3 a) $\nabla_x = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$ Given: x, a are column vectors of size $n \times 1$

Let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$\bar{x}^T \bar{a} = [x_1 \ x_2 \ \cdots \ x_n] \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2 + x_3 a_3 + \cdots + x_n a_n \quad \cdots \quad \mathrm{I}$$

$$\bar{a}^T \bar{x} = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n \quad \cdots \quad \mathrm{II}$$

$$\mathrm{I} = \mathrm{II}$$

$$\therefore \nabla_x(\mathrm{I}) = \nabla_x(\mathrm{II}) \quad \cdots \quad \mathrm{III}$$

$$\therefore \nabla_x(\bar{x}^T \bar{a}) = \nabla_x(\bar{a}^T \bar{x})$$

$$\nabla_x(\bar{x}^T \bar{a}) = \nabla_x(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n)$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1}(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n) \\ \vdots \\ \frac{\partial}{\partial x_n}(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n) \end{bmatrix}$$

Consider $\frac{\partial (x^T a)}{\partial x_i}$ where $1 \leq i \leq n$

$$= \frac{\partial}{\partial x_i}(x_1 a_1 + x_2 a_2 + \cdots + x_i a_i + \cdots + x_n a_n)$$

$$= (0 + 0 + \cdots + a_i + \cdots + 0)$$

$$= a_i$$

$$\therefore \frac{\partial}{\partial x_i}(x^T a) \quad \forall\, i : 1 \leq i \leq n = a_i$$

$$\therefore \nabla_x (x^T a) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \bar{a} \quad --- \quad \underline{\mathrm{IV}}$$

$$\therefore \nabla_x (x^T a) = \nabla_x (a^T x) \ [\text{by } \underline{\mathrm{III}}] = \bar{a} \ [\text{by } \underline{\mathrm{IV}}]$$

Hence Proved

b) To Prove : $\nabla_x (x^T A x) = (A + A^T) x$ $\qquad x : n \times 1$

$A : n \times n$

Let $x = [x_1 \ x_2 \ x_3 \ \cdots \ x_n]^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n \end{bmatrix}$$

$$\therefore x^T (Ax) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$$

$$\begin{aligned} = \{ & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots + a_{1n}x_1x_n \\ & + a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \cdots + a_{2n}x_2x_n \\ & + \vdots \\ & + a_{n1}x_1x_n + a_{n2}x_2x_n + \cdots + a_{nn}x_n^2 \end{aligned}$$

$$\therefore \nabla_x (x^T A x) = \begin{bmatrix} \frac{\partial}{\partial x_1}(x^T A x) \\ \frac{\partial}{\partial x_2}(x^T A x) \\ \vdots \\ \frac{\partial}{\partial x_n}(x^T A x) \end{bmatrix}$$

Consider $\frac{\partial (x^T A x)}{\partial x_i}$ where $1 \le i \le n$

$$\begin{aligned} \frac{\partial}{\partial x_i} (& a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1i}x_1x_i + \cdots + a_{1n}x_1x_n + \\ & + a_{21}x_1x_2 + a_{22}x_2^2 + \cdots + a_{2i}x_2x_i + \cdots + a_{2n}x_2x_n + \\ & + \cdots \\ & + a_{i1}x_1x_i + a_{i2}x_2x_i + \cdots + a_{ii}x_i^2 + \cdots a_{in}x_ix_n \\ & + \vdots \\ & + a_{n1}x_1x_n + a_{n2}x_2x_n + \cdots + a_{ni}x_ix_n + \cdots + a_{nn}x_n^2) \end{aligned}$$

$$\begin{aligned} = (& 0 + 0 + \cdots + a_{1i}x_1 + \cdots + 0) \\ & + 0 + 0 + \cdots + a_{2i}x_2 + \cdots + 0 \\ & + \cdots \\ & + a_{i1}x_1 + a_{i2}x_2 + \cdots + 2a_{ii}x_i + \cdots + 0 \} \end{aligned}$$

$+ \quad 0 + 0 + \cdots + a_{ni} x_n + \cdots + 0 \Big)$

Rearranging the terms

$$= x_1(a_{1i} + a_{i1}) + x_2(a_{i2} + a_{2i}) + \cdots + x_i(2a_{ii}) + \cdots + x_n(a_{in} + a_{ni})$$

$$\therefore \frac{\partial (x^T A x)}{\partial x_i} = \begin{bmatrix} a_{1i}+a_{i1} & a_{i2}+a_{2i} & \cdots & a_{ii}+a_{ii} & \cdots & a_{in}+a_{ni} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$\forall i : 1 \leq i \leq n$

$$\therefore \nabla_x (x^T A x) = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \cdots & & a_{1n}+a_{n1} \\ a_{12}+a_{21} & a_{22}+a_{22} & \cdots & & a_{2n}+a_{n2} \\ \vdots & & & & \\ a_{1i}+a_{i1} & a_{2i}+a_{i2} & \cdots & a_{ii}+a_{ii} \cdots & a_{in}+a_{ni} \\ \vdots & & & & \\ a_{1n}+a_{n1} & a_{2n}+a_{n2} & \cdots & & a_{nn}+a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$= \boxed{M_1 x}$$

Consider $\cancel{A^T A}$ $A + A^T = M_2$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \cdots & a_{1n}+a_{n1} \\ a_{21}+a_{12} & a_{22}+a_{22} & \cdots & a_{2n}+a_{n2} \\ \vdots & & & \vdots \\ a_{n1}+a_{1n} & a_{n2}+a_{2n} & \cdots & a_{nn}+a_{nn} \end{bmatrix}$$

Observe $M_2 = M_1 = (A + A^T)$

$\therefore \nabla_x (x^T A x) = M_1 x = M_2 x = (A + A^T) x$

Hence Proved

c) Given : A is symmetric $\Rightarrow A^T = A$ I

To prove : $\nabla_x (x^T A x) = 2Ax$

From Part (b) $\nabla_x (x^T A x) = (A + A^T) x$

Using Statement I $A^T = A$

$\therefore A + A^T = A + A = 2A$

$\therefore \nabla_x (x^T A x) = 2Ax$

Hence Proved

d) $\nabla_x((Ax+b)^T(Ax+b)) = 2A^T(Ax+b)$

Consider $(Ax+b)^T(Ax+b)$

Using $(M_1+M_2)^T = M_1^T + M_2^T$ --- (I)

$(Ax+b)^T = (Ax)^T + b^T$

$\therefore (Ax+b)^T(Ax+b) = ((Ax)^T + b^T)(Ax+b)$

Using $(M_1M_2)^T = M_2^T M_1^T$ --- II

Using $(M_1^T)^T = M_1$ ---- III

$(x^TA^T + b^T)(Ax+b)$

Expand the product

$x^TA^TAx + x^TA^Tb + b^TAx + b^Tb$

$\nabla_x(x^TA^TAx + x^TA^Tb + b^TAx + b^Tb)$

$= \underbrace{\nabla_x(x^TA^TAx)}_{1} + \underbrace{\nabla_x(x^TA^Tb)}_{2} + \underbrace{\nabla_x(b^TAx)}_{3} + \underbrace{\nabla_x(b^Tb)}_{4}$

Using $\nabla_x(M_1+M_2) = \nabla_x M_1 + \nabla_x M_2$

4 $= 0$ $(\nabla_x(\text{constant}) = 0)$

For 1, using theorem in part (b)

$\nabla_x(x^TA^TAx) = ((A^TA) + (A^TA)^T)x$

Simplify using II and III

$(A^TA + A^TA)x = 2A^TAx$

For ② Using theorem in part (a)

$\nabla_x (x^T A^T b) = A^T b \quad (\because \nabla_x(x^T a) = a)$

For ③ ~~Using~~

$\nabla_x(b^T A x) = \nabla_x((A^T b)^T x) \quad$ (Using II and III)

Now applying theorem from part ~~b~~ (a)

$\nabla_x(b^T A x) = \nabla_x((A^T b)^T x) = A^T b \quad (\because \nabla_x(a^T x) = a)$

$\therefore$ ① + ② + ③ + ④

$= 2A^T A x + A^T b + A^T b + 0$

$= 2A^T A x + 2A^T b$

$= 2A^T (Ax + b)$

Hence Proved