HW5 Aditya Shirwatkar 21232

Q.1 $\Lambda \in \mathbb{R}^{n \times n}$

1) To prove : $g e^{\Lambda} g^{-1} = e^{g \Lambda g^{-1}}$

$\therefore e^{\Lambda} = I + \Lambda + \frac{\Lambda^2}{2!} + \cdots\cdots$

$\therefore \hat{e} g^{-1} = I g^{-1} + \Lambda g^{-1} + \frac{\Lambda^2 g^{-1}}{2!} + \cdots\cdots$

$\therefore g \hat{e} g^{-1} = g I g^{-1} + g \Lambda g^{-1} + \frac{g \Lambda^2 g^{-1}}{2!} + \cdots\cdots \quad \text{--- (1)}$

$\therefore (g \Lambda g^{-1})^2 = (g \Lambda g^{-1})(g \Lambda g^{-1})$ & similarly $(g \Lambda g^{-1})^3 = (g \Lambda g^{-1})^2 (g \Lambda g^{-1})$

$= g \Lambda^2 g^{-1}$ $\qquad = g \Lambda^2 g^{-1} g \Lambda g^{-1}$

through induction $\qquad = g \Lambda^3 g^{-1}$

$(g \Lambda g^{-1})^n = g \Lambda^n g^{-1}$ for $n \in \{0, 1, 2, \cdots\cdots\}$

$\therefore$ In (1), $g e^{\Lambda} g^{-1} = g I g^{-1} + (g \Lambda g^{-1})^1 + \frac{(g \Lambda g^{-1})^2}{2!} + \cdots\cdots$

$= e^{g \Lambda g^{-1}}$

Hence proved!

2) $\frac{d}{dt}[e^{\Lambda\theta}] = \Lambda\dot{\theta} e^{\Lambda\theta} = e^{\Lambda\theta} \Lambda\dot{\theta}$ } To prove

$\therefore e^{\Lambda\theta} = I + \Lambda\theta + \frac{(\Lambda\theta)^2}{2!} + \cdots\cdots$ & assuming only θ is function of t

$= I + \Lambda\theta + \frac{\Lambda^2\theta^2}{2!} + \cdots\cdots$

$\therefore \frac{d}{dt}(e^{\Lambda\theta}) = 0 + \Lambda + \frac{2 \cdot \Lambda^2 \theta \cdot \dot{\theta}}{2!} + \frac{3 \Lambda^3 \theta^2 \dot{\theta}}{3!} + \cdots\cdots$

$= \Lambda\dot{\theta}\left[I + \Lambda\theta + \frac{\Lambda^2\theta^2}{2!} + \frac{\Lambda^3\theta^3}{3!} + \cdots\cdots \right]$

$= \Lambda\dot{\theta} e^{\Lambda\theta}$

(or)

$$= \left[I + \wedge\theta + \frac{\wedge^2 \theta^2}{2!} + \cdots \right] \wedge\dot{\theta}$$

$$= e^{\wedge\theta} \wedge\dot{\theta}$$

Q.2 $\quad g(t) \in SE(3) \qquad \hat{V}^s = \dot{g} g^{-1}$

$$\hat{V}^b = g^{-1} \dot{g}$$

& $V^s = Ad_g V^b$ where $Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$

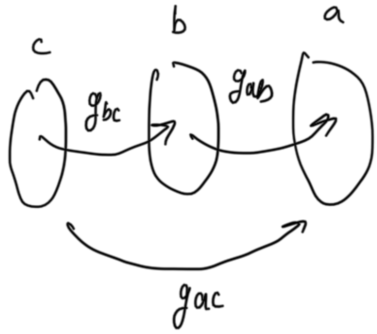
$\therefore g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ where $R \in SO(3)$ & $p \in \mathbb{R}^3$

$\therefore g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$ & $\dot{g} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$

$a \rightarrow b \rightarrow c$

$\therefore$ we know that we can compose rigid body transformation between coordinate frames a, b, c

i.e. $g_{ac} = g_{ab} \, g_{bc}$



$$\therefore \dot{g}_{ac} = \dot{g}_{ab} \, g_{bc} + g_{ab} \, \dot{g}_{bc}$$

$$\& \; g_{ac}^{-1} = g_{bc}^{-1} \, g_{ab}^{-1}$$

$$\therefore V_{ac}^{S} = \dot{g}_{ac} \cdot g_{ac}^{-1}$$

$$= (\dot{g}_{ab} \, g_{bc} + g_{ab} \, \dot{g}_{bc})(g_{bc}^{-1} g_{ab}^{-1})$$

$$= (\dot{g}_{ab} g_{bc} + g_{ab} \dot{g}_{bc})(g_{bc}^{-1} g_{ab}^{-1})$$

$$= \dot{g}_{ab} \cancel{g_{bc} g_{bc}^{-1}} g_{ab}^{-1} + g_{ab} \underbrace{\dot{g}_{bc} g_{bc}^{-1}}_{\hat{V}^s_{bc}} g_{ab}^{-1}$$

$$\hat{V}^s_{ac} = \hat{V}^s_{ab} + g_{ab} \hat{V}^s_{bc} g_{ab}^{-1}$$

$$\therefore V^s_{ac} = V^s_{ab} + \left(g_{ab} \hat{V}^s_{bc} g_{ab}^{-1}\right)^\vee$$

Let $\hat{V}^s_{bc} = \begin{bmatrix} \hat{\omega}^s_{bc} & v^s_{bc} \\ 0 & 0 \end{bmatrix}$ i.e $V^s_{bc} = \begin{bmatrix} v^s_{bc} \\ \omega^s_{bc} \end{bmatrix}$

Let $g_{ab} \hat{V}_{bc} g_{ab}^{-1} = \hat{X} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^s_{bc} & v^s_{bc} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} R_{ab} \hat{\omega}^s_{bc} & R_{ab} v^s_{bc} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$\hat{X} = \begin{bmatrix} R_{ab} \hat{\omega}^s_{bc} R_{ab}^T & -R_{ab} \hat{\omega}^s_{bc} R_{ab}^T p_{ab} + R_{ab} v^s_{bc} \\ 0 & 0 \end{bmatrix}$$

$$\therefore (\hat{X})^\vee = \begin{bmatrix} -R_{ab} \hat{\omega}^s_{bc} R_{ab}^T p_{ab} + R_{ab} v^s_{bc} \\ \left(R_{ab} \hat{\omega}^s_{bc} R_{ab}^T\right)^\vee \end{bmatrix}$$

$\therefore$ Now $\left(R_{ab} \hat{\omega}^s_{bc} R_{ab}^T\right) + \left(R_{ab} \hat{\omega}^s_{bc} R_{ab}^T\right)^T$

$$= R_{ab} \hat{\omega}^s_{bc} R_{ab}^T + R_{ab} (\hat{\omega}^s_{bc})^T R_{ab}^T$$

$$= 0$$

$$\therefore R_{ab} \hat{\omega}^s_{bc} R_{ab}^T \in so(3)$$

$\therefore \; R_{ab} \hat{\omega}_{bc} R_{ab}^T \in so(3)$

$\therefore$ we can see that $R_{ab} \hat{\omega}^s_{bc} R_{ab}^T = \widehat{R_{ab}\omega_{bc}}$

because for any vector y : $R_{ab} \hat{\omega}^s_{bc} R_{ab}^T y$

$$\Leftrightarrow \widehat{R_{ab} \omega^s_{bc}}\, y$$

since

$$R_{ab}\left[\omega^s_{bc} \times (R_{ab}^T y)\right] \Rightarrow \left[R_{ab}\omega^s_{bc} \times (R_{ab} R_{ab}^T y)\right]$$

$$\Rightarrow \widehat{R_{ab} \omega^s_{bc}}\, y$$

$$= \begin{bmatrix} -\left(\widehat{R_{ab} \omega^s_{bc}}\right) p_{ab} + R_{ab} v^s_{bc} \\ \left(\widehat{R_{ab}\omega^s_{bc}}\right)^{\vee} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{p}_{ab} R_{ab} \omega^s_{bc} + R_{ab} v^s_{bc} \\ R_{ab} \omega^s_{bc} \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} v^s_{bc} \\ \omega^s_{bc} \end{bmatrix} = Ad_{g_{ab}} V^s_{bc}$$

Thus $V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} V^s_{bc}$

for $\hat{V}^b_{ac} = g_{ac}^{-1} \dot{g}_{ac}$

$$= (g_{bc}^{-1} g_{ab}^{-1})(\dot{g}_{ab} \cdot g_{bc} + g_{ab} \cdot \dot{g}_{bc})$$

$$= g_{bc}^{-1} g_{ab}^{-1} \dot{g}_{ab} g_{bc} + g_{bc}^{-1} \dot{g}_{bc}$$

$$= g_{bc}^{-1} \hat{V}^b_{ab} g_{bc} + \hat{V}^b_{bc}$$

$\hat{V}^b_{ab}$

$\therefore$ Let $g_{bc}^{-1} \hat{V}_{ab}^{b} g_{bc} = \hat{X} = \begin{bmatrix} R_{bc}^{T} & -R_{bc}^{T} p_{bc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{ab}^{b} & v_{ab}^{b} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} R_{bc}^{T} \hat{\omega}_{ab}^{b} & R_{bc}^{T} v_{ab}^{b} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{bc}^{T} \hat{\omega}_{ab}^{b} R_{bc} & R_{bc}^{T} \hat{\omega}_{ab}^{b} p_{bc} + R_{bc}^{T} v_{ab}^{b} \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} R_{bc}^{T} \hat{\omega}_{ab}^{b} p_{bc} + R_{bc}^{T} v_{ab}^{b} \\ (R_{bc}^{T} \hat{\omega}_{ab}^{b} R_{bc})^{\vee} \end{bmatrix}$$

$\therefore$ using the proofs from before

$$X = \begin{bmatrix} R_{bc}^{T} (\hat{\omega}_{ab}^{b} p_{bc} + v_{ab}^{b}) \\ (\widehat{R_{bc}^{T} \omega_{ab}^{b}})^{\vee} \end{bmatrix} = \begin{bmatrix} R_{bc}^{T} (-\hat{p}_{bc} \hat{\omega}_{ab}^{b} + v_{ab}^{b}) \\ R_{bc}^{T} \omega_{ab}^{b} \end{bmatrix}$$

$$= \begin{bmatrix} R_{bc}^{T} & -R_{bc}^{T} \hat{p}_{bc} \\ 0 & R_{bc}^{T} \end{bmatrix} \begin{bmatrix} v_{ab}^{b} \\ \omega_{ab}^{b} \end{bmatrix} = Ad_{g_{bc}^{-1}} V_{ab}^{b}$$

Hence $V_{ac}^{b} = Ad_{g_{bc}^{-1}} V_{ab}^{b} + V_{bc}^{b}$

Q.3 (i) $(Ad_g)^{-1} = Ad_{g^{-1}}$ for all $g \in SE(3)$

$\therefore Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$ & $R^T R = RR^T = I$

$\because R \in SO(3)$

& $Ad_{g^{-1}} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$

$\therefore$ if we prove $(Ad_g)(Ad_{g^{-1}}) = (Ad_{g^{-1}})(Ad_g) = I$

then it will mean that $Ad_{g^{-1}} = (Ad_g)^{-1}$

$$\therefore Ad_g \cdot Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

$$= \begin{bmatrix} RR^T & -RR^T\hat{p} + \hat{p}RR^T \\ 0 & RR^T \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_3 \end{bmatrix}$$

$$= I_6$$

$$\therefore Ad_{g^{-1}} Ad_g = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

$$= \begin{bmatrix} R^T R & R^T\hat{p}R - R^T\hat{p}R \\ 0 & R^T R \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_3 \end{bmatrix}$$

$$= I_6$$

$$\therefore Ad_{g^{-1}} Ad_g = Ad_g Ad_{g^{-1}} = I$$

Hence $Ad_{g^{-1}} = (Ad_g)^{-1}$

(2) To prove : $Ad_{g_1 g_2} = Ad_{g_1} Ad_{g_2} \quad \forall \, g_1, g_2 \in SE(3)$

$\therefore$ we know that a composistion of rigid body transformation

$$(g_1 g_2) = g_1 \times g_2$$

if $g_1 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$ & $g_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore Ad_{g_1 g_2} = \begin{bmatrix} (R_1 R_2) & (\widehat{R_1 p_2 + p_1}) R_1 R_2 \\ 0 & (R_1 R_2) \end{bmatrix}$$

$$= \begin{bmatrix} R_1 R_2 & (\widehat{R_1 p_2} + \hat{p}_1) R_1 R_2 \\ 0 & (R_1 R_2) \end{bmatrix}$$

$\therefore$ as proved in Q.2 : $\widehat{Rx} = R \hat{x} R^T$

$$= \begin{bmatrix} R_1 R_2 & R_1 \hat{p}_2 R_1^T R_1 R_2 + \hat{p}_1 R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 R_2 & R_1 \hat{p}_2 R_2 + \hat{p}_1 R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} \quad \text{--- (1)}$$

$\therefore$ $Ad_{g_1} = \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix}$ & $Ad_{g_2} = \begin{bmatrix} R_2 & \hat{p}_2 R_2 \\ 0 & R_2 \end{bmatrix}$

$$\therefore Ad_{g_1} \times Ad_{g_2} = \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix}\begin{bmatrix} R_2 & \hat{p}_2 R_2 \\ 0 & R_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 R_2 & R_1 \hat{p}_2 R_2 + \hat{p}_1 R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix}$$

$$\text{——} \quad (2)$$

$\therefore$ from (1) & (2)

$$Ad_{g_1 g_2} = Ad_{g_1} \cdot Ad_{g_2}$$

Q.4 $\hat{\xi} \in se(3)$ & $\xi \in \mathbb{R}^6$

$\therefore$ To show that $g\hat{\xi}g^{-1}$ is a valid twist $\in se(3)$ with $Ad_g \xi \in \mathbb{R}^6$ if $g \in SE(3)$

$$\therefore g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad \text{where } R \in SO(3) \text{ \& } p \in \mathbb{R}^3$$

$$\therefore g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

& $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$ for some $\hat{\omega} \in so(3)$, $v \in \mathbb{R}^3$

$$\therefore g\hat{\xi}g^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R\hat{\omega} & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R\hat{\omega}R^T & -R\hat{\omega}R^T p + Rv \\ 0 & 0 \end{bmatrix}$$

Now $R\hat{\omega}R^T \in so(3)$ $\because$ $R\hat{\omega}R^T + (R\hat{\omega}R^T)^T$

$$= R\hat{\omega}R^T + R\,\hat{\omega}^T R^T = 0$$

& $-R\hat{\omega}R^T p + Rv \in \mathbb{R}^3$

$g\hat{\xi}g^{-1}$ can be a valid twist $\in se(3)$

with twist coordinates $(g\hat{\xi}g^{-1})^\vee = \begin{bmatrix} -R\hat{\omega}R^T p + Rv \\ (R\hat{\omega}R)^\vee \end{bmatrix}$

$$= \begin{bmatrix} -\widehat{R\omega}\,p + Rv \\ (\widehat{R\omega})^\vee \end{bmatrix}$$

(as proved in Q.2
$R\hat{\omega}R^T \Leftrightarrow \widehat{R\omega}$)

$$= \begin{bmatrix} \hat{p}R\omega + Rv \\ R\omega \end{bmatrix} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$= Ad_g\,\xi$$