Bhaat iszzh the ekchuahl beloo oph $\frac{1}{\infty}$

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January 12, 2019

1 Motivation

So about four years ago, Pranshu Malik, aka Bhai, misinformed you about the nature of $\frac{1}{\infty}$, but never fear: here I am, to set the record straight.

2 The original statement

Dekh bhai, is maal ki ham *limit* nikaal saktein hain. Iske liye pen-pencil ka intezaam karle phata phat. Train nikal rahi hai.

$$\lim_{\chi \to \infty} \left(\frac{1}{\chi} \right) = 0$$

This can be phroobed by the phollowing:

$$\lim_{\chi \to \infty^+} \left(\frac{1}{\chi} \right) = \lim_{h \to 0} \left(\frac{1}{\infty + h} \right) = 0 \tag{1}$$

$$\lim_{\chi \to \infty^-} \left(\frac{1}{\chi} \right) = \lim_{h \to 0} \left(\frac{1}{\infty - h} \right) = 0 \tag{2}$$

Strangely, the original statement uses χ (the Greek letter chi) and not x as the variable, so I'm going to switch so that this makes much more sense.

3 The problem

Basically, Pranshu's statement does not actually prove anything. All he did was a change of variable on the limit, rather than provide a proof. If it were intuitive that $\lim\limits_{h \to 0} \frac{1}{\infty + h} = 0$, then the simpler fact that $\lim\limits_{h \to \infty} \frac{1}{h} = 0$ should also be considered intuitive, which it is not. If it were, this would be a futile exercise. The fact that I'm writing this four years later may mean that it was a futile exercise, but I'm doing it anyway. (Also, ∞^+ and ∞^- don't mean anything as far as I know; what would it even mean to approach positive infinity from the right?)

4 Ab bhai tu *asli* maths pe aaja

The intuitive notion of a limit is formalized by an *epsilon-delta* statement, which states the following:

Theorem. *If* $\lim_{x \to a} f(x) = L$, *then for every* $\epsilon > 0$, *there exists a* $\delta > 0$, *such that for every* x, $0 < |x - a| < \delta$ *implies* $|f(x) - L| < \epsilon$.

Intuitively, look at the $|x - a|$ and $|f(x) - L|$ as statements about distances on a number line: if x is within δ of the target value a, then it is guaranteed that $f(x)$ will be within ϵ of the target value L.

5 Example

For a simple example, consider a linear function of the form $f(x) = bx + c$. Then, the limit we want to show is

$$\lim_{x \to a} bx + c = ba + c \tag{3}$$

which, in terms of the epsilon-delta statement, says that for every $\epsilon > 0$, there exists a $\delta > 0$, such that for every x, $0 < |x - a| < \delta$ implies $|(bx + c) - (ba + c)| < \epsilon$.

The trick in these proofs is the selection of δ in terms of ϵ, which turns out to be complicated for nonlinear functions but is straightforward here. We want to choose δ such that the following is true:

$$|(bx + c) - (ba + c)| < \epsilon \implies |b(x - a)| < \epsilon \tag{4}$$

We can make this into a statement about x plus a constant just by dividing by b:

$$|x - a| < \frac{\epsilon}{b} \tag{5}$$

which is of the form $|x - a| < \delta$ as we require, for the choice $\delta = \frac{\epsilon}{b}$. So we can begin the proof now.

Proof. Let $\epsilon > 0$. We need to prove that a δ exists for an arbitrary choice of ϵ, so suppose the value of this δ is given by $\delta = \frac{\epsilon}{b}$ and we will show that this choice satisfies the definition of a limit.

Suppose $0 < |x - a| < \delta$, that is,

$$|x - a| < \frac{\epsilon}{b} \tag{6}$$

Then, multiplying by b,

$$|bx - ba| < \epsilon \tag{7}$$

Finally, we can rewrite this in the form of the original function and show the final statement on ϵ,

$$|(bx + c) - (ba + c)| < \epsilon \tag{8}$$

as required. This completes the proof. $\square$

6 The specific case: $\frac{1}{\infty}$

We can use this kind of argument to actually prove, rigorously, that $\frac{1}{\infty}$ tends to zero. Epsilon-delta arguments are interesting in that they require you to sort of assume what you need to prove. To find a suitable δ in terms of ϵ, you need to know through more conventional limit solving methods (factorization, L'Hôpital's rule, numerical methods, or just direct substitution) what the limit actually is before you can prove it. Note that you shouldn't actually assume the limit; that's why the part of the $\epsilon - \delta$ argument where you find δ isn't part of the proof and is designated as scratch work. For the purposes of the proof, we assume we can just conjure up a valid δ out of thin air or pure instinct, and the universe will go "well all right then".

Anyway, I've procrastinated on this long enough. Let's see how we deal with $\frac{1}{x}$, a nonlinear function.

$$\lim_{x \to \infty} \frac{1}{x} = 0 \tag{9}$$

Since a is not a determinate real number here, we will have to modify the statement slightly. In the language of the $\epsilon - \delta$ proof, this translates to: *for every $\epsilon > 0$, there exists an N such that if $x > N$, then $\left|\frac{1}{x}\right| < \epsilon$.* We have switched δ to N just so that we can have clear terminology; δ usually refers to a small number, which is not the case here, whereas N has no such connotations.

The choice of N is fairly easy here, since changing the inequality at the start from one on x to one on $\frac{1}{x}$ requires only one simple operation and no strange logic as you will see in the next example. We want to choose N such that

$$\left|\frac{1}{x}\right| < \epsilon \tag{10}$$

which we can quickly change to become

$$|x| > \frac{1}{\epsilon} \tag{11}$$

This is equivalent to the required $|x| > N$, therefore we can say $\epsilon = \frac{1}{N}$. Now, we begin the proof.

Proof. Let $\epsilon > 0$ and let $N(\epsilon) = \frac{1}{\epsilon}$. Starting with the condition on x, that is,

$$|x| > N \tag{12}$$

we can convert this into a statement on $f(x) = \frac{1}{x}$:

$$\left|\frac{1}{x}\right| < \frac{1}{N} = \epsilon \tag{13}$$

This completes the proof. $\square$

It is not very clear just from the mathematical line of reasoning why that completes the proof. In words, this says that if x is sufficiently large (greater than some N), then the distance of its reciprocal from zero is less than some ϵ related to N; a condition on the input to the function determines the bounds on the output, and as x gets arbitrarily large (i.e. it approaches infinity), its reciprocal gets arbitrarily close to zero.

It may seem like $\epsilon - \delta$ logic can be twisted to conclude whatever you want. Suppose we had started with the assumption that the limit is 1, not 0; that is, we state that as a number gets arbitrarily large, its reciprocal gets arbitrarily close to 1. To refute this, we will need to study how to deal in more generality with the limit of a nonlinear function.

7 Nonlinearity in $\epsilon - \delta$ in general

It may be interesting to look at an $\epsilon - \delta$ limit that is not resolved so easily, by taking a reciprocal.

$$\lim_{x \to 1} \frac{1}{x} = 1 \tag{14}$$

This can be restated as: *for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then* $\left|\frac{1}{x} - 1\right| < \epsilon$.

To prove this, we select a δ in the same way as we did for the linear function; via scratch work that is not part of the proof but will allow us to magically choose the right δ based on the conclusion.

$$\left|\frac{1}{x} - 1\right| < \epsilon \tag{15}$$

We want to change this into a statement on $|x - 1|$, which we can partially do by taking a common denominator of x and rewriting a little bit:

$$\frac{|x - 1|}{|x|} < \epsilon \tag{16}$$

To resolve the problem with the $|x|$, we can set bounds on δ. Say, semi-arbitrarily, $\delta = \frac{1}{2}$, i.e. we start our proof with the expression

$$|x - 1| < \frac{1}{2} \tag{17}$$

which can be rewritten as

$$\frac{1}{2} < |x| < \frac{3}{2} \tag{18}$$

or

$$\frac{2}{3} < \frac{1}{|x|} < 2 \tag{19}$$

We only care about the upper limit here, because we are using this expression in the statement equivalent to $|f(x) - L| < \epsilon$ where only the upper limit matters. So we can say

$$\frac{|x-1|}{|x|} = |x-1|\frac{1}{|x|} < 2\delta \tag{20}$$

using the limits placed on both $|x-1|$ and $\frac{1}{|x|}$ previously. Therefore, we can say $\delta = \frac{\epsilon}{2}$ in this case, where δ is $\frac{1}{2}$ or less. If $\frac{\epsilon}{2} > \frac{1}{2}$, then we can take δ as $\frac{1}{2}$ and this will still be valid. More simply, we can say

$$\delta = \min\left(\frac{\epsilon}{2}, \frac{1}{2}\right) \tag{21}$$

Now, we can begin the proof.

Proof. Let $\epsilon > 0$ and let $\delta = \min\left(\frac{\epsilon}{2}, \frac{1}{2}\right)$. Then, we start with a condition on δ,

$$|x-1| < \delta = \min\left(\frac{\epsilon}{2}, \frac{1}{2}\right) \tag{22}$$

Multiplying by 2,

$$2|x-1| < 2\delta = \min(1, \epsilon) \tag{23}$$

We want to translate this into a statement on $|f(x) - L|$, which we do as follows:

$$\left|\frac{1}{x} - 1\right| = \left|\frac{x-1}{x}\right| < 2|x-1| \tag{24}$$

for the particular choice of $\delta = \frac{1}{2}$ or greater. If less, we know that the tighter bound of $\delta = \frac{\epsilon}{2}$ is required. If $\delta < \frac{1}{2}$, then the same logic holds, because

$$|x-1| < \delta < \frac{1}{2} \implies \frac{1}{|x|} < 2 \tag{25}$$

Therefore, we can connect these two inequalities via the $2|x-1|$ term:

$$\left|\frac{1}{x}-1\right| < \min(1,\epsilon) \leq \epsilon \tag{26}$$

which had to be shown. This completes the proof. $\square$

8 An Incorrect Initial Assumption

Above, I addressed the problem of the $\epsilon - \delta$ reasoning not feeling airtight. With the "magical" nature of the choice of δ, what's to stop you from going through this same process to find a δ corresponding to every ϵ surrounding an incorrect limit? Let's try this (I'm not too sure of my reasoning here, so take this bit with a grain of salt.) We want to prove the hypothesis that

$$\lim_{x\to\infty}\frac{1}{x} \neq x_0 \tag{27}$$

for some $x_0 \neq 0$. This will be done via contradiction. We set out to prove the converse, which in $\epsilon - \delta$ terms states that *for every* $\epsilon > 0$ *there exists an* N *such that if* $x > N$*, then* $\left|\frac{1}{x} - x_0\right| < \epsilon$.

The "scratch work" to find N can proceed similarly:

$$\left|\frac{1}{x} - x_0\right| < \epsilon \tag{28}$$

$$-\epsilon + x_0 < \frac{1}{x} < \epsilon + x_0 \tag{29}$$

$$\frac{1}{x_0-\epsilon} > x > \frac{1}{x_0+\epsilon} \tag{30}$$

Immediately we see something is weird, because the initial assumption is that x is unbounded upwards; you would expect that if that were the condition, we wouldn't immediately find an upper bound by assuming the fact to be proved. For now, we ignore this and continue. This argument gives us $N = \frac{1}{x_0+\epsilon}$, or $\epsilon = \frac{1}{N} - x_0$. Now, we attempt the proof.

Proof. Assume that the converse of what we want to show is true, i.e. that for every $\epsilon > 0$ there exists an N such that if $x > N$, then $\left|\frac{1}{x} - x_0\right| < \epsilon$ for some $x_0 \neq 0$.

Start, as usual, with the condition on the input variable:

$$|x| > N \tag{31}$$

and rewrite it to be similar to the function limit condition,

$$\left|\frac{1}{x}\right| < \frac{1}{N} \tag{32}$$

$$\frac{-1}{N} < \frac{1}{x} < \frac{1}{N} \tag{33}$$

$$\frac{-1}{N} - x_0 < \frac{1}{x} - x_0 < \frac{1}{N} - x_0 \tag{34}$$

We select the stronger condition for a bound on the absolute value assuming without loss of generality that $x_0 > 0$, i.e.

$$\left|\frac{1}{x} - x_0\right| < \frac{1}{N} + x_0 = \epsilon \tag{35}$$

which is not the ϵ that was originally selected, and so it cannot be shown that for *any* choice of ϵ there exists a valid N, because this does not follow from the original inequality. In fact, a stronger statement can be made: certain ϵ can be selected such that there is no valid N. If $\epsilon = x_0$, which is a valid choice because we have assumed $x_0 > 0$ strictly and it still satisfies the initial condition $\epsilon > 0$, then no real N exists that is valid. Therefore the $\epsilon - \delta$ proof is invalid unless $x_0 = 0$, at which point this becomes an invalid choice for ϵ. However, this contradicts the initial assumption. Therefore, we can say that $\lim_{x\to\infty} \frac{1}{x} = 0$ and the proof is complete. □

Without the assumption that $x_0 > 0$, the proof is similar; in the case $x_0 < 0$, the other choice of limit (that is, $\frac{1}{N} - x_0$) is taken as being greater and the choice $\epsilon = -x_0$ has no corresponding real N.

This proof is not actually complete, as the method is flawed; just because the choice of $N(\epsilon)$ that flows naturally from the arguments we are familiar with did not work correctly, we cannot conclude that *no such* $N(\epsilon)$ exists. A *much* better way to do this, however, instead of invoking proof by contradiction and trying to do $\epsilon - \delta$ on an incorrect hypothesis, is to say that a function cannot have two limits at a point, and to show the actual limit. It turns out that you can also use $\epsilon - \delta$ to show this.

Theorem. *If* $\lim_{x\to a} f(x) = L_1$*, and* $\lim_{x\to a} f(x) = L_2$*, then* $L_1 = L_2$*.*

Proof. Both limits can be stated in terms of $\epsilon - \delta$ notation:

$$\forall \epsilon > 0 \exists \delta_1 s.t. |x - a| < \delta_1 \implies |f(x) - L_1| < \epsilon \tag{36}$$

$$\forall \epsilon > 0 \exists \delta_2 s.t. |x - a| < \delta_2 \implies |f(x) - L_2| < \epsilon \tag{37}$$

Taking both of these as true, we take $\delta = \min(\delta_1, \delta_2)$. Therefore both statements are true; by definition, $\delta \leq \delta_1$ therefore $|f(x) - L_1| < \epsilon$, and by definition, $\delta \leq \delta_2$ therefore $|f(x) - L_2| < \epsilon$.

We can rewrite both inequalities to get rid of the absolute values,

$$-\epsilon < f(x) - L_1 < \epsilon \tag{38}$$

$$-\epsilon < f(x) - L_2 < \epsilon \tag{39}$$

Subtracting the two,

$$-\epsilon - (-\epsilon) \leq L_2 - L_1 \leq \epsilon - \epsilon \tag{40}$$

$$0 \leq L_2 - L_1 \leq 0 \tag{41}$$

Note the switch to $\leq$ from $<$ in this step. This is because both of the initial statements were about error bars on $f(x)$; that $f(x)$ was in an interval $[L - \epsilon, L + \epsilon]$. When subtracting, these include the possibility of equality because the error bars may be the same and cancel one another out, which turns out to be the case.

This last statement can be easily seen to be equivalent to

$$L_1 = L_2 \tag{42}$$

This completes the proof. $\square$

Basically, don't be dumb, and do the limit correctly before you try and prove it.

9 Nonexistent Limits

Unlike what Pranshu asserted, the inverse question to the one that motivated the original discussion does not follow quite the same line of reasoning as that one. Namely, the limit under discussion is

$$\lim_{x \to 0} \frac{1}{x} \tag{43}$$

which diverges to positive or negative infinity. We can see this using $\epsilon - \delta$.

First, we try and select a δ. This seems strange because there is no limiting value of the function. To resolve this, we assume there is one. Suppose $\lim\limits_{x \to 0} \frac{1}{x} = L$, where L is finite. It turns out that without even selecting a δ the proof is possible, so let's do it:

Proof. Suppose $L > 0$. We want to show that there exists a δ for every ϵ that satisfies the usual property; however, since we are proving the converse, we can choose a specific ϵ and show that it does not work. Say $\epsilon = L$; then,

$$0 < |x| < \delta \implies \left| \frac{1}{x} - L \right| < L \implies x > \frac{1}{2L} \tag{44}$$

This is false for any negative value of x $(-\delta < x < 0)$, so no positive value of L can be valid.

Similarly, suppose $L < 0$ and say $\epsilon = -L$. Then,

$$0 < |x| < \delta \implies \left| \frac{1}{x} - L \right| < -L \implies x < \frac{1}{2L} = -\left| \frac{1}{2L} \right| \tag{45}$$

This is false for any positive value of x $(0 < x < \delta)$, so no negative value of L can be valid.

The only remaining possibility is $L = 0$, that is, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x| < \delta$ implies $\left|\frac{1}{x}\right| < \epsilon$. This can be easily disproved by taking any positive ϵ and showing that $\frac{1}{\epsilon}$ is a lower, not an upper, bound on the magnitude of x. For example, $\epsilon = 1$ has $|x| > 1$ and so no valid δ exists. Therefore, L cannot be finite and so the limit does not exist. This completes the proof. $\square$

10 Indeterminate Forms

Another thing that Pranshu got wrong is his statement that $\frac{1}{\infty}$ and $\frac{1}{0}$ are *indeterminate forms*, which he reiterates a few times.

"it izzh indeterminate, therephore the answer hasz no phixzed beloo. Okay?"
"Yeh indeterminate to hai hee, (...)"
"Therefore, Limit doesn't exist and the beloo was already indeterminate, hence this is an unsolvable mystery."

However, this is incorrect. There is no possible way to write a limit that evaluates to $\frac{1}{0}$ that does not diverge to infinity, and there is no possible way to write a limit that evaluates to $\frac{1}{\infty}$ that does not evaluate to zero. For example, consider the following limit:

$$\lim_{x\to 1} \frac{x^2}{x-1} \tag{46}$$

The numerator is 1 in the limit, and the denominator is 0. Therefore the quotient diverges. To the left, it diverges to $-\infty$, because the denominator is negative and so the term is essentially 1 divided by a very small negative number; to the right, it diverges to $+\infty$, because the denominator is positive and so the term is essentially 1 divided by a very small positive number. In this way, it behaves the same as $\lim_{x\to 0} \frac{1}{x}$, another $\frac{1}{0}$ form. It is definitely not indeterminate; even though it may be undefined over the reals, it can be determined, and no more complicated terms that evaluate to 1 and 0 for the numerator and denominator respectively can have a limit that is anything else.

To find true indeterminate forms, an operation (multiplication or exponentiation) must be applied between two "conflicting" very large or very small number terms. It helps to think about these in terms of magnitudes of numbers, rather than arbitrary terms. Indeterminate forms simplify to one of $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$ (all four of these can be algebraically simplified down to any of them), 1^∞, ∞^0, and 0^∞ (which you can possibly also simplify into each other but I'm not so sure about that).

As an example of indeterminate forms, consider the following two functions under limits as $x \to 1$ that are $\frac{0}{0}$ and resolve to two different values:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2 \tag{47}$$

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2} = \lim_{x\to 1} \frac{x+1}{x-1} \to \infty \tag{48}$$

Since we get more than one value after we evaluate different expressions of the same form, its value is not determined only by its simple numerical form by directly substituting the input variable, so it can be called indeterminate, unlike *everything* that Pranshu said was indeterminate.

This completes the proof that I've had too much time on my hands over this break.