proved

5) $X \sim \text{Indicator}(E) = I(E)$

$$X = \begin{cases} 1 & \text{if event } E \text{ occurs} \\ 0 & \text{if event } E \text{ does not occur.} \end{cases}$$

$$\therefore p_X(x) = \begin{cases} Pr(E) & \text{if } x = 1 \\ 1 - Pr(E) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

a) $E[I_E] = \sum_{i=0,1} i \cdot Pr[X=i]$

$= 0 \cdot Pr[X=0] + 1 \cdot Pr[X=1]$

$= Pr[X=1]$

$\boxed{E[I_E] = Pr(E)}$ — 3

b) $Var(I_E) = E[I_E^2] - \left(E[I_E]\right)^2$ — 1

$E[I_E^2] = \sum_{i=0,1} i^2 Pr[X=i]$

$= 0^2 \cdot Pr[X=0] + 1^2 Pr[X=1]$

$E[I_E^2] = Pr[E]$ —— 2

$\therefore$ Substituting eq. 2 & 3 in 1 :

$$Var[I_E] = Pr(E) - (Pr(E))^2$$

$$\boxed{Var[I_E] = Pr(E)(1 - Pr(E))}$$

c) $X \sim Geometric(p)$ with $p < 1$

p.m.f for geometric RV $= (1-p)^{i-1} \cdot p$ for $1 \leq i < \infty$

$$\therefore E[X] = \sum_{i=1}^{\infty} i \, Pr[X = i]$$

$$= \cancel{X \, Pr[X = X]}$$

$$= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p$$

$$= p \left( \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \right) \text{ ——— } 1$$

$$= p \times s$$

Let, $s = \sum_{i=1}^{\infty} i (1-p)^{i-1}$

$\Rightarrow s = (1-p)^0 + 2(1-p)^1 + 3(1-p)^2 + \text{---}$ — 2

Multiplying eq$^n$ 1 by $(1-p)$

$\Rightarrow (1-p)s = (1-p) + 2(1-p)^2 + \text{---}$ — 3

Substracting eqⁿ ③ from ② -

$$S - S(1-p) = (1-p)^0 + (1-p) + (1-p)^2 + \cdots$$
$$= 1 + (1-p) + (1-p)^2 + \cdots$$

Using summation of infinite G.P. -

$$\Rightarrow Sp = \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$\Rightarrow \boxed{S = \frac{1}{p^2}} \quad — ④$$

Substituting ④ in ① -

$$\Rightarrow E[X] = p \times \frac{1}{p^2} = \frac{1}{p}$$

$$\boxed{E[X] = 1/p} \quad — ⑤$$

d) $Var[X] = E[X^2] - (E[X])^2$

$\Rightarrow E[X^2] = \sum_{i=1}^{\infty} i^2 p(1-p)^{i-1} = \sum_{i=1}^{\infty} i^2 P[X=i]$

$= p\left(\sum_{i=1}^{\infty} i^2 (1-p)^{i-1}\right)$

Let $S = \sum_{i=1}^{\infty} i^2 (1-p)^{i-1}$

$\Rightarrow E[X^2] = pS$ — ①

$\Rightarrow S = 1^2(1-p)^0 + 2^2(1-p)^1 + 3^2(1-p)^2 + \cdots$ — ②

Multiplying ② by $(1-p)$

$\Rightarrow S(1-p) = 1^2(1-p)^1 + 2^2(1-p)^2 + \cdots$ — ③

Substracting eq^n ③ from eq^n ②

$\Rightarrow S - S(1-p) = 1^2(1-p)^0 + (2^2-1^2)(1-p)^1 + (3^2-2^2)(1-p)^2 + \cdots$

$= 1 + (2-1)(2+1)(1-p) + (3-2)(3+2)(1-p)^2 + \cdots$

$\Rightarrow Sp = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + \cdots$ — ④

Multiplying eqⁿ 4 by $(1-p)$

$$\Rightarrow Sp(1-p) = (1-p) + 3(1-p)^2 + 5(1-p)^3 + \cdots \quad -(5)$$

Substracting eqⁿ 5 by 4 -

$$\Rightarrow Sp - Sp(1-p) = 1 + 2(1-p) + 2(1-p)^2 + 2(1-p)^3 + \cdots$$

$$= 1 + 2(1-p)\left(1 + (1-p) + (1-p)^2 + \cdots\right)$$

$$= 1 + 2(1-p)\left(\frac{1}{1-(1-p)}\right) \quad - \left\{\text{using sum of infinite GP}\right\}$$

$$= 1 + \frac{2(1-p)}{p}$$

$$\Rightarrow Sp^2 = \frac{2-p}{p}$$

$$\Rightarrow \boxed{S = \frac{2-p}{p^3}} \quad -(6)$$

Substituting 6 in 1 -

$$E[x^2] = p\frac{(2-p)}{p^3} = \frac{2-p}{p^2} \quad -(7)$$

Since from que (c) -

$$E[x] = \frac{1}{p}$$

and, from eq$^n$ (7)

$$E[x^2] = \frac{2-p}{p^2}$$

$$\Rightarrow Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{2-p-1}{p^2}$$

$$\Rightarrow \boxed{var[x] = \frac{1-p}{p^2}}$$

6) $X \sim Poisson(\lambda)$

$$p_X(i) = \frac{e^{-\lambda} \cdot \lambda^i}{i!}, \quad i \geq 0$$

a) To prove: $\sum_{i=0}^{\infty} p_X(i) = 1$

$$\therefore \sum_{i=0}^{\infty} p_X(i) = \sum_{i=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \left( \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \infty \right)$$

$$= e^{-\lambda} \left( 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \infty \right) \quad - (1)$$

~~$= e^{-\lambda}$~~ We Know that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty \quad - (2)$$

Substituting (2) in (1)

$$\sum_{i=0}^{\infty} p_X(i) = e^{-\lambda} (e^{\lambda})$$

$$\boxed{\sum_{i=0}^{\infty} p_X(i) = 1}$$

Hence Proved

b) $$E[X] = \sum_{i=0}^{\infty} \cancel{x}\, i \, P_r(X=i)$$

$$= \sum_{i=0}^{\infty} i \cdot \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$$= 0 + e^{-\lambda} \sum_{i=1}^{\infty} \frac{i\lambda^{i}}{i!}$$

$$= e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i}}{(i-1)!}$$

$$= e^{-\lambda} \left[ \frac{\lambda^1}{0!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots \right]$$

$$= e^{-\lambda} \cdot \lambda \left[ 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} \quad - \text{From } 2$$

$$\boxed{E[X] = \lambda}$$

7) $X \sim \text{Pareto}(\alpha)$, $1 < \alpha < 2$

$$f_X(x) = \alpha x^{-\alpha-1}, \quad x \geq 1$$

a) To prove $\int_1^\infty f_X(x)dx = 1$

$$\int_1^\infty f_X(x)dx = \int_1^\infty \alpha . x^{-\alpha-1} dx$$

$$= \alpha \int_1^\infty x^{-\alpha-1} dx$$

$$= \alpha \left. \frac{x^{-\alpha-1+1}}{-\alpha-1+1} \right|_1^\infty \qquad \because \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \alpha . \left. \frac{x^{-\alpha}}{-\alpha} \right|_1^\infty$$

To remove negative sign, invert limits

$$= \frac{\alpha}{\alpha} \left. x^{-\alpha} \right|_\infty^1$$

$$= 1^{-\alpha} - \infty^{-\alpha}$$

$$= 1 - \frac{1}{\infty^\alpha} = 1 - 0 \qquad \because \frac{1}{\infty^n} = 0 \text{ for } n \geq 0$$

$$\boxed{\int_1^\infty f_X(x)dx = 1}$$

Hence Proved

b) $E[X] = \int_1^{\infty} x . f_X(x)\, dx$

$= \int_1^{\infty} x . \alpha . x^{-\alpha-1}\, dx$

$= \alpha \int_1^{\infty} x^{-\alpha-1+1}\, dx$

$= \alpha \int_1^{\infty} x^{-\alpha}\, dx = \alpha \left[ \frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^{\infty}$

$= \frac{\alpha}{1-\alpha} \left[ \infty^{1-\alpha} - 1^{1-\alpha} \right]$

$= \frac{\alpha}{1-\alpha} \left[ 0 - 1 \right] \quad \because 1 < \alpha < 2$

$= \frac{-\alpha}{1-\alpha} = \frac{\alpha}{\alpha-1}$

$\therefore \boxed{E[X] = \frac{\alpha}{\alpha-1}}$

c) $Var[X] = E[X^2] - (E[X])^2$ — 1

$$E[x^2] = \int_1^\infty x^2 . f_x(x)\,dx$$

$$= \int_1^\infty x^2 . \alpha . x^{-\alpha-1}\,dx$$

$$= \alpha \int_1^\infty x^{-\alpha-1+2}\,dx = \alpha \int_1^\infty x^{1-\alpha}\,dx$$

$$= \alpha \left[\frac{x^{1-\alpha+1}}{1-\alpha+1}\right]_1^\infty$$

$$= \alpha \left[\frac{x^{2-\alpha}}{2-\alpha}\right]_1^\infty$$

$$= \frac{\alpha}{2-\alpha}\left[x^{2-\alpha}\right]_1^\infty$$

$$= \frac{\alpha}{2-\alpha}\left[\operatorname{lt}_{x\to\infty}\left[x^{2-\alpha}\right] - 1^{2-\alpha}\right]$$

$$= \frac{\alpha}{2-\alpha}\left(\operatorname{lt}_{x\to\infty} x^{2-\alpha} - 1\right)$$

for $1<\alpha<2$, $\operatorname{lt}_{x\to\infty} x^{2-\alpha} \Rightarrow \infty$

$\therefore \boxed{Var[x] \Rightarrow \infty \quad \text{for } 1<\alpha<2}$

8) F is a CDF fn. → strictly increasing

(a) $U \sim$ Uniform $(0,1)$

pdf of Uniform $(a,b)$ is –

$$f(x) = \begin{cases} 0 & x \le a \\ 1/(b-a) & a < x < b \\ 0 & x \geq b \end{cases}$$

∴ Pdf of Uniform $(0,1)$ is –

$$\Rightarrow \quad f_U(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Cdf of Uniform $(0,1)$ –

$$F_u(\alpha) = \int_{-\infty}^{\alpha} f(x)\,dx$$

$$= \int_{0}^{\alpha} 1\,dx = \alpha$$

$$\boxed{F_u(\alpha) = \alpha} \Rightarrow \boxed{Pr(U \le \alpha) = \alpha} \quad -(1)$$

Given, $X = F^{-1}(U)$

Cdf of X, $F_X(\alpha) = Pr(X \le \alpha)$

$$= Pr(F^{-1}(U) \le \alpha)$$

$$= Pr(U \le F(\alpha)) \quad \{\because F \text{ is strictly increasing}$$

$-(2)$

From eq$^n$ 2 and 1, we can say that –

$\Rightarrow F_X(\alpha) = F_U(F(\alpha)) = F(\alpha)$

$\Rightarrow \therefore \boxed{\text{c.d.f. of } X = F}$

(b) $Y \rightarrow$ RV with cdf $F$ $\Rightarrow F_Y(\alpha) = Pr(Y \leq \alpha)$ — 1

To Prove – $F(Y) \sim$ Uniform$(0,1)$
using property of Uniform $(0,1)$ RV.
cdf of $F_U(\alpha) = \alpha$.

So, to prove $F(Y)$ to be uniform$(0,1)$
we can prove $\boxed{Pr[F(Y) \leq \alpha] = \alpha}$.

L.H.S. $Pr[F(Y) \leq \alpha]$
Since, F is strictly increasing.

$\Rightarrow Pr[Y \leq \underbrace{F^{-1}(\alpha)}_{\alpha'}]$

$\Rightarrow \alpha' = F^{-1}(\alpha)$

$\Rightarrow F[\alpha'] = F[F^{-1}(\alpha)] = \alpha =$ R.H.S.

= Thus, $\boxed{Pr[F(Y) \leq \alpha] = \alpha}$

$\Rightarrow$ Thus, we can say that $F(Y)$ is Uniform$(0,1)$