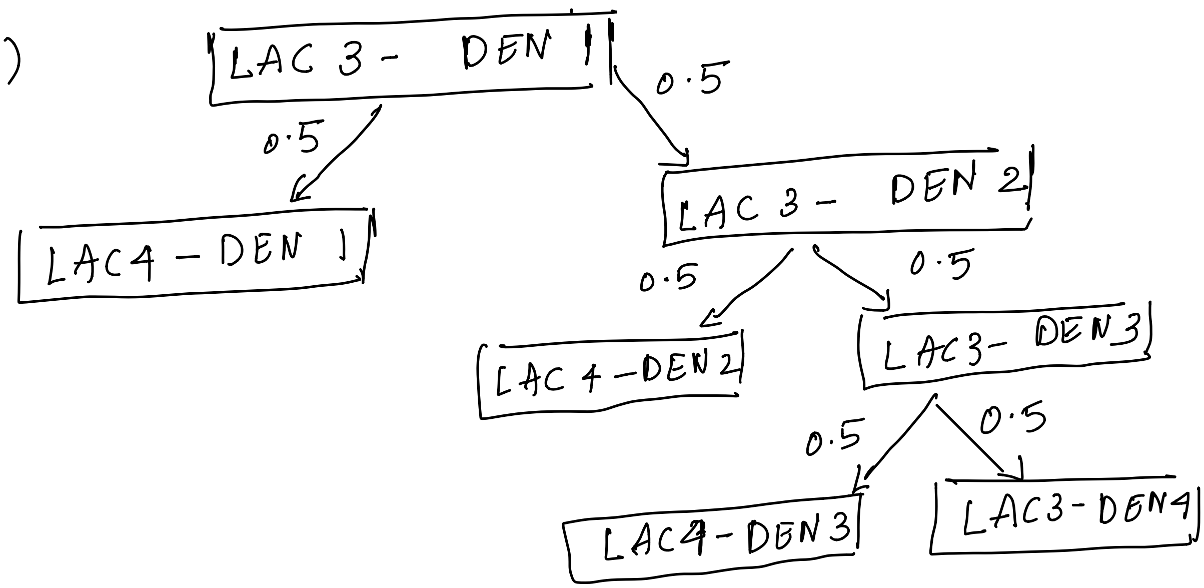
problem 1.   GSW   OKC

(a)   LAC   DEN — now

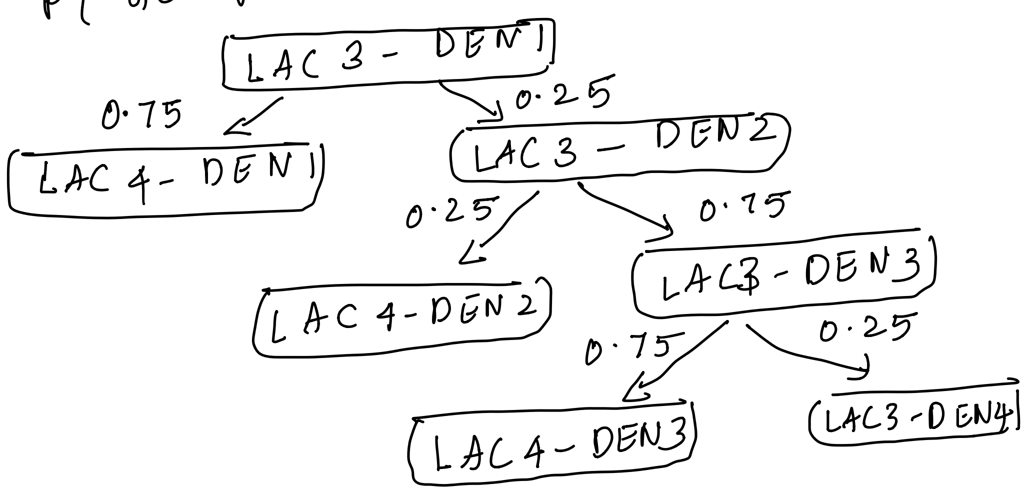
$$P = \binom{4}{3} (0.5)^3\, 0.5 = 0.25$$

(b)



(c) $p(GSW) = 0.5^3 = 0.125$

(d)



(e) $P(DEN) = 0.25 \times 0.75 \times 0.25$

$= 0.046875$

Problem 2.

Let E_i be the event that you pick iPhone i at the ith step. We need to find $P(\cup_{i=1}^n E_i)$. By the principle of inclusion-exclusion we have

$$P(\cup_{i=1}^n E_i) = \sum_{i=1}^{n} \sum_{i} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \cdots + (-1)^{n+1} P(E_i \cap E_j \cdots \cap E_n)$$

Note that $P(E_i) = 1/n$ for all i. One way to see this is by using the full sample space: there are $n!$ possible orderings of the iPhones, all equally likely, and $(n-1)!$ of these are favorable to E_i.

Similarly

$$P(E_i \cap E_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

,

$$P(E_i \cap E_j \cap E_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

and so on.

In the inclusion-exclusion formula, there are n terms involving one event, $\binom{n}{2}$ terms involving two events, $\binom{n}{3}$ terms involving three events, and so forth. By the symmetry of the problem, all n terms of the form $P(E_i)$ are equal, all $\binom{n}{2}$ terms of the form $P(E_i \cap E_j$ are equal. Therefore one has

$$\begin{aligned} P(\cup_{i=1}^n E_i) &= \frac{n}{n} - \frac{\binom{n}{2}}{n(n-1)} + \frac{\binom{n}{3}}{n(n-1)(n-2)} - \cdots + (-1)^{n+1} \frac{1}{n!} \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \frac{1}{n!} \end{aligned}$$

problem 3

(a)

Let A and R be the variables for "the owner have an above-average lifespan" and "the ring is the One Ring", respectively. Then,

$P(A|R) = 0.95$, $P(\bar{A}|R) = 0.05$, $P(\bar{A}|\bar{R}) = 0.75$ and $P(A|\bar{R}) = 0.25$. Then,

$$\begin{aligned}
P(R|A) &= \frac{P(A|R)P(R)}{P(A)} \\
&= \frac{P(A|R)P(R)}{P(A|R)P(R) + P(A|\bar{R})P(\bar{R})} \\
&= \frac{0.95 \cdot 10^{-4}}{0.95 \cdot 10^{-4} + 0.25 \cdot (1 - 10^{-4})} \\
&= \frac{0.95}{0.95 + 0.25 * 9999} \\
&= 0.00038
\end{aligned}$$

(b)

Now Let W be the variable for "writing will apear on it". Thus $P(W|R) = 0.9$, $P(\bar{W}|R) = 0.1$, $P(\bar{W}|\bar{R}) = 0.95$ and $P(W|\bar{R}) = 0.05$. Since A and W are conditionally independent tests, we have

$$P(R|WA) = \frac{P(WA|R)P(R)}{P(WA)} = \frac{P(WA|R)P(R)}{P(WA|R)P(R) + P(WA|\bar{R})P(\bar{R})}$$

Since A and W are conditionally independent, therefore,

$$P(R|WA) = \frac{P(W|R)P(A|R)P(R)}{P(WA|R)P(R) + P(WA|\bar{R})P(\bar{R})} = \frac{P(W|R)P(A|R)P(R)}{P(W|R)P(A|R)P(R) + P(W|\bar{R})P(A|\bar{R})P(\bar{R})}$$

Thus,

$$P(R|WA) = \frac{0.9 \cdot 0.95 \cdot 10^{-4}}{0.9 \cdot 0.95 \cdot 10^{-4} + 0.05 \cdot 0.25 \cdot (1 - 10^{-4})} \approx 0.006794$$

Problem 4

Way 1

First introduce a random variable I_x, where $I_x = \begin{cases} 1, & if\ X > x \\ 0, & otherwise \end{cases}$.
Now assume $X = \sum_{x=0}^{\infty} I_x$. Then $E[X] = E[\sum_{x=0}^{\infty} I_x] = \sum_{x=0}^{\infty} E[I_x] = \sum_{x=0}^{\infty} P(X > x)$
QED.

Way 2

$$\begin{aligned}
\sum_{x=0}^{\infty} Pr[X > x] &= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} Pr[X = y] \\
&= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} P_x(y) \\
&= \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} Pr[X = y] \\
&= \sum_{y=1}^{\infty} P_x(y) \sum_{x=0}^{y-1} 1 \\
&= \sum_{y=1}^{\infty} P_x(y) y \\
&= E[X]
\end{aligned}$$

Problem 5

(a)

$$
\begin{aligned}
E[I_E] &= 1.Pr(I_E = 1) + 0.Pr(I_E = 0) \\
&= Pr(I_E = 1) \\
&= Pr(E)
\end{aligned}
$$

(b)

$$
\begin{aligned}
Var(I_E) &= E[I_E^2] - E[I_E]^2 \\
E[I_E^2] &= 1^2.Pr(I_E = 1) + 0^2.Pr(I_E = 0) \\
&= pr(I_E = 1) \\
&= Pr(E) \\
Var(I_E) &= pr(E) - pr(E)^2 \\
&= Pr(E)(1 - Pr(E)
\end{aligned}
$$

(c)

$$
P_X = (1-p)^{x-1}.p
$$

$$
\begin{aligned}
E[X] &= \sum_{x=1}^{\infty} x(1-p)^{x-1}.p \\
&= p.[1 + 2.(1-p) + 3.(1-p)^2 + ...]\ Say\ eq(1) \\
(1-p).E[X] &= p.[0 + 1.(1-p) + 2.(1-p)^2 + 3.(1-p)^3 + ...]\ Say\ eq(2)
\end{aligned}
$$

$let's\ take\ eq(1)\ -\ eq(2)$

$$
\begin{aligned}
E[X] - (1-p).E[X] &= p.[1 + (1-p) + (1-p)^2 + ...] \\
p.E[X] &= p.[\frac{1}{1-(1-p)}] \\
E[X] &= \frac{1}{p}
\end{aligned}
$$

(d)

To find the variance we need to find E[x^2], we will start With E[X] and will find E[X^2]

$$\begin{aligned}
E[X] &= p.\sum_{x=1}^{\infty} x(1-p)^{x-1} \\
(1-p).E[X] &= p.\sum_{x=1}^{\infty} x(1-p)^{x} \\
\frac{1-p}{p^2} &= \sum_{x=1}^{\infty} x(1-p)^{x} \\
\frac{d}{dp}(\frac{1-p}{p^2}) &= \frac{d}{dp}(\sum_{x=1}^{\infty} x(1-p)^{x}) \\
\frac{d}{dp}(\frac{1-p}{p^2}) &= \sum_{x=1}^{\infty} \frac{d}{dp}(x(1-p)^{x}) \\
\frac{d}{dp}(\frac{1-p}{p^2}) &= -1.\sum_{x=1}^{\infty} x^2(1-p)^{x-1} \\
\frac{-2}{p^3}+\frac{1}{p^2} &= -1.\sum_{x=1}^{\infty} x^2(1-p)^{x-1} \\
\frac{2-p}{p^3} &= \sum_{x=1}^{\infty} x^2(1-p)^{x-1} \\
p.\frac{2-p}{p^3} &= p.\sum_{x=1}^{\infty} x^2(1-p)^{x-1} \\
\frac{2-p}{p^2} &= E[X^2]
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

Problem 6

(a)

$$\begin{aligned}\sum_{i=0}^{\infty} P_X(i) &= \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\\ &= e^{-\lambda}(1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+...)\\ &= e^{-\lambda}e^{\lambda} \text{ (From Taylor series expansion of } e^{\lambda})\\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}E[X] &= \sum_{i=0}^{\infty} iP_X(i) = \sum_{i=0}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!}\\ &= \sum_{i=1}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=1}^{\infty} i\frac{\lambda^i}{i!}\\ &= \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!}\\ &= \lambda e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\\ &= \lambda e^{-\lambda}e^{\lambda} = \lambda\end{aligned}$$

Problem 7

(a)

$$\int_1^{+\infty} f_X(x) = \int_1^{+\infty} \alpha x^{-\alpha-1} dx = -x^{-\alpha}|_1^{+\infty} = 0 + 1 = 1$$

(b)

$$E[x] = \int_1^{+\infty} x \cdot \alpha \cdot x^{-\alpha-1} dx = \int_1^{+\infty} \alpha x^{-\alpha} dx = \frac{\alpha}{-\alpha+1} x^{-\alpha+1}|_1^{+\infty} = \frac{\alpha}{\alpha-1}$$

(c)

Similarly, we have

$$E[x^2] = \int_1^{+\infty} \alpha x^{-\alpha+1} dx = \frac{\alpha}{-\alpha+2} x^{-\alpha+2}|_1^{+\infty}$$

Since $1 < \alpha < 2$, then $0 < -\alpha + 2 < 1$. Thus $x^{-\alpha+2} \to +\infty$, so $E[x^2] = +\infty$ and $Var[x] = +\infty$

Problem 8.

(a) Let F be the CDF of U.

Let $F_X : \mathbb{R} \to [0,1]$ be the CDF of X.

For any $x \in \mathbb{R}$, we have

$$F_X(x) = P(X \leq x) = P(F^{-1}(U) \leq x)$$

$$= P(U \leq F(x)) = F(x)$$

(since $0 \leq F(x) \leq 1$ for all x, $P(U \leq F(x)) = F(x)$)

Since $F_X(x) = F(x)$ $\forall x \in \mathbb{R}$, F is the CDF of X.

(b). Let Z be a random variable such that $Z = F(Y)$.

Since $F(Y)$ takes values in $(0,1)$,

$P(Z \leq z) = P(F(Y) \leq z) = 0$ for all $z \leq 0$

& $P(Z \leq z) = P(F(Y) \leq z) = 1$ for all $z \geq 1$

For $0 < z < 1$, we have

$$P(Z \leq z) = P(F(Y) \leq z) = P(Y \leq F^{-1}(z))$$

$$= F(F^{-1}(z)) = z$$

Thus Z has Uniform $(0,1)$ CDF.