Q 231 : 1

$\det(A) = 1/2$

$\therefore \det(2A) = 2^4 \cdot 1/2 = 8$

$\therefore \det(-A) = (-1)^4 \cdot 1/2 = 1/2$

$\therefore \det(A^{-1}) = \frac{1}{\det(A)} = 2$

$\therefore \det(A^2) = (\det A)^2 = 1/4$

Q. 232 : 7

a) $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$

$\therefore$ since rank = 1 it means 2 of the rows must be zero. $\therefore \det(A) = 0$

b) $\because$ Upper Δle : $\det(A) = 4 \cdot 1 \cdot 2 \cdot 2 = 16$

c) $\therefore U^T$: $\det(U^T) = \det(U) = 16$ ($\because$ Δgular)

d) $\therefore U^{-1}$: $\det(U^{-1}) = \frac{1}{\det U} = \frac{1}{16}$

e) $\therefore$ Row exchange matrix $= P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ since 2 exchanges happen, $\therefore$ no effect on determinant

$\therefore \det(M) = \det(PU) = \det(U) = 16$

Q. 233: 10

$Q^T Q = I$ $\therefore$ To prove $\det Q = \pm 1$

$\therefore$ for det to exist Q must be square, hence the

$$\det(Q^T Q) = \det(Q^T)\det(Q)$$

$$\therefore \det(I) = \det(Q^T)\det(Q)$$

$$\det(Q^T)\det(Q) = 1$$

$$\therefore \det(Q) = \frac{1}{\det(Q^T)} \quad \text{— (1)}$$

Now any matrix can be represented as $PQ = LU$

$$\therefore \det(PQ) = \det(LU)$$

$$\therefore \det(P)\cdot\det(Q) = \det(L)\cdot\det(U) \quad \text{— (2)}$$

$$\therefore (PQ)^T = (LU)^T$$

$$Q^T P^T = U^T L^T$$

$$\therefore \det(Q^T)\det(P^T) = \det(U^T)\det(L^T)$$

$$\therefore \qquad = \det(U)\det(L) \text{ ....(since diagonal)} \quad \text{— (3)}$$

$$\therefore \det(Q^T)\cdot\det(P^T) = \det(Q)\cdot\det(P) \text{ ---(from (2), (3))}$$

$$\therefore [\det(Q)]^2 = \frac{\det(P^T)}{\det(P)}$$

& $P = E_t \text{ ------ } E_1$ (for t row switches)

$$\det(P) = \det(E_t \text{ ---- } E_1) = \det(E_t)\cdot\det(E_{t-1}) \ldots \det(E_1)$$

& $\det(P^T) = \det(E_1^T E_2^T \ldots E_t^T) = \det(E_1^T) \ldots\ldots \det(E_t^T)$

$\therefore$ since elementary matrices have $\det = -1$

$$[\det(Q)]^2 = \frac{(-1)^t}{} = 1$$

$$\overline{(-1)^t}$$

$\therefore \det Q = \pm 1$

Q.233:12

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & (b-a)(b+a) \\ 0 & (c-a) & (c-a)(c+a) \end{bmatrix}$$

$$= (b-a)\cdot(c-a)\cdot \det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & (c+a) \end{bmatrix} = (b-a)\cdot(c-a)\cdot\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (b-c) \end{bmatrix}$$

$$= (b-a)(c-a)(b-c)$$

Q.233:14

a) $\begin{bmatrix} a_{11} & \cdots \\ \vdots & \ddots \end{bmatrix}$ & $\begin{bmatrix} b_{11} & \cdots \\ \vdots & \ddots \end{bmatrix}$

False because $\det(A) = a_{11}\det[\text{cofactor}(a_{11})] + \Delta_A$

& $\det(B) = b_{11}\det[\text{cofactor}(b_{11})] + \Delta_B$

Now $\det[\text{cofactor}(a_{11})] = \det[\text{cofactor}(b_{11})] = x$

& $\Delta_A = \Delta_B = \Delta$

$\therefore \det(A) = x + \Delta$

$\therefore \det(B) = \frac{x}{2} + \Delta$

b) False ; only if matrix is upper triangular / lower triangular or diagonal

c) $\det(A) \neq 0$ & $\det(B) = 0$ then what is $\det(A+B)$? False

we can't say anything ! because $\det(A+B) \neq \det(A) + \det(B)$

d) True ; $\det(AB) = \det(A) \cdot \det(B) = 0$

$\therefore$ AB is singular

e) $\det(AB) = \det(A) \cdot \det(B)$ if A & B is square

$\det(AB - BA) = 0$ if A & B is square

True under a condition

Q. 234:18

$$\det(A) = \det\begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{bmatrix} = \det = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 5 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= -4$$

$\therefore \det(B) = 4$ & $\det(C) = \det\begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 9 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 6 \end{bmatrix}$

$$= \det\begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore \det(AB) = \det(A) \cdot \det(B) = -4 \cdot 4 = -16$

$\therefore \det(A^T A) = \det(A^T) \cdot \det(A) = \det(A)^2 = 16$

$\therefore \det(C^T) = \det(C) = 0$

Q. 234 : 21

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det A^{-1} = \dfrac{1}{\det(A)} = \dfrac{1}{ab - bc}$

if $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

then $\det(A^{-1}) = \det\left(\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

$= \dfrac{1}{(ad-bc)^2} \cdot \det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \dfrac{1}{(ab - bc)}$

was wrong in the question

Q. 235 : 29

$\det P = 1$?

$\therefore P = A(A^TA)^{-1}A^T$

$\therefore \det(P) = \det(A(A^TA)^{-1}A^T)$

$= \underbrace{\det(A) \cdot \det((A^TA)^{-1}) \det(A^T)}$

if & only if A is square matrix

$= 1$

Q. 242:6

a) $D_1 = 1$; $D_2 = 0$

$D_3 = -1$

$\therefore D_3 = D_2 - D_1$

$$\therefore D_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & \ddots & 1 \\ \vdots & \vdots & \vdots & & 1 \end{bmatrix} \quad n$$

$$\therefore D_n = 1 \cdot \underbrace{\det \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ 0 & 1 & \ddots & \\ 0 & & \ddots & \ddots \\ \vdots & \vdots & & \end{pmatrix}}_{\substack{(n-1)\times(n-1) \\ \text{tridiagonal}}} - 1 \underbrace{\det \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}}_{(n-1)\times(n-1)}$$

$$= D_{n-1} - \left[\underbrace{\det \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \\ \vdots & 0 & 1 & \ddots & \\ \vdots & \vdots & & & \end{pmatrix}}_{\substack{(n-2)\times(n-2) \\ \text{tri diagonal}}} - \underbrace{\det \begin{pmatrix} 0 & 1 & 1 & 0 & \cdots \\ 0 & & \ddots & & \\ 0 & & & \ddots & \\ \vdots & & & & \ddots \end{pmatrix}}_{1 \text{ column is zero}} \right] + 0$$

$$D_n = D_{n-1} - D_{n-2}$$

b)

D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}
1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0

after every 6^{th} n the pattern repeats

$D_{1000} = ?$ we know at $D_{1002} = 1$

$\because 1002 \,\%\, 6 = 0$

$$\begin{array}{cccccc} \cdots & D_{1000} & D_{1001} & D_{1002} & \cdots \\ & -1 & 0 & 1 & \end{array}$$

$\therefore D_{1000} = -1$

Q. 242: 7

a) $\det(A) = 4 \cdot \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} - 4 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 4(4-1) - 4(2-1) + 4(-2-2\cdot(4-2)+1\cdot 2) - 4(-1-2(-1))$

$= 4\cdot(3-1-4-1) = -12$

b) $\det(A) = -\det\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & -2 & -1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} = -4\det\begin{pmatrix} 1 & -1 & 0 \\ -2 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$= 4\det\begin{pmatrix} 1 & 0 & 0 \\ -2 & -3 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$= -12$

Q

Q.242 : 8

$$\det A_2 = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \cdot \det([0]) - 1 \cdot \det([1]) = -1$$

$$\therefore \det(A_3) = \det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 0 \cdot \det(A_2) - 1 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= -2 \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = 2$$

$$\therefore \det(A_4) = \det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 0 \cdot \det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$+ 1 \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= -2 \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= -3 \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = -3$$

Through induction $A_n = (-1)^{n-1} \cdot (n-1)$

Q.243 : 9

a) $$\det A = \sum_{\text{all } P's} (a_{1\alpha} a_{2\beta} \cdots\cdots a_{n\nu}) \det P \quad \Big\} \text{ \& } A = n \times n$$

\# multipli $\Rightarrow$ $n! \times n$

b) $$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots\cdots + a_{in} C_{in}$$

$$\text{\& } C_{ij} = (-1)^{i+j} \det M_{ij}$$

multi $\Rightarrow$ $n \cdot (n-1) \cdot (n-2) \cdots\cdots 1 \Rightarrow n!$

c) No. of steps in gaussian elim + n

Q. 243 : 12

$$\det A = \det\begin{pmatrix} a_{11} & \cdots \\ \vdots & \ddots \end{pmatrix} = \det\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \vdots & \ddots \end{pmatrix} + \det\begin{pmatrix} 0 & a_{12} & \cdots \\ \vdots & \ddots \end{pmatrix}$$

$\Rightarrow 0$ when all 1^{st} row is 0 or all are multiples of a_{11}

$\therefore$ [OR] $\det A$

$= a_{11} C_{11} + a_{12} C_{12} + \cdots$

$\Rightarrow 0$ (when all $(a_{12} C_{12} + \cdots) = 0$) & $(a_{11}$ or $C_{11} = 0)$

$\neq 0$ otherwise

Q 269 : 9

$$\Rightarrow \det(A - \lambda I) = (\lambda_1 - \lambda) \cdot (\lambda_2 - \lambda) \cdots\cdots (\lambda_n - \lambda)$$

$$= (\lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)\lambda + \lambda^2)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda)$$

$$= (\lambda_1 \lambda_2 \lambda_3 - (\;\;)\lambda + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - \lambda^3) \cdots$$

$$= O(\lambda^{n-2}) + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-1)^{n-1}\lambda^{n-1} + (-1)^n \lambda^n$$

& $\Rightarrow$

$$\therefore \det(A - \lambda I) = \det\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) - (a_{11} - \lambda) \; a_{23} \; (a_{33} - \lambda) \cdots (a_{nn} - \lambda)$$

$$= O(\lambda^{n-2}) + (a_{11} + \cdots + a_{nn})(-1)^{n-1}\lambda^{n-1} + (-1)^n \lambda^n + \cdots$$

$$\therefore (a_{11} + \cdots + a_{nn}) = (\lambda_1 + \cdots + \lambda_2)$$

Q. 269 : 12

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \therefore \lambda_1 + \lambda_2 = 3 - 3 = 0$$

$$\lambda_1 \lambda_2 = -9 - 16 = -25$$

$$\Rightarrow \lambda_2 = 5 \ \& \ \lambda_1 = -5$$

$$\therefore (A - \lambda I)x = 0$$

$\lambda_2 = 5$

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} x_2 = 0$$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} x_2 = 0$$

eigen vector $\left\{ \Rightarrow x_2 = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} \right.$ where $\alpha \in \mathbb{R}$

$\lambda_1 = -5$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} x_1 = 0$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} x_1 = 0$$

$$\Rightarrow x_1 = \begin{bmatrix} \beta \\ -2\beta \end{bmatrix}$$

where $\beta \in \mathbb{R}$

Q. 269 : 14

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

rank = 1

$$\therefore \det(A - I\lambda) = \det \begin{bmatrix} (1-\lambda) & 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 & 1 \\ 1 & 1 & (1-\lambda) & 1 \\ 1 & 1 & 1 & (1-\lambda) \end{bmatrix}$$

$$0 = \det \begin{bmatrix} (1-\lambda) & 0 & 0 & 0 \\ 1 & (1-\lambda) - \frac{1}{(1-\lambda)} & 1 - \frac{1}{(1-\lambda)} & 1 - \frac{1}{(1-\lambda)} \\ 1 & 1 - \frac{1}{(1-\lambda)} & (1-\lambda) - \frac{1}{(1-\lambda)} & 1 - \frac{1}{(1-\lambda)} \\ 1 & 1 - \frac{1}{(1-\lambda)} & 1 - \frac{1}{1-\lambda)} & (1-\lambda) - \frac{1}{(1-\lambda)} \end{bmatrix}$$

$$= (1-\lambda) \det \begin{bmatrix} \frac{(1-\lambda)^2 - 1^2}{(1-\lambda)} & \frac{-\lambda}{(1-\lambda)} & x \\ x & \cdot & x \\ x & x & \cdot \end{bmatrix}$$

$$= (1-\lambda) \det \begin{bmatrix} \frac{-\lambda \cdot (2-\lambda)}{(1-\lambda)} & \frac{-\lambda}{(1-\lambda)} & x \\ x & \cdot & x \\ x & \cdot & \cdot \end{bmatrix}$$

$$= -\frac{(1-\lambda)\, \lambda^3}{(1-\lambda)^3} \det \begin{bmatrix} (2-\lambda) & 1 & 1 \\ 1 & (2-\lambda) & 1 \\ 1 & 1 & (2-\lambda) \end{bmatrix}$$

$$= \frac{-\lambda^3}{(1-\lambda)^2} \left[(2-\lambda) \cdot \left[(2-\lambda)^2 - 1^2 \right] - 1 \left[(2-\lambda) - 1 \right] + 1 \left[1 - (2-\lambda) \right] \right]$$

$$= \frac{-\lambda^3}{(1-\lambda)^2} \times \left[(2-\lambda)\,(3-\lambda)\,(1-\lambda) - 2\,(1-\lambda) \right]$$

$$= \frac{-\lambda^3}{(1-\lambda)} \left[(2-\lambda)(3-\lambda) - 2 \right] = \frac{-\lambda^3}{(1-\lambda)} \cdot \left[4 - 5\lambda + \lambda^2 \right]$$

$$= \frac{-\lambda^3}{(1-\lambda)} \, (4-\lambda) \cdot (1-\lambda)$$

$$\Rightarrow \lambda^3 (\lambda - 4) = 0$$

$$\left. \begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 4 \\ \lambda_1 \lambda_2 \lambda_2 \lambda_4 &= 0 \end{aligned} \right\}$$

$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$ & $\lambda_4 = 4$

$$\text{Null}\left(\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}\right) \Rightarrow N\left(\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & \overset{-8/3}{-3+1/3} & 1+1/3 & 1+1/3 \\ 0 & \overset{4/3}{1+1/3} & -3+1/3 & 1+1/3 \\ 0 & 1+1/3 & 1+1/3 & -3+1/3 \end{bmatrix}\right)$$

$$\Rightarrow N\left(\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}\right)$$

$$\therefore \Rightarrow N\left(\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$\therefore$ If α is last element of eigenvector then

$v = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{bmatrix}$ is the corresponding eigenvector for $\lambda = 4$

where $\alpha \in \mathbb{R}$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ rank $= 2$

& $\sum\lambda = 0$ & $\prod\lambda = 0$

$$\therefore \det(C - \lambda I) = \det\begin{bmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{bmatrix}$$

$$= \det\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 1 & -\lambda + \frac{1}{\lambda} & 1 & 1/\lambda \\ 0 & 1 & -\lambda & 1 \\ 1 & 1/\lambda & 1 & -\lambda + \frac{1}{\lambda} \end{bmatrix} = -\lambda \det\begin{pmatrix} \frac{1-\lambda^2}{\lambda} & 1 & \frac{1}{\lambda} \\ 1 & -\lambda & 1 \\ 1/\lambda & 1 & \frac{1-\lambda^2}{\lambda} \end{pmatrix}$$

$$= -\lambda\left[\frac{(1-\lambda^2)}{\lambda}\left[-(1-\lambda^2)-1\right] - 1\cdot\left[\frac{1-\lambda^2}{\lambda}-\frac{1}{\lambda}\right] + \frac{1}{\lambda}\left[1+1\right]\right]$$

$$= (1-\lambda^2)\cdot(\lambda^2-2) + \lambda^2 + 2$$

$$= \lambda^2 - 2 - \lambda^4 + 2\lambda^2 + 2 = 3\lambda^2 - \lambda^4 = (\sqrt{3}-\lambda)(-\sqrt{3}-\lambda)\cdot\lambda^2$$

$$\begin{bmatrix}\sqrt{3} & 1 & 0 & 1\\ 1 & \sqrt{3} & 1 & 0\\ 0 & 1 & \sqrt{3} & 1\\ 1 & 0 & 1 & \sqrt{3}\end{bmatrix}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{V_{\sqrt{3}}} = \sqrt{3}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{V_{\sqrt{3}}} \Rightarrow \text{By guessing } V_{\sqrt{3}} = \begin{bmatrix}\alpha\\ \alpha\\ -\alpha\\ -\alpha\end{bmatrix}$$

Similarly $V_{-\sqrt{3}} = \begin{bmatrix}-\alpha\\ -\alpha\\ \alpha\\ \alpha\end{bmatrix}$

Q. 269:16

$$A = \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \quad \text{then} \quad A - I = \begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$$

$\therefore \det(A-I) = -3$ (calculated in previous questions)

$$\therefore \text{eig}(A-I) \Rightarrow \begin{vmatrix}-\lambda & 1 & 1 & 1\\ 1 & -\lambda & 1 & 1\\ 1 & 1 & -\lambda & 1\\ 1 & 1 & 1 & -\lambda\end{vmatrix}$$

$$\Rightarrow \begin{vmatrix}-\lambda & 0 & 0 & 0\\ 1 & -\lambda+\frac{1}{\lambda} & 1+\frac{1}{\lambda} & 1+\frac{1}{\lambda}\\ 1 & 1+\frac{1}{\lambda} & -\lambda+\frac{1}{\lambda} & 1+\frac{1}{\lambda}\\ 1 & 1+\frac{1}{\lambda} & 1+\frac{1}{\lambda} & -\lambda+\frac{1}{\lambda}\end{vmatrix}$$

$$\Rightarrow -\lambda \begin{vmatrix} \frac{1-\lambda^2}{\lambda} & \frac{1+\lambda}{\lambda} & \cdot \\ \cdot & \times & \cdot \\ \cdot & \cdot & \times \end{vmatrix} \Rightarrow \frac{-\lambda}{\lambda^3}(1+\lambda)^3 \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$\Rightarrow -\frac{(1+\lambda)^3}{\lambda^2}\Big[(1-\lambda)\big[(1-\lambda)^2-1\big] - 1\big[(1-\lambda)-1\big] + 1\big[1-(1-\lambda)\big]\Big]$$

$$\Rightarrow -\frac{(1+\lambda)^3}{\lambda^2}\Big[-(1+\lambda)\,\lambda\,(2-\lambda) + 2\lambda\Big] \Rightarrow -\frac{(1+\lambda)^3}{\lambda}\Big[\lambda-\lambda^2\Big]$$

$$\Rightarrow (1+\lambda)^3(1-\lambda)$$

$\Sigma\lambda = 0$ $\quad \Pi\lambda = 0$ $\qquad$ if 2 are 0 then one is -1 & 1 ?

Q. 270:21

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow \det\begin{bmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$-\lambda(3-\lambda) - 4 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0 \Rightarrow (\lambda-4)(\lambda+1) = 0$$

$$A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$\therefore A v_A = \lambda_A v_A$$

$$\therefore A^{-1} A v_A = \lambda_A A^{-1} v_A$$

$$A^{-1} v_A = \frac{1}{\lambda_A} v_A$$

$\therefore$ A & A^{-1} have the same eigen vectors

whereas A & A^{-1} have eigen values reciprocal to each other

Q. 279:5

$$A_1 = \begin{bmatrix} 2 & -2 \end{bmatrix} \quad \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -2 \end{vmatrix} = -(4-\lambda^2) + 4$$

$$\begin{bmatrix} 2 & -2 \end{bmatrix} \quad \begin{vmatrix} 2 & -2-\lambda \end{vmatrix} \Rightarrow \lambda^2 = 0$$

$$\therefore \text{Null}\left(\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}\right) \Rightarrow \text{Null}\left(\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\right)$$

$$\therefore v = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$$

$\therefore$ dimension of eigen subspace $= 1$ $\Rightarrow$ geometric multiplicity $= 1$

& No. of repeated $= 2$

$\therefore$ Non diagonalizable

$$A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \Rightarrow \det(A - I\lambda) = \begin{vmatrix} 2-\lambda & 0 \\ 2 & -2-\lambda \end{vmatrix} = -(4-\lambda^2)$$

$$\Rightarrow (\lambda+2)(\lambda-2) \Rightarrow 0$$

$\therefore$ eigen values are distinct thus diagonalizable

$$A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} \Rightarrow (\lambda-2)^2 = 0$$

$$\therefore N\left(\begin{bmatrix} 2-2 & 0 \\ 2 & 2-2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right) \Rightarrow v = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

$\therefore$ Not diagonalizable

Q. 279 : 8

$A = uv^T$ Let $u \Rightarrow n \times 1$ & $v \Rightarrow 1 \times n$ where $n \in \mathbb{R}$

a) $Au = u\underbrace{v^T u}_{\text{some scalar } = \lambda} = \lambda u$

where $\lambda = v^T u$

b) $Ax = uv^T x = \lambda x$ for some x

if $x \in \text{Null}(A)$ then $uv^T x = 0$

thus $\lambda = 0$ are also valid eigen values

$\because$ rank $= 1$; dim of eigensubspace will be $(n-1)$ $\Leftrightarrow$ same as null(A)

$\lambda = 0$ is repeated $(n-1)$ times, where $(n \times n)$ is size of A

c) if $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\quad v^T = [v_1 \; \cdots \; v_n]$

$$\therefore A = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} [v_1 \cdots v_n] = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_1v_2 & u_2v_2 & & \\ \vdots & & \ddots & \end{bmatrix}$$

$\therefore$ Trace $(A) = (u_1v_1 + u_2v_2 + \cdots) = v^Tu$

& Sum of eigenvalues $= \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n$

$= 0 + 0 \cdots + 0 + v^Tu$

$= v^Tu$

Q.279: 11 $\quad \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$

a) False $\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ which is invertible

b) Could be True or could be False

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow$ rank : 2

& we have 2 repeated eig vals

$\therefore n - p = 1 \neq$ rank

c) Could be true, could be false

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Q. 281 : 27

det $A = 25$ trace 10 & $\lambda = 5$ is repeated

there can be multiple sol^n's

a) $A = \begin{bmatrix} 8 & d \\ c & 2 \end{bmatrix} \Rightarrow 25 = 16 - cd$

$cd = -9$

& $\lambda_1 = \lambda_2 = 5$

Let $c = 3$ & $d = -3$

$\therefore Ax = 5x \Rightarrow N\left(\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}\right) \Rightarrow$ rank 1

$\therefore x = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$

b) $A = \begin{bmatrix} 9 & 4 \\ c & 1 \end{bmatrix} \Rightarrow 9 - 4c = 25 \Rightarrow c = -4$

$\therefore Ax = 5x \Rightarrow N\left(\begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}\right) \Rightarrow$ rank 1

$x = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$

c) $A = \begin{bmatrix} 10 & 5 \\ -5 & c \end{bmatrix} \Rightarrow 10c + 25 = 25$

$c = 0$

$\therefore Ax = 5x \Rightarrow N\left(\begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}\right) \Rightarrow$ rank 2

$x = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$

Q. 281 : 32

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$

$= 4 - 4\lambda + \lambda^2 - 1$

$0 = (\lambda - 3)(\lambda - 1)$

$Ax_1 = 3x_1$

$\therefore \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ax_2 = 1x_1$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ & $x_2 = \begin{bmatrix} \beta \\ -\beta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad S^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$\therefore \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \therefore A = S \Lambda S^{-1}$

$$\therefore A^k = S \Lambda^k S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 1^k \\ -3^k & 1^k \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^k = \frac{1}{2} \begin{bmatrix} 3^k + 1^k & -3^k + 1^k \\ -3^k + 1^k & 3^k + 1^k \end{bmatrix}$$

Q: 282 : 34

$A = S \Lambda S^{-1}$

$\therefore \det(S \Lambda S^{-1}) = \det(S) \det(\Lambda) \det(S^{-1})$

& since S^{-1} exists

$$= \det(\Lambda) \cdot \frac{\cancel{\det(S)}}{\cancel{\det(S)}}$$

& since its a diagonal matrix ;

$= \lambda_1 \lambda_2 \lambda_3 \cdots$

& This quick proof only works when A is diagonalizable

Q 282 : 40

$A = S\Lambda S^{-1}$

$$\therefore M = (A-\lambda_1 I)\cdot(A-\lambda_2 I)\cdots$$

$$= (S\Lambda S^{-1}-\lambda_1 I)\cdot(S\Lambda S^{-1}-\lambda_2 I)\cdots$$

$$= (S\Lambda S^{-1}-\lambda_1 SS^{-1})\cdot(S\Lambda S^{-1}-\lambda_2 SS^{-1})\cdots$$

$$= \left[(S\Lambda-\lambda_1 S)S^{-1}\right]\cdot\left[(S\Lambda-\lambda_1 S)S^{-1}\right]\cdots$$

$$= \left[S(\Lambda-\lambda_1 I)S^{-1}\right]\left[S(\Lambda-\lambda_2 I)S^{-1}\right]\cdots$$

$$= S\left[(\Lambda-\lambda_1 I)\cdot(\Lambda-\lambda_2 I)\cdots\right]S^{-1}$$

$$= \begin{bmatrix} 0 & & \\ & \lambda_2-\lambda_1 & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \lambda_1-\lambda_2 & & \\ & 0 & \\ & & \ddots \end{bmatrix}$$

$$\begin{bmatrix} 0 & & & \\ & \lambda_2-\lambda_1 & & \\ & & \lambda_3-\lambda_1 & \\ & & & \ddots \end{bmatrix}\begin{bmatrix} \lambda_1-\lambda_2 & & & \\ & 0 & & \\ & & \lambda_3-\lambda_2 & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & (\lambda_3-\lambda_2)(\lambda_3-\lambda_1) & \\ & & & \ddots \end{bmatrix}$$

similarly we can see that as we progress multiplication till n, we will have something like this at last

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & (\lambda_n-\lambda_{n-2})(\lambda_n-\lambda_{n-1})\cdots\lambda_n \end{bmatrix}\begin{bmatrix} \lambda_1-\lambda_n & & \\ & \ddots & \\ & & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots\ 0 \end{bmatrix}}_{\text{all diagonals become 0}}$$

Thus all eigen...

$\therefore M = 0$

Q.295:25

$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \Rightarrow \lambda_1, \lambda_2 = 1, 9$

$B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \Rightarrow \lambda_1, \lambda_2 = -1, 9$

$\therefore v_1^A \Rightarrow \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \Rightarrow v_1^A = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad v_2^A \Rightarrow \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 \\ -1/2 & 3/2 \end{bmatrix}$

$\therefore$ Since B has an eigen val < 0 it won't have a real square root

Q.295:27

$A = S\Lambda_1 S^{-1} \qquad B = S\Lambda_2 S^{-1}$

$\therefore AB = (S\Lambda_1 S^{-1})(S\Lambda_2 S^{-1})$

$= S\Lambda_1\Lambda_2 S^{-1}$

$\therefore BA = (S\Lambda_2 S^{-1})(S\Lambda_1 S^{-1})$

$= S\Lambda_2\Lambda_1 S^{-1}$

Now if $\Lambda_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$

$$\begin{bmatrix} & & \\ 0 & & \ddots \end{bmatrix}$$

& $\Lambda_2 = \begin{bmatrix} {}^2\lambda_1 & & 0 \\ & {}^2\lambda_2 & \\ 0 & & \ddots \end{bmatrix}$

then $\Lambda_1 \Lambda_2 = \begin{bmatrix} {}^1\lambda_1 {}^2\lambda_1 & & 0 \\ & {}^1\lambda_2 {}^2\lambda_2 & \\ 0 & & \ddots \end{bmatrix}$

similarly $\Lambda_2 \Lambda_1 = \begin{bmatrix} {}^2\lambda_1 {}^1\lambda_1 & & 0 \\ & {}^2\lambda_2 {}^1\lambda_2 & \\ 0 & & \ddots \end{bmatrix}$

& since scalars are commutative ; $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$

thus $AB = BA$