ASSIGNMENT

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1]

a) X = no. of heads in first 2 flips
Y = no. of tails in last 2 flips.

Since the events are equally probable, we can use counting method.

Following are the possible outcomes.

F1	F2	F3	X	Y
H	H	H	2	2
H	H	T	2	1
H	T	H	1	1
T	H	H	1	2
H	T	T	1	0
T	T	H	0	1
T	H	T	1	1
T	T	T	0	0

~~Size~~ Size of sample space = 8

$\therefore$ Probability of each outcome = $1/8$

$\therefore$ $p(x,y) = 1/8$ for $x \in \{0,1,2\}$, $y = \{0,1,2\}$.

$$E[X] = \sum_{x \in \{0,1,2\}} x \cdot p(x)$$

$$= 0 \cdot p(0) + 1 \cdot p(1) + 2 p(2)$$

$$= 0 \cdot (2/8) + 1 \cdot (4/8) + 2 \cdot (2/8)$$

$$= 4/8 + 4/8 = 1$$

$\therefore E[X] = 1$ — 1

Similiarly:

$$E[Y] = \sum_{y=\{0,1,2\}} y \cdot p(y)$$

$$= 0 \cdot p(0) + 1 \cdot p(1) + 2p(2)$$

$$= 0 \cdot (2/8) + 1 \cdot (4/8) + 2 \cdot (2/8)$$

$$= 4/8 + 4/8 = 1$$

$\therefore E[Y] = 1$ —— 2

Considering joint distribution in x & y, we know that:

$$E[g(X,Y)] = \sum_{x \in \{0,1,2\}} \sum_{y=\{0,1,2\}} g(x,y) \cdot p(x,y)$$

$\therefore$ ~~let~~ let $g(X,Y) = XY$

$$\therefore E[XY] = \sum_{x=0}^{2} \sum_{y=0}^{2} XY \cdot p(x,y)$$

$$= (0\times0)p(0,0) + (0\times1)p(0\times1) + (0\times2)p(0,2)$$
$$+ (1\times0)p(1,0) + (1\times1)p(1,1) + (1\times2)p(1\times2)$$
$$+ 2\times0\, p(2,0) + (2\times1)p(2,1) + (2\times2)p(2,2)$$

$$= 0 + 0 + 0 + 0 + 1(2/8) + 2(1/8)$$
$$+ 0 + 2(1/8) + 4(1/8)$$

$E[XY] = 10/8$ — 3

Since
$cov(X,Y) = E[XY] - E[X].E[Y]$

from eqns. 1 2 & 3

$cov(X,Y) = 10/8 - (1\times 1)$

$= \frac{10-8}{8} = \cancel{2/8}\ 2/8$

$\therefore \boxed{cov(X,Y) = 1/4 = 0.25}$

1.b) ~~x~~ ~~l~~ X - fair sided dice with face values $\{-5,-2,0,2,5\}$

$Y = X^2$

$\Omega_x = \{-5,-2,0,2,5\}$ $\quad\Omega_y = \{25,4,0,4,25\}$

Probability of each sample $= 1/5$

$E[X] = \sum_{x\in\Omega_x} x.p(x)$

$= -5.(1/5) + (-2)(1/5) + (0)(1/5) + 2(1/5) + 5(1/5)$

$\boxed{E[X] = 0}$ - 1

$E[Y] = \sum_{y\in\Omega_y} y\,p(y)$

$$= \frac{1}{5}(25) + \frac{1}{5}(4) + \frac{1}{5}(0) + \frac{1}{5}(4) + \frac{1}{5}(25)$$

$$E[Y] = \frac{58}{5} \quad - (2)$$

We know that for joint distribution in x & y.

$$E[g(x)] = \sum_{x \in \Omega_x} \sum_{y \in \Omega_y} g(x) \cdot p(x,y)$$

$$\therefore \cancel{let}\ E[XY] = \sum_{x \in \Omega_x} \sum_{y \in \Omega_y} xy\, p(x,y)$$

Since $Y = X^2$, we know that possible pairs are:
$\{-5, 25\}, \{-2, 4\}, \{0, 0\}, \{2, 4\}, \{5, 25\}$

$$\therefore E[XY] = (-5 \times 25) \cdot p(-5, 25) + (-2 \times 4) p(-2, 4) + (0 \times 0)\, p(0,0) + (2 \times 4)\, p(2,4) + (5 \times 25)\, p(5, 25)$$

$$= -125(1/5) + (-8) 1/5 + 0 + 8(1/5) + 125(1/5)$$

↳ Since each event has probability = 1/5

$$E[XY] = 0 \quad - (3)$$

$$cov(X,Y) = E[XY] - E[X] \cdot E[Y]$$

from eqn. (1), (2) & (3)

$$cov(X,Y) = 0 - 0 \times 58/5$$

$$\boxed{cov(X,Y) = 0}$$

1c) Zero covariance doesnot imply that the RV's are independent.

$$cov(X,Y) = E[XY] - E[X].E[Y]. \rightarrow (i)$$

As seen in part 1b):

$Y = X^2$, which clearly shows that X and Y are not independent.

However $cov(X,Y) = 0$. — From 1b)

While calculating XY, it may happen that the ~~negative~~ products of ~~XY~~ different values of X and Y are such that the sum of negative products equal the sum of positive products, hence making $E[XY] = 0$.

If either of $E[X]$ or $E[Y] = 0$,

$Cov(X,Y) = 0$. — From (i).

2. Given, $X \sim RV(\mu, \sigma^2)$

and, X is non-negative

thus, $x \geq 0$

Also, $t > 0$, where $t \in \mathbb{R}$

(a) To prove - $E[X] \geq \int_t^\infty x f(x)\,dx$

Sol^n:- $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$ [By definition]

Since, $\Omega_X \geq 0$, thus -

$$\Rightarrow E[X] = \int_0^\infty x f(x)\,dx \quad \text{--- (1)}$$

Since, $t > 0$, thus eq^n (1) can also be written as -

$$\Rightarrow E[X] = \int_0^t x f(x)\,dx + \int_t^\infty x f(x)\,dx \quad \text{--- (2)}$$

Since, $x \geq 0$, $t \geq 0$, and by definition of pdf $f(x) \geq 0$

thus, $\int_0^t x f(x)\,dx$ will also be greater than equal to zero

So, if we remove a positive value from R.H.S. of eq^n (2), then, it will be smaller than the R.H.S. → Removing $\int_0^t x f(x)\,dx$ from (2) -

$$\Rightarrow \boxed{E[X] \geq \int_t^\infty x f(x)\,dx} \quad \text{--- (3)}$$

2. b) In eq^n 3 -

$$E[X] \geq \int_t^\infty x f(x)\, dx$$

The lowest value that 'x' can take in this eq^n is 't', so if we remove 'x' from the integral and replace it with its minimum value i.e. 't', then the inequality should still hold, thus -

$$E[X] \geq t \int_t^\infty f(x)\, dx$$

$$\Rightarrow \frac{E[X]}{t} \geq \int_t^\infty f(x)\, dx \quad — 4$$

by definition of Pdf → ~~Pr~~

$$\int_t^\infty f(x)\, dx = Pr[X > t] \quad — 5$$

Substituting eq^n 5 in 4 -

$$\Rightarrow \boxed{\frac{E[X]}{t} \geq Pr[X > t]} \quad — 6$$

2. (c) To prove - $Pr(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$

Solⁿ :- from eqⁿ (6) - (2b)

$$\frac{E[X]}{t} \geq Pr(X > t) \quad -(6) \quad \left\{ \begin{array}{l} \text{Given, } X \geq 0 \\ t > 0 \end{array} \right\}$$

Let, $t = t^2$, then this eqⁿ will still hold as $t^2 > 0$

Re-writing eqⁿ (6), with $t = t^2$ -

$$\frac{E[X]}{t^2} \geq Pr(X > t^2) \quad -(7)$$

Let Y be another RV, such that -

$$Y = (X-\mu)^2$$

$\because$ Y is a functⁿ of X & a square, so it will always be (+) ve

Thus, eqⁿ (7) will also hold for RV - 'Y' -

$$\Rightarrow \frac{E[Y]}{t^2} \geq Pr(Y > t^2) \quad -(8)$$

From, definition of var (σ^2) -

$$var(x) = \sigma^2 = E[(x - E(x))^2]$$

$$\Rightarrow \sigma^2 = E[(x-\mu)^2]$$

$\because$ we assumed, $Y = (x-\mu)^2$

thus, $\boxed{\sigma^2 = E[Y]}$

Substituting $E[Y]$ in eq^n 8 -

$$\Rightarrow \frac{\sigma^2}{t^2} \geq Pr(Y > t^2) \quad - 9$$

Substituting $Y = (x-\mu)^2$ in 9 -

$$\Rightarrow \frac{\sigma^2}{t^2} \geq Pr((x-\mu)^2 \geq t^2)$$

taking sq. root inside probability -

$$\Rightarrow \boxed{\frac{\sigma^2}{t^2} \geq Pr[|x-\mu| \geq t]}$$

Hence, proved

Q.3) a) Given,

$X_1, X_2, \ldots, X_k$ be k independent exponential RV.

$$f_{X_i}(x) = \lambda_i e^{-\lambda_i x}, \quad x \geq 0, \forall i \in \{1, 2, \ldots k\}$$

$$Z = \min\{X_1, X_2, \ldots X_k\}$$

i) pdf of Z ⇒

To find the pdf, we compute the cdf first and then take derivative of it to calculate pdf.

$$\frac{d}{d\alpha} F_Z(\alpha) = f_Z(z).$$

Now,

$$F_Z(\alpha) = P_r(Z \leq \alpha)$$

$$= P_r(\min(X_1, X_2, \ldots X_k) \leq \alpha)$$

$$= 1 - P_r(\min(X_1, X_2, \ldots X_k) > \alpha)$$

∵ $\min(X_1, X_2, \ldots X_k)$ should be greater α, every X_i should be $> \alpha$. ~~and~~

$$= 1 - P_r(X_1 > \alpha \cap X_2 > \alpha \ldots \cap X_k > \alpha)$$

∵ X_i are independent

$$= 1 - [P_r(X_1 > \alpha) \times P_r(X_2 > \alpha) \ldots P_r(X_k > \alpha)].$$

$$= 1 - [(1 - P_r(X_1 \leq \alpha)) \times (1 - P_r(X_2 \leq \alpha)) \times \dots \times (1 - P_r(X_k \leq \alpha))]$$

$$F_Z(\alpha) = 1 - [(1 - F_{X_1}(\alpha)) \times (1 - F_{X_2}(\alpha)) \times \dots (1 - F_{X_k}(\alpha))]$$

— Eq 1.

Let's find $F_{X_1}(\alpha)$.

$$F_{X_1}(\alpha) = P_r(X_1 \leq \alpha)$$

$$= \int_0^\alpha \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \lambda_1 \int_0^\alpha e^{-\lambda_1 x}\, dx$$

$$= \lambda_1 \times \left[\frac{e^{-\lambda_1 x}}{-\lambda_1}\right]_0^\alpha$$

$$= \frac{\lambda_1}{-\lambda_1} \times [e^{-\lambda_1 \alpha} - 1]$$

$$= -[e^{-\lambda_1 \alpha} - 1]$$

$$= (1 - e^{-\lambda_1 \alpha})$$

$$1 - F_{X_1}(\alpha) = e^{-\lambda_1 \alpha} \rightarrow \text{Eq } 2.$$

Substituting Eq 2 in Eq 1 for $X_1, X_2, \dots X_k$.
i.e. $\forall i \in \{1, 2, \dots k\}$.

$$F_Z(\alpha) = 1 - [e^{-\lambda_1 \alpha} \times e^{-\lambda_2 \alpha} \times e^{-\lambda_3 \alpha} \dots \times e^{-\lambda_k \alpha}]$$

$$F_Z(\alpha) = 1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3 + \dots \lambda_k)\alpha} \quad \text{— Eq } 3.$$

Taking derivative of $F_2(\alpha)$ from Eq(3).

$$\frac{d}{d\alpha} F_2(\alpha) = \frac{d}{d\alpha}\left(1 - e^{-(\lambda_1 + \dots + \lambda_k)\alpha}\right)$$

$$= 0 + (\lambda_1 + \dots + \lambda_k)\, e^{-(\lambda_1 + \lambda_2 + \dots \lambda_k)\alpha}$$

$$\therefore f_2(\alpha) = (\lambda_1 + \lambda_2 + \dots \lambda_k)\, e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k)\alpha}$$

Replacing α with x.

pdf → $$\boxed{f_2(x) = (\lambda_1 + \lambda_2 + \dots \lambda_k)\, e^{-(\lambda_1 + \lambda_2 + \dots \lambda_k)x}.} \quad \text{--- } (4)$$

So, $Z = \min(X_1, X_2, \dots, X_k)$ is itself an exponential distribution.

(iii) Let $(\lambda_1 + \lambda_2 + \dots \lambda_k)$ be a constant c --- (5)

$$\therefore E[Z] = \int_0^\infty x \times c e^{-cx}\, dx.$$

By Integration by Parts, we have

$$\int f\, dg = fg - \int g\, df.$$

Here, $f = x$, $dg = c e^{-cx} dx$.

$$\therefore g = -e^{-cx},$$

$$\therefore E[Z] = x \int_0^\infty c e^{-cx}\, dx - \int_0^\infty -e^{-cx}\, dx$$

$$= x\cdot(-e^{-cx})\Big|_0^\infty + \frac{e^{-cx}}{-c}\Big|_0^\infty$$

$$= 0 + \left(\frac{-1}{c}\right)[0-1]$$

$$= \frac{1}{c}.$$

Substituting the value of c from Eq ⑤,

$$\therefore \boxed{E[Z] = \frac{1}{(\lambda_1+\lambda_2+\dots\lambda_k)}}$$

iii $E[X^2] = \int_0^\infty x^2 c e^{-cx}\, dx.$

Here, $f = x^2$, $dg = ce^{-cx}\,dx$, $g = -e^{-cx}$

From Integration by Parts,

$$= x^2 \times (-e^{-cx})\Big|_0^\infty - \int_0^\infty -e^{-cx} \times 2x\, dx.$$

$$= 0 + 2\int_0^\infty x e^{-cx}\,dx$$

$$= 2\left[x\int e^{-cx}dx - \int 1 \times \frac{1}{-c} \times e^{-cx}dx \right]_0^\infty$$

$$= 2\left[\frac{x}{-c}e^{-cx} + \frac{1}{c}\int e^{-cx}dx \right]_0^\infty$$

$$= 2\left[\frac{-x}{c}e^{-cx}\Big|_0^\infty - \frac{1}{c^2}\times\left[e^{-cx}\right]_0^\infty \right]$$

$$= 2\left[0 - \frac{1}{c^2}\left[0-1\right] \right]$$

$$= \frac{2}{c^2}.$$

$$\therefore \quad Var(Z) = E[Z^2] - (E[Z])^2$$

$$= \frac{2}{c^2} - \frac{1}{c^2}$$

$$= \frac{1}{c^2}$$

$$\boxed{Var(Z) = \frac{1}{(\lambda_1 + \lambda_2 + \cdots + \lambda_k)^2}} - \text{From } (5)$$

3. (b) $X, Y \sim RVs$

Joint distributⁿ -

$$f_{XY}(x, y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Given, $Z = XY$, | pdf $(z) = f_Z(z) = ?$

Solⁿ - c.d.f. of 'Z' -

$$F_Z(\alpha) = Pr(Z \le \alpha)$$
$$= 1 - Pr(Z > \alpha)$$
$$= 1 - \underbrace{Pr(XY > \alpha)} \quad \{\because Z = XY\}$$

for, $Pr(XY \geq \alpha) \rightarrow$ ~~Lim~~

Let's find the limits of 'y' -

$\Rightarrow \quad 0 \le x \le y \le 1$

$\Rightarrow$ for $(XY \geq \alpha)$ to be true -

$$XY \geq \alpha \quad \text{— (1)}$$

$$\cancel{\boxed{Y \geq \alpha / X}} \quad \text{~~→ lower limit of Y~~}$$

and, $Y \geq X$ — (2) {Given}

Substituting, maximum value of 'x' from eqⁿ ② to ① -

$y \cdot y \geq \alpha$

$\Rightarrow y^2 \geq \alpha$

$\Rightarrow y \geq \sqrt{\alpha}$ — ③ ← lower limit of y

And, as given, max value of 'y' is '1' — ④

For limits of 'x' -

$xy \geq \alpha$

$x \geq \alpha/y$ — ⑤ ← lower limit of 'x'

And, as given, max value of 'x' is 'y' — ⑥

Thus, substituting limits of (x,y) from eqⁿ ③, ④, ⑤, ⑥ in C.D.F of z

$$F_z(\alpha) = 1 - \Pr(xy > \alpha)$$

$$= 1 - \int_{\sqrt{\alpha}}^{1} \int_{\alpha/y}^{y} f(x,y)\, dx\, dy$$

$$= 1 - \int_{\sqrt{\alpha}}^{1} \int_{\alpha/y}^{y} 2\, dx\, dy$$

$$\Rightarrow F_Z(\alpha) = 1 - 2\int_{\sqrt{\alpha}}^{1}\int_{\alpha/y}^{y} dx\, dy$$

$$= 1 - 2\int_{\sqrt{\alpha}}^{1} [x]_{\alpha/y}^{y}\, dy$$

$$= 1 - 2\int_{\sqrt{\alpha}}^{1} \left(y - \frac{\alpha}{y}\right) dy$$

$$= 1 - 2\left[\frac{y^2}{2} - \alpha \ln(y)\right]_{\sqrt{\alpha}}^{1}$$

$$= 1 - 2\left(\frac{1 - (\sqrt{\alpha})^2}{2} - \left(\alpha \cancelto{0}{\ln(1)} - \alpha \ln(\sqrt{\alpha})\right)\right)$$

$$= 1 - 2\left(\frac{1-\alpha}{2} + \alpha \ln(\sqrt{\alpha})\right)$$

$$= 1 - \left[1 - \alpha + \cancel{2}\alpha \frac{\ln(\alpha)}{\cancel{2}}\right]$$

$$\boxed{F_Z(\alpha) = \alpha - \alpha \ln(\alpha)}$$

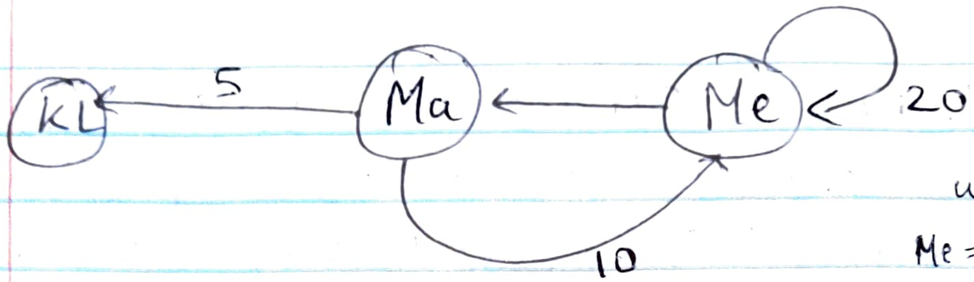
Pdf, $f_Z(\alpha) = \dfrac{dF_Z(\alpha)}{d\alpha} = 1 - \left[\ln(\alpha) + \dfrac{\cancel{\alpha}}{\cancel{\alpha}}\right]$

$$= -\ln(\alpha)$$

Thus, pdf of $Z \Rightarrow \boxed{f_Z(z) = -\ln(z)}$ $\forall\ z \in (0,1)$

4)

a)



where
Me = Meereen
Ma = Mantarys
KL = Kings Landing.

There are ~~choices~~ 2 decisions that need to be made, one at Me & the other at Ma

let I_1 = step 1 (from Me) & I_2 = step 2 (from Ma)

$$\therefore I_1 = \begin{cases} \text{Ma} & \text{w.p. } 1/2 \\ \text{Me} & \text{w.p. } 1/2 \end{cases} \qquad I_2 = \begin{cases} \text{KL} & \text{wp } 1/2 \\ \text{Me} & \text{wp } 1/2 \end{cases}$$

(Destination above each case list)

~~From~~ let X = number of days to reach KL.
From law of total expectation ∴

$$E[A] = \sum_{x \in \Omega_x} E[A \mid X = x] \cdot p_X(x)$$

$$E[X] = E[X \mid I_1 = Me] \cdot P(I_1 = Me) + E[X \mid I_1 = Ma] \cdot p(I_1 = Ma) \quad -(1)$$

where $E[X \mid I_1 = Me] = E[\#\text{ of days to reach KL} \mid I_1 = Me]$
$E[X \mid I_1 = Ma] = E[\#\text{ of days to reach KL} \mid I_1 = Ma]$

~~Further:~~
$\cancel{E[X \mid I_1 = Ma] = E[X \mid I_2 =}$

$E[X \mid I_1 = Me] \cdot p^{(I_1 = Me)} = \cancel{20 + E[X]}$ $E[X + 20]\, p(I_1 = Me)$

as she returned to the same spot after 20 days and she doesn't remember her choices, it would be the same as starting over again after 20 days

~~by loe~~ $= \frac{1}{2}(E[20] + E[X])$ — by linearity of ~~exp~~ E

$E[X \mid I_1 = Me] \times p(I_1 = Me) = (20 + E[X])\frac{1}{2}$ — 2

~~From~~ 1. Further

$E[X \mid I_1 = Ma] = E[X \mid I_2 = KL] \cdot p(I_2 = KL) + E[X \mid I_2 = Me] \cdot p(I_2 = Me)$ — 5

— by law of total expectation

$E[X \mid I_2 = KL] \cdot p(I_2 = KL) = 5 \times 1/2$

— as if $I_2 = KL$, she will reach in 5 days

$= 5/2$ — 3

$E[X \mid I_2 = Me] \cdot p(I_2 = Me) = \frac{1}{2}(E[X + 10])$

as she returned to Me after 10 days and she doesnot remember her choices, it would be the same as starting over again after 10 days

$E[X \mid I_2 = Me] \cdot p(I_2 = Me) \overset{loe}{=} \frac{1}{2}(10 + E[X])$ — 4 from (3), (4) and (5)

$$E[X \mid I_1 = Ma] = E[X \mid I_2 = KL] \cdot p(I_2 = KL) + E[X \mid I_2 = Me] \cdot p(I_2 = Me)$$

$$= 5/2 + \frac{1}{2}(10 + E[X]) \quad -(6)$$

Substituting (2) & (6) in (1).

$$E[X] = E[X \mid I_1 = Me] \cdot p(I_1 = Me) + E[X \mid I_1 = Ma] \cdot p(I_1 = Ma)$$

$$E[X] = \frac{1}{2}(20 + E[X]) + \frac{1}{2}\left(5/2 + \frac{1}{2}(10 + E[X])\right)$$

$$= \frac{40 + 2E[X]}{4} + \frac{\cancel{5/2}\ 5 + 10 + E[X]}{4}$$

$$4E[X] = 55 + 3E[X]$$

$$\therefore \boxed{E[X] = 55}$$

4b) $Var[X] = E[X^2] - E[X]^2$ — by def^n.

From law of total expectation

$$E[A^2] = \sum_{x=\Omega_x} E[A^2 | X=x] \cdot p_X(x)$$

~~So~~ We ~~assume~~ know that:

I_1 = Step 1 from Me Dest

I_2 = Step 2 from Ma Dest

$$I_1 = \begin{cases} Ma & wp\ 1/2 \\ Me & wp\ 1/2 \end{cases} \qquad I_2 = \begin{cases} KL & wp\ 1/2 \\ Me & wp\ 1/2 \end{cases}$$

$$\therefore E[X^2] = E[X^2 | I_1 = Me] \cdot P(I_1 = Me) + E[X^2 | I_1 = Ma] \cdot P(I_1 = Ma) \quad — ①$$

where $E[X^2 | I_1 = Me] = E[(\#\text{ of days to reach KL})^2 | I_1 = Me]$

$E[X^2 | I_1 = Ma] = E[(\#\text{ of days to reach KL})^2 | I_2 = Ma]$

$$E[X^2 | I_1 = Me] \cdot P(I_1 = Me) = E[(X+20)^2] \times 1/2$$

↳ as she returns to Me after 20 days, and doesnot remember her choices,

$$\therefore \overset{\text{loe}}{=} \frac{1}{2}\left(E[X^2] + E[40X] + E[400]\right)$$

$$= \frac{1}{2}\left(E[x^2] + 40E[x] + 400\right) \quad (2)$$

Since
$E[const] = const$
$E[const \times X] = const\, E[x]$

Further:

$$E[x^2 / I_1 = Ma] = E[x^2 | I_2 = KL] \cdot p(I_2 = KL) + E[x^2 | I_2 = Me] \cdot P(I_2 = Me)$$

↳ by law of total expectation (5)

$$E[x^2 | I_2 = KL] \cdot p(I_2 = KL) = E[5^2] \cdot 1/2$$
$$= \frac{25}{2} \quad (3)$$

as it takes 5 days to reach KL from Ma.

$$E[x^2 | I_2 = Me] \cdot p(I_2 = Me) = \frac{1}{2}\left(E[(x+10)^2]\right)$$

as she returned to Meereen after 10 days and she doesnot remember her choices, it would be the same as starting again after 10 days.

$$E[x^2 | I_2 = Me] \cdot p(I_2 = Me) \stackrel{loe}{=} 1/2\left(E[x^2] + E[20x] + E[100]\right)$$

$$= \frac{1}{2}\left(E[x^2] + 20E[x] + 100\right) \quad (4)$$

Since
$E(const) = const$
$E(const \times X) = const\, E[x]$

Substituting (a) & (b) in 5.

$$E[x^2 | I_1 = Ma] = E[x^2 / I_2 = Me] . P(I_2 = Me) + E[x^2 / I_2 = KL) P(I_2 = KL)$$

$$= \cancel{\frac{25}{2}} + \frac{1}{2}(E[x^2] + 20E[x] + 100) + \frac{25}{2} \rightarrow (6)$$

Substituting (6) & (2) in (1)

$$E[x^2] = \frac{1}{2}\left(\frac{1}{2}(E[x^2] + 20E[x] + 100) + \frac{25}{2}\right) + \frac{1}{2}(E[x^2] + 40E[x] + 400)$$

$$= \frac{1}{4}(E[x^2] + 20E[x] + 100 + 25) + \frac{1}{4}(2E[x^2] + 80E[x] + 800)$$

$$4E[x^2] = 3E[x^2] + 100E[x] + 925$$

$$E[x^2] = 100E[x] + 925$$

from part a, we know that $E(x) = 55$

$$\therefore E[x^2] = 100 \times 55 + 925$$

By defn of variance

$$Var(x) = 100 \times 55 + 925 - (55)^2$$

$$\boxed{Var(x) = 3400}$$

Q5 a] The given probability is represented as

$P[X_{i+1} \mid X_i \, X_{i-1}]$

weather tomorrow — weather today — weather yesterday

The given probabilities are :

$Pr[C \mid CC] = 0.9$, $Pr[C \mid CS] = 0.8$, $Pr[C \mid SC] = 0.5$,

$Pr[C \mid SS] = 0.1$

This can be represented in the form of four states CC, CS, SC, SS as follows

$CC \xrightarrow{0.9} CC \qquad CC \xrightarrow{0.1} SC$

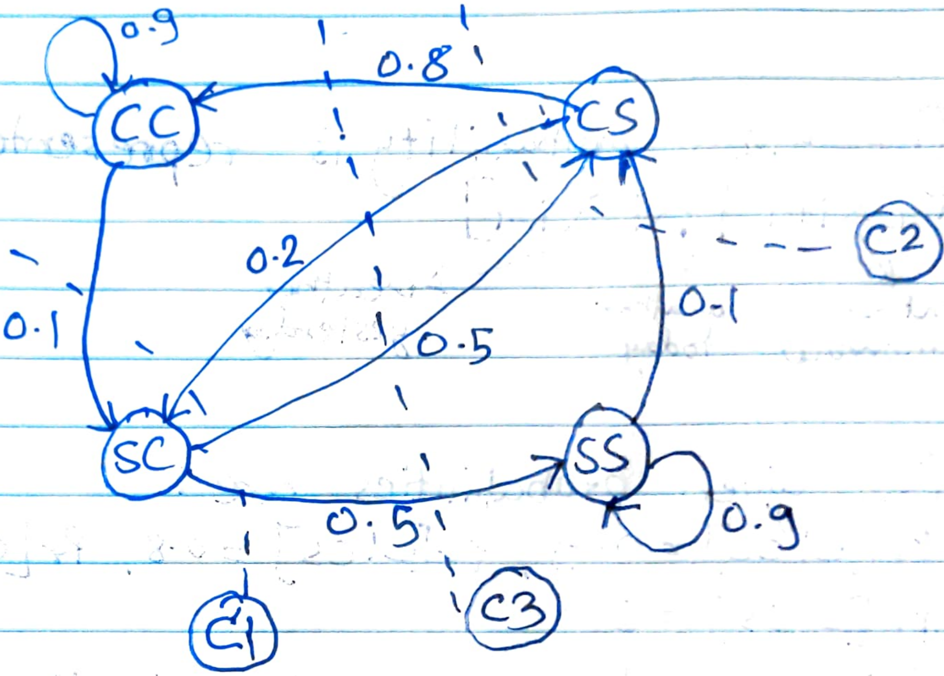
Today Yesterday Tomorrow Today

$CS \xrightarrow{0.8} CC \qquad CS \xrightarrow{0.2} SC$

$SC \xrightarrow{0.5} CS \qquad SC \xrightarrow{0.5} SS$

$SS \xrightarrow{0.1} CS \qquad SS \xrightarrow{0.9} SS$

From the given states the Markov chain will be :-



From the above ~~Markov~~ Markov chain diagram we have made 3 cuts which are C1, C2, C3.

Let π_{cc}, π_{cs}, π_{sc}, π_{ss} be the steady state probability.

Now by validity test.

$$\pi_{cc} + \pi_{cs} + \pi_{sc} + \pi_{ss} = 1 \quad \ldots \text{(i)}$$

Applying local ~~local~~ balancing on cut C1

$$0.5\pi_{sc} + 0.5\pi_{sc} = 0.1\pi_{cc} + 0.2\pi_{cs}$$

$$\pi_{sc} = 0.1\pi_{cc} + 0.2\pi_{cs} \quad \ldots \text{(ii)}$$

Applying local ~~local~~ balancing on cut C2

$$0.2\pi_{cs} + 0.8\pi_{cs} = 0.5\pi_{sc} + 0.1\pi_{ss}$$

$$\pi_{cs} = 0.5\pi_{sc} + 0.1\pi_{ss} \quad \ldots \quad (iii)$$

Applying ~~local~~ local balancing on cut C3

$$0.8\pi_{cs} + 0.2\pi_{cs} = 0.5\pi_{sc} + 0.5\pi_{sc}$$

$$\pi_{cs} = \pi_{sc} \quad \ldots \quad (iv)$$

Now from eq. (iii) & (iv)

$$\pi_{cs} = 0.5\pi_{sc} + 0.1\pi_{ss}$$

ie $\pi_{cs} = 0.5\pi_{cs} + 0.1\pi_{ss}$

$\therefore$ $0.5\pi_{cs} = 0.1\pi_{ss}$

$$5\pi_{cs} = \pi_{ss} \quad \ldots \quad (v)$$

Now from eq. (ii) & (iv)

$$\pi_{sc} = 0.1\pi_{cc} + 0.2\pi_{cs}$$

ie $\pi_{cs} = 0.1\pi_{cc} + 0.2\pi_{cs}$

$$0.8\pi_{cs} = 0.1\pi_{cc}$$

$$8\pi_{cs} = \pi_{cc} \quad \ldots \quad (vi)$$

By substituting values from (iv), (v) & (vi) in (i)

$$\pi_{cc} + \pi_{cs} + \pi_{sc} + \pi_{ss} = 1$$

ie. $8\pi_{cs} + 5\pi_{cs} + \pi_{cs} + \pi_{cs} = 1$

$$15\pi_{cs} = 1$$

$$\pi_{cs} = \frac{1}{15} \quad \ldots \quad (vii)$$

From (iv) & (vii)

$\pi_{cs} = \pi_{sc} = \frac{1}{15}$

From (v) & (vii)

$5\pi_{cs} = \pi_{ss} = 5 \times \frac{1}{15} = \frac{1}{3}$

From (vi) & (vii)

$8\pi_{cs} = \pi_{cc} = 8 \times \frac{1}{15} = \frac{8}{15}$

Ans $\pi_{cs} = \frac{1}{15}$, $\pi_{sc} = \frac{1}{15}$, $\pi_{cc} = \frac{8}{15}$, $\pi_{ss} = \frac{1}{3}$

Q5 b] Now it's given that the system has reached steady state.

$\therefore$ From question (i)

$\pi_{cs} = \frac{1}{15}$, $\pi_{sc} = \frac{1}{15}$, $\pi_{cc} = \frac{8}{15}$, $\pi_{ss} = \frac{1}{3}$

$\therefore$ By the property of global balance

Probability that it will be snowy after 3 day

= ~~Proba~~ steady state probability of snowing

$= \pi_{sc} + \pi_{ss}$

snowy today, clear yesterday; snowy today, clear yesterday

$$= \frac{1}{15} + \frac{1}{3}$$

$$= \frac{6}{15} = \frac{2}{5}$$

Q5 c] For the given question the transition matrix will be as follows

	CC	CS	SC	SS
CC	0.9	0	0.1	0
CS	0.8	0	0.2	0
SC	0	0.5	0	0.5
SS	0	0.1	0	0.9

From submitted program,
Steady State: After ~~873~~ 1761 iteration
$\Rightarrow$ [0.53333333, 0.06666667, 0.06666667, 0.3333333]

6. For $X = (x_1, x_2 ---, x_k)$ to be Multi-variate Normal distribution, -
$t_1 x_1 + t_2 x_2 + --- + t_k x_k$ should be a Normal distribution

ⓐ Given, X is Normal distⁿ -
thus,

$$t_1 x_1 + t_2 x_2 + --- + t_k x_k \sim Nor(\mu, \sigma^2) \quad \text{—(1)}$$

Let all t_i are zero, except t_j, then -

since, eqⁿ (1) hold for all real values of t_i

$$\Rightarrow t_j x_j \sim Nor(\mu, \sigma^2)$$

$$\Rightarrow x_j \sim \frac{1}{t_i} Nor(\mu, \sigma^2)$$

Using transformation property of Normal distribuⁿ

$$\Rightarrow \boxed{x_j \sim Nor\left(\frac{\mu}{t_i}, \frac{\sigma^2}{t_i^2}\right)}$$

Thus, x_j is also a Normal distribution.

6. (b) $\Rightarrow X \sim Nor(0,1)$

$$S = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$$

$\Rightarrow Y = SX$

$$= \begin{cases} Nor(0,1) \\ -\cancel{Nor(0,1)} \end{cases}$$

$\Rightarrow$ In order to show that (X, Y) is not a multi-variate Normal, we need to proof that for at-least one linear combⁿ of X & Y for which it is not normal.

let, $f(x,y) = t_1 X + t_2 Y$ be linear combⁿ of (X, Y)

$f(x, sx) \Rightarrow t_1 X + t_2 (SX)$ $\{$Given, $Y = SX\}$

$\Rightarrow$ let, $t_1 = t_2 = t$ -

$f(x,sx) \Rightarrow tX + t(SX)$

$f(x,sx) \Rightarrow = tX(1+S)$

$$f(x,sx) = \begin{cases} 2tX & S = 1 \\ 0 & S = -1 \end{cases}$$

Not Normal distributⁿ

Since, linear combination of $(X, Y) \rightarrow f(x, y)$ can be zero, when $S = -1$ with $1/2$ probability and when their linear multiples (t_i) are equal thus, we can say that (X, Y) are not Multi-variate Normal distribution.

6. (c) $Z, W \overset{iid}{\sim} Nor(0, 1)$

$(Z, W) \overset{?}{=} Nor()$ & $(Z+2W, 3Z+5W) \overset{?}{=}$

$\Rightarrow$ for (Z, W) to be multivariate Normal Distn.

$$t_1 Z + t_2 W \quad \text{--- (1)}$$

$$\Rightarrow t_1 Nor(0,1) + t_2 Nor(0,1)$$

{Since, Z, W are iid, thus their μ, σ^2 will be same}

Using transformatn property of Normal Distributn.

Eqn (1) –

$$t_1 Nor(0,1) + t_2 Nor(0,1)$$

$$\Rightarrow Nor(0, t_1^2) + Nor(0, t_2^2) \quad \text{--- (2)}$$

Since, Z & W are independent, thus, using weighted sum of Normal distributn on eqn (2) –

eqn (2)

$$\Rightarrow Nor(0+0, t_1^2 + t_2^2) \Rightarrow \text{Normal distn.}$$

thus, $\boxed{(Z, W) \sim Nor(0, t_1^2 + t_2^2)}$, thus Multivariate Normal distn.

6. (d) For, $(Z+2W,\ 3Z+5W)$

linear combⁿ -

$\Rightarrow t_1(Z+2W) + t_2(3Z+5W)$

$\Rightarrow Z(t_1+3t_2) + W(2t_1+5t_2)$

$\Rightarrow \cancel{t_1(Nor(0,1) + 2Nor(0,1)) + t_2(3Nor(0,1) + 5Nor(0,1))}$

~~Using linear transformatⁿ~~ -

$\Rightarrow (t_1+3t_2)\,Nor(0,1) + (2t_1+5t_2)\,Nor(0,1)$ — ①

Using linear transformatⁿ of Normal distributⁿ eqⁿ ① -

$\Rightarrow Nor(0,(t_1+3t_2)^2) + Nor(0,(2t_1+5t_2)^2)$ — ②

Now, using weighted sum of independent Normal as Z, W are independent -

eqⁿ ② -

$\Rightarrow Nor(0+0,\ (t_1+3t_2)^2 + (2t_1+5t_2)^2)$

$\Rightarrow Nor(0,\ 5t_1^2 + 34t_2^2 + 26t_1t_2)$

Thus, linear combⁿ of $(Z+2W,\ 3Z+5W)$ is a Normal distⁿ with ~~mean = 0~~ ~~variance =~~

Thus, $(Z+2W,\ 3Z+5W)$ is Multivariate Normal distⁿ

6.(d) $X = (X_1 \cdots X_n) \sim$ Multivariate ND
$Y = (Y_1 \cdots Y_n) \sim$ Multivariate ND

$\Rightarrow X \perp Y$

$\Rightarrow X = t_1 X_1 + t_2 X_2 + \cdots + t_n X_n \sim Nor(\mu_X, \sigma_X^2)$
(Since multivariate)

$\Rightarrow Y = r_1 Y_1 + r_2 Y_2 + \cdots + r_n Y_n \sim Nor(\mu_Y, \sigma_Y^2)$
(Since Multivariate)

Thus, concatenated vector $W = (X_1 \cdots X_n Y_1 \cdots Y_n)$

$$W = \underbrace{t_1 X_1 + \cdots t_n X_n}_{X} + \underbrace{r_1 Y_1 + \cdots r_n Y_n}_{Y}.$$

$$= X + Y$$
$$= Nor(\mu_X, \sigma_X^2) + Nor(\mu_Y, \sigma_Y^2)$$

Since, $X \perp Y$ thus, using weighted sum of independent Normal property

$$\Rightarrow \boxed{W = Nor(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)}$$

Thus, W is also Multivariate Normal distributn.

6.

(e) Fact-1 :- Uncorrelated implies independence

$X = (X_1, X_2) \Rightarrow$ Multi-variate ND

if every component of X_1 & X_2 are uncorrelated then, X_1 and X_2 will be independent.

Fact-2 -

$$\text{cov}(aX+bY, cW+dV) = ac\,\text{cov}(X,W) + ad\,\text{cov}(X,V) + bc\,\text{cov}(Y,W) + bd\,\text{cov}(Y,V)$$

Given $\rightarrow$ $X, Y \overset{iid}{\sim} Nor(0,1)$

To prove $\rightarrow$ $(X+Y, X-Y)$ is Multivariate

$\Rightarrow$ Since, $X \perp Y$ (Given)

$$\text{cov}(XY) = \text{cov}(YX) = 0$$

$\Rightarrow$ Using Fact-2, to calculate $\text{Cov}(X+Y, X-Y)$ -

$$\Rightarrow \text{Cov}(X+Y, X-Y) = \text{cov}(XX) - \text{cov}(XY) + \text{cov}(Y,X) - \text{cov}(Y,Y) \quad \text{--- (1)}$$

~~= cov(X Y~~

From, ~~cov(x property~~, ~~Fact-2~~ $\left[\begin{array}{l} \text{Cov}(XX) = \text{Var}(X) \\ \text{cov}(YY) = \text{var}(Y) \end{array}\right.$

$\Rightarrow \text{cov}(X,Y) = E[XY] - E(X)E(Y)$ {from Assignment #2 que(1)}

$\Rightarrow \text{cov}(X,X) = E[X^2] - (E[X])^2$

$= \text{var}(X)$

$\Rightarrow \text{cov}(Y,Y) = \text{var}[Y]$

Substituting in eq^n (1) -

$\Rightarrow \text{cov}(X+Y, X-Y) = \text{var}[X] - \text{var}[Y]$

Since, X & Y are ~~identical~~ iid, thus $\text{var}[X] = \text{var}[Y]$

$\Rightarrow \text{cov}[X+Y, X-Y] = 0$

Correlation is given by -

$$\text{corr}[X+Y, X-Y] = \frac{\text{cov}[X+Y, X-Y]}{\sqrt{\text{var}(X+Y)\,\text{var}(X-Y)}}$$

Since, $\text{cov}(X+Y, X-Y) = 0$, thus,

$$\boxed{\text{corr}(X+Y, X-Y) = 0}$$

Thus, $(X+Y)$ & $(X-Y)$ are uncorrelated,

Now, using Fact - 1, we can say that $(X+Y)$ and $(X-Y)$ will be independent.

$$(X+Y, X-Y) = t_1(X+Y) + t_2(X-Y) \quad - (3)$$

$\Rightarrow (X+Y) = Nor(0,1) + Nor(0,1)$

Using weighted sum of Normal as X & Y are iid -

$\Rightarrow (X+Y) = Nor(0,2)$

$\Rightarrow (X-Y) = Nor(0,1) - Nor(0,1)$

Using weighted sum of ND.

$= Nor(0,2)$

$\Rightarrow$ Substituting in eq^n (3) -

$= t_1 Nor(0,2) + t_2 Nor(0,2)$

$\xrightarrow[\text{property}]{\text{Transf}^n}$ $Nor(0, 2t_1^2) + Nor(0, 2t_2^2)$

Since, we have proved that $(X+Y)$ & $(X-Y)$ are independent

$(X+Y, X-Y) = Nor(0+0, 2t_1^2 + 2t_2^2)$

$$\boxed{(X+Y, X-Y) = Nor(0, 2(t_1^2 + t_2^2))}$$

Thus, Multi-variate Normal Distribut^n.

7a). Number of distinct types of pokemon $= n$.
let $T_1, T_2, T_3 \dots T_n$ be the n-distinct types of pokemon

$\therefore \Omega_T = \{T_1, T_2, \dots T_n\}$, where size$(\Omega_T) = n$.

let X = no. of days to capture atleast one pokemon of all n distinct types.

→ To catch the 1st type T_1, no. of days required $= 1$ — ①
as the first pokemon captured would be distinct.

→ To catch the 2nd type T_2, we can capture any pokemon except T_1 i.e $\Omega_T - \{T_1\}$

$\therefore$ probability of capturing 2nd distinct type $= \dfrac{\text{size}(\Omega_T - \{T_1\})}{\text{size}(\Omega_T)}$

$$= \frac{n-1}{n} \quad \text{— ②}$$

# of days required to capture the 2nd new type would be numbers of days to capture T_e, where $T_e \in (\Omega_T - \{T_1\})$

$\therefore$ We see that the number of days required to capture the 2nd distinct type follows a geometric RV with probability of success $= \dfrac{n-1}{n}$ — from ②

Similiarly, number of days required to capture 3rd distinct pokemon $\sim Geo\left(\frac{n-2}{n}\right)$

$\vdots$

Number of days required to capture n^{th} distinct type $\sim Geo\left(\frac{1}{n}\right)$

$$\therefore X = 1 + Geo\left(\frac{n-1}{n}\right) + Geo\left(\frac{n-2}{n}\right) + \cdots Geo\left(\frac{1}{n}\right) \quad (3)$$

$$\therefore E[X] = E\left[1 + Geo\left(\frac{n-1}{n}\right) + Geo\left(\frac{n-2}{n}\right) + \cdots Geo\left(\frac{1}{n}\right)\right]$$

By linearity of expectation :

~~Since there ar~~

$$E[X] = E[1] + E\left[Geo\left(\frac{n-1}{n}\right)\right] + E\left[Geo\left(\frac{n-2}{n}\right)\right] + \cdots E\left[Geo\left(1/n\right)\right]$$

$\because E[\text{constant}] = \text{constant}$

We know that $E[X] = 1/p$ for $X \sim Geo(p)$

$$E[X] = 1 + \frac{1}{\left(\frac{n-1}{n}\right)} + \frac{1}{\left(\frac{n-2}{n}\right)} + \cdots \frac{1}{1/n}$$

$$\boxed{E[X] = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots \frac{1}{n}}$$

b) $X = 1 + Geo\left(\frac{n-1}{n}\right) + Geo\left(\frac{n-2}{n}\right) + \dots Geo\left(\frac{1}{n}\right)$

↳ From ③

Since there are infinite pokemon of each type, capturing a pokemon of type T_i does not affect the probability of capturing a pokemon of type T_j.

∴ let x_i = no. of days required to capture i^{th} distinct pokemon

and x_j = no. of days required to capture j^{th} distinct pokemon

We know that $X_i \sim Geo$ and $X_j \sim Geo$.

$X_i = Geo\left(\frac{n-(i-1)}{n}\right)$ and $X_j = Geo\left(\frac{n-(j+1)}{n}\right)$

Since ~~Since~~ X_i and X_j are independent, we can use linearity of variance

$Var(X) = Var\left(1 + Geo\left(\frac{n-1}{n}\right) + Geo\left(\frac{n-2}{n}\right) + \dots Geo\left(1/n\right)\right)$

$= Var(1) + Var\left(Geo\left(\frac{n-1}{n}\right)\right) + Var\left(Geo\left(\frac{n-2}{n}\right)\right) \dots$

$+ Var\left(Geo\left(\frac{1}{n}\right)\right)$

$= 0 + Var\left(Geo\left(\frac{n-1}{n}\right)\right) + Var\left(Geo\left(\frac{n-2}{n}\right)\right) \dots$

$+ Var\left(Geo\left(1/n\right)\right)$

∵ since var[const] = constant.

We know that $Var(Geo(p)) = \frac{1-p}{p^2}$

$$\therefore Var(X) = \frac{1-\left(\frac{n-1}{n}\right)}{\left(\frac{n-1}{n}\right)^2} + \frac{1-\left(\frac{n-2}{n}\right)}{\left(\frac{n-2}{n}\right)^2} + \cdots\cdots + \frac{1-1/n}{(1/n)^2}$$

$$= \frac{n^2(n-n+1)}{n(n-1)^2} + \frac{2n^2}{n(n-2)^2} + \frac{3n^2}{(n)(n-3)^2} + \cdots n(n-1)$$

$$\boxed{Var(X) = \frac{n}{(n-1)^2} + \frac{2n}{(n-2)^2} + \frac{3n}{(n-3)^2} + \cdots (n)(n-1)}$$