

Notes for EE C128: Feedback Control Systems
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1 Introduction and Overview

1.1 Dynamic Systems and Feedback Control

A dynamical system is a system whose behaviour changes over time. This is often in response to some external input/stimulus. If the system does not have this, it is called autonomous. Feedback is a situation in which two or more dynamical systems are connected together, and each system influences the behaviour of the other one. We usually show feedback systems in the form of block diagrams, which can denote closed-loop or open-loop systems. For example, if systems 1 and 2 are in series to one another with a loop back, we observe the output of system 2 and use it to revise/refine the input to system 1. Open-loop systems do not have this revision step.

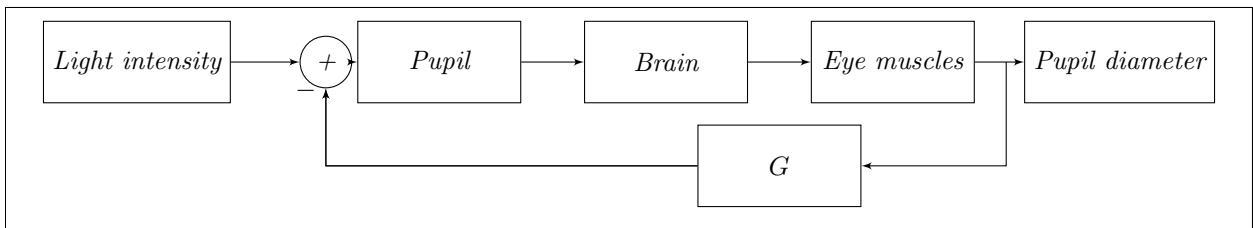
Biological systems make use of feedback on different scales (molecules, ecosystems). The human body's glucose regulation system is an example of this: there are mechanisms to detect high or low levels of glucose, and release hormones to either suppress or favour it, to bring glucose levels into normal ranges. Another example of a feedback control system is a thermostat. It is set at a desired temperature, and a furnace adjusts the heating level based on this input. The system then measures the deviation from the desired temperature, and sends this back to the furnace. Other than this, cruise control is another example, where speed is regulated based on feedback.

The “control” aspect means these systems have wide applications, such as in electronic amplifiers. Control is a form of information science; it uses information in analog and digital representations. A modern controller

1. senses the operation of a system and compares against desired behaviour
2. computes corrective actions
3. applies signals or forces to correct the system

The key issues in designing control systems are ensuring that the dynamics of the closed loop system are stable, so that small disturbances cannot blow up the system, and ensuring that the system has the desired behaviour. This includes disturbance attenuation and responsiveness to a change in the operating point. This course will cover the analysis and synthesis of linear feedback control systems and transform and time domains; control system design by root locus, frequency response, and state space methods, and applications to electromechanical systems.

Example 1.1. (HW1 Q1) The human eye has a biological control system that varies the pupil diameter to maintain constant light intensity to the retina. As the light intensity increases, the optical nerve sends a signal to the brain, which commands internal eye muscles to decrease the pupil's eye diameter. When the light intensity decreases, the pupil diameter increases. Draw a functional block diagram of the light-pupil system indicating the input, output, and intermediate signals; the sensor, the controller; and the actuator.



where G is a function that converts pupil diameter to light intensity.

1.2 Classical Techniques

1. Root locus analysis is a graphical method for examining how the roots of a system change when a system parameter is varied.
2. A Bode plot describes a system's behaviour in the frequency domain in terms of magnitude and phase.

Another important term to describe a type of controller is LQR: linear quadratic regulator. A controller is LQR when system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function.

2 Modelling in the Frequency Domain

2.1 The Laplace Transform

2.1.1 Definition and Basics

Consider a system with an input $r(t)$ and output $c(t)$. There are two ways to represent this system; in time and in frequency. The frequency domain representation in continuous time relies on the Laplace transform. The idea of the Laplace transform is: given a time domain signal $f(t)$, the Laplace transform returns a frequency domain signal $F(s)$, where $s = \sigma + j\omega$ is a complex number. Given $f(t)$, we find $F(s)$ by

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

The integral starts from an ϵ to the left of 0. We represent this by $F(s) = \mathcal{L}(f(t))$.

The inverse Laplace transform is

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

where $u(t)$ is the unit step,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Some common Laplace transform pairs are

Description	$f(t)$	$F(s)$
Impulse	$\sigma(t) = \delta(t)$	1
Step	$u(t)$	$\frac{1}{s}$
Ramp	$tu(t)$	$\frac{1}{s^2}$
	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$

2.1.2 Laplace Transform Properties

Linearity Linearity means the Laplace transform preserves scalar multiplication and addition,

$$\mathcal{L}(kf(t)) = k \cdot F(s)$$

$$\mathcal{L}(f_1(t) + f_2(t)) = F_1(s) + F_2(s)$$

Frequency shift Multiplying a time-domain signal by an exponential is equivalent to shifting it in frequency,

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

Time shift Shifting a signal in time is equivalent to multiplying it by an exponential in frequency,

$$\mathcal{L}(f(t-T)) = e^{-sT}F(s)$$

Scaling Scaling the time axis of a signal is equivalent to the reverse scaling on the frequency axis and on the frequency result,

$$\mathcal{L}(f(at)) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Differentiation Differentiation in time multiplies the signal by s in frequency, adjusted for a zero error(?):

$$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0^-)$$

Integration Integration in time divides the signal by s in frequency:

$$\mathcal{L}\left(\int_{0^-}^t f(\tau)d\tau\right) = \frac{F(s)}{s}$$

Final Value Theorem

$$\lim_{t \rightarrow \infty} \mathcal{L}(f(t)) = \lim_{s \rightarrow 0} sF(s)$$

In this context, the stability of a signal means all of the roots of the denominator of $F(s)$ must have a strictly negative real part. For example, consider

$$F(s) = \frac{10}{s+15}$$

This has a root at -15. Therefore we can apply the FVT and say

$$\mathcal{L} \left| \lim_{t \rightarrow \infty} f(t) \right| = \lim_{s \rightarrow 0} \frac{10s}{s+15} = 0$$

Initial Value Theorem Consider a function f such that $f(0^-) = 0$ and $f(0^+) = 7$. The value at 0 exactly is not well defined, but we can say

$$\mathcal{L}(f(0^+)) = \lim_{s \rightarrow \infty} sF(s)$$

Example 2.1. Find $\mathcal{L}^{-1} \left(\frac{1}{(s+5)^2} \right)$.

We know that $\mathcal{L}^{-1} \left(\frac{1}{s^2} \right)$ is the integral of $\mathcal{L}^{-1} \left(\frac{1}{s} \right)$, so it is $tu(t)$. Now we can use the shift property to shift the s by 5, and get

$$\mathcal{L}^{-1} \left(\frac{1}{(s+5)^2} \right) = te^{-5t}u(t)$$

2.1.3 Inverse Transform of an Arbitrary Rational Function

Consider $F(s) = \frac{N(s)}{D(s)}$. Because the functions N and D are polynomials, we know that an expansion into roots exists:

$$F(s) = \sum_{i=1}^n \frac{k_i}{s+p_i}$$

where the p_i s are the zeros of $D(s)$. The k_i s can be found by evaluating

$$F(s)(s+p_i)|_{s=-p_i} = k_i$$

Then, we take $\mathcal{L}^{-1} \left(\sum_i \frac{k_i}{s+p_i} \right)$ and use superposition to find $f(t)$.

2.1.4 Example Problems

Example 2.2. (HW1 Q2) A system can be modelled using the following differential equation:

$$2\frac{d^4w}{dt^4} + 17\frac{d^3w}{dt^3} + 6\frac{d^2w}{dt^2} + 8\frac{dw}{dt} + 3w = \frac{d^3r}{dt^3} + 3\frac{d^2r}{dt^2} + 2\frac{dr}{dt} + 7r$$

Determine the transfer function from $R(s)$ to $W(s)$.

We know that the transfer function of a system governed by a linear constant-coefficient differential equation $\sum a_i \frac{d^i c}{dt^i} = \sum b_j \frac{d^j r}{dt^j}$ is

$$\frac{W(s)}{R(s)} = \frac{\sum b_k s^k}{\sum a_i s^i}$$

which in this case is

$$\frac{W(s)}{R(s)} = \frac{s^3 + 3s^2 + 2s + 7}{2s^4 + 17s^3 + 6s^2 + 8s + 3}$$

Example 2.3. (HW1 Q3) For each transfer function below, determine $h(t)$.

(a) $H_1(s) = \frac{1}{s^2 + 4s + 20}$

We rewrite this as $H_1(s) = \frac{1}{(s+2)^2 + 16}$, which in inverse Laplace is

$$\begin{aligned} h_1(t) &= \mathcal{L}^{-1}(H_1(s)) = e^{-2t} \mathcal{L}^{-1}\left(\frac{1}{s^2 + 16}\right) \\ &= \frac{1}{4} e^{-2t} \mathcal{L}^{-1}\left(\frac{4}{s^2 + 16}\right) \\ &= \frac{1}{4} e^{-2t} \sin(4t) u(t) \end{aligned} \tag{1}$$

using the time-shift property and the usual transform for a time-domain sine.

(b) $H_2(s) = \frac{s}{s^2 + 4s + 20}$

This is a multiplication by s on the previous expression, which by the differentiation property gives us

$$h_2(t) = \frac{dh_1(t)}{dt} = \frac{1}{4} e^{-2t} (-2 \sin(4t) + 4 \cos(4t)) u(t)$$

(c) $H_3(s) = \frac{2s+3}{s^2 + 4s + 20}$

By linearity, we can say that $h_3(t) = 2h_2(t) + 3h_1(t)$, and so

$$h_3(t) = \frac{1}{2}e^{-2t} \sin(4t)u(t) + 3e^{-2t} \cos(4t)u(t)$$

(d) $H_4(s) = \frac{s^2}{s^2+4s+20}$

We differentiate $h_2(t)$ as $H_4(s) = sH_2(s) \implies h_4(t) = \frac{dh_2(t)}{dt}$,

$$h_4(t) = -e^{-2t} (4 \cos 4t + 3 \sin 4t) + \delta(t)$$

(e) $H_5(s) = \frac{5s+1}{(s+1)^2(s-3)}$

By partial fraction decomposition, we can rewrite this as

$$H_5(s) = \frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s-3}$$

and the Laplace inverse of this is the sum of the inverses of each one term-by-term,

$$h_5(t) = te^{-t} - e^{-t} + e^{3t}$$

2.2 Transfer Function Representation

A transfer function is the ratio of the output to the input of a system. Suppose a system has paired inputs/outputs $(v_1(t), c_1(t))$ and $(v_2(t), c_2(t))$. Then in the frequency domain, we know that

$$\frac{C_1(s)}{r_1(s)} = \frac{C_2(s)}{r_2(s)}$$

which is not the case in the time domain. (Convolution in time translates to multiplication in frequency.) The transfer function models the system, and should not depend on the value of an input.

Assume that a linear constant-coefficient differential equation describes a system:

$$\sum_{i=0}^n a_i \frac{d^i c(t)}{dt^n} = \sum_{k=0}^m b_k \frac{d^k r(t)}{dt^m}$$

Then, the transfer function can be found by taking the Laplace transform of both sides and rearranging for the required ratio:

$$\frac{C(s)}{R(s)} = \frac{\sum_{k=0}^m b_k s^k}{\sum_{i=0}^n a_i s^i}$$

with the implicit assumption that the system starts at zero input, zero output, and zeroes in all the derivatives of the input and output.

2.2.1 Circuit Transfer Functions

Assume that an electrical system takes current as its input and has voltage as the output. If there is a resistor between this input and output, the transfer function is given by

$$V(t) = R i(t) \implies TF = \frac{V(t)}{i(t)} = R$$

For a capacitor, this is

$$V(t) = \frac{1}{C} \int_0^t i(\tau) d\tau \implies V(s) = \frac{1}{Cs} \cdot i(s) \implies TF = \frac{1}{Cs}$$

And for an inductor, we get

$$V(t) = L \frac{di(t)}{dt} \implies V(s) = L s i(s) \implies TF = Ls$$

For more complicated systems involving many of these components, we use KVL and KCL. KVL states that the sum of voltages around a closed path is zero, and KCL states that the sum of currents flowing from any node is zero.

2.2.2 Examples

Example 2.4. Consider a circuit with, in order, a battery (voltage $V(t)$), an inductor, a resistor, and a capacitor. (Circuitikz TBD, or not). Find the transfer function.

$$TF = \frac{V_c(s)}{V(s)}$$

To find this in terms of circuit parameters, we write the KVL equation:

$$L \frac{di(t)}{dt} + R i(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = V(t)$$

Differentiating, we get

$$LC \frac{d^2V_c(t)}{dt^2} + RC \frac{dV_c(t)}{dt} + V_c(t) = V(t)$$

Then we can take the Laplace transform of both sides, and we get

$$\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Example 2.5. (HW1 Q4) Find the transfer function that describes the voltage output across the inductor ($V_L(t)$) with the voltage source ($V_s(t)$) as the input. (Picture tbd).

We write the KVL equations for the circuit:

$$\begin{aligned} V_s(t) - Ri_1(t) - Ri_2(t) - V_L(t) - \frac{1}{C} \int_0^t i_1(\tau) d\tau &= 0 \\ V_s(t) - Ri_1(t) - \frac{1}{C} \int_0^t i_3(\tau) d\tau - Ri_3(t) - \frac{1}{C} \int_0^t i_1(\tau) d\tau &= 0 \end{aligned}$$

Additionally, the voltage drop across the inductor is related to the current, and the following equation governs the split of current:

$$\begin{aligned} V_L(t) &= L \frac{di_2(t)}{dt} \\ i_1 &= i_2 + i_3 \end{aligned}$$

Taking the Laplace transform of each equation, we get

$$\begin{aligned} V_s(s) - RI_1(s) - RI_2(s) - V_L(s) - \frac{1}{Cs} I_1(s) &= 0 \\ V_s(s) - RI_1(s) - \frac{1}{Cs} I_3(s) - RI_3(s) - \frac{1}{Cs} I_1(s) &= 0 \\ V_L(s) &= LsI_2(s) \\ I_1(s) &= I_2(s) + I_3(s) \end{aligned}$$

which can be translated into the following matrix equation:

$$\begin{bmatrix} -R - \frac{1}{Cs} & -R & 0 & 1 \\ -R - \frac{1}{Cs} & 0 & -\frac{1}{Cs} & 1 \\ 0 & Ls & 0 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_L \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \\ V_s \\ 0 \end{bmatrix}$$

From solving this, we get the following relation for $V_s(s)$:

$$V_s(s) = V_L(s) \frac{2CLs^2 + 3CRs + 1}{CLs^2}$$

and therefore the transfer function is

$$\frac{V_L(s)}{V_s(s)} = \frac{CLs^2}{2CLs^2 + 3CRs + 1} = \frac{12s^2}{24s^2 + 60s + 1}$$

Operational Amplifiers An op-amp has two inputs of opposite signs - one is positive, one is negative - with input voltages V_1 and V_2 , and one single output V_0 . Op-amps have a gain characterized by the parameter A . The output is related to the input by

$$V_0(t) = A(V_2(t) - V_1(t))$$

If A is high, then the output signal is much larger than the input signal, which is why A is called the *gain*.

An inverting op-amp has its positive terminal connected to ground, i.e. $V_2(t) = 0$. We see that $V_0(t) = -AV_1(t)$. Consider a system with a resistor of impedance $Z_1(s)$ in series with a parallel combination of an inverting op-amp of gain A and a resistor of impedance $Z_2(s)$. This system has an input $V_i(s)$ and an output $V_o(s)$, or the equivalents in time.

Let A go to ∞ . We get I_a (the current just after the parallel split before the op-amp) and V_1 (the voltage at the same point) both going to 0. We can apply KCL and find that $I_1(s) = -I_2(s)$. Also, we know that

$$I_1(s) = \frac{V_i(s)}{Z_1(s)}, I_2(s) = \frac{V_0(s)}{Z_2(s)}$$

and therefore we find the transfer function

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

The transfer function of a similar circuit without the connection to ground on the op-amp, and instead with a connection to ground before the first resistor, with the input at the positive of the op-amp, has a transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

Example 2.6. (HW1 Q5) Find the transfer function, $G(s) = \frac{V_o(s)}{V_i(s)}$, for the following operational amplifier: (picture tbd)

We get the following KVL equations for this circuit:

$$\begin{aligned} v_i(t) - Ri(t) - \frac{1}{C} \int i(\tau) d\tau &= 0 \\ 0 - Ri(t) - Ri_1(t) &= v_0(t) \\ Ri_1(t) &= \frac{1}{C} \int i_2(\tau) d\tau \\ i(t) &= i_1(t) + i_2(t) \end{aligned}$$

The Laplace transformed equations can be translated into matrix form,

$$\begin{bmatrix} -R - \frac{1}{Cs} & 0 & 0 & 0 \\ -R & -R & 0 & -1 \\ 0 & R & -\frac{1}{Cs} & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} I \\ I_1 \\ I_2 \\ V_0 \end{bmatrix} = \begin{bmatrix} -v_i \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives us the following equation for the transfer function,

$$\frac{V_0(s)}{V_i(s)} = \frac{R^2 C^2 s^2 + 2 R C s}{R^2 C^2 s^2 + 1 + 2 R C s} = 1 - \frac{1}{(R C s + 1)^2} = 1 - \frac{1}{\left(\frac{s}{2} + 1\right)^2}$$

In the form of a rational function, this is

$$\frac{V_o(s)}{V_i(s)} = \frac{s^2 + 4s}{s^2 + 4s + 4}$$

2.2.3 Mechanical System Transfer Functions

Consider a spring of spring constant k with one end fixed to a wall, with force $f(t)$ being applied to it. The force is related to the position of the spring by

$$f(t) = kx(t) = k \int_0^t v(\tau) d\tau$$

Here, the transfer function is

$$Z(s) = \frac{F(s)}{x(s)} = k$$

Note that this is the equivalent of the impedance $\frac{V(s)}{I(s)}$ in a circuit.

Another example is a viscous damper, with an input force $f(t)$ and output displacement $x(t)$. The force-velocity relationship for this system is determined by a system constant f_ν , by

$$f(t) = f_\nu v(t) = f_\nu \frac{dx(t)}{dt}$$

The transfer function would therefore be

$$\frac{F(s)}{X(s)} = f_\nu s$$

A third example is a mass M to which a force $f(t)$ is applied, resulting in a displacement $x(t)$. Newton's second law tells us

$$f(t) = m \frac{d^2x}{dt^2} \implies TF = \frac{F(s)}{X(s)} = ms^2$$

2.2.4 Multi-Part Mechanical Systems

Consider a system consisting of two springs in series, with an applied force on the second one and a wall before the first. Find the transfer function $\frac{F(s)}{X_2(s)}$.

The total displacement is $x_1(t) + x_2(t)$, and both the springs experience the same force. If we consider the springs separately, one is the earlier case and the other experiences a net displacement of $x_2(t) - x_1(t)$. Therefore

$$\begin{aligned} f(t) &= k_1 x_1 = k_2(x_2 - x_1) \implies x_1 = x_2 \frac{k_2}{k_1 + k_2} \\ f(t) &= k_1 x_1 = k_1 \frac{k_2}{k_1 + k_2} \\ \frac{F(s)}{X_2(s)} &= \frac{k_1 k_2}{k_1 + k_2} \end{aligned}$$

Then, with two springs in parallel, the force splits but the displacement remains the same: $f(t) = f_1(t) + f_2(t)$. Therefore we get

$$f(t) = f_1(t) + f_2(t) = k_1 x(t) + k_2 x(t) // \frac{F(s)}{X_2(s)} = k_1 + k_2$$

Consider a system with a spring of constant k in parallel with a dampener of constant f_v , both in series with a mass m . Find the transfer function from X to F .

We start with the total force,

$$f(t) = f_s(t) + f_d(t) + f_m(t) = kx(t) + f_v \frac{dx(t)}{dt} + m \frac{d^2x(t)}{dt^2}$$

This gives us a second-order differential equation, which is easy to understand with the Laplace transform,

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + f_v s + k}$$

Example 2.7. (HW1 Q6) Find the transfer function, $G(s) = \frac{X_2(s)}{F(s)}$, for the translational mechanical network below. (picture tbd)

Consider the forces on M_1 due to the displacement x_1 . We get the input force and the motion of the block directed to the right, and the force from the dampener and the spring directed to the left. Due to the displacement x_2 , we get the forces from the dampener and the spring directed to the right. Now, consider the forces on M_2 due to the displacement x_1 . We get the forces from the dampener and the spring directed to the right. Finally, due to the displacement x_2 , we get forces from the dampener and the spring directed to the left, along with the motion of the block itself. Combining all of this, we get

$$\begin{aligned} f(t) + f_v \frac{d(x_2 - x_1)}{dt} + k(x_2 - x_1) - M_1 \frac{d^2x_1}{dt^2} &= 0 \\ k(x_1 - x_2) + f_v \frac{d(x_1 - x_2)}{dt} - M_2 \frac{d^2x_2}{dt^2} &= 0 \end{aligned}$$

Rearranging and performing the Laplace transform, we get

$$\begin{aligned} X_1(s)(-M_1s^2 - f_v s - k) + X_2(s)(f_v s + k) + F(s) &= 0 \\ X_1(s)(f_v s + k) + X_2(s)(-M_2s^2 - f_v s - k) &= 0 \end{aligned}$$

which we translate to the following matrix problem,

$$\begin{bmatrix} -M_1s^2 - f_v s - k & f_v s + k \\ f_v s + k & -M_2s^2 - f_v s - k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -F(s) \\ 0 \end{bmatrix}$$

From this, we get X_2 and therefore the transfer function,

$$\frac{X_2(s)}{F(s)} = \frac{f_v s + k}{M_1 M_2 s^4 + (M_1 + M_2) f_v s^3 + (M_1 + M_2) k s^2}$$

With the given parameters all set to unity, we get

$$\frac{X_2(s)}{F(s)} = \frac{s+1}{s^4 + 2s^3 + s^2}$$

2.2.5 Mechanical-Electrical Equivalence

It is possible to convert mechanical systems to electrical circuits and vice versa, while still getting the same input and output. The equation of motion for the mechanical system above is

$$(ms^2 + f_v s + k)X(s) = F(s)$$

This is equivalent to

$$\left(ms + f_v + \frac{k}{s} \right) v(s)$$

Now, consider velocity to be equivalent to current. The counterpart of force is voltage (denoted here by $E(s)$.) We get

$$E(s) = \left(ms + f_v + \frac{k}{s} \right) I(s)$$

Replace m by L , f_v by R , and k by $\frac{1}{C}$. Then we get

$$E(s) = \left(Ls + R + \frac{1}{Cs} \right) I(s)$$

This is a circuit with input current $i(t)$, consisting of an inductor, resistor, and capacitor in series.

With different initial assumptions, we may get different circuits. Let velocity be analogous to voltage and force to current. Then we get

$$I(s) = \left(ms + f_v + \frac{k}{s} \right) E(s)$$

Here, the analogy in constants is $m \rightarrow C$, $f_v \rightarrow \frac{1}{R}$, $k \rightarrow \frac{1}{L}$.

Example 2.8. (HW2 Q1) Draw the equivalent electrical circuit for the following system: (picture TBD)

Balancing forces on the first mass gives us

$$k_1(x_2 - x_1) + f_{v_1}(\dot{x}_2 - \dot{x}_1) + k_2(x_3 - x_1) - m_1\ddot{x}_1 = 0$$

and on the second mass,

$$f(t) - k_1(x_2 - x_1) - f_{v_1}(\dot{x}_2 - \dot{x}_1) - f_{v_2}(\dot{x}_2 - \dot{x}_3) - m_2\ddot{x}_2 = 0$$

Also, due to equal and opposite forces on the right spring and dampener, we have

$$k_2(x_1 - x_3) = f_{v_2}(\dot{x}_3 - \dot{x}_2)$$

Let x_1, x_2, x_3 be analogous to the time integral of current. Then, any term that directly involves these can be considered to be describing a capacitor, any term in their first derivative a resistor, and any term in their second derivative an inductor. To reflect this, we also change k_1, k_2 to $\frac{1}{C_1}, \frac{1}{C_2}$, f_{v_1}, f_{v_2} to R_1, R_2 , and m_1, m_2 to L_1, L_2 .

The first equation becomes

$$\frac{1}{C_1} \int (i_2 - i_1)(\tau) d\tau + R_1(i_2 - i_1) + \frac{1}{C_2} \int (i_3 - i_1)(\tau) d\tau - L_1 \frac{di_1}{dt} = 0$$

This suggests a potential drop across C_1 and R_1 in series is equal to the potential drop across L_1 and C_2 (flip the order of the currents across C_2 when it is taken to the other side of the equation) in series, with a branch between them changing the current.

The second equation becomes

$$v_i(t) - \frac{1}{C_1} \int (i_2 - i_1)(\tau) d\tau - R_1(i_2 - i_1) - R_2(i_2 - i_3) - L_2 \frac{di_2}{dt} = 0$$

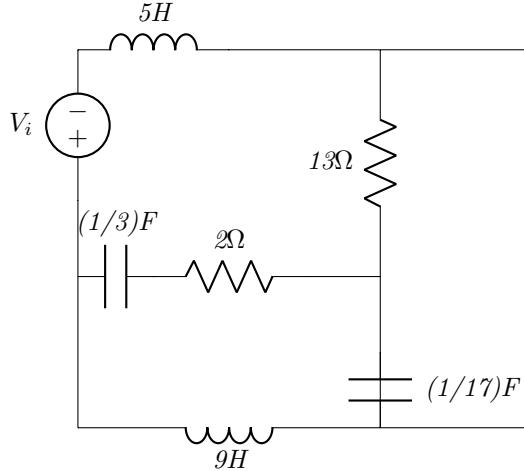
This suggests a loop containing the current source and the same elements R_1 and C_1 in series, after which the path changes from the first one, crossing a resistor R_2 and an inductor L_2 .

The third equation becomes

$$\frac{1}{C_2} \int (i_1 - i_3)(\tau) d\tau + R_2(i_2 - i_3) = 0$$

This suggests that there is no potential drop between these two elements, which must be adjacent by the above constraints, i.e. there is a short between them. This resolves the issue of there being a disconnected source and sink for i_3 in the above equations.

Combining all of this, we can draw the resultant circuit.



2.2.6 Cascaded Transfer Functions

Transfer functions are transitive, and it is possible to derive the differential equation governing a system based on its component-wise transfer functions, or the intermediate transfer function based on one overall. Say a system output is Y and input is U with intermediate step Z , and let $\frac{Y(s)}{U(s)}$ be $\frac{2s+3}{3s^2+4s+5}$. We are interested in the transfer function $U \rightarrow Z$.

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{U(s)} \frac{Y(s)}{Z(s)} = \frac{1}{3s^2 + 4s + 5} \cdot (2s + 3)$$

Then

2.2.7 Nonlinear Systems

Linear systems are characterized by two properties:

Superposition: $f(x) + f(y) = f(x + y)$

Homogeneity: $f(\alpha x) = \alpha f(x)$

Any function that does not obey these properties is called nonlinear. When applied to a system, we assume it is a function that itself takes a function as input and returns a function. A system with input $r(t)$ and output $c(t)$, denoted by $c(t) = \mathcal{S}(r(t))$, is linear if $\alpha_1 c_1(t) + \alpha_2 c_2(t) = \mathcal{S}(\alpha_1 r_1(t) + \alpha_2 r_2(t))$ for any valid input-output pairs $(r_1, c_1), (r_2, c_2)$ and any real coefficients α_1, α_2 .

A system can be *linearized* by taking the order-1 term in the Taylor series of the governing equation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + O(x^2)$$

Example 2.9. A system is described by $f(t) = m\ddot{x}(t) + b\dot{x}(t) + f_s(x, t)$ where f_s is the force from a non-linear spring. The spring force is defined by $f_s(x, t) = k(1 - e^{-x})$, where $x(t)$ is the spring displacement and k is some constant. Find the transfer function, $X(s)/F(s)$, for small excursions around $x = 0$.

The differential equation governing the system is

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + k(1 - e^{-x})$$

This can be linearized by Taylor expanding the exponential term around $x = 0$,

$$e^{-x} \approx e^0 + \frac{\partial(-e^{-x})}{\partial x} \Big|_{x=0} (x - 0) = 1 - x$$

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + k(1 - (1 - x(t))) = m\ddot{x}(t) + b\dot{x}(t) + kx(t)$$

We take a Laplace transform,

$$F(s) = ms^2X(s) + bsX(s) + kX(s)$$

and therefore the transfer function is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Linearization is only valid locally: the farther away from the center x_0 you get, the less accurate the linear equation is.

Consider the differential equation

$$\frac{dx}{dt} + \cos x = 0$$

We want to linearize this around $x_0 = \frac{\pi}{4}$. To do this, we linearize the cosine function:

$$\cos x \approx \cos \frac{\pi}{4} + (\cos x)' \Big|_{\frac{\pi}{4}} (x - \frac{\pi}{4}) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \frac{\pi}{4})$$

or

$$\cos x \approx \frac{1}{\sqrt{2}} \left(1 - \left(x - \frac{\pi}{4} \right) \right)$$

Linearization allows a system to be represented as a linear constant-coefficient differential equation and therefore as an input-output pair described by a transfer function.

3 Modelling in the Time Domain

3.1 System States

Let the *state* of a system be a set of parameters that fully characterize the behaviour of a system. For instance, a car moving in one direction along a road may have its state determined by its position and its speed. These are not unique variables; any two linearly independent variables spanning the same space are valid state parameters.

Let \vec{x} be a state vector containing all the state variables.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = x_i(t)$$

The formal definition of the state is as follows: $x(t)$ is a state if, given $x(t_0)$ and the system, we can solve for $x(t) \forall t > t_0$. We can use states to describe systems. Consider a dampener with constant c in parallel with a spring of constant k , both in series with a mass M . The differential equation governing their motion is

$$M\ddot{y} + c\dot{y} + ky = 0$$

To fully solve this differential equation, we need $y(0)$ and $\dot{y}(0)$. Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. Then, we can write a coupled systems of differential equations on these variables:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{M}x_1 - \frac{C}{M}x_2 \end{aligned}$$

This allows us to write

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{C}{M} \end{bmatrix} x(t) \end{aligned}$$

We can write any differential equation describing input-output behaviour as

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

where x is a vector of state variables, u is a control signal, and y is a vector of measurements. Note that f and h could both be nonlinear.

When $u(t)$ is zero, we call the system *autonomous*.

We have $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$. Therefore the transformations are $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^n$, and $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^r$.

For example, a coupled set of differential equations that describes a system is

$$\begin{aligned}\dot{x}(t) &= \sin tx(t) + \cos(u(t)) \\ y(t) &= x(t)\end{aligned}$$

3.2 State-Space Representation

LTI systems can be represented in *state space* as a pair of matrix equations. A system is LTI if it has linearity and time-invariance. Formally, let a system be described by $Y = G(X)$. The system is linear if the sum of the system outputs corresponding to inputs X_1 and X_2 is equal to the system response to an input $X_1 + X_2$, for any X_1, X_2 . Explicitly,

$$G(X_1 + X_2) = G(X_1) + G(X_2)$$

The transfer-function notion of multiplying frequency responses holds only if linearity holds.

The system is time-invariant if a time-shifted input creates a correspondingly time-shifted output:

$$G(X_1(t - T)) = [G(X_1)](t - T)$$

3.2.1 Example: Linear State-Space Representation

We can create this state-space representation of an LTI system.

Example 3.1. Consider a circuit with a voltage source, and components characterized by R, L, C in that order in series. The differential equation governing this circuit is, by KVL,

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t) dt = V(t)$$

To get rid of the integral, we say the variable is $q = \int i(t) dt$, the charge. Then this becomes a second-order equation in q ,

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t)$$

or alternatively, paired equations:

$$\begin{aligned}\dot{q} &= i \\ \dot{i} &= -\frac{1}{LC}q - \frac{R}{L}i + \frac{V}{L}\end{aligned}$$

This can be decomposed as

$$\begin{aligned}x(t) &= \begin{bmatrix} q(t) \\ i(t) \end{bmatrix} \\ u(t) &= V(t) \\ y(t) &= V_L(t)\end{aligned}$$

which allows for a matrix representation

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \\ C &= \begin{bmatrix} -\frac{1}{C} & -R \end{bmatrix}, D = [1]\end{aligned}$$

3.2.2 Example: Nonlinear State-Space Representation

Example 3.2. Consider a cart being driven by a force F . Its position is given by P , the distance from the base to the center of mass of the pendulum is L (along the string), M is the mass of the cart and m is the mass to be balanced. The matrix representation of the system is

$$\begin{bmatrix} M+m & -mL \cos \theta \\ -mL \cos \theta & y + mL^2 \end{bmatrix} \begin{bmatrix} \ddot{P} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} + mL \sin \theta \dot{\theta}^2 \\ \gamma \dot{\theta} - mgL \sin \theta \end{bmatrix}$$

This is a nonlinear time-invariant system.

3.3 Phase Variable Representation

Consider a system of the form

$$\sum_i a_i \frac{d^i y}{dt^i} = b_0 u$$

where $a_n = 1$ by definition. We can define a series of state variables,

$$x_1 = y, x_2 = \dot{y}, \dots, x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

Then, we have relationships between these. Each variable x_{i-1} is the derivative of x_i , and the derivative of the last coefficient is

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + b_0u$$

We can make a sparse matrix representation of this relationship, A , to describe the first part of the relationship $\dot{x}(t) = Ax(t) + Bu(t)$. Here the $u(t)$ is the constant term that is left over, i.e. $B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$. By definition also, $C = [1 \ 0 \ \dots \ 0]$, and $D = [0]$.

3.4 Transfer Function to State-Space

Since an underlying LTI system is the same no matter whether it is being represented in time or frequency, it is possible to translate between them. Suppose we have a transfer function as given below, that we convert into a second-order differential equation:

$$\frac{Z(s)}{U(s)} = \frac{1}{3s^2 + 4s + 5} \implies \ddot{z} + \frac{4}{3}\dot{z} + \frac{5}{3}z = \frac{1}{3}u$$

We can convert this to a system of first-order equations,

$$\begin{aligned} x_1 &= z \implies \dot{x}_1 = x_2 \\ x_2 &= \dot{z} \implies \dot{x}_2 = \frac{1}{3}u - \frac{4}{3}x_2 - \frac{5}{3}x_1 \end{aligned}$$

In general, we consider an LTI system specified by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

We take a Laplace transform and get

$$sX(s) - x(0) = AX(s) + BU(s), \quad Y(s) = CX(s) + DU(s)$$

We can drop the $x(0)$ by assuming that the zero state is zero. Then, we can rewrite to get

$$\begin{aligned} (sI - A)x(s) &= BU(s) \\ X(s) &= (sI - A)^{-1}BU(s) \end{aligned}$$

When s is an eigenvalue of A , we get a zero or a pole of the system, to be covered later. For now, we can disregard this and substitute into the expression for Y ,

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s) \implies TF = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

4 Time Response

4.1 Poles and Zeros

The poles and zeros of a system can help us gain a qualitative understanding of stability, oscillatory dynamics, etc. of a system without having to explicitly solve for the time-domain system output. They require analyzing the system in frequency by its transfer function, which we consider to be rational. Consider a system given by

$$G(s) = \frac{N(s)}{D(s)}$$

Then, the poles of G are the values of s that make $N(s) = \infty$, or those that make $D(s) = 0$. In case $N(s)$ and $D(s)$ have a common root, meaning that $G(s)$ takes on a 0/0 form, that value of s is still considered a pole.

For example, take

$$G(s) = \frac{s+2}{s+5}$$

Here, $s = -5$ is a pole. Another example is

$$G(s) = \frac{(s+2)(s+5)}{(s+5)(s+3)}$$

Here, $s = -5, -3$ are poles.

Consider the first system, with an input $R(s) = \frac{1}{s}$. The output in frequency space is

$$C(s) = R(s)G(s) = \frac{s+2}{s(s+5)} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

on which the inverse Laplace transform can be applied, to get

$$c(t) = \left(\frac{2}{5} + \frac{3}{5}e^{-5t} \right) u(t)$$

At $s = -5$, we get an output that decays as $t \rightarrow \infty$, which we can later learn to recognize as a consequence of the pole being in the left half-plane (having a real component less than zero.)

Poles can come either from the input, which is called a forced response (such as from an input of $u(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right)$), or from the system, which is called a natural response. Poles and zeroes are the amplitudes of both the forced and natural responses.

4.2 First-Order System Time Analysis

The generic (not the *most* general, but the most general case is just a scaled version of this) form of a first-order system with no zero is

$$G(s) = \frac{a}{s + a}$$

We give this system an input $\frac{1}{s}$, and get the output

$$C(s) = \frac{a}{s(s + a)} \implies c(t) = 1 - e^{-at}$$

We can analyze the time-domain behaviour of this system by a few parameters: rise time, settling time, peak time, and overshoot are the main ones. $\frac{1}{a}$ in the above expression is referred to as the system's *time constant*, the time required for the system to decay to 37% of its original value. We define the *rise time* as the time it takes the output to go from 0.1 to 0.9 of the final value. For the general first-order system, this is

$$\begin{aligned} C(t_1) &= 1 - e^{-at_1} = 0.1 \\ C(t_2) &= 1 - e^{-at_2} = 0.9 \\ T_r &= \frac{2.2}{a} \end{aligned}$$

We define the *settling time* as the time after which the response reaches and settles to within 2% of the final value. In this case, that comes out to $T_s = \frac{4}{a}$.

Example 4.1. (HW3 Q1) Find the output response, time constant, rise time, and settling time for (a) $G(s) = \frac{4}{s+4}$ (b) $G(s) = \frac{7}{s+7}$ with a system input $\frac{1}{s}$.

- (a) Fitting the transfer function to the usual first-order form, we see that $c(t) = (1 - e^{-4t})u(t)$. The time constant is $\frac{1}{4} = 0.25s$, the rise time is $\frac{2.2}{4} = 0.55s$, and the settling time is $\frac{4}{4} = 1s$.
- (b) Fitting the transfer function to the usual first-order form, we see that $c(t) = (1 - e^{-7t})u(t)$. The time constant is $\frac{1}{7} \approx 0.142s$, the rise time is $\frac{2.2}{7} = 0.314s$, and the settling time is $\frac{4}{7} = 0.571s$.

4.3 Second-Order System Time Analysis

In a second-order system, of the general form

$$G(s) = \frac{b}{s^2 + as + b}$$

there are a few different types of possible responses due to the presence of two poles.

4.3.1 Overdamped

If both poles are real, negative and distinct, the system is called *overdamped*. Let the poles be defined by $s - (-\sigma + j\omega)$. In this case $\sigma > 0$ (negative real part with the negation up front) and $\omega = 0$ so that the poles look like $s + \sigma$. Then the system frequency response can be decomposed into partial fractions as

$$C(s) = \frac{1}{s}G(s) = \frac{1}{s} + \frac{k_1}{s + \sigma_1} + \frac{k_2}{s + \sigma_2}$$

where the last two terms together are called the system's *natural response*, $C_N(s)$. We can do an inverse transform on this to get

$$c_N(t) = k_1 e^{-\sigma_1 t} + k_2 e^{-\sigma_2 t}$$

As $t \rightarrow \infty$, we see that $c(t) \rightarrow 1$.

4.3.2 Underdamped

If both poles are in the left half-plane with nonzero imaginary parts, the system is called *underdamped*.

For example, say $G(s) = \frac{9}{s^2 + 2s + 9}$. The poles are at $-1 \pm 2\sqrt{2}j$. Then, we can find the time-domain output by the inverse Laplace transform:

$$C(t) = \mathcal{L}^{-1}(R(s)G(s)) = 1 - e^{-t} \left(\cos 2\sqrt{2}t + \frac{1}{2\sqrt{2}} \sin 2\sqrt{2}t \right)$$

Here, the real part gives the decay factor of e^{-t} and the imaginary part shows the oscillation through the sine and cosine terms.

4.3.3 Undamped

This case is characterized by two distinct poles on the imaginary axis. For example, let $G(s) = \frac{9}{s^2 + 9}$. Then,

$$c(t) = \mathcal{L}^{-1}\left(\frac{1}{s}G(s)\right) = 1 - \cos 3t$$

Here, there is no growth or decay term, and the system just oscillates.

4.3.4 Critically damped

In this case, there are 2 poles that are real and repeated, in the left half-plane (both are negative). Consider the case $G(s) = \frac{9}{s^2 + 6s + 9}$. Both the poles are at -3, so the solution comes out to

$$c(t) = 1 - 3te^{-3t} - e^{-3t}$$

This is a special case of an overdamped system.

Example 4.2. (HW3 Q2) For each of the transfer functions shown below, find the locations of the poles and zeros, plot them on the s-plane, and then write an expression for the general form of the step response without solving for the inverse Laplace transform. State the nature of each response (overdamped, underdamped, and so on).

$$(a) G(s) = \frac{1}{s+3}$$

$$(b) G(s) = \frac{2(s+1)}{(s+4)(s+7)}$$

$$(c) G(s) = \frac{2}{(s+3)(s+4)}$$

$$(d) G(s) = \frac{6}{s^2+8s+41}$$

(leaving out the s-plane plots)

- (a) The only pole is at $s = -3$ and there are no zeros. The step response is the inverse Laplace transform of $\frac{1}{s(s+3)}$ which has the form $(A + Be^{-3t})u(t)$. (Since this is not a second order system it cannot be said to be damped at all.)
- (b) The poles are at $s = -4$ and $s = -7$, and the zero is at $s = -1$. The step response is the inverse Laplace transform of $\frac{2(s+1)}{s(s+4)(s+7)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+7}$, which has the form $(A + Be^{-4t} + Ce^{-7t})u(t)$. This system is overdamped.
- (c) The poles are at $s = -3$ and $s = -4$, and there are no zeros. The step response has the form $(A + Be^{-3t} + Ce^{-4t})u(t)$. The system is overdamped.
- (d) The poles are at $-4 \pm j\sqrt{5}$, and there are no zeros. The step response is the inverse Laplace transform of $\frac{6}{s(s^2+8s+41)} = \frac{A}{s} + \frac{B}{(s+4)^2+25}$ which has the form $A + Be^{-4t} \sin(5t)$ (using time-shift and the standard Laplace transform for a sine.)

4.4 General Second-Order Systems

We have seen in the above cases that second-order systems are characterized by two factors: exponential growth or decay, and oscillations. We characterize them by the damping ratio ζ and the natural frequency ω_n . We can find expressions for these parameters from the general form of a second-order system,

$$G(s) = \frac{b}{s^2 + as + b}$$

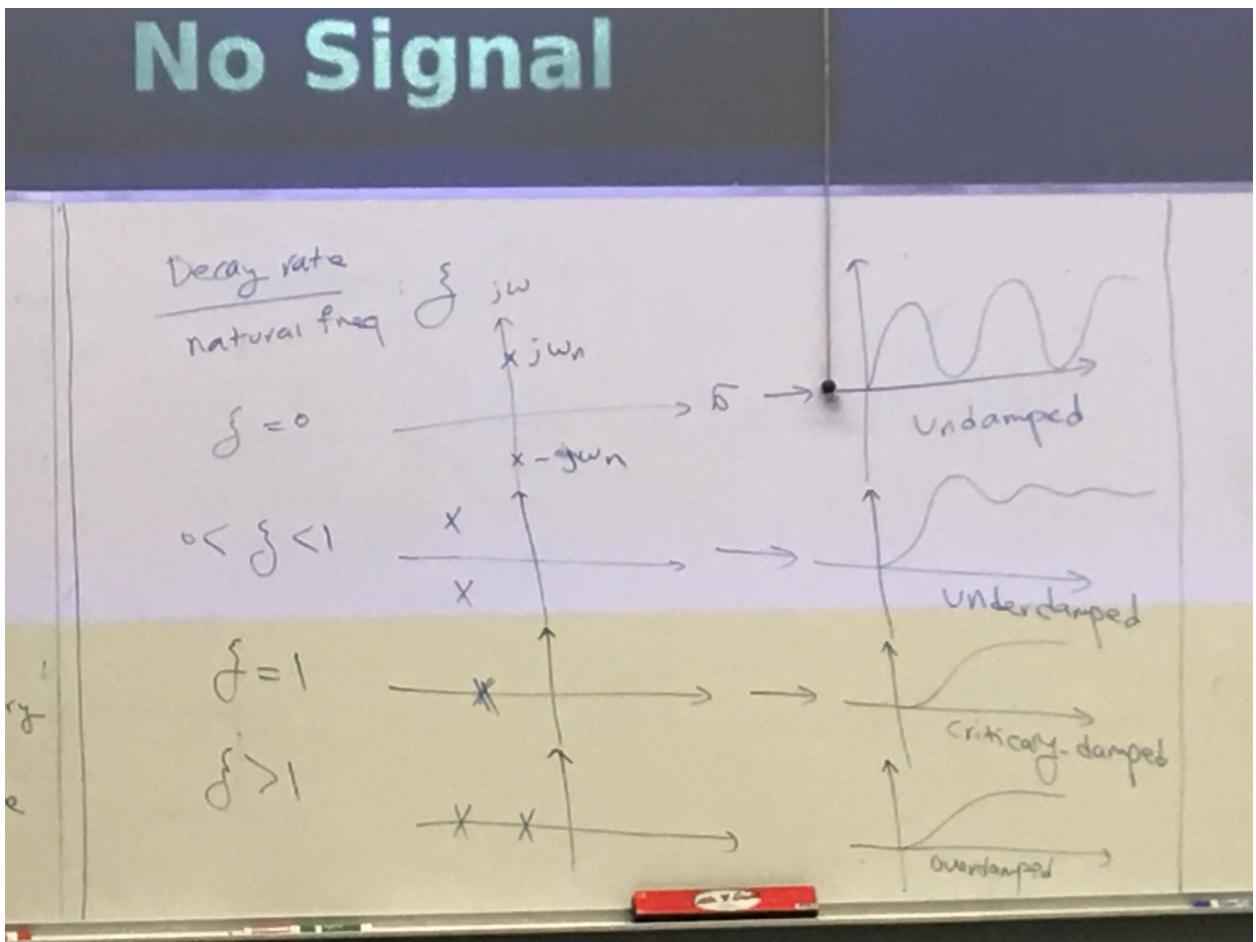
Assuming $a = 0$, the poles are at $\pm\sqrt{bj}$ so $\omega_n = \sqrt{b}$. Therefore we can write

$$G(s) = \frac{\omega_n^2}{s^2 + as + \omega_n^2}$$

A convenient definition of the damping ratio (that allows us to complete the square) gives us $a = 2\zeta\omega_n$. This means the poles are at

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

If $\zeta < 1$, we have oscillatory behaviour, and $-\zeta\omega_n$ gives the decay rate. Based on the value of ζ , we can characterize the system and make general graphs of the response.



4.5 Second-Order Time Response Parameters

Consider an underdamped second-order system with input $\frac{1}{s}$. Parameters in time relate to the locations of poles in the Laplace domain. Let the second-order system be described by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}j$$

We denote the real and imaginary parts of this by $s_{1,2} = -\sigma_d \pm j\omega_d$. Intuitively, we see that $\sigma_d = \zeta\omega_n$, the exponential damping ratio, and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, the frequency of oscillation. The magnitude of the pole is ω_n .

We can carry out the inverse Laplace transform to explicitly solve for the time response:

$$C(s) = G(s)R(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{s}$$

$$c(t) = \mathcal{L}^{-1}(C(s)) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n\sqrt{1 - \zeta^2}t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1 - \zeta^2}}\right)$.

By plotting the time response for different values of ζ , we see that for lower ζ , the response is more oscillatory.

The underdamped response in the time domain is specified by the rise time, the time required to go from 0.1 to 0.9 of the final value. For a second-order system, this does not have an analytic expression, but it can be approximated by

$$T_r = \frac{1}{\omega_n} (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$$

The peak time is the time required to reach the maximum. We can find the peak time by taking a derivative and finding the smallest (earliest time) solution. In general, we get

$$c'(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-j\omega_n t} \sin(\omega_n\sqrt{1 - \zeta^2}t) = 0$$

This is zero when the sine term is zero, i.e.

$$t_n = \frac{n\pi}{\omega_n\sqrt{1 - \zeta^2}} \implies T_p = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

The percentage overshoot is defined as

$$\%OS = \frac{C_{max} - C_{final}}{C_{final}} \times 100 = \frac{C(T_p) - C(\infty)}{C(\infty)} \times 100$$

Using the above formula for T_p , we get

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

This is useful as it only has ζ dependence, so if overshoot is set as a design requirement then that can completely specify ζ :

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

The settling time is the time required to get to and stay within 2% of the final value. This is the time it takes for the sinusoidal part of the time response to have an amplitude 0.02,

$$e^{-\zeta\omega_n T_s} \frac{1}{\sqrt{1 - \zeta^2}} = 0.02$$

which allows us to solve for T_s ,

$$T_s = \frac{-\ln(0.02\sqrt{1 - \zeta^2})}{\zeta\omega_n}$$

The numerator varies from 3.91 to 4.74 as ζ varies from 0 to 0.9. For simplicity we set the settling time conventionally to

$$T_s = \frac{4}{\zeta\omega_n}$$

The peak time is inversely proportional to the imaginary part of the poles, and the settling time is inversely proportional to the real part. If poles move vertically (to increase magnitude), the imaginary part of the poles increases in magnitude, and therefore the peak time goes down. If poles move horizontally (again to increase magnitude), the real part of the poles increases in magnitude, and therefore the settling time goes down.

4.6 Approximations

We can approximate higher-order systems to second-order by considering only the two poles closest to the origin. These are referred to as dominant poles. In general, we have to intuitively see whether an approximation is good or not. Consider a system with poles on the real axis at -1 and -5. The output is

$$c(t) = \alpha e^{-t} + \beta e^{-5t}$$

The βe^{-5t} term dies much faster than the αe^{-t} , so modelling it as a first-order system is a good approximation. However, if there are several poles close together, then taking just the single closest pole would be a bad approximation.

If a system has zeroes, we can use partial fraction expansion,

$$T(s) = \frac{s+a}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

where $A = \frac{a-b}{c-b}$ and $B = \frac{a-c}{b-c}$.

If $|a| \gg |b|$ we can approximate this to $a - b \approx a$, and the same for c . So the transfer function becomes:

$$T(s) \approx \frac{\frac{a}{c-b}}{s+b} + \frac{\frac{a}{b-c}}{s+c} = \frac{a}{(s+b)(s+c)}$$

If $|a|$ is sufficiently large, we can approximate this to the system having no zeroes. The initial system found using partial fractions has a zero, and coefficients A and B; the final system with the approximations on the size of a has no zero, and coefficients A' and B'.

4.7 Finding poles from state-space

Suppose we have a system representation given by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

We take the Laplace transform and get

$$\begin{aligned}sX(s) - x(0) &= AX(s) + BU(s) \\ X(s) &= (sI - A)^{-1}(x(0) - BU(s)) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

Consider a 2×2 example of this, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned}sI - A &= \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix} \\ (sI - A)^{-1} &= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}\end{aligned}$$

The points at which this is not defined are the eigenvalues of A . Based on partial fraction expansion (in general), we see that the response of the system depends on $\lambda_1, \dots, \lambda_n$, where the lambdas are poles or modes of the system.

Example 4.3. Consider a system whose state-space representation is given by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \\ y &= [1 \ 1 \ 0] x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\end{aligned}$$

We can find the transfer function using the above method; we get

$$Y(s) = \frac{s^3 + 12s^2 + 16s + 5}{(s+1)(s+2)(s+3)(s+4)}$$

-2, -3, and -4 are eigenvalues of A . These are modes of the system. -1 is both a zero and a pole, so these two effects cancel.

By performing a partial fraction decomposition, we get

$$Y(s) = \frac{-6.5}{s+2} + \frac{19}{s+3} + \frac{-11.5}{s+4}$$

or

$$y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}$$

Example 4.4. (HW4 Q1) A system is represented by the state and output equations that follow. Without solving the state equation, find the poles of the system.

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -3 & -1 \\ -5 & -7 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= [2 \ 5] x(t)\end{aligned}$$

We take the Laplace transform of the time-evolution equation to solve explicitly for $X(s)$,

$$\begin{aligned}(sI - A)X(s) &= \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} \\ X(s) &= \begin{bmatrix} s+3 & 1 \\ 5 & s+7 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} = \frac{1}{s^2 + 10s + 16} \begin{bmatrix} s+7 & -1 \\ -5 & s+3 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} \\ X(s) &= \frac{1}{s(s+2)(s+8)} \begin{bmatrix} s+7 \\ -5 \end{bmatrix}\end{aligned}$$

No new poles are introduced by $Y(s) = [2 \ 5] X(s)$; from this we see the system has poles at $s = -2, -8$.

In the above example, it is not necessary to explicitly take the Laplace transform; in general the system poles are the roots of the characteristic polynomial $\det(sI - A)$.

Example 4.5. (HW4 Q2) Given the following state-space representation of a system, find $Y(s)$:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} x(t) + [1 \ 0]^T \sin(3t) \\ y(t) &= [3 \ 2] x(t), x(0) = [0 \ 1]^T\end{aligned}$$

We take the Laplace transform of both sides of the time-evolution equation, noting that the initial condition is not zero,

$$sX(s) - x(0) = AX(s) + BU(s) \implies \begin{bmatrix} s-1 & -2 \\ 3 & s-4 \end{bmatrix} X(s) = \begin{bmatrix} \mathcal{L}^{-1}(\sin(3t)) \\ 1 \end{bmatrix} = \begin{bmatrix} 3/(s^2+9) \\ 1 \end{bmatrix}$$

We can get an expression for $Y(s)$ in terms of known quantities from this,

$$Y(s) = [3 \ 2] X(s) = \frac{1}{(s^2 - 5s + 10)(s^2 + 9)} [3 \ 2] \begin{bmatrix} s-4 & 2 \\ -3 & s-1 \end{bmatrix} \begin{bmatrix} 3 \\ s^2 + 9 \end{bmatrix}$$

Multiplying everything out, we get

$$Y(s) = \frac{2s^3 + 4s^2 + 27s - 18}{(s^2 - 5s + 10)(s^2 + 9)}$$

4.8 State-space from time output

Consider a one-dimensional system with $y = x$. The system is given by $\dot{x} = ax + bu$ as usual, where everything is a scalar because the system is one-dimensional. Multiply both sides of this by e^{-at} . Rearranging, we get

$$\begin{aligned}e^{-at} \frac{dx(t)}{dt} &= ae^{-at}x + e^{-at}bu \\ e^{-at}bu(t) &= e^{-at} \left(\frac{dx(t)}{dt} - ax(t) \right) = e^{-at} \frac{d}{dt} (e^{-at}x(t))\end{aligned}$$

We can integrate both sides and apply the Fundamental Theorem of Calculus,

$$\int_0^t e^{-a\tau} bu(\tau) d\tau = \int_0^t \frac{d}{d\tau} (e^{-a\tau} x(\tau)) d\tau = e^{-at} x(t) - x(0)$$

Therefore

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

The expression e^{at} would not make sense in a multidimensional system where the state-space equations are not scalar. To deal with this, we can Taylor expand the exponential function,

$$a \in \mathbb{R} : e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots$$

and we can define exponentiation of a matrix in terms of this sum,

$$A \in \mathbb{R}^{n \times n} : e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

If A is diagonalizable, i.e. it can be written as PDP^{-1} , then the exponential becomes much simpler:

$$e^A = P \begin{bmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n} \end{bmatrix} P^{-1}$$

With this definition, we can write the state-space representation of the system,

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

We call e^{At} the *state transition matrix*.

Example 4.6. (HW4 Q3) Using classical (not Laplace) methods only, solve for (a) the state-transition matrix $\Phi(t)$, (b) the state vector $x(t)$, (c) the output of the system $y(t)$.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 2 & 3 \end{bmatrix} x(t) \\ x(0) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

(a) The state-transition matrix can be found by diagonalizing the A matrix.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \implies \det(\lambda I - A) = \lambda^2 + 1.$$

The eigenvalues are $\pm i$. The null space of the i -eigenspace is $\begin{bmatrix} i \\ -1 \end{bmatrix}$ and that of the $-i$ -eigenspace is $\begin{bmatrix} i \\ 1 \end{bmatrix}$. Therefore the diagonal representation is

$$A = \frac{1}{2} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix},$$

and we exponentiate by raising the diagonal matrix elements to the required power,

$$\Phi(t) = \frac{1}{2} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

(b) The state vector can be found as follows,

$$x(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(t-\tau) \\ \sin(t-\tau) \end{bmatrix} d\tau.$$

We integrate to get

$$x(t) = \begin{bmatrix} 2 \sin t \\ 2 \cos t - 1 \end{bmatrix}.$$

(c) The output is

$$\begin{bmatrix} 2 & 3 \end{bmatrix} x(t) = 4 \sin t + 6 \cos t - 3.$$

5 Stability

5.1 Stability States

Stability is the most important specification for a control system. It can be naturally defined in terms of a system's time-domain response. We can express the response of a system as the sum of the forced and natural response, $c(t) = c_f(t) + c_n(t)$. For an LTI system, there are three kinds of stability states:

1. An LTI system is stable if the natural response goes to zero as time goes to infinity.
2. An LTI system is unstable if the natural response grows unbounded as time goes to infinity.
3. An LTI system is marginally stable if the natural response remains constant or oscillates as time goes to infinity.

We can identify some stability states graphically, such as in Figure 1.

An alternate definition of stability is BIBO (bounded-input bounded-output) stability. A system is BIBO stable if every bounded input yields a bounded output, and BIBO unstable otherwise. For LTI systems, marginal stability and BIBO stability mean the same thing. In general, asymptotic stability \subseteq marginal stability \subseteq BIBO stability.

5.2 Identifying Stability from State-Space

We can represent a system in state-space as the system of equations $\dot{x} = Ax + Bu$, $y = Cx + Du$. The poles of an LTI system are the eigenvalues of the A matrix, $\lambda_1, \lambda_2, \dots, \lambda_n$. Assume $\lambda_i \neq \lambda_j$ for a first case. The output is a combination of exponentials with these eigenvalues as the exponent coefficients,

$$c(t) = \sum_i \alpha_i e^{\lambda_i t}$$

Asymptotic stability means the λ_i s must all be in the left half-plane. A second case is when there are repeated poles in the left half-plane; we then use solutions $e^{\lambda t}$, $te^{\lambda t}$, etc. A third case is when there is at least one $\lambda_i > 0$. This system is unstable because the $e^{\lambda_i t}$ term grows unbounded. In a fourth case, λ_i is on the imaginary axis and all distinct. This gives rise to bounded but oscillatory motion. A system is marginally stable if and only if the real part of all their eigenvalues is zero or negative, and poles on the $j\omega$ axis are distinct. In case of repeated imaginary components, i.e. λ_i on the imaginary axis and appearing at least twice, we have an unbounded solution $C_1 e^{0t} + C_2 t e^{0t} = C_1 + C_2 t$.

To identify stability, we can look at the eigenvalues of A , i.e. solve for the roots in $\det(\lambda I - A) = 0$. This requires that we solve an n -degree polynomial, which is in general difficult. An easier method is to find roughly where the eigenvalues are located. Specifically, we need to find whether or not all poles are in the left half-plane, for which it is not essential that we find the exact poles. The Routh-Hurwitz criterion is a method to do this.

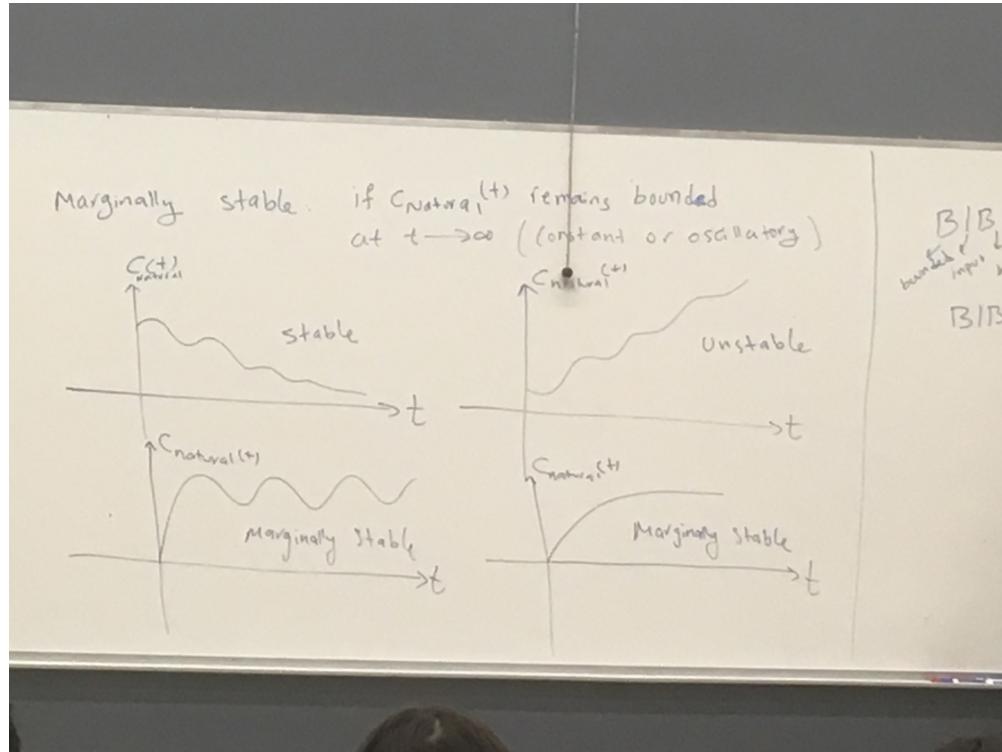


Figure 1: Graphically identifying stability states

5.3 Routh-Hurwitz Criterion

We start from a polynomial $\sum_i a_i s^i$. We construct the following table,

$$\begin{array}{c|ccccc} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & \dots & \dots \\ s^{n-2} & b_1 & b_2 & b_3 & \dots \end{array}$$

We start by filling in the coefficients, then we fill in more according to the following rule: consider a 2×2 matrix obtained by the first column of the preceding rows, and the column right about b_i and to the right, then take its determinant and divide it by the element right above b_i . Figure 2 (from the textbook) shows this.

For example,

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

In this way, we fill in $n - 2$ rows. Going from top to bottom, we see how many times the sign changes. The number of sign changes is the number of right half-plane poles.

For example, consider $a_2 s^2 + a_1 s + a_0$. The table is

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 2: Generic fourth-order Routh table

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & -\frac{a_2 \cdot 0 - a_0 a_1}{a_1} = a_0 & 0 \end{array}$$

If a_0, a_1, a_2 all have the same sign, then the system is stable. This is a simpler way of seeing the stability of the system than explicitly solving for the eigenvalues.

For a third-order system, we get

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & a_1 - \frac{a_3 a_0}{a_2} & 0 \\ s^0 & a_0 & 0 \end{array}$$

If $a_3 > 0$, then for stability, we know that $a_2, a_0 > 0$, and also $a_2 a_1 > a_3 a_0$.

Suppose there were a fourth-order system with a leftmost column that is, in order, $1, -5, 1.2, 2, -6$. This has three sign changes, and so there are three right half-plane poles.

Example 5.1. (HW4 Q4) Consider a unity negative feedback system with plant $G(s) = \frac{K(s+6)}{s(s-2)(s+3)}$. Find the range of K for closed-loop stability.

The transfer function for this system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substituting in the given $G(s)$, we get

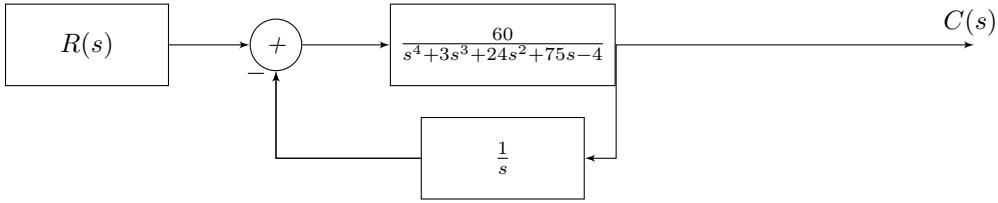
$$\frac{C(s)}{R(s)} = \frac{K(s+6)}{s^3 + s^2 + (K-6)s + 6K}$$

We analyze the stability of this system using the Routh-Hurwitz criterion.

$$\begin{array}{c|ccc} s^3 & 1 & K-6 & 0 \\ s^2 & 1 & 6K & 0 \\ s^1 & -5K-6 & 0 & 0 \\ s^0 & 6K & 0 & 0 \end{array}$$

The condition for the sign not to switch from the s^2 to the s^1 row is $-5K-6 > 0 \implies K < -\frac{6}{5}$, and the condition for the sign not to switch from the s^1 to the s^0 row is $6K > 0 \implies K > 0$. These two cannot be true simultaneously. Therefore, no matter the value of K , there is at least one pole in the right half-plane of the system, and so there does not exist a K such that the system is stable.

Example 5.2. (HW4 Q5) Using the Routh-Hurwitz criterion, find how many closed-loop poles of system shown below lie in the left half-plane, in the right half-plane, and on the $j\omega$ axis.



The closed-loop transfer function is

$$TF = \frac{G(s)}{1 + \frac{1}{s}G(s)} = \frac{60}{s^5 + 3s^4 + 24s^3 + 75s^2 - 4s + 60}$$

We set up the Routh table for this transfer function:

s^5	1	24	-4
s^4	3	75	60
s^3	-1	24	0
s^2	3	60	0
s^1	-4	0	0
s^0	60	0	0

There are four inversions, so the closed-loop system has four RHP poles and one LHP pole.

5.4 Zero in Routh-Hurwitz

If we get a zero in the leftmost column, this corresponds to imaginary axis poles and has to be dealt with differently. Suppose the column corresponding to s^k for some $0 < k < n$ is zero. To deal with this, we replace zero with some ϵ , which is either positive or negative 10^{-6} . For example, suppose the transfer function is

$$G(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

The table becomes

s^5	1
s^4	2
s^3	ϵ
s^2	$\frac{6\epsilon - 7}{\epsilon}$
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$
s^0	3

This has two sign changes whether ϵ is positive or negative, so there are two poles in the right half-plane.

Suppose there is an entire row of zeroes. More concretely, let there be a row corresponding to s^k with coefficients e_k, e_{k-1}, \dots and below that a row of all zeroes corresponding to s^{k-1} . We construct a polynomial as follows,

$$P(s) = e_k s^k + e_{k-1} s^{k-2} + e_{k-2} s^{k-4} + \dots$$

$P(s)$ either contains only odd degrees or only even degrees. Then, $P(s)$ multiplied by some other polynomial, say $Q(s)$, is the original polynomial again. We take a derivative, to get

$$P'(s) = k e_k s^{k-1} + (k-2) e_{k-1} s^{k-3} + \dots$$

We replace the row of zeroes with the coefficients of this polynomial. Then, all the rows above the row of zeroes describe the roots of $Q(s)$, and those below/including the row of zeroes describe the roots of $P(s)$. To find the roots on the $j\omega$ axis that the zero in the leftmost column indicates are present, we express $P(s)$ either as an even polynomial or s multiplied by an even polynomial. We then let $y = s^2$, and note that the zeroes are symmetric. If y is complex, there will be zeroes in all four quadrants, placed symmetrically in a rectangle about the origin. If y is real, then the zeroes will either be symmetric on the real axis or on the $j\omega$ axis. The number of zeroes on the $j\omega$ axis is equal to k minus twice the number of RHP poles. If there is no zero in the first column, there are no poles on the imaginary axis. If there is no change in sign, we have asymptotic stability.

Example 5.3. Consider the following transfer function, which we want to analyze for stability

$$G(s) = \frac{5}{s^5 + s^4 + 6s^3 + 6s^2 + 8s + 8}$$

The Routh-Hurwitz table up to a zero row is

$$\begin{array}{c|ccc} s^5 & 1 & 6 & 8 \\ s^4 & 1 & 6 & 8 \\ s^3 & 0 & 0 & 0 \end{array}$$

We follow the above procedure, and normalize the result, to get

$$\begin{array}{c|ccc} s^5 & 1 & 6 & 8 \\ s^4 & 1 & 6 & 8 \\ s^3 & 1 & 3 & 0 \\ s^2 & 3 & 8 & 0 \\ s^1 & \frac{1}{3} & 0 & 0 \\ s^0 & 8 & 0 & 0 \end{array}$$

In the first two rows, there are no sign changes, therefore the polynomial $Q(s)$ has no right half-plane zeroes. $P(s)$ has no RHP zeroes, so there are $4 - 0 \times 2 = 4$ zeroes on the $j\omega$ axis.

Example 5.4. (HW4 Q6) Consider the following Routh table. Notice that the s^5 row was originally all zero. Identify how many roots of the original polynomial were in the right half-plane, in the left half-plane, and on the $j\omega$ axis.

s^7	1	2	-1	-2		
s^6	1	2	-1	-2		
s^5	3	4	-1	0		
s^4	1	-1	-3	0		
s^3	7	8	0	0		
s^2	-15	-21	0	0		
s^1	-9	0	0	0		
s^0	-21	0	0	0		

From the Routh table, we see that the even polynomial that is being analyzed from s^6 on down has one inversion, and therefore it has one RHP pole. By symmetry it must have one LHP pole corresponding to that one. Therefore, all four of its other poles must be on the $j\omega$ axis. So the overall polynomial has two LHP poles, one RHP pole, and four $j\omega$ poles.

If we look at the top two rows, we notice that the polynomial can be stated explicitly as $(s+1)(s^6 + 2s^4 - s^2 - 2) = (s+1)(s^2 + 2)(s^3 - 1)$, so this can actually be explicitly factorized as $(s+1)^2(s + j\sqrt{2})(s - j\sqrt{2})(s - 1)(s^2 + 1)$, from which we see that there actually are four poles on the $j\omega$ axis ($\pm j, \pm j\sqrt{2}$), one in the right half plane (1), and two in the left half plane (both at -1).

Example 5.5. (HW5 Q1) Let the plant

$$G(s) = \frac{9}{s(s^7 + 5s^6 + 7s^5 + 20s^4 + 19s^3 + 35s^2 + 24s + 15)}$$

be placed in unity negative feedback. Identify the number of poles of the closed-loop transfer function in the right halfplane, in the left half-plane, and on the $j\omega$ -axis.

The Routh-Hurwitz table is shown below.

s^8	1	7	19	24	9	0
s^7	1	4	7	3	0	0
s^6	1	4	7	3	0	0
s^5	0	0	0	0	0	0

The s^5 row is all zeroes, so there must be an even polynomial factor. This comes out to

$$P(s) = s^6 + 4s^4 + 7s^2 + 3$$

which we differentiate to get the new s^5 row. The Routh-Hurwitz table is then as follows.

s^8	1	7	19	24	9	0
s^7	1	4	7	3	0	0
s^6	1	4	7	3	0	0
s^5	3	8	7	0	0	0
s^4	4	14	9	0	0	0
s^3	-10	1	0	0	0	0
s^2	8	5	0	0	0	0
s^1	1	0	0	0	0	0
s^0	1	0	0	0	0	0

Since there are two inversions below the even polynomial, there are two right half-plane poles. Due to symmetry about the origin, there are two left half-plane poles, as well as two left half-plane poles for the original polynomial. The remaining two roots of the even polynomial are on the $j\omega$ axis.

6 Steady-State Error

6.1 Representing and Calculating Error

The *steady state* of a system is its state as $t \rightarrow \infty$. The steady-state error in a system is the difference between its input and output in this limit. Steady-state error is an important design parameter in control systems. It usually arises from nonlinear effects (e.g. friction of a cart on a track), but we will only study steady-state errors induced by controller design.

Suppose we want to make a system whose output is as close to the input as possible. We can measure the quality of tracking by the quantity

$$\lim_{t \rightarrow \infty} |c(t) - r(t)|$$

If the steady-state error is zero, we have perfect tracking, but we can show that with proportional control this is never the case. Consider a unity negative feedback system with plant $G(s)$ having a gain K . For a finite and nonzero system output $c(t)$, or in frequency $C(s)$, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}$$

The error can also be represented in the frequency domain. In time, let the error be $e(t) = r(t) - c(t)$, which translates to frequency as

$$E(s) = R(s) - C(s) = \left(\frac{1}{1 + KG(s)} \right) R(s)$$

This is nonzero for all finite values of K , therefore steady-state error is present.

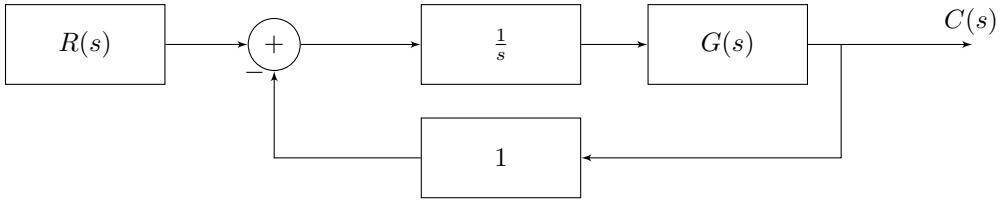
If $E(s)$ has no right half-plane pole, then we can use the final value theorem (from the Laplace transform properties) to relate this frequency-domain representation to the time-domain steady-state error:

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} sR(s) (1 - G(s))$$

For example, if the transfer function is $\frac{5}{s^2+7s+10}$, then the error signal is $E(s) = \frac{s^2+7s+5}{s(s^2+7s+10)}$ which is marginally stable as there is one pole at the origin. In the limit as $s \rightarrow 0$, we get $\frac{1}{2}$.

6.2 Integration

Steady-state error can be removed by adding an integrator in the feed-forward arm (in series with the plant). Assume the system has no zero at the origin. Then, we can set up a feedback system for $e(t)$ to accumulate the error over time, by integrating it before the system.



(there's a 1 in feedback here because I can't figure out how to *draw a straight line* in tikz kill me)

$$u(t) = \int_0^t r(\tau) - c(\tau) d\tau \implies \frac{U(s)}{E(s)} = \frac{1}{s}$$

Overall, we get

$$C(s) = U(s)G(s) = \frac{1}{s}E(s)G(s) = \frac{1}{s}(R(s) - C(s))G(s)$$

Therefore

$$C(s) = R(s) \left(\frac{\frac{1}{s}G(s)}{1 + \frac{1}{s}G(s)} \right)$$

The transfer function here is called the *closed-loop transfer function*, and it is equal to the *feed-forward transfer function* divided by 1 plus the feed-forward transfer function.

Based on this, the error function is

$$E(s) = R(s) - C(s) = \frac{R(s)}{1 + \frac{1}{s}G(s)}$$

In the limit,

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{1}{1 + \frac{G(s)}{s}} = 0$$

Therefore, introducing one integrator leads to perfect tracking. This will later inform the designs of lag, lead, and lag-lead compensators.

6.3 Steady-State Error in Unity Negative Feedback Systems

In a unity negative feedback system, the closed-loop transfer function with plant $G(s)$ is

$$TF = \frac{G(s)}{1 + G(s)}$$

and so, using the final value theorem, the steady-state error is

$$e(\infty) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

This allows us to calculate the steady-state error for different types of input signals. Consider the case of a unit step, $R(s) = \frac{1}{s}$, and so the error is

$$e_{step}(\infty) = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)}$$

i.e. 1 over 1 plus the DC gain of the forward transfer function.

Example 6.1. (HW5 Q3) What is the steady-state error for a step input of 10 units with plant

$$G(s) = \frac{(s+7)(s+17)(s+27)}{(s+14)(s+20)(s+33)}$$

placed in unity negative feedback?

We use the final value theorem, noting that the system is stable because the poles are all negative,

$$e(\infty) = \lim_{s \rightarrow 0} \frac{10(s+14)(s+20)(s+33)}{(s+7)(s+17)(s+27) + (s+14)(s+20)(s+33)} = \frac{10 \times 14 \times 20 \times 33}{7 \times 17 \times 27 + 14 \times 20 \times 33} = 7.4198$$

To drive the steady-state error to zero, the system needs to have overall order $\frac{1}{s^n}$ for positive n . This is equivalent to saying the forward transfer function needs to have at least one pure integral, as we just saw that an integrator drives steady-state error to zero. If there are no integrals, there will be a nonzero steady-state error equal to $\prod z_i / \prod p_i$, the product of zero locations divided by the product of pole locations.

For a ramp input, with $R(s) = \frac{1}{s^2}$, we get

$$e_{ramp}(\infty) = \lim_{s \rightarrow 0} \frac{1/s}{1 + G(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)}$$

Similarly to the above, the response to a ramp input has no error if the forward transfer function has two pure integrals.

For a parabolic input, with $R(s) = \frac{1}{s^3}$, we get

$$e_{parabola}(\infty) = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)}$$

The response to a parabolic input has no error if the forward transfer function has three pure integrals.

Example 6.2. (HW5 Q2) For a unity negative feedback system with plant

$$G(s) = \frac{20(s+7)(s+11)(s+18)}{s(s+25)(s^2+4s+40)}$$

find the steady-state errors for the test inputs $u(t)$, $tu(t)$, $t^2u(t)$.

We employ the Final Value Theorem (noting that the system is stable because the poles of the denominator are all less than zero),

$$e(\infty) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{s(s+25)(s^2+4s+40)}{s(s+25)(s^2+4s+40) + 20(s+7)(s+11)(s+18)}$$

Taking the limit in each case, we get

$$\begin{aligned} R(s) &= \frac{1}{s} \implies e(\infty) = 0 \\ R(s) &= \frac{1}{s^2} \implies e(\infty) = \frac{1000}{27720} = 0.036 \\ R(s) &= \frac{1}{s^3} \implies e(\infty) \rightarrow \infty \end{aligned}$$

Therefore the case $u(t)$ has zero error, $tu(t)$ has an error of 0.036, and $t^2u(t)$ has an error that blows up to infinity.

Note that this matches what we would expect: the feed-forward transfer function has one pure integral, so the step input response has zero error ($n = 1$), the ramp input response has a finite error ($n = 0$) and the parabola input response blows up to infinity ($n = -1$).

6.4 Static Error Constants

The three terms in the denominator of the error for the step, ramp, and parabola inputs are collectively referred to as *static error constants*. They are the position constant,

$$K_p = \lim_{s \rightarrow 0} G(s)$$

the velocity constant,

$$K_v = \lim_{s \rightarrow 0} sG(s)$$

and the acceleration constant,

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)$$

The static error constants allow us to find the steady-state error without directly making use of the final value theorem, by $e_{step}(\infty) = \frac{1}{1+K_p}$, $e_{ramp}(\infty) = \frac{1}{K_v}$, and $e_{parabola}(\infty) = \frac{1}{K_a}$.

6.5 Steady-State Error for Disturbances

The theory of steady-state error can be used to compensate for external disturbances added to the system other than at the input. Consider a system with two separate feed-forward blocks, a controller and a plant, and a disturbance being added between them.

(Tikz TBD if I get to root locus by midnight)

The output is now

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

which allows us to solve for the error,

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

If we let $D(s) \rightarrow 0$ we see this reduces to the usual error for a unity negative feedback system.

To find the steady-state error, we apply the final value theorem to this error frequency representation, and get

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s}{1 + G_1(s)G_2(s)}R(s) - \lim_{s \rightarrow 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

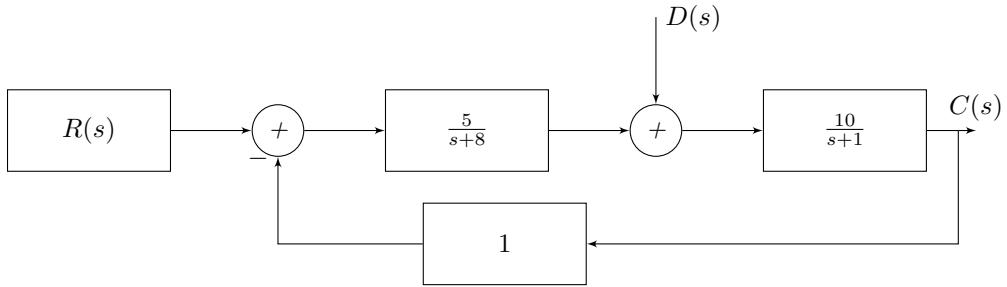
We can separate this into terms $e_R(\infty) + e_D(\infty)$.

For a step disturbance, this reduces to

$$e_{D,step}(\infty) = -\frac{1}{\lim_{s \rightarrow 0} \frac{1}{G_2(s)} + \lim_{s \rightarrow 0} G_1(s)}$$

Example 6.3. (HW5 Q4)

Find the total steady-state error due to a unit step input and a unit step disturbance in the system of the following figure.



The steady-state error can be split into that due to the input and the disturbance,

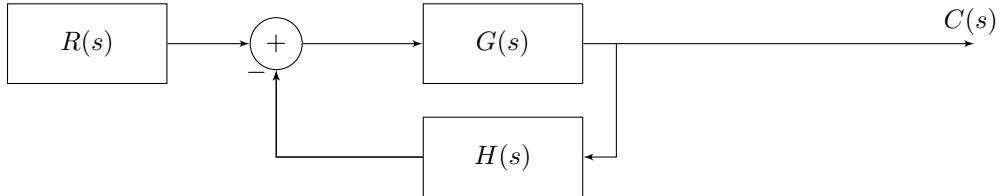
$$e_R(\infty) = \lim_{s \rightarrow 0} \frac{1}{1 + G_1(s)G_2(s)} = \frac{1}{1 + (5/8)10} = \frac{8}{58}$$

$$e_D(\infty) = \frac{-1}{\lim_{s \rightarrow 0} \frac{1}{G_2(s)} + \lim_{s \rightarrow 0} G_1(s)} = \frac{-1}{\frac{1}{10} + \frac{5}{8}} = \frac{-40}{29}$$

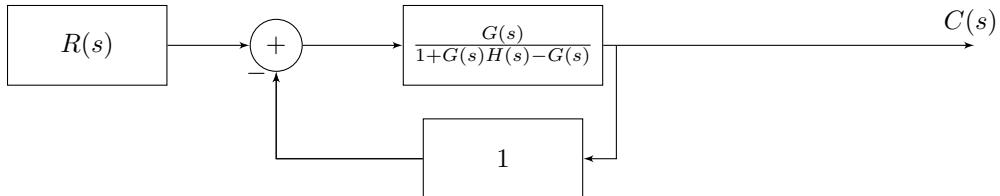
Therefore the total steady-state error is the sum of these, which comes out to -1.2414.

6.6 Steady-State Error for Nonunity Feedback Systems

So far we have only looked at systems in unity negative feedback, i.e. with a feedback arm correction equal to the output (feedback transfer function is 1). In general, though, we can say there is a feedback arm with transfer function $H(s)$ placed in feedback with a plant $G(s)$. Then, we can most easily analyze its steady-state error by forming the equivalent unity feedback system.

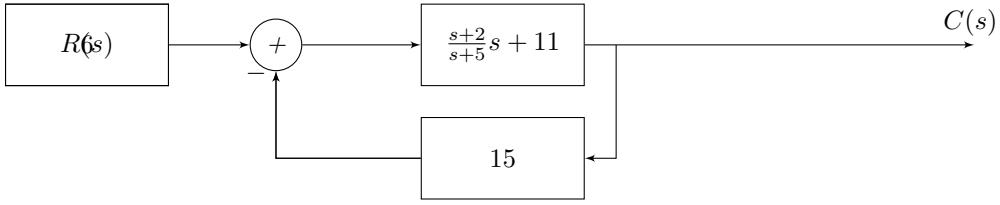


This is equivalent to the following unity feedback system:



Therefore the steady-state error can be found as usual for unity feedback systems.

Example 6.4. (HW5 Q5) For the system shown below, find the steady-state error for unit step, ramp, and parabolic inputs.



To find the error, we find the overall loop transfer function. We do this by finding the Laplace domain expression at the output and setting it equal to $C(s)$. Before the feed-forward transfer block, we have $6R(s) - 15C(s)$, so

$$\frac{s+2}{(s+5)(s+11)} (6R(s) - 15C(s)) = C(s)$$

Rearranging for $C(s)$, we find

$$C(s) = \frac{6(s+2)/(s+5)(s+11)}{1 + \frac{15(s+2)}{(s+5)(s+11)}} R(s)$$

So the error function in the Laplace domain is

$$E(s) = R(s) - C(s) = \left(1 - \frac{6(s+2)}{(s+5)(s+11) + 15(s+2)} \right) R(s)$$

For a unit-step input, we get

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = 1 - \frac{6(2)}{(5)(11) + 15(2)} \approx 0.859$$

and for the other two, the limits are this factor with an additional multiplicative factor of $\frac{1}{s}$ and $\frac{1}{s^2}$ respectively, so both of these have their error blow up to infinity.

For a system with both a disturbance and a nonunity feedback arm (say with controller $G_1(s)$, plant $G_2(s)$, rectifier $H(s)$, and disturbance between the controller and plant $D(s)$) we can combine the previous two results to get a steady-state error expression,

$$e(\infty) = \lim_{s \rightarrow 0} s \left(\left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \right] R(s) - \left[\frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \right] D(s) \right)$$

7 Root Locus Techniques

7.1 Poles of a feedback system

The stability and steady-state error of a control system are mostly determined by the positions of its poles in the frequency representation. These poles can be shifted around as desired by varying a number of parameters, mainly the feed-forward gain K . Root locus diagrams are a graphical representation of where these poles are for different values of K .

Consider a feedback loop with a gain of K followed by a system characterized by $G(s)$ in the forward arm, and a feedback component with transfer function $H(s)$ in the feedback arm. Without the feedback component, the transfer function is

$$\frac{C(s)}{R(s)} = KG(s)$$

The tuneable parameter K does not affect the locations of the poles; the only poles the system has are the poles of $G(s)$, and K only scales the output. However, with the feedback arm, we get

$$W(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

This can have its poles move based on the value of K . The poles satisfy

$$1 + KG(s)H(s) = 0$$

7.2 Root Locus Diagrams

Assume $K \geq 0$. Then, we define the *root-locus* of the system as the locations of all the poles of the closed loop system for all values $K \geq 0$. Since K changes continuously from zero to infinity, we get a number of continuous curves, one for each open-loop pole (that changes with varying gain).

To construct a root-locus diagram, we need to ask a few questions:

1. What parts of the real axis belong to the root locus? (Look between the poles and see which regions are and are not allowed)
2. How many branches would go to infinity?
3. What are the asymptotic behaviours?
4. What are the break-in and break-away points (intersections with the real axis going towards and away from negative values of the imaginary component)?

5. At what angle does the root locus depart or arrive at an end point?
6. Where do we hit the imaginary axis?

To start constructing the diagram, we write $G(s)H(s)$ as a product of zero and pole terms. In the complex plane, a complex number $\sigma + j\omega$ can be represented as a vector $M\angle\theta$. Any expression $(s + a)$, evaluated at $s = \sigma + j\omega$ can be represented as a combination of a magnitude and phase. Therefore, consider the product $G(s)H(s)$ as a combination of pole and zero distances and directions,

$$G(s)H(s) = \frac{\prod_{i=1}^m (s - z_i)}{\prod_{j=1}^n (s - p_j)}$$

We take the phase of both sides, to get

$$\angle k + \sum_{i=1}^m \angle(\bar{s} - z_i) - \sum_{j=1}^n \angle(\bar{s} - p_j) = 180(2n + 1)$$

where n is any integer.

Then, starting from each pole or zero, imagine an arc to \bar{s} , an arbitrary point. Then, we know that the sum of the angles of the zeroes, minus those of the poles, is an integer multiple of 180. (Note that the angle of K is zero, as it is purely along the real axis.)

Once we have found a point on the root locus using the phases, we can use the magnitude information to find the corresponding gain. Recall the expression $1 + KG(s)H(s) = 0$; we can rearrange this to get

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$$

Consider the expression $1 + KG(s)H(s) = 0$, which we showed above determines the closed-loop pole locations. If $K \rightarrow 0$, then \bar{s} goes to the poles of $G(s)H(s)$, because $G(\bar{s})H(\bar{s})$ goes to infinity. Similarly, if $K \rightarrow \infty$, then \bar{s} goes to the zeroes of $G(s)H(s)$. If G and H are both proper, then the number of finite zeroes is less than the number of finite poles.

When $K = 0$, the trajectories start at the n poles, and when $K \rightarrow \infty$, m trajectories converge to z_1, \dots, z_m . $n - m$ poles go towards infinity.

7.3 Branches to infinity

As $k \rightarrow \infty$, i.e. as a branch gets further and further away, all of the poles and zeroes can be approximated as being at the same point. Call this point σ and let it be the intersection of the x axis and the asymptotic line to \bar{s} . Then, in the approximation, we get

$$\begin{aligned} 1 + KG(\bar{s})H(\bar{s}) &= 0 \\ 1 + K(\bar{s} - \sigma)^{m-n} &= 0, \end{aligned}$$

the phase of which is

$$\angle = \frac{(2r+1)180}{n-m},$$

i.e. any odd multiple of π radians.

We can precisely locate σ by

$$\sigma = \frac{\sum_{j=1}^n p_j - \sum_{i=1}^m z_i}{n-m}.$$

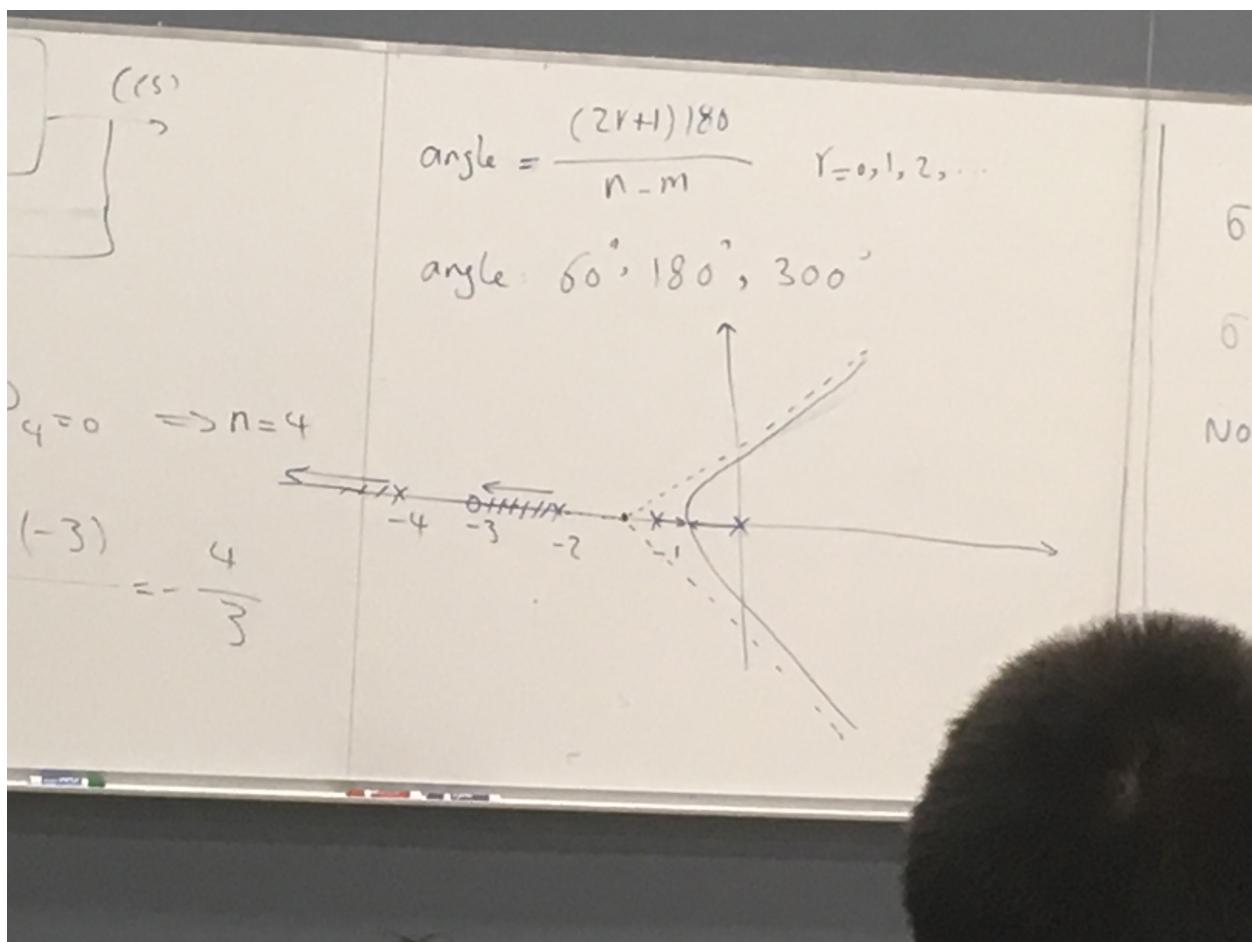
We call σ the intercept. Note that $n-m$ branches go to ∞ , all of which have the same σ . For this reason, it is also called the centroid.

7.4 Example of locating the centroid

Consider a proportional control system with feed-forward transfer function $G(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$. The poles are -1, -2, -4 and 0, and the only zero is -3. Therefore the centroid has an x coordinate of

$$\frac{0 - 1 - 2 - 4 - (-3)}{4 - 1} = -\frac{4}{3}$$

The valid angles to the position at infinity from the measured centroid come out to 60, 180, and 300 degrees. We get a plot that looks like this:



7.5 Root locus rules

Precisely finding the root locus seems to require checking every point in the complex plane for whether it has angles that add up to an odd multiple of 180° . This can be done computationally but is too laborious to give us a good idea of what the root locus looks like by hand. However, there are techniques to sketch the root locus and find out how the poles change with different gain values without this computation. To do this, we follow these five rules:

1. The number of branches is equal to the number of closed-loop poles.
2. The root locus is symmetric about the real axis (if it weren't, the closed-loop polynomial would have complex coefficients, which is unphysical.)
3. On the real axis, the root locus exists to the left of an odd number of open-loop poles or zeros. (Due to real-axis symmetry, the only angular contribution on the real axis itself is from the open-loop poles or zeros, and to the left of an even number of these, the overall angle is zero so it cannot be on the root locus, but to the left of an odd number the overall angle is 180° as required.)
4. The root locus begins at the poles of $G(s)H(s)$ and ends at the zeros of $G(s)H(s)$, the open-loop poles and zeros. (We can see this by substituting in arbitrarily small and arbitrarily large values of K , and taking limits to show that the poles and zeros tend to the open-loop ones.)

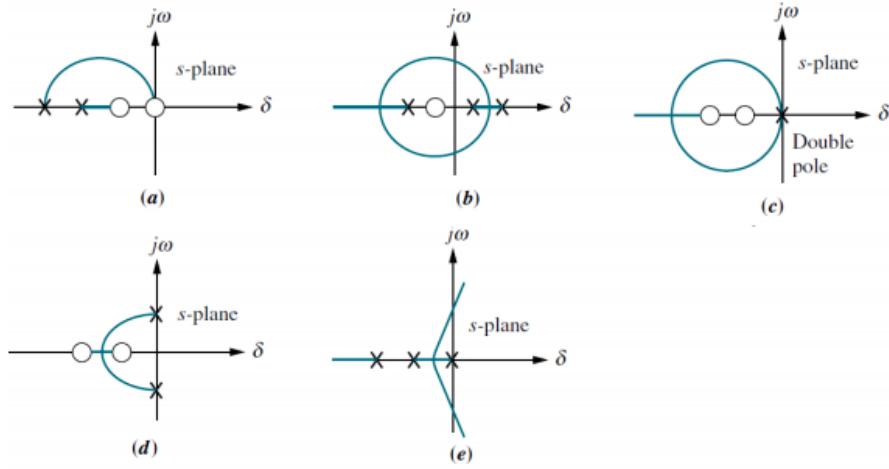


Figure 3: Example problem RL sketches

5. The root locus approaches straight-line asymptotes as it approaches the infinite zeros or leaves the infinite poles. These asymptotes are completely defined by their real-axis intercept and polar angle,

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#FP - \#FZ}$$

$$\theta_a = \frac{(2k+1)\pi}{\#FP - \#FZ}$$

Example 7.1. (HW6 Q1) For each of the given root loci in Figure 3, identify whether or not the sketch is a valid root locus, and if it is not, explain.

- (a) This is not a valid root locus, as it is not symmetric about the real axis, and it exists on the real axis to the left of an even number of open-loop poles or zeros.
- (b) This is not a valid root locus, as it exists on the real axis to the left of an even number of open-loop poles or zeros. Additionally, it ends at an open-loop pole instead of starting there.
- (c) This is not a valid root locus, as it exists on the real axis to the left of an even number of open-loop poles or zeros.
- (d) This is a valid root locus.
- (e) This is a valid root locus.

7.6 Break-away and break-in points

We know that

$$G(\bar{\sigma})H(\bar{\sigma}) = -\frac{1}{k}$$

To find break-away and break-in points (where the root locus leaves and enters the real axis), which are respectively local minima and maxima of σ , we take a derivative and set it equal to zero. Using the product expansion into poles and zeros, we can show that this is equivalent to the sum of reciprocals of the pole terms being equal to the sum of reciprocals of the zero terms.

$$\frac{d}{d\sigma} \left(\frac{\prod_i (\sigma - z_i)}{\prod_i (\sigma - p_i)} \right) = 0$$

We can apply the quotient rule, and notice that since the right hand side is zero, it is sufficient to say the numerator must be zero, i.e.

$$\prod_i (\sigma - p_i) \frac{d}{d\sigma} \left(\prod_i (\sigma - z_i) \right) = \prod_i (\sigma - z_i) \frac{d}{d\sigma} \left(\prod_i (\sigma - p_i) \right)$$

Using the product rule, we can see that the derivative of a product of n factors of the form $(\sigma - a_i)$ is equal to the product itself times the sum of the reciprocals of each term in turn:

$$\begin{aligned} \frac{d}{d\sigma} (\sigma - a_1)(\sigma - a_2) \dots (\sigma - a_n) &= (\sigma - a_2) \dots (\sigma - a_n) + (\sigma - a_1) \dots (\sigma - a_n) + \dots + (\sigma - a_1) \dots (\sigma - a_{n-1}) \\ &= ((\sigma - a_1)(\sigma - a_2) \dots (\sigma - a_n)) \left(\frac{1}{\sigma - a_1} + \frac{1}{\sigma - a_2} + \dots + \frac{1}{\sigma - a_n} \right) \end{aligned} \quad (2)$$

Therefore we can say

$$\prod_i (\sigma - p_i) \left(\sum_j \frac{\prod_k (\sigma - z_k)}{\sigma - z_j} \right) = \prod_k (\sigma - z_k) \left(\sum_j \frac{\prod_i (\sigma - p_i)}{\sigma - p_j} \right)$$

or, dividing by both of the products,

$$\sum_j \frac{1}{\sigma - z_j} = \sum_j \frac{1}{\sigma - p_j}$$

Example 7.2. Find the break-in and break-away points for a system with open-loop zeros at $s = 3, 5$ and poles at $s = -1, -2$.

$$G(s)H(s) = \frac{(s-3)(s-5)}{(s+1)(s+2)}$$

$$\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2}$$

Solving this gives us a break-in point of $\sigma = 3.82$, and a break-away point of $\sigma = -1.45$.

7.7 Sketching the root locus

Based on the above rules and methods, we can define a simple procedure to make an approximate sketch of a root locus diagram:

1. Place the given open-loop poles and zeros.
2. Fill in the real-axis portions to the left of an odd number of open-loop poles and zeros.
3. If the finite poles and zeros do not match, calculate and sketch the asymptotes to or from the infinite poles or zeros.
4. Connect any finite open-loop poles or zeros that are disconnected, such that they are symmetric about the real axis, and such that the correct break-in and break-away points are present.

Example 7.3. (HW6 Q2) For a system with open-loop zeros at $s = -8, -5$ and poles at $s = 1 \pm 2j$, sketch the root locus and find the break-in point.

The feed-forward transfer function is

$$G(s) = \frac{(s+8)(s+5)}{s^2 - 2s + 5}$$

We place the poles and zeros as specified, and fill in the root locus from -5 to -8 . There are no infinite poles or zeros. We find the break-in point:

$$\begin{aligned} \frac{1}{\sigma+8} + \frac{1}{\sigma+5} &= \frac{1}{\sigma-1-2j} + \frac{1}{\sigma-1+2j} \\ \frac{2\sigma+13}{\sigma^2+13\sigma+40} &= \frac{2\sigma-2}{\sigma^2-2\sigma+5} \end{aligned}$$

From this we get $\sigma = -6.2206, 1.5540$. We reject the latter as it is not on the root locus. Therefore we can draw curves from both of the open-loop poles to this break-in point, which coupled with the real axis portion completes the root locus.

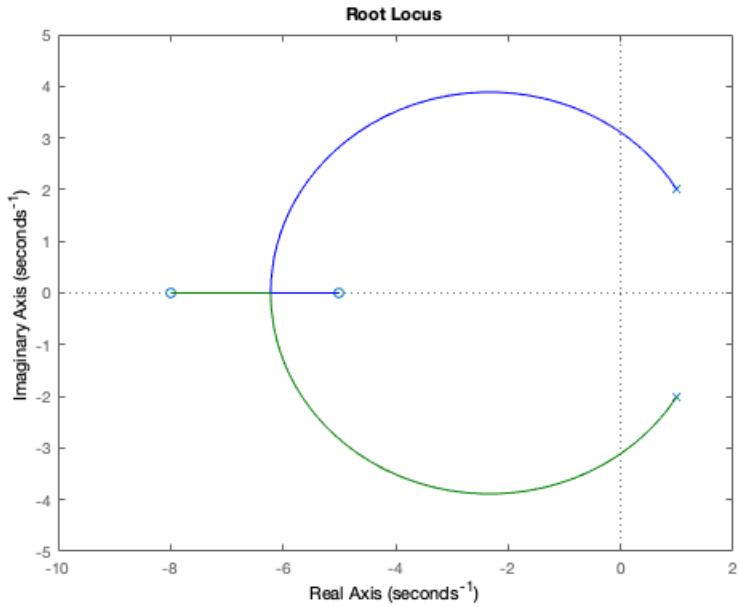


Figure 4: Root locus for $G(s) = \frac{(s+8)(s+5)}{(s^2 - 2s + 5)}$

Figure 4 shows this root locus.

Example 7.4. (HW6 Q3) For a unity feedback system with plant

$$G(s) = \frac{K(s^2 - 25)}{s^2 + 9}$$

sketch the root locus and identify the values of K for which the system is marginally stable and unstable.

The open-loop zeros are $s = \pm 5$ and the open-loop poles are $s = \pm 3j$. The real axis between $s = -5$ and $s = 5$ is filled in, as it is to the left of one open-loop zero, and the break-in point where the root locus reaches the real axis can be calculated by

$$\frac{1}{s+5} + \frac{1}{s-5} = \frac{1}{s+3j} + \frac{1}{s-3j}$$

This is true at $s = 0$. Therefore the root locus starts at $s = \pm 3j$, moves directly to the origin, and then moves along the real axis to $s = \pm 5$.

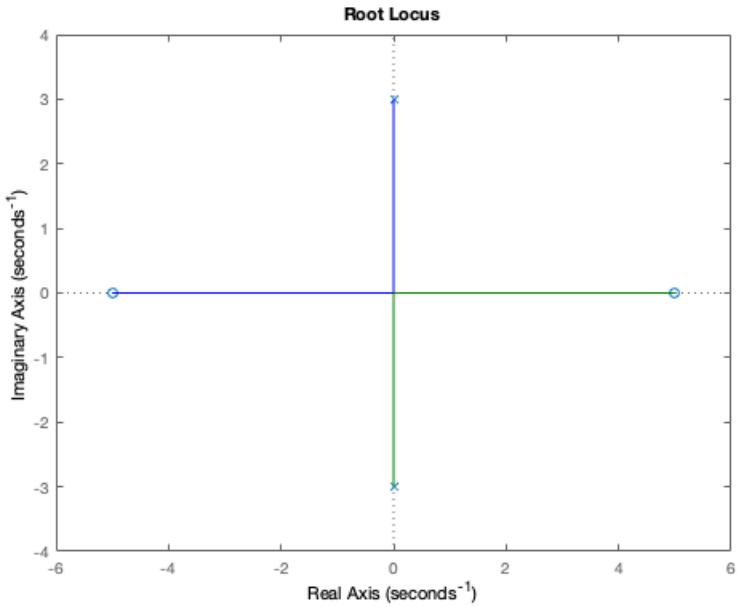


Figure 5: Root locus for $G(s) = \frac{K(s^2 - 25)}{s^2 + 9}$

Combining all of this, we get the root locus plot shown in Figure 5.

While the root locus is on the imaginary axis, the system is marginally stable. This is the case from $K = 0$, where the poles are the open-loop poles, up until the break-in point. We can calculate K at the break-in point,

$$K_{BI} = \frac{1}{|G(0)|} = \frac{9}{25}$$

After this value of K , the system becomes unstable as one of its poles moves into the right half-plane. Therefore we have marginal stability for $0 < K < \frac{9}{25}$, and instability for $K \geq \frac{9}{25}$.

Example 7.5. (HW6 Q4) Do the following for a unity feedback system with plant

$$G(s) = \frac{K(s + 4)}{s(s + 5)(s^2 + 4s + 20)}$$

- (a) Sketch the root locus.
- (b) Find the asymptotes.
- (c) Find the value of gain that will make the system marginally stable.
- (d) Find the value of gain for which the closed-loop transfer function will have a pole on the real axis at -0.5.

(a) Shown in Figure 6. Finding this requires finding the asymptotes first, so see part (b).

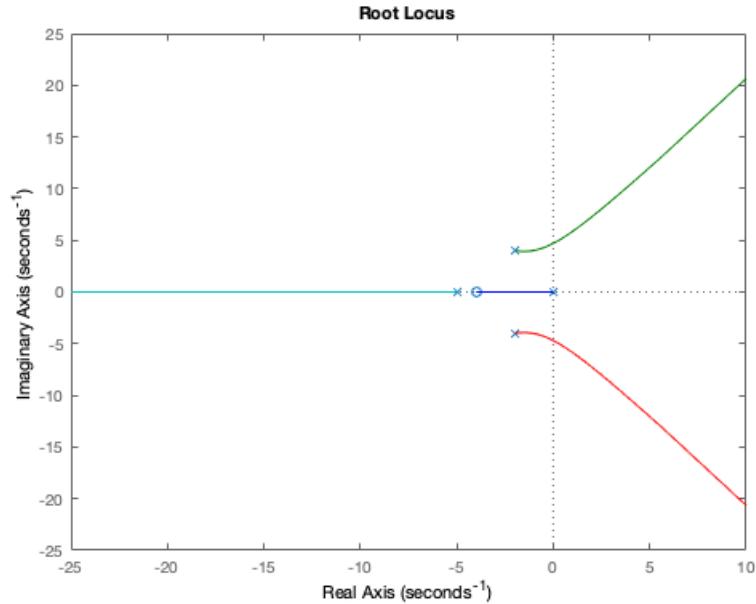


Figure 6: Root locus for $G(s) = \frac{K(s+4)}{s(s+5)(s^2+4s+20)}$

(b) We calculate the real-axis intercept for the asymptotes,

$$\sigma_a = \frac{0 + (-5) + (-2 + 4j) + (-2 - 4j) - (-4)}{4 - 1} = \frac{-5}{3}$$

and the angles,

$$\theta_a = \frac{(2k+1)\pi}{3} = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

The pole at $s = -5$ goes off to the asymptote at $-\infty$ (angle π), and the two poles at $s = -2 \pm 4j$ go asymptotically to those at $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ respectively.

(c) For the system to be marginally stable, we want poles on the $j\omega$ axis. Let $s = j\omega$. We can constrain this to points on the root locus by requiring that the angle is an odd multiple of 180° . Take the phase of $G(s)$,

$$\begin{aligned} \angle(j\omega + 4) - \angle(j\omega) - \angle(j\omega + 5) - \angle(20 - \omega - 2 + j4\omega) &= (2k+1)\pi \\ \arctan \frac{\omega}{4} - \arctan \frac{\omega}{5} - \arctan \frac{4\omega}{20 - \omega^2} &= (4k+1)\frac{\pi}{2} \end{aligned}$$

Simplifying further, we get

$$\arctan \frac{\frac{\omega}{20+\omega^2} + \frac{4\omega}{20-\omega^2}}{1 + \frac{4\omega^2}{(20-\omega^2)(20+\omega^2)}} = (4k+1)\frac{\pi}{2}$$

We expect the denominator to go to zero, in order to get $\arctan \infty = \frac{\pi}{2}$. Therefore this reduces to

$$\begin{aligned} \frac{4\omega^2}{(20+\omega^2)(20-\omega^2)} &= -1 \\ \omega^4 - 4\omega^2 - 400 &= 0 \\ \omega &\approx \pm 4.701 \end{aligned}$$

We find the corresponding gain,

$$K = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}} = \frac{\omega \sqrt{5^2 + \omega^2} \sqrt{(\omega - 4)^2 + 2^2} \sqrt{(\omega + 4)^2 + 2^2}}{\sqrt{4^2 + \omega^2}} \approx 98.8961$$

- (d) We can find the gain required for a pole at -0.5 by computing the inverse magnitude of $G(s)H(s)$ at $s = -0.5$,

$$K = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}} = \frac{0.5 \times 4.5 \times (4^2 + 1.5^2)}{3.5} = 11.73$$

7.8 Dominant Poles

It is often helpful in a second-order approximation to define the dominant poles. We can approximate the higher-order system to second order by only selecting the two dominant poles. We say that poles p_1 and p_2 are dominant if they are closed-loop poles, and if the real part of any other closed loop poles is at least 5 times as much as the real parts of p_1 and p_2 . More concretely, if p_1 and p_2 have real parts σ_1 and σ_2 , then the constraint on any other pole p_i with real part σ_i is $|\sigma_i| \geq 5\sigma_1$ and the same for σ_2 .

In this case, the transfer function is approximated to

$$T(s) \approx \frac{L}{(s - p_1)(s - p_2)}$$

If p_1 and p_2 are dominant poles, and every zero of $T(s)$ that is not far away in the left side of (p_1, p_2) is close to other poles of $T(s)$ for cancellation, then we can design a system with variable gain based on the root locus.

Recall that the real part of a pole is given by $\zeta\omega_n$ and the imaginary part is $\omega_n\sqrt{1-\zeta^2}$. The angle to the pole is given by $\theta = \tan^{-1}\frac{\omega}{\sigma}$ (if we write the pole as $\sigma + j\omega$); the settling time is $\frac{4}{\sigma}$, the peak time is $\frac{\pi}{\omega}$; and the overshoot is a function of ζ , specifically $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ which is therefore a function of θ .

We can therefore design for a transient response via gain adjustments. Given a desirable overshoot, we can find the corresponding angle θ , and draw lines to find intersections with the root locus.

Example 7.6. (HW7 Q1) For a unity feedback system in which the plant is

$$G(s) = \frac{K(s+2)}{s(s+7)(s+12)}$$

find K to yield closed-loop complex poles with a damping ratio of 0.5. Does the solution require a justification of a second-order approximation?

The desired asymptotes have an angle $\theta_a = \arccos 0.5 = \pm\frac{\pi}{3}$. Therefore, we can numerically find the intersections of the asymptotic lines with the root locus and get $K = 246.0125$.

We can also approximately solve for the required gain by approximating the system to second order, cancelling $s+2$ and s . The root locus for this system consists of straight lines, with a breakaway point of $\sigma = -9.5$. This root locus matches up with the asymptotes at $s = -9.5 \pm 9.5\sqrt{3}i$. Finally, by substituting into $G(s)$, we can find the value of gain.

$$K = 1/|G(s)| = |(-2.5 \pm 9.5\sqrt{3}i)(2.5 \pm 9.5\sqrt{3}i)| = 277$$

8 Design via Root Locus

8.1 Compensators

Often, it is impractical to directly change the poles and zeros of a plant in order to get better performance in terms of steady-state error and transient response. The pole and zero positions of the plant are usually important to the specifics of the control system, and cannot be changed without changing the performance of the plant. Therefore, in order to design for improved steady-state error and transient response, a feedback control system can be *augmented* with a compensator, either cascaded with it (in feed-forward) or placed in feedback.

We have already seen that integration improves steady-state error; similarly, differentiation improves transient response. We can intuitively understand this by recognizing that the transient response is a fast change that when differentiated is a large input signal, and when the system reaches a steady state the derivative action does very little to move the system away from that desired steady state.

Ideal compensators, using pure integration and differentiation, need active networks (i.e. require op-amps, additional power sources, etc.), but compensators that are not ideal but provide similar performance can be implemented using passive networks (using only resistors, capacitors, etc.)

8.2 Compensating for Steady-State Error

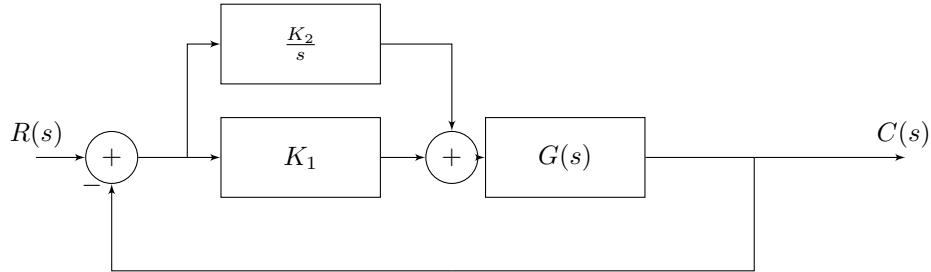
8.2.1 Proportional-Integral (PI) Control

Ideal integral compensation involves placing an open-loop pole at the origin, effectively multiplying the system by $\frac{1}{s}$. To keep the original closed-loop poles roughly where they were before compensation, a zero is placed close to the pole. (The impact of the new pole on the root locus is minimal if there is a zero with similar angles and magnitudes compensating for it at every point.) This improvement does not affect the transient response much, because the closed-loop poles and gain do not change appreciably due to the introduction of a pole-zero pair that almost perfectly cancels each other's effect on the root locus.

In general, an ideal integral compensator has a transfer function of the form

$$G_c(s) = K_1 + \frac{K_2}{s} = \frac{K_1 \left(s + \frac{K_2}{K_1} \right)}{s}$$

The location of the pole can be set by varying $\frac{K_2}{K_1}$. In the system block diagram, this can be split into two compensating blocks, a proportional block and an integral block:



Example 8.1. (HW7 Q2) Design a PI controller to drive the step response error to zero for a unity feedback system with plant

$$G(s) = \frac{K}{(s+2)(s+3)(s+12)}$$

without appreciably affecting transient response. The system operates with a damping ratio of 0.5. Compare the specifications of the uncompensated and compensated systems, i.e. percent overshoot, settling time.

To drive the steady-state error to 0, we introduce a factor of $(s+0.1)/s$ into the open-loop feed forward transfer function. $G(s)$ becomes

$$G(s) = \frac{K(s+0.1)}{s(s+2)(s+3)(s+12)}$$

and the steady-state error goes to

$$e(\infty) = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = 0$$

because of the $1/s$ factor introduced to G . By plotting the root locus before and after introducing the integrating factor, we find that the gain for the proportional case and the PI case is $K = 122$ for a damping ratio of 0.5. Using MATLAB we can find the percentage overshoot and rise time at this point for both compensated and uncompensated system. We first do this for the uncompensated system.

```

G = tf([122], [1 17 66 72]);
sys = feedback(G, 1);
stepinfo(sys)

ans =

struct with fields:

    RiseTime: 0.4545
    SettlingTime: 2.1382
    SettlingMin: 0.5827
    SettlingMax: 0.7226
    Overshoot: 14.9085
    Undershoot: 0

```

```

Peak: 0.7226
PeakTime: 1.0365

```

Then, we do this for the compensated system. Similarly as above, we get an overshoot of 15.8% and a settling time of 2.1s. So without much change in the transient response parameters, the PI system is able to drive the steady-state error to 0.

8.2.2 Lag Compensation

Introducing $\frac{1}{s}$ to the forward transfer function requires active elements. It is possible to construct a passive compensator with similar performance, by placing a pole near the origin and a corresponding zero at some desired position. To drive the steady-state error down, we want to place the zero near the origin. The compensator has the form

$$G_c(s) = \frac{K(s + z_c)}{s + p_c}$$

The static error constant for the uncompensated system, in which we take $G_c(s) = K$ and $G(s) = \prod(s + z_i)/\prod(s + p_i)$ is

$$K_p = \frac{K z_1 z_2 \dots}{p_1 p_2 \dots}$$

Introducing the new pole and zero, we get

$$K_p = \frac{(K z_1 z_2 \dots)(z_c)}{(p_1 p_2 \dots)(p_c)}$$

To improve the steady-state error, we want to increase the static error constant, so we select a z_c greater than p_c by some desired factor, and a p_c close to the origin. Similar to the PI case, we still want the pole and zero to be close to each other relative to the closed-loop poles, so that the pair has no net angular contribution to the root locus. Therefore z_c must be placed close to the origin as well. For example, $z_c = -0.01$ and $p_c = -0.001$ is a valid pair that creates a factor of 10 reduction in the steady-state error.

A lag compensator can be designed given a required reduction in steady-state error. For example, say a system has a known $e_{old}(\infty)$ and a desired $e_{new}(\infty)$. Then

$$K_{p,old} = \frac{1 - e_{old}(\infty)}{e_{old}(\infty)}$$

$$K_{p,new} = \frac{1 - e_{new}(\infty)}{e_{new}(\infty)}$$

and the desired improvement in K_p is directly proportional to the pole-zero ratio,

$$\frac{z_c}{p_c} = \frac{K_{p,new}}{K_{p,old}}$$

Then, by arbitrarily selecting a p_c close to the origin, we get z_c and this specifies the lag compensator.

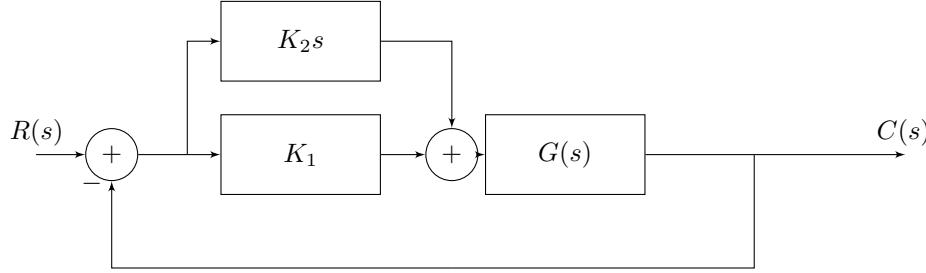
8.3 Compensating for Transient Response

Transient response can be quantified using a few different metrics. Mostly these are overshoot and settling time. Having improved steady-state error without changing the transient response much, we now want to improve the transient response.

8.3.1 Proportional-Derivative (PD) Control

In ideal derivative compensation, a pure differentiator is added to the forward arm of a feedback control system. This adds a zero to the forward transfer function,

$$G_c(s) = s + z_c$$



This case is different from the steady-state error in that we now want to change the root locus significantly so that we can cause transient response improvements. Recall that the overshoot is determined by the system's damping ratio, which in turn is determined by the angular position θ of a pole on the root locus.

Recall the relationships determining transient response for a second-order system,

$$T_s = \frac{4}{\zeta \omega_n}$$

$$\%OS = e^{-\zeta \pi / \sqrt{1-\zeta^2}} \times 100$$

8.3.2 Lead Compensation

Similarly to lag compensation, it is possible to cause derivative action without active elements (and improving the noise problems with derivatives) by slightly changing the system. This is done by introducing a pole to the system such that the angles due to the new pole plus those due to the new zero are 180° . We design a lead compensator by arbitrarily selecting either a pole or zero and finding the angular contribution of that

pole or zero, then placing a zero or pole such that this effect occurs with the overall sum of angles. (See textbook example 9.4, which I'm too *lazy* pressed for time to transcribe.)

Example 8.2. (*HW7 Q3*) A unity feedback system with plant

$$G(s) = \frac{K(s+5)}{(s+1)(s+3)(s+6)}$$

is operating with a dominant-pole damping ratio of 0.707. Design a PD controller so that the settling time is reduced by a factor of 2. Compare the transient (%OS, T_s) and steady-state ($e(\infty)$) performance of the uncompensated and compensated systems. Describe any problems with your design.

For the uncompensated system, we draw intersections with the root locus on MATLAB and find that $K = 7.1516$ for the given damping ratio. To compensate the system, we want to move the pole along this asymptote (the one at 45 degrees, or $\zeta = \frac{1}{\sqrt{2}}$) to double its distance away, from $-2.185 + 2.185i$ to $-4.37 + 4.37i$. We calculate the change in the sum of pole angles caused by this shift, and place a zero to cancel out this angular effect. (from the official solutions):

```
CL_poles = pole(feedback(K*G, 1));
target_pole = CL_poles(2) * 2;
w_d = imag(target_pole);
sigma_d = real(target_pole);
p1 = atan2(w_d, sigma_d-(-1)) * 180/pi;
p2 = atan2(w_d, sigma_d-(-3)) * 180/pi;
p3 = atan2(w_d, sigma_d-(-6)) * 180/pi;
z1 = atan2(w_d, sigma_d-(-5)) * 180/pi;
pole_angle = z1 - (p1+p2+p3);
z2 = -180 - pole_angle;
compensated.zero = -sigma_d - w_d/tand(180 - z2);
C = s + compensated.zero;
```

From this, we get that the compensated zero is at -9.0904 . Simulating a step response, we get that the settling time reduced by a factor of 1.9 and the overshoot increased by 2.5%. The settling time reduction is not exactly 2 because the approximation to a second order system is slightly inaccurate.

8.4 Compensating for Steady-State Error and Transient Response

8.4.1 Proportional-Integral-Derivative (PID) Control

PI control was useful in order to reduce steady-state error without affecting transient response much, and PD control was useful in order to reduce overshoot and rise time without affecting steady-state error much. Now, naturally, we combine the two, so as to reduce both. Use your imagination for the PID block diagram because I'm not dealing with three arrows to a sum block.

The transfer function for a PID controller is

$$G_c(s) = K_1 + \frac{K_2}{s} + K_3 s = \frac{K_3 \left(s^2 + \frac{K_1}{K_3} s + \frac{K_2}{K_3} \right)}{s}$$

This has two zeros, and a pole at the origin. The pole and one of the zeros can be designed to carry out integral compensation, and the other zero can be set to carry out derivative compensation.

To design a PID controller, we do the following:

1. Design a PD controller to meet transient response specifications, fixing one of the zeros and the overall gain.
2. Check that the requirements have been met, and adjust.
3. Design a PI controller to yield the required steady-state error.
4. Determine the gain values K_1, K_2, K_3 based on these designs.
5. Check that the requirements have been met, and adjust.

Example 8.3. *For a unity feedback system with plant*

$$G(s) = \frac{K}{(s+2)(s+5)}$$

design a PID controller that will yield a peak time of 0.785s and a damping ratio of 0.7809, with zero steady-state error for a step input.

(to be done later)

8.4.2 Lag-Lead Compensation

The natural analogue of the PID controller as a combination of PI and PD is the lag-lead compensator, in which we first design a lead compensator, moving a pole to adjust transient response, and then a lag compensator, placing a pole-zero pair to improve steady-state error. (Example 9.6 in the textbook)

8.5 Physical Realization of Compensation

We can physically implement the compensators described above in two types of electrical networks: active and passive. Active networks can be implemented using an op-amp with two impedances (networks of resistors and capacitors), and passive networks can be implemented just with the passive resistors and capacitors.

8.5.1 Active-Circuit Realization

The transfer function of an inverting op-amp is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

where Z_1 is the impedance connected in series with the input, and Z_2 is that connected in parallel with the op-amp. The structure in which the two impedances are being divided allows us to easily pattern-match with the forms of compensators.

Note that the setup for a lag or a lead compensator is the same. The additional constraint specifying one is in the values of the passive elements; for a lag compensator $R_2C_2 > R_1C_1$ and for a lead compensator $R_1C_1 > R_2C_2$.

Function	$Z_1(s)$	$Z_2(s)$	Overall TF
Gain	R_1	R_2	$-\frac{R_2}{R_1}$
Integration	R	C	$-\frac{1}{RCs}$
Differentiation	C	R	$-RCs$
PI controller	R_1	$R_2 + \frac{1}{Cs}$	$-\frac{R_2}{R_1} \frac{(s + \frac{1}{R_2 C})}{s}$
PD controller	$\frac{R_1}{1+R_1C_1s}$ (C and R_1 in parallel)	R_2	$-R_2 C \left(s + \frac{1}{R_1 C} \right)$
PID controller	$\frac{R_1}{1+R_1C_1s}$	$R_2 + \frac{1}{C_2 s}$	$- \left[\left(\frac{R_2}{R_1} + \frac{C_1}{C_2} \right) + R_2 C_1 s + \frac{1}{R_1 C_2 s} \right]$
Lag/lead compensation	$\frac{R_1}{1+R_1C_1s}$	$\frac{R_2}{1+R_2C_2s}$	$- \frac{C_1}{C_2} \frac{(s + \frac{1}{R_1 C_1})}{(s + \frac{1}{R_2 C_2})}$

Table 1: Active realization of controllers and compensators

Example 8.4. (Textbook Example 9.9) Implement a PID controller with transfer function

$$G_c(s) = \frac{(s + 55.92)(s + 0.5)}{s}$$

We can match terms in the transfer function to those in the general form of an active-circuit realization of a PID controller. The transfer function is

$$G_c(s) = s + 56.42 + \frac{27.96}{s}$$

Therefore, matching terms to the general form of a PID controller, we get

$$\begin{aligned} R_2 C_1 &= 1 \\ \frac{R_2}{R_1} + \frac{C_1}{C_2} &= 56.42 \\ \frac{1}{R_1 C_2} &= 27.96 \end{aligned}$$

Note that there are three constraints on the four variables, so we can arbitrarily select one and pick the others based on it. Let $C_2 = 0.1\mu F$. Then $R_1 = 357.65k\Omega$, which leaves us with

$$\frac{R_2}{357650} + \frac{10^7}{R_2} = 56.42$$

which gives us $R_2 = 178.891k\Omega$, and $C_1 = 5.6\mu F$.

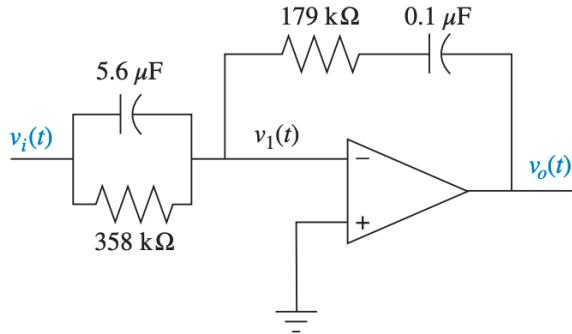


Figure 7: The implemented PID controller

8.5.2 Passive-Circuit Realization

Lag, lead, and lag-lead compensators can all be implemented with passive networks that do not require an op-amp. To do this, we design a network such that its transfer function as determined by KVL and KCL is of the required form of one of these compensators.

This is too much circuitikz for too little useful or satisfying output, so

TABLE 9.11 Passive realization of compensators

Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation		$\frac{R_2}{R_1 + R_2} \frac{s + \frac{1}{R_2 C}}{s + \frac{1}{(R_1 + R_2)C}}$
Lead compensation		$\frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}}$
Lag-lead compensation		$\frac{\left(s + \frac{1}{R_1 C_1}\right)\left(s + \frac{1}{R_2 C_2}\right)}{s^2 + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1}\right)s + \frac{1}{R_1 R_2 C_1 C_2}}$

Figure 8: Passive-circuit realization of compensators

Example 8.5. (HW8 Q1) Identify and realize the following compensators with passive networks. Assume the reference passive component is $10\mu F$.

(a)

$$0.1 \frac{s + 0.1}{s + 0.01}$$

(b)

$$\frac{s + 2}{s + 6}$$

(c)

$$\left(\frac{s + 0.1}{s + 0.01} \right) \left(\frac{s + 1}{s + 10} \right)$$

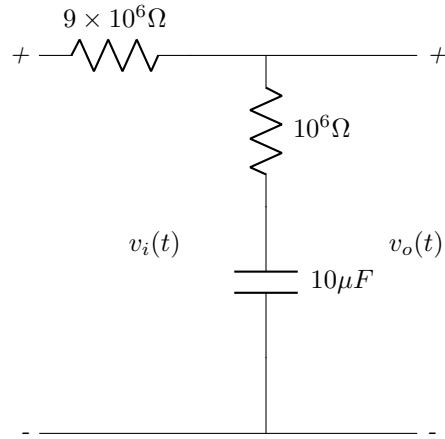
(a) The system can be realized using a lag compensator. Matching the terms to the form of a lag compensator, we get

$$0.1 = \frac{R_2}{R_1 + R_2}$$

$$\frac{1}{R_2 C} = 0.1 \implies R_2 = 10^6 \Omega$$

$$(R_1 + R_2)C = 100 \implies R_1 + R_2 = 10^7 \Omega \implies R_1 = 9 \times 10^6 \Omega$$

Therefore the transfer function is achieved using a lag compensator with these values.

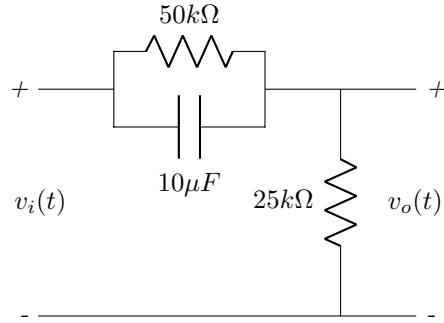


(b) The system can be realized using a lead compensator. Matching the terms to the form of a lead compensator, we get

$$\frac{1}{R_1 C} = 2 \implies R_1 = \frac{1}{2 \times 10 \times 10^{-6}} = 50k\Omega$$

$$\frac{1}{R_1 C} + \frac{1}{R_2 C} = 6 \implies R_2 = \frac{1}{4 \times 10 \times 10^{-6}} \implies R_2 = 25k\Omega$$

Therefore the transfer function is achieved using a lead compensator with these values.



- (c) The system can be realized using a lag-lead compensator. Matching the terms to the form of a lag-lead compensator, we get

$$\frac{1}{R_1 C_1} = 1$$

$$\frac{1}{R_2 C_2} = 0.1$$

$$\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} = 10.01$$

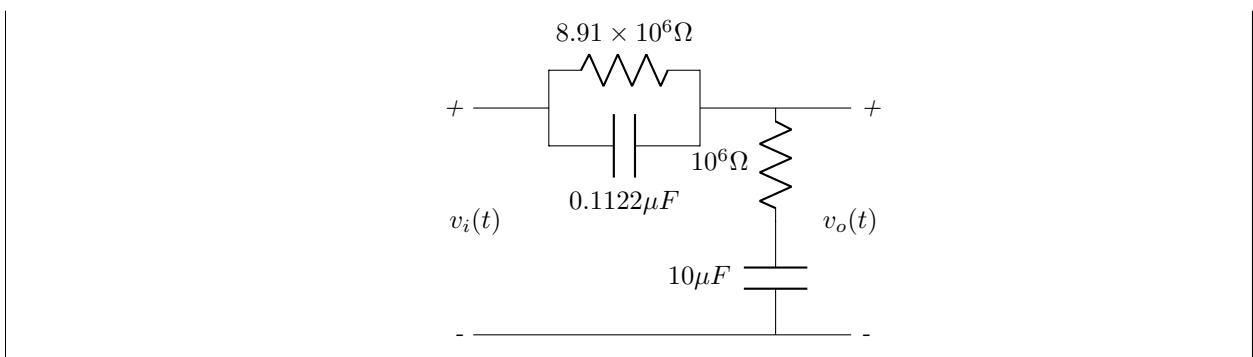
We get

$$R_2 = \frac{1}{10^{-6}} = 10^6\Omega$$

$$\frac{1}{R_2 C_1} = 10.01 - 0.1 - 1 \implies C_1 = 0.1122\mu F$$

$$R_1 = \frac{1}{C_1} = 8.91 \times 10^6\Omega$$

Therefore the transfer function is achieved using a lag-lead compensator with these values.



9 Frequency Response Techniques

9.1 Analytically Finding Magnitude and Phase

The frequency response of a system is the combination of magnitude and phase responses to a pure frequency input such as a sinusoid. We define the magnitude frequency response to be the ratio of the magnitude of the output sinusoid to that of the input sinusoid, and the phase response to be the difference in phase angle between the output and input sinusoids.

We can obtain the magnitude and phase frequency response by substituting in $s \rightarrow j\omega$ into a transfer function, and solving for the magnitude and phase of the resultant complex number.

Example 9.1. (HW8 Q2) Find analytical expressions for the magnitude and phase response for each $G(s)$ below.

(a)

$$G(s) = \frac{1}{s(s+2)(s+6)}$$

(b)

$$G(s) = \frac{(s+3)}{s(s+2)(s+6)}$$

(c)

$$G(s) = \frac{(s+3)(s+5)}{s(s+2)(s+6)}$$

(a) Using the fact that the magnitude of a product of terms is the product of magnitudes, we get

$$|G| = \frac{1}{\omega\sqrt{\omega^2 + 4\sqrt{\omega^2 + 36}}}$$

and using the fact that phases add, we get

$$\angle G = -\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{6}\right)$$

(b) We introduce one term each to the magnitude and phase of the above,

$$|G| = \frac{\sqrt{\omega^2 + 9}}{\omega\sqrt{\omega^2 + 4\sqrt{\omega^2 + 36}}}$$

$$\angle G = \tan^{-1}\left(\frac{\omega}{3}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{6}\right)$$

(c) We introduce one term each to the magnitude and phase of the above,

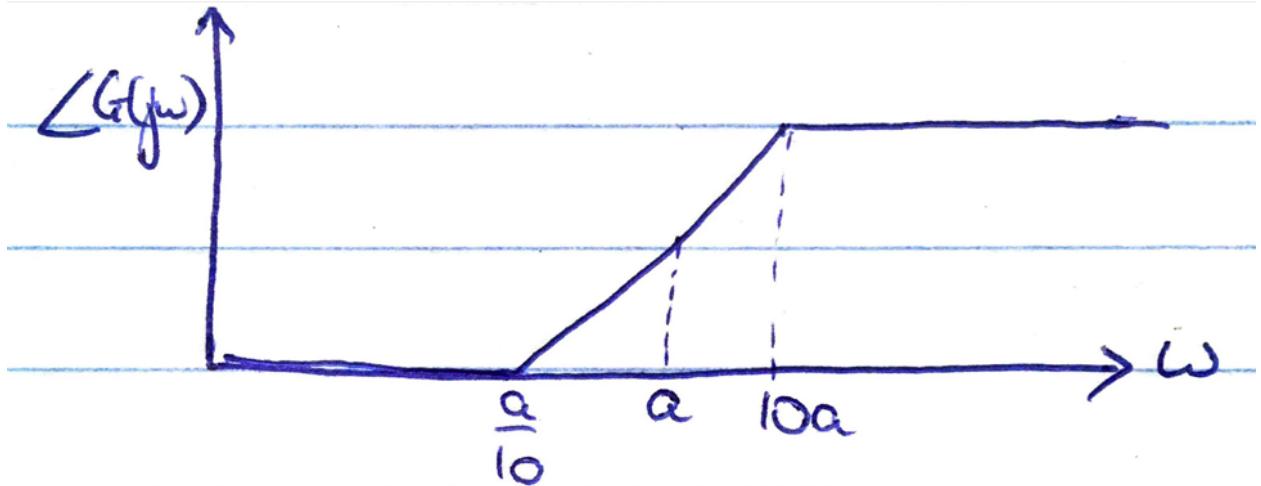
$$|G| = \frac{\sqrt{\omega^2 + 9}\sqrt{\omega^2 + 25}}{\omega\sqrt{\omega^2 + 4}\sqrt{\omega^2 + 36}}$$

$$\angle G = \tan^{-1}\left(\frac{\omega}{5}\right) + \tan^{-1}\left(\frac{\omega}{3}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{6}\right)$$

9.2 Bode Plots

Bode plots allow us to represent $20 \log |G(j\omega)|$ and $\angle G(j\omega)$ in terms of $\log(\omega)$. Consider a function $G(s) = s + a$; $G(j\omega) = j\omega + a = a(j\frac{\omega}{a} + 1)$. If ω is small then $G(j\omega) \approx a$, and if ω is very large then $G(j\omega) \approx j\omega = \omega \angle 90^\circ$.

A Bode plot can be constructed entirely with straight lines. These are simplifications of true curves, but are fairly accurate. For a Bode plot of $s + a$, low frequencies have an angle of 0, and high frequencies have an angle of 90° . At a medium point, say $\omega = a$, then the angle is 45 degrees, so the linear approximation looks like this:



Suppose $a = 1$. Then $G(s) = s + 1$, and $20 \log_{10} a = 20 \log 1 = 0$. So to the left of 1 on the dependent ($\log \omega$) axis, the Bode plot is just 0.

We can draw Bode plots for higher-order transfer functions by splitting them up into a product of first-order transfer functions, and separately considering each component's magnitude and phase.

$$G(s) = \frac{s+3}{s(s+1)(s+2)} = \frac{(3/2)\left(\frac{s}{3} + 1\right)}{s(s+1)\left(\frac{s}{2} + 1\right)}$$

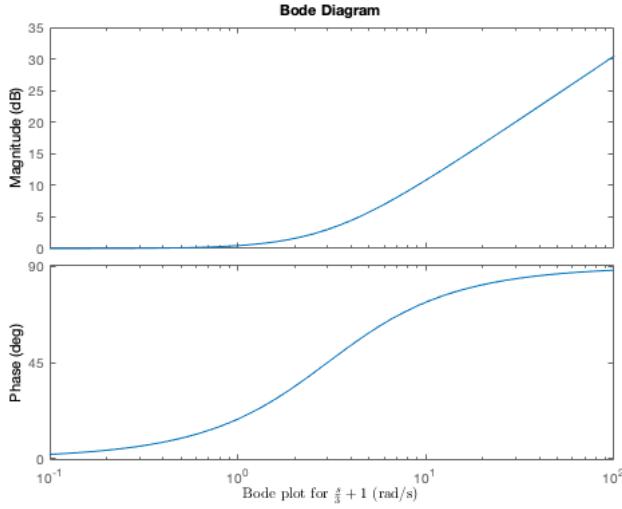


Figure 9: Bode plot for $\frac{s}{3} + 1$

Then, we separately plot each factor. First we consider $\frac{s}{3} + 1$. Figure 9 shows the Bode plot for this.

Then, $\frac{1}{s+2}$, as shown in Figure 10.

Then, $\frac{1}{s+1}$, as shown in Figure 11.

Finally, we look at $\frac{3}{2} \times \frac{1}{s}$, in Figure 12.

To combine all of these, we find the breakpoints, i.e. the points at which the slope on the magnitude plot changes. These turn out to be 1, 2, 3. Before 1, we calculate the sum of slopes. The only term to have a nonzero slope is $\frac{3}{2} \times \frac{1}{s}$, which has a slope of 20. Between 1 and 2, similarly, the slope is -40, between 2 and 3 it is -60, and after 3 it is -40.

Therefore based on that we can construct a composite graph, shown in Figure 13.

This is only a magnitude plot. Looking at the above phase plots, we can do something similar for the composite phase.

Example 9.2. (HW8 Q3) Sketch the Bode asymptotic magnitude and asymptotic phase plots for

$$G(s) = \frac{1}{s(s+2)(s+6)}$$

We rewrite $G(s)$ with a phase input as

$$G(j\omega) = \frac{1}{12} \frac{1}{j\omega} \frac{1}{1+j\omega/2} \frac{1}{j\omega/6}$$

The magnitude is

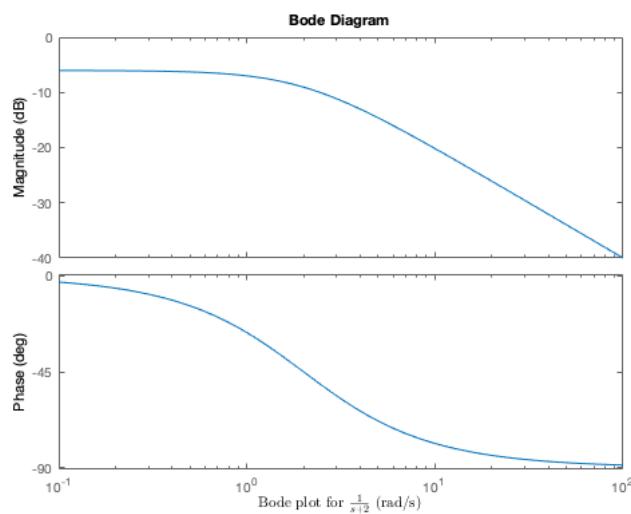


Figure 10: Bode plot for $\frac{1}{s+2}$

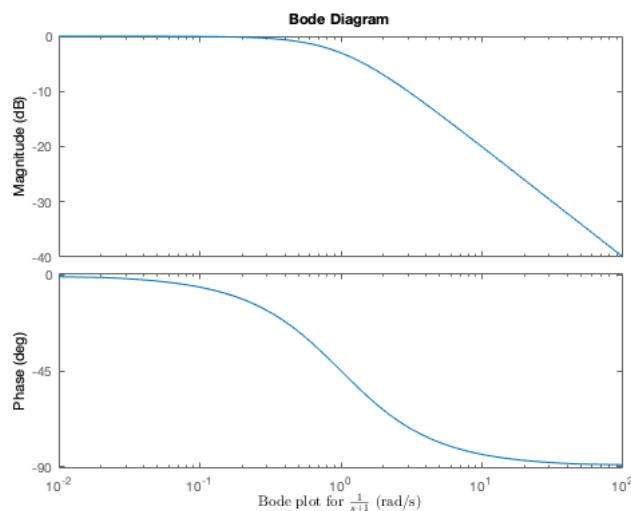


Figure 11: Bode plot for $\frac{1}{s+1}$

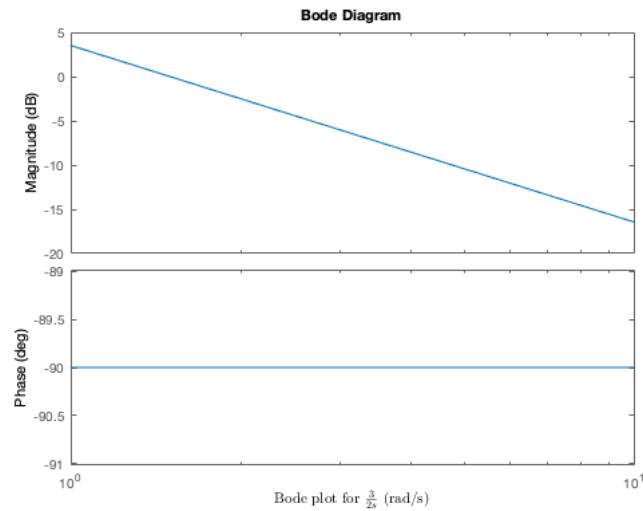


Figure 12: Bode plot for $\frac{3}{2} \times \frac{1}{s}$

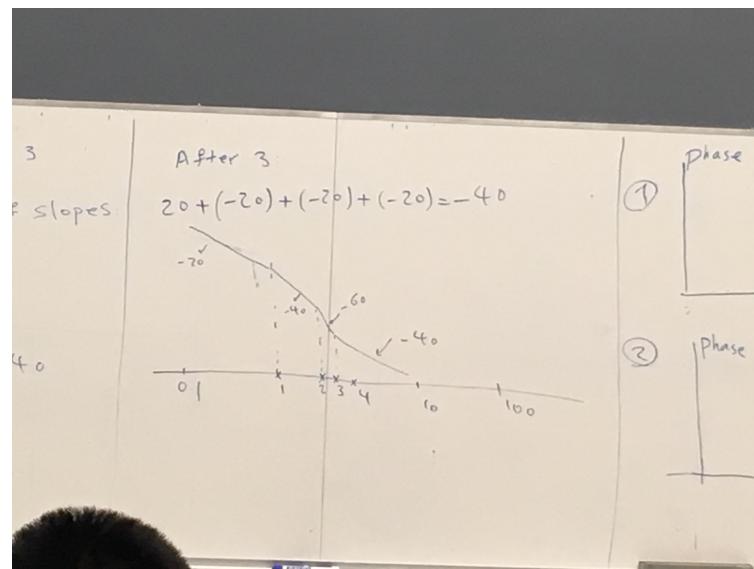


Figure 13: Bode plot for the whole function $G(s)$

$$M = 20 \log |G(j\omega)| = -20 \log(12) - 20 \log |j\omega| - 20 \log \left| j \frac{\omega}{2} + 1 \right| - 20 \log \left| j \frac{\omega}{6} + 1 \right|$$

The breakpoints are 2 and 6. For $\omega < 2$ the slope is -20dB/decade , as we approximate the last two components as having no magnitude; for $2 < \omega < 6$ the slope is -40dB/decade , as we approximate only the $j\omega/6$ component as having no appreciable magnitude; for $\omega > 6$ the slope is -60dB/decade , as all the components are significant.

From the slope information and one point on the graph, we can construct the approximate Bode magnitude plot. By substituting in $\omega = 1$ and considering the magnitude approximation, we get $M(\omega = 1) \approx -20 \log(12) \approx -21.6$. Therefore we can construct a magnitude plot.

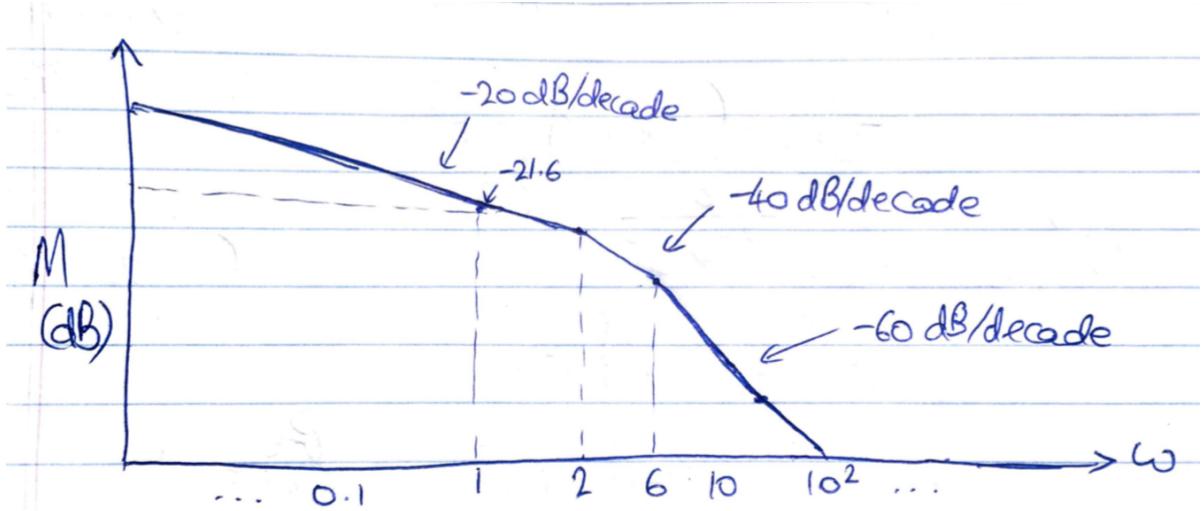
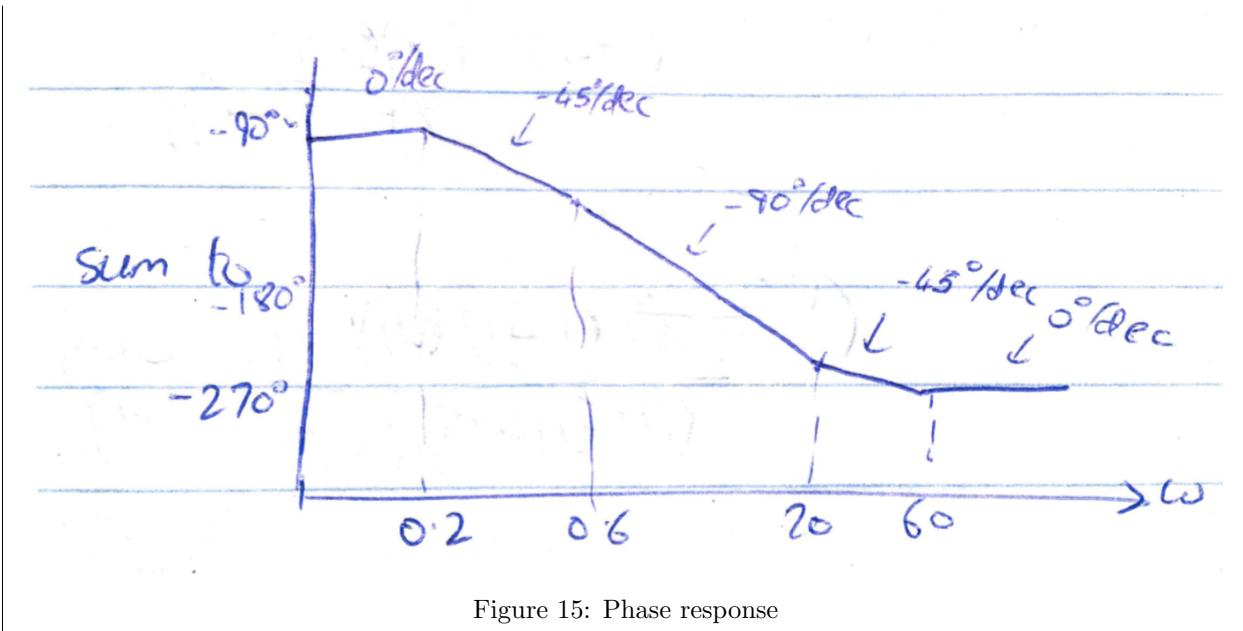


Figure 14: Magnitude response

For the phase, we use the aforementioned approximation, in which any term of the form $s/a + 1$ has zero phase up until $\frac{a}{10}$, a phase of 45 degrees from $\frac{a}{10}$ until $10a$, and a phase of 90 degrees thereafter. Therefore the phase changes at $\omega = 0.2$, $\omega = 0.6$, $\omega = 20$, and $\omega = 60$. Note that here the phases are negative because the terms are in the denominator.



9.3 The Nyquist Criterion

The Nyquist criterion is another graphical method of analyzing a system in frequency. It relates the stability of a closed-loop system to the open-loop frequency response, and can also be extended to transient response and steady-state errors, providing an alternate approach to the root locus.

The Nyquist criterion works by plotting a diagram, and the construction of this diagram requires the concept of mapping contours. Consider a collection of points in the s -plane, called a contour, and consider a function of the form

$$F(s) = \frac{(s - z_1)(s - z_2)\dots}{(s - p_1)(s - p_2)\dots}$$

A contour can be mapped to another contour through a function, by substituting every point of the input contour into s and considering the points $F(s)$ for each one. If we assume a clockwise direction for mapping the points on an input contour A , then contour B maps in a clockwise direction if $F(s)$ has just zeros, or has just poles that are not encircled by the contour, and B maps in a counterclockwise direction if $F(s)$ has just poles that are encircled by the contour.

This concept helps us define the Nyquist criterion. We know that the stability of a system is defined by whether it has any poles in the right half-plane, which for a feedback system with forward transfer function $G(s)$ and feedback transfer function $H(s)$, corresponds to solutions to $1 + G(s)H(s) = 0$. As a consequence of the above relationship between the direction of a mapping and poles and zeros, we know that each pole or zero factor of $1 + G(s)H(s)$ whose vector undergoes a complete rotation around contour A yields a complete rotation in contour B . We can count these complete rotations N , and from that relate the number of counterclockwise rotations about the origin in contour B to the number of poles P and zeros Z of $1 + G(s)H(s)$: $N = P - Z$.

To examine stability, we let the contour A cover the entire right half-plane. We do this by letting A span the extent of the $j\omega$ axis, and a semicircle surrounding the right half-plane from $j\infty$ to $-j\infty$. The mapping of

this contour through $F(s) = G(s)H(s)$ is called the system's Nyquist diagram. Since the $+1$ was dropped, we actually look for encirclements of $s = -1$.

9.4 Sketching the Nyquist Diagram

To sketch the Nyquist diagram, we have to map the right half-plane contour through the function $G(s)H(s)$. To do this, we first let $s \rightarrow j\omega$ to cover the imaginary axis. Then, for the rest, we look at the vectors of $G(s)H(s)$ and let their magnitude go to infinity, analyzing their angle. By finding real-axis crossings on the $j\omega$ axis and substituting back into $G(j\omega)H(j\omega)$, we can find the real-axis closed-loop poles, which will be important for stability.

Example 9.3. (HW9 Q1) Sketch the Nyquist diagram for a unity feedback system with $G(s) = \frac{10}{s(s+3)(s+6)}$.

Let $s \rightarrow j\omega$, and solve for the Cartesian representation of the function:

$$G(j\omega)H(j\omega) = \frac{-90\omega^2 - 10(18\omega - \omega^3)j}{81\omega^4 + (18\omega - \omega^3)^2}$$

This has real-axis crossings (i.e. points where the imaginary part is zero) at $\omega = \pm 3\sqrt{2}$. Substituting back in, we get $G(j\omega)H(j\omega)|_{\omega=\pm 3\sqrt{2}} = -\frac{5}{81}$. Substituting in points in between, such as $-1, 0, 1$, we see that for $-3\sqrt{2} < \omega < 0$, $\text{Im}(GH) > 0$ and for $0 < \omega < 3\sqrt{2}$, $\text{Im}(GH) < 0$. This allows us to draw the diagram around -1 and find the encirclements.

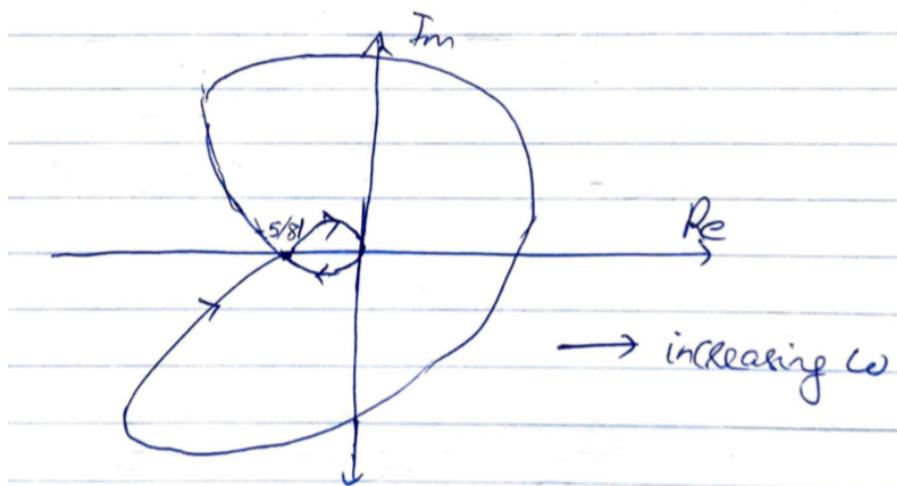


Figure 16: Nyquist diagram for the given system

If there are open-loop poles on the $j\omega$ axis, we can construct an alternative input contour that moves around them in either direction. These must be infinitesimally small detours, otherwise some right half-plane poles may be excluded or some left half-plane poles may be included.

9.5 Using the Nyquist Criterion for Stability

An advantage of the Nyquist criterion is that the function being used is $G(s)H(s)$, i.e. any feed-forward gain is linear in the relevant function. We can imagine the Nyquist diagram growing and shrinking with varying values of K until it intersects with -1, at which point it is unstable.

Example 9.4. Find the range of K for stability for both of the systems given below.

- (a) $G(s) = K/(s+2)$, $H(s) = 1/(s+4)(s+6)$
- (b) $G(s) = K(s^2 - 4s + 13)/(s+2)(s+4)$, $H(s) = 1/s$

(a) The polynomial to analyze is

$$G(s)H(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

To apply the Nyquist criterion, we let $s \rightarrow j\omega$:

$$G(j\omega)H(j\omega) = \frac{K(j\omega - 2)(j\omega - 4)(j\omega - 6)}{-(\omega^2 + 4)(\omega^2 + 16)(\omega^2 + 36)} = \frac{K((12\omega^2 - 48) + j(-\omega^3 + 44\omega))}{-(\omega^2 + 4)(\omega^2 + 16)(\omega^2 + 36)}$$

To get the range of K , we compute the real-axis crossings by setting the imaginary part to 0 and setting $K = 1$. We get $\omega = 0$ and $\omega = \pm 2\sqrt{11}$. Substituting back in, we get $G(j0)H(j0) = \frac{48}{4 \times 16 \times 36} = \frac{1}{48}$ which does not affect stability as it is positive, and $G(j2\sqrt{11}) = \frac{12(44)-48}{-(44+4)(44+16)(44+36)} = -\frac{1}{480}$. Therefore the range of stability is $K < 480$ as up until this gain value the Nyquist plot does not cross -1, i.e. the closed-loop transfer function $GH + 1$ does not blow up.

(b) The polynomial to analyze is

$$G(s)H(s) = \frac{K(s^2 - 4s + 13)}{s(s+2)(s+4)}$$

To apply the Nyquist criterion, we let $s \rightarrow j\omega$:

$$\left(\frac{10\omega^2 - 110}{(\omega^2 + 4)(\omega^2 + 16)} \right) + j \left(\frac{-\omega^4 + 45\omega^2 - 104}{\omega(\omega^2 + 4)(\omega^2 + 16)} \right)$$

To get the range of K , we compute the real-axis crossings by setting the imaginary part to 0 and setting $K = 1$. We get $\omega = \pm 1.5633$ and $\omega = \pm 6.5235$. The corresponding real-axis positions are -0.719902 and 0.115754 . Therefore the range of stability is $0 < K < 1/(0.7199) = 1.389$.

The above can also be done numerically using MATLAB, instead of via analytically finding real-axis crossings and shifting them with a gain.

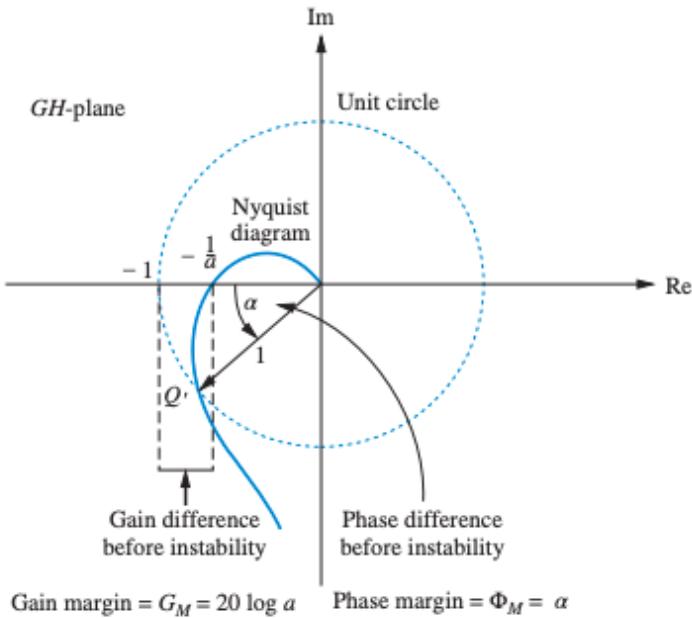


Figure 17: Gain and phase margin

9.6 Gain and Phase Margin

The Nyquist diagram allows us to define two qualitative measures of system stability: gain margin and phase margin. The gain margin is the change in open-loop gain, in dB (i.e. $M = 20 \log |GH|$, required at 180° of phase shift to make the closed-loop system unstable, and the phase margin is the change in open-loop phase shift required at unity gain to make the closed-loop system unstable. The higher both of them are, the more stable the system. Figure 17 shows the gain and phase margin graphically.

The given definitions of gain and phase margin provide intuitively-correct ways to calculate them. For gain margin, we calculate the K required to make the system unstable as in the above example, and convert to dB. For phase margin, we solve by finding the ω input that yields a unity gain frequency, and substitute this ω back into the transfer function and compute the resultant phase. If no valid ω exists, we say the phase margin is infinite.

Example 9.5. (HW9 Q3) For each system of the above example, find the gain margin and phase margin if the value of K in each part is (a) $K = 1000$ (b) $K = 100$ (c) $K = 0.1$.

We will solve for phase margin in general for both systems, and then substitute in specific values of K . To find the phase margin, we solve the transfer function for the unity gain frequency, i.e. find ω such that $|G(j\omega)H(j\omega)| = 1$. In the first case, this is

$$\frac{K}{(j\omega + 2)(j\omega + 4)(j\omega + 6)}$$

The magnitude is equal to the magnitude of the numerator terms divided by those on the denominator, so we want

$$1 = \frac{K}{\sqrt{\omega^2 + 4\sqrt{\omega^2 + 16\sqrt{\omega^2 + 36}}}}$$

$$K^2 = (\omega^2 + 4)(\omega^2 + 16)(\omega^2 + 36)$$

We can solve this for specific values of K .

In the second case, we get

$$1 = \frac{K\sqrt{(13 - \omega^2)^2 + 16\omega^2}}{\omega\sqrt{\omega^2 + 4\sqrt{\omega^2 + 16}}}$$

$$K^2 = \frac{\omega^2(\omega^2 + 4)(\omega^2 + 16)}{169 + \omega^4 - 10\omega^2}$$

- (a) In the first system, at $K = 1000$, there is a real-axis crossing at $-\frac{1000}{480}$, therefore the gain for a real-axis crossing at -1 is 0.48. Therefore the gain margin is $20 \log 0.48$. In the second system, at $K = 1000$, there is a real-axis crossing at -719.9 , therefore the gain for a real-axis crossing at -1 is 0.001389. Therefore the gain margin is $20 \log 0.001389$.

For phase margin, in the first system, we find $\omega = 9.0654$ rad/s (solving with MATLAB's fzero). Substituting this back into the transfer function, we get a phase of -20.28° . In the second system, we find $\omega \approx 1000$ rad/s. Substituting this back into the transfer function, we get a phase of 90.6° .

- (b) In the first system, at $K = 100$, there is a real-axis crossing at $-\frac{100}{480}$, therefore the gain for a real-axis crossing at -1 is 4.8. Therefore the gain margin is $20 \log 4.8$. In the second system, at $K = 100$, there is a real-axis crossing at -71.99 , therefore the gain for a real-axis crossing at -1 is 0.01389. Therefore the gain margin is $20 \log 0.01389$.

For phase margin, in the first system, we find $\omega = 2.54$ rad/s. Substituting this back into the transfer function, we get a phase of 72.7° . In the second system, we find $\omega = 99.8$ rad/s. Substituting this back into the transfer function, we get a phase of 95.7° .

- (c) In the first system, at $K = 0.1$, there is a real-axis crossing at $-\frac{0.1}{480}$, therefore the gain for a real-axis crossing at -1 is 4800. Therefore the gain margin is $20 \log 4800$. In the second system, at $K = 0.1$, there is a real-axis crossing at -0.07199 , therefore the gain for a real-axis crossing at -1 is 13.89. Therefore the gain margin is $20 \log 13.89$.

For phase margin, in the first system, we find no valid ω , so the phase margin is said to be infinite. In the second system, we find $\omega = 0.162$ rad/s. Substituting this back into the transfer function, we get a phase of 80.2° .

10 Design via State Space

Designing control systems via transfer functions is a limited approach, because transfer functions are necessarily linear and single-input single-output. The state-space formulation, in which a system is represented as a set of first-order differential equations, is much more general. We will use state-space methods to control the locations of closed-loop poles.

10.1 Different state-space representations

There are three forms in which a system is usually represented in state space: phase-variable form, observer canonical form, and cascade form. (There are others but they aren't relevant for this class, as far as I know.)

In phase-variable form, the state variables are the output and $n - 1$ of its derivatives. The matrix formulation looks like

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} u$$

and

$$y = [1 \ 0 \ 0 \ \dots \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

We can translate a system to phase-variable form by constructing its transfer function and reading off the required coefficients from that.

Example 10.1. (HW10 Q1) Given the system

$$\dot{x} = \begin{bmatrix} -30 & -10 \\ -20 & -35 \end{bmatrix} x + \begin{bmatrix} 1 \\ 5 \end{bmatrix} u, y = [2 \ 1] x$$

find the transformation P such that (\bar{A}, \bar{B}) is in phase variable form, where $\bar{A} = P^{-1}AP$ and $\bar{B} = P^{-1}B$, and find $\bar{A}, \bar{B}, \bar{C}$ such that $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$ and $y = \bar{C}\bar{x}$.

The transfer function is

$$C(sI - A)^{-1}B = \frac{7s + 100}{s^2 + 65s + 850}$$

from which we get the phase-variable form,

$$\begin{aligned}\bar{A} &= \begin{bmatrix} 0 & 1 \\ -850 & -65 \end{bmatrix} \\ \bar{B} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \bar{C} &= [100 \quad 7]\end{aligned}$$

We can get the transformation by solving for P in $\bar{A} = P^{-1}AP$; we get

$$P = \begin{bmatrix} -15 & 1 \\ 130 & 5 \end{bmatrix}$$

The second common form for a state-space representation of a system is the observer canonical form. Here, the nontrivial values are down the left column, and can be found using the coefficients of the transfer function directly. We do this by setting up the system as a sequence of integrators with gains and feedback in between, and setting the state variables as the output of each integrator.

As an example, consider the transfer function

$$\frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Placing everything in numerators by cross-multiplying and dividing throughout by the highest power of s (to get everything in the form of an integral) gives us

$$\begin{aligned}C(s) &= \frac{1}{s}(R(s) - 9C(s)) + \frac{1}{s^2}(7R(s) - 26C(s)) + \frac{1}{s^3}(2R(s) - 24C(s)) \\ C(s) &= \frac{1}{s} \left(R(s) - 9C(s) + \frac{1}{s} \left(7R(s) - 26C(s) + \frac{1}{s} (2R(s) - 24C(s)) \right) \right)\end{aligned}$$

By selecting the state variables as the outputs of the integrators, we get

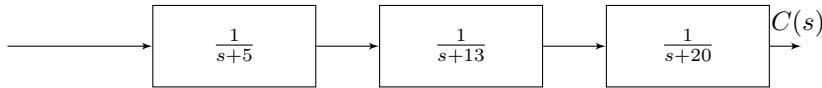
$$\begin{aligned}\dot{x}_1 &= -9x_1 + x_2 + r \\ \dot{x}_2 &= -26x_1 + x_3 + 7r \\ \dot{x}_3 &= -24x_1 + 2r\end{aligned}$$

which is represented in matrix form as

$$\dot{x} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r, y = [1 \ 0 \ 0] x$$

The third common form for a state-space representation is cascade form. Consider a forward transfer function with multiple factors, and select the state variables to be the output of each block in turn.

Example 10.2. (HW10 Q3 setup) Consider the following third-order transfer function represented as three cascaded first-order blocks,



If we let the state variables be the output of each block in turn, we get

$$X_3(s) = \frac{R(s)}{s+5} \implies \dot{x}_3 + 5x_3 = r$$

by doing an inverse Laplace transform. Similarly we get $\dot{x}_2 + 13x_2 = x_3$ and $\dot{x}_1 + 20x_1 = x_2$. This is a set of first-order differential equations in the state variables, as required, so the cascade form of the system is

$$\dot{x} = A_z x + B_z r = \begin{bmatrix} -20 & 1 & 0 \\ 0 & -13 & 1 \\ 0 & 0 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = C_z x = [1 \ 0 \ 0] x$$

10.2 Similarity transforms

These are essentially changes of coordinates between the different representations of a control system. A system represented as

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

can be translated to a similar system as described by a transformation matrix P by

$$z = P^{-1}APz + P^{-1}Bu$$

$$y = CPz + Du$$

The analogues between the matrices $A - D$ before and after the transformation are clear from the forms of the equations.

A similarity transform can be selected such that the resultant A matrix is diagonal; in this case the transformed system is the same as that obtained using partial fraction decomposition on the transfer function.

10.3 Controller Design

We can design a controller by specifying the system input as an element-wise gain on the system state rather than an arbitrary function. That is, $\dot{x} = Ax + Bu = Ax - BKx$. Then, the system poles are completely specified by the characteristic polynomial $\det(sI - (A - BK))$. A control system can then be designed by computing a desired characteristic polynomial based on given system parameters, and the two can be matched term-by-term to specify desired gain values.

Example 10.3. (HW10 Q2) Given the following open-loop plant,

$$G(s) = \frac{1}{(s+2)(s+3)(s+10)}$$

design a controller to yield a 15% overshoot and a settling time of 0.75 second. Place the third pole 10 times as far from the imaginary axis as the dominant pole pair. Use the phase variables for state-variable feedback.

First, we translate the system parameters to a desired closed-loop characteristic polynomial. From an overshoot of 15%, we get the required ζ ,

$$\zeta = -\frac{\ln 0.15}{\sqrt{\pi^2 + \ln^2(0.15)}} = 0.51693$$

and from the rise time of 0.75s, we get

$$\omega_n = \frac{4}{0.75 \times 0.51693} = 10.2637$$

Therefore, the desired closed-loop poles are

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -5.333 \pm 8.786i$$

Place the third closed-loop pole at -53.3. Then, the desired closed-loop polynomial is

$$(s + 53.3)(s + 5.333 - 8.786i)(s + 5.333 + 8.786i) = s^3 + 64s^2 + 674.527s + 5634.04$$

To find the gains to match this to, we write down the linear constant-coefficient differential equation for the system by doing an inverse Laplace transform,

$$C(s)(s^3 + 15s^2 + 56s + 60) = R(s) \implies \frac{d^3c(t)}{dt^3} + 15\frac{d^2c(t)}{dt^2} + 56\frac{dc(t)}{dt} + 60c(t) = r(t)$$

Pick state variables $x_1(t) = c(t), x_2(t) = \frac{dc(t)}{dt}, x_3(t) = \frac{d^2c(t)}{dt^2}$. Then the state-space time-evolution equation is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -56 & -15 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Let $u = -Kx$. Then the system's characteristic polynomial is the determinant of $sI - (A - BK)$, which is

$$\det(sI - (A - BK)) = s^3 + (15 + k_3)s^2 + (56 + k_2)s + (60 + k_1)$$

Matching this to the desired closed-loop characteristic polynomial, we get $K = [k_1 \ k_2 \ k_3] = [5619.04 \ 618.527 \ 49]$. These gains on each of x_1, x_2, x_3 placed in feedback with the system introduced to the input will create the desired performance specifications.

10.4 Controllability

A system is controllable if an input can be found that takes every state variable from a desired initial state to a desired final state. A system can only be designed using state-space pole placement methods if it is controllable.

Concretely, we check that an n th order plant with time-evolution equation $\dot{x} = Ax + Bu$ is controllable by checking that the controllability matrix,

$$C_M = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

is full-rank.

If a system is represented in state-space in a form other than phase-variable, the above method of controller design requires that we build the transformation from that form to phase-variable form. (If this is not the case, the gains do not match the nontrivial elements of the A matrix and the system is underspecified or inconsistent.)

Let there be a plant in some form that is not phase-variable, $\dot{z} = Az + Bu, y = Cz$. Let its controllability matrix be $C_{Mz} = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$. Suppose the system can be transformed into phase-variable form by the transformation $z = Px$. Then

$$\begin{aligned}\dot{x} &= P^{-1}APx + P^{-1}Bu \\ y &= CPx\end{aligned}$$

and the controllability matrix is

$$C_{Mx} = P^{-1} [B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

which gives us P in terms of the two controllability matrices,

$$P = C_{Mz}C_{Mx}^{-1}$$

Once the system is in phase-variable form, the gains can be designed as in the example, and this system can be translated back to the original representation by $K_z = K_x P^{-1}$.

10.5 Observer Design

The above method of controller design relies on full access to the state of the system at any given time. This is not always guaranteed; for example, in the inverted pendulum lab, the state was $[x \ \dot{x} \ \theta \ \dot{\theta}]^T$, but the only accessible state variables were x and θ as they could be physically measured. The best approximations to the derivatives were found by numerically differentiating the two output signals, which is a noisy process that proved to be highly error-prone.

When part of a system state cannot be directly accessed, we construct an *observer* to calculate the rest of the state.

Let the plant have dynamics $\dot{x} = Ax + Bu, y = Cx$ and let the corresponding observer have dynamics $\dot{\hat{x}} + A\hat{x} + Bu, \hat{y} = C\hat{x}$. Then the error between the estimated and actual states converges slowly, at the rate of the plant's own transient response. To make it converge faster, we employ a similar strategy of multiplying a constant gain matrix, say L , with the output error as we did the state for a controllable system. Therefore the observer ends up having a time-evolution equation $\dot{\hat{x}} + A\hat{x} + Bu + L(y - \hat{y})$. This allows us to express the time-evolution of the state error and from there that of the output error,

$$\begin{aligned}\dot{e}_x &= (A - LC)e_x \\ e_y &= Ce_x\end{aligned}$$

What remains is to design L . We do this in the same way that we designed K , by computing the characteristic equation and matching it to a desired characteristic equation. In this case the characteristic equation is $\det(sI - (A - LC))$. We compute a desired characteristic equation based on desired performance specifications, and match term by term to gain values l_i to determine the observer.

10.6 Observability

Similarly to the controllability matrix, the observability matrix gives a condition under which a system represented in state-space is observable. The observability matrix is defined by

$$O_M = [C \quad CA \quad \dots \quad CA^{n-1}]^T$$

Similarly to the controllability case, the design process for an observer described above only works if the system is in observer canonical form. Let a system be represented in some other form, $\dot{z} = Az + Bu, y = Cz$ and let this system have observability matrix O_{Mz} . Then, let the transformation $z = Px$ transform the system to observer canonical form. As before, we have $\dot{x} = P^{-1}APx + P^{-1}Bu, y = CPx$. The difference is now that the transform can be found using the two observability matrices, $P = O_{Mz}^{-1}O_{Mx}$. Also similarly, the observer gain vector transforms by $L_z = PL_x$.

Example 10.4. (HW10 Q3)

The system in cascade form was calculated in a previous example,

$$\begin{aligned}\dot{x} &= A_z x + B_z r = \begin{bmatrix} -20 & 1 & 0 \\ 0 & -13 & 1 \\ 0 & 0 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \\ y &= C_z x = [1 \quad 0 \quad 0] x\end{aligned}$$

This allows us to find the observability matrix for this form.

Multiplying out the given transfer function to convert to observer canonical form, we get

$$G(s) = \frac{1}{s^3 + 38s^2 + 425s + 1300}$$

In observer canonical form, this is

$$\hat{\dot{x}} = \begin{bmatrix} -38 & 1 & 0 \\ -425 & 0 & 1 \\ -1300 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \hat{y} = [1 \quad 0 \quad 0] \hat{x}$$

The time-evolution of the error is given by

$$\dot{e}_x = (A - LC)e_x = \begin{bmatrix} -(38 + l_1) & 1 & 0 \\ -(425 + l_2) & 0 & 1 \\ -(1300 + l_3) & 0 & 0 \end{bmatrix} e_x$$

which gives us the gain matrix in observer canonical form

$$L = \begin{bmatrix} 562 \\ 39575 \\ 1498700 \end{bmatrix}$$

To transform it back to cascade form, we calculate the transformation matrix using the observability matrices,

$$P = O_{MZ}^{-1} O_{MX} = \begin{bmatrix} 1 & 0 & 0 \\ -18 & 1 & 0 \\ 25 & -5 & 1 \end{bmatrix}$$

and so

$$L_z = PL_x = \begin{bmatrix} 562 \\ 29459 \\ 1314875 \end{bmatrix}$$