HW 6 Aditya Shirwatkar CP 220 21232

Pg 150

Q.14 $\therefore T$ is R^3 to R^3 & linear

for T^2 to be linear,

1> $T^2(0) = 0$ must be true

$\therefore \quad T^2(0) = T \cdot T(0) = T \cdot 0 = 0$

2> $\therefore$ if ~~x~~ $\Rightarrow x'$ then so must $\alpha x \Rightarrow \alpha x'$ where α is const.

if $T^2(x) = x'$

$\therefore \quad T^2(\alpha x) = T \cdot T(\alpha x) = T \cdot \alpha(Tx) = \alpha(T^2(x)) = \alpha x'$

linear transformation

3> $\therefore$ if $x \Rightarrow x'$ & $y \Rightarrow y'$ then $x+y \Rightarrow x'+y'$

$\therefore \quad T^2 x = x'$ & $T^2 y = y'$

$\therefore \quad T^2(x+y) = T \cdot T(x+y) = T \cdot (Tx + Ty) = T^2 x + T^2 y = x' + y'$

linear transformation

Hence 1>, 2> & 3> are satisfied, so T^2 is a linear transformation

Pg: 151

Q. 18 $\therefore$ if vector space is P_3 of all $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

& $S \subset P_3$ with $\int_0^1 p(x)\,dx = 0$

$\therefore$ for S to be a subspace it must pass through zero

& we know $\int_0^1 p(x)\,dx = \left[a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} \right]_0^1 = 0$

$$\Rightarrow a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0 \quad \text{——} (1)$$

$\therefore$ Polynomials with this condition are subset of P_3

$\therefore$ if we choose our basis for P_3 as $\{1, x, x^2, x^3\}$

$\therefore$ if $[a_0, a_1, a_2, a_3] = [0,0,0,0]$

then it satisfies the eq^n (1) $\therefore$ It is a subspace !

Pg 153

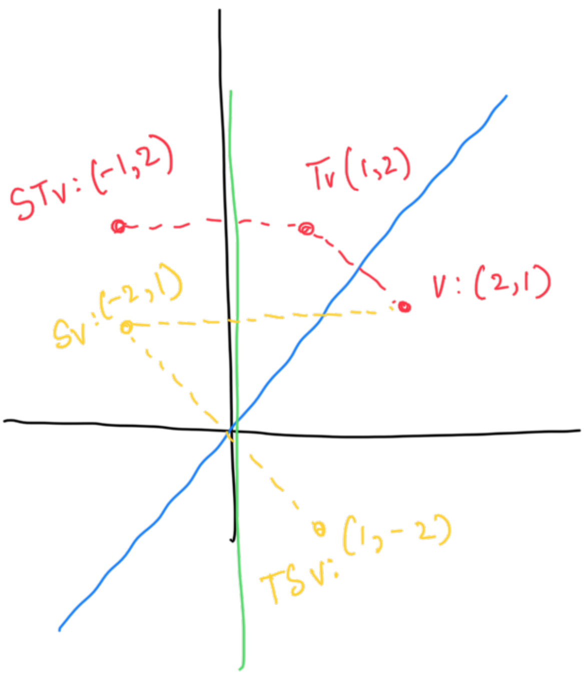
Q. 45

T : Reflection about 45° line

S : Reflection about 90° line

V : domain xy plane

What is $S(T(V))$?



$\therefore S(T(v)) = (-1, 2)$ (from geometry)

& $Sv = (-2,1)$

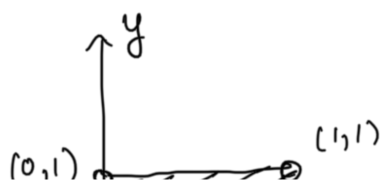
& $TSv = (1,-2)$

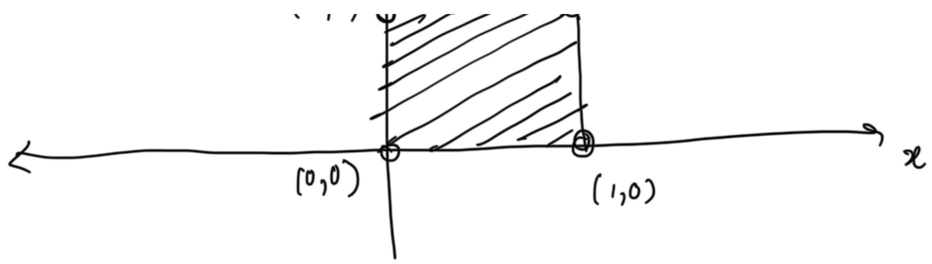
$\therefore ST \neq TS$

Pg 154

Q. 50

$A_{2\times2}$





a) A quadrilateral : can be square, rectangle or parallelogram

b) $A\begin{bmatrix}0\\1\end{bmatrix} = \alpha\begin{bmatrix}0\\1\end{bmatrix}$ & $A\begin{bmatrix}1\\0\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix}$ & $A\begin{bmatrix}1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix}$ & $A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Downarrow$ $a_{12} = 0$, $a_{22} = \alpha$

$\Downarrow$ $a_{11} = \alpha$, $a_{21} = 0$

$\therefore A = \begin{bmatrix}\alpha & 0\\0 & \alpha\end{bmatrix}$ where $\alpha \in \mathbb{R} - \{0\}$

c) A in which rank < 2 : $A = \begin{bmatrix}a & 0\\b & 0\end{bmatrix}$ or $\begin{bmatrix}a & b\\0 & 0\end{bmatrix}$

where $a, b \in \mathbb{R}$

d) $\therefore$ $|\det(A)|$ will represent the area of the transformed region

$\det A = \pm 1$

$\Rightarrow$ Rotation or Reflection Matrices are one such eg:

$\Rightarrow$ There can be others as well

Pg 167

Q. 5

$\therefore$ for orthogonality : $\langle a, b\rangle^T = 0$

$\therefore \langle v_1, v_2\rangle = \begin{bmatrix}1 & 2 & -2 & 1\end{bmatrix}\begin{bmatrix}4\\0\\4\end{bmatrix} = 4 - 8 \neq 0$

$\lfloor 0 \rfloor$

$\therefore \langle v_1, v_3 \rangle = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = 1 - 2 + 2 - 1 = 0$ ✓

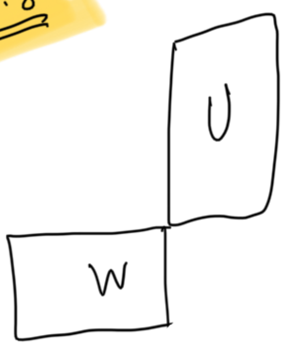
$\therefore \langle v_1, v_4 \rangle = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 + 2 - 2 + 1 \neq 0$

$\therefore \langle v_2, v_3 \rangle = \begin{bmatrix} 4 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = 4 - 4 = 0$ ✓

$\therefore \langle v_2, v_4 \rangle = \begin{bmatrix} 4 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4 + 4 \neq 0$

$\therefore \langle v_3, v_4 \rangle = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 1 - 1 - 1 \neq 0$

Q.8



if $u \in U$ & $w \in W$

we know that subspaces are orthogonal

i.e $u^T w = 0$

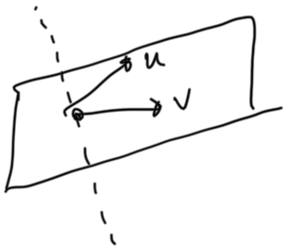
$\therefore$ if there is some element $x \in U \cap W$

the it should also satisfy $x^T x = 0$ property

& this is only possible if $x = \mathbf{0}$ vector

Hence $U \cap W = \{0\}$

Q.9



$u : (1, 1, 2)$ $\quad$ $A = \begin{bmatrix} u^T \\ v^T \end{bmatrix}$

$v : (1, 2, 3)$

$\therefore Ax = 0$

$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} x = 0$ $\quad$ if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$\therefore x_1 + x_2 + 2x_3 = 0$

$\therefore -x_3 = 0$ $\Rightarrow$ $x_1 + x_2 = 0$ $\quad$ if $x_1 = \lambda$ then $x_2 = -\lambda$

$\therefore x = \begin{bmatrix} \lambda \\ -\lambda \\ 0 \end{bmatrix}$ $\Rightarrow$ orthogonal complent $\forall \lambda \in \mathbb{R}$

Pg : 168

Q. 11

① $Ax = b$ $\quad$ ② $A^T y = 0$; $y^T b \neq 0$

$\therefore$ if $Ax = b$ true

$\therefore$ Let $A^T y = 0$; $y^T b \neq 0$ be true for now

$\therefore$ if we take $y^T A = 0$

$y^T A x = 0 x$

$\therefore y^T b = 0$

This contradicts $y^T b \neq 0$

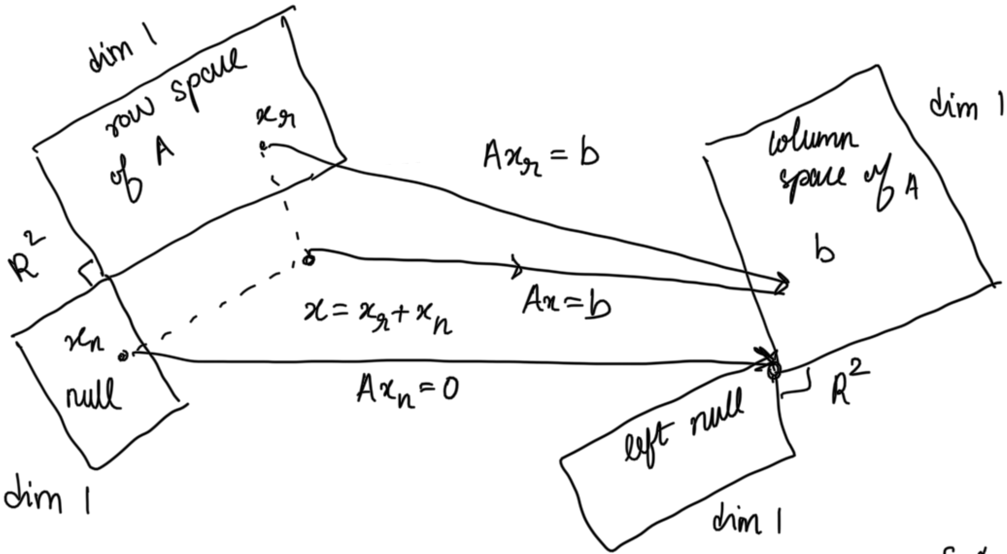
Hence only one of ① or ② can be true

Pg 169

Q.32

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\quad x_r = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



$x_r = \begin{bmatrix} \alpha \\ \frac{b_1 - \alpha}{2} \end{bmatrix}; \alpha \in \mathbb{R}$

or $x_r = \begin{bmatrix} \alpha \\ \frac{\frac{b_2}{2} - \alpha}{2} \end{bmatrix}$

& $b_1 = \frac{b_2}{2}$ for sol^n

$Ax_n = 0$

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} x_n = 0$

$x_n = \begin{bmatrix} \lambda \\ -\frac{\lambda}{2} \end{bmatrix}; \lambda \in \mathbb{R}$

$\therefore x = \begin{bmatrix} \alpha \\ \frac{b - \alpha}{2} \end{bmatrix} + \begin{bmatrix} \lambda \\ -\lambda/2 \end{bmatrix}$

Similarly for B : $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} x_r = b \Rightarrow x_r = \begin{bmatrix} b_1 \\ \alpha \end{bmatrix}$ or $x_r = \begin{bmatrix} b_2/3 \\ \alpha \end{bmatrix}$ & $b_1 = b_2/3$ for solution

$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} x_n = 0 \Rightarrow x_n = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$ $\qquad \therefore x = \begin{bmatrix} b \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$

Pg 170

Q.43

P is the plane of vectors in R^4 : $x_1 + x_2 + x_3 + x_4 = 0$

$\therefore P^\perp$ should satisfy $(\alpha x_1 + \beta x_2 + \gamma x_3 + \Delta x_4)^T e_i = 0$

where e_1, e_2, e_3, e_4 are basis of $P^\perp$

$\therefore [\alpha_1 x_1 + \beta x_2 + \gamma x_3 - \Delta(x_1 + x_2 + x_3)]^T e_i = 0$

$\therefore$ if $P^\perp = [\, e_1 \quad e_2 \quad e_3 \quad e_4 \,]$

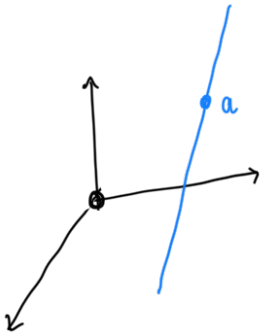
$\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix} x_n = \begin{bmatrix} 0 \end{bmatrix}$ $\qquad \therefore x_n = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$$\begin{bmatrix} \lambda_2 \\ \lambda_3 \\ -(\lambda_1+\lambda_2+\lambda_3) \end{bmatrix}$$

$$\lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 - (\lambda_1+\lambda_2+\lambda_3) m_4 = 0$$

Pg 178

Q.16



a] $\therefore \langle x, Py \rangle = x^T P y$

$\therefore \langle Px, y \rangle = (Px)^T y = x^T P^T y$

we know for any matrix A : $P = aa^T / a^Ta$ scalar

$$\therefore P^T = \frac{(aa^T)^T}{a^Ta} = \frac{aa^T}{a^Ta}$$

$\therefore \langle x, Py \rangle = \langle Px, y \rangle$

b] $a^Ta = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3$ $\quad aa^T = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$x = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\quad y = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\quad \therefore Px = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad \therefore Py = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\therefore \cos\theta_1 = \frac{\langle x, Py \rangle}{\|x\| \, \|Py\|} = \frac{\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\sqrt{5}\,\sqrt{3}} = \frac{1}{\sqrt{5}\sqrt{3}}$$

$$\therefore \cos\theta_2 = \frac{\langle Px, y \rangle}{\|Px\| \, \|y\|} = \frac{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}{\sqrt{3}\,\sqrt{9}} = \frac{1}{\sqrt{3} \cdot 3}$$

$\theta_1 \neq \theta_2$

c) $\therefore \langle Px, Py \rangle = (Px)^T (Py) = x^T P^T P y = x^T P^2 y$

$$= x^T \frac{(aa^T)(aa^T)}{(a^Ta)(a^Ta)} y = x^T \frac{aa^T}{(a^Ta)} y$$

$$= x^T P y$$

$$\therefore \cos\theta_3 = \frac{\langle Px, Py \rangle}{\|Px\| \|Py\|} = \frac{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\sqrt{3}\,\sqrt{3}} = \frac{3}{3} = 1 \Rightarrow \theta_3 = 0^\circ$$

Pg 191

Q.9

a) if $P = P^T P$ & $P^T = P^T P$

if $P = A(A^TA)^{-1}A^T$ for any matrix A

then

$$P^T = (A^T)^T\left((A^TA)^{-1}\right)^T (A)^T$$

$$= A\left((A^TA)^{-1}\right)A^T = P$$

$$\therefore P^TP = P \cdot P = \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right)$$

$$= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T$$

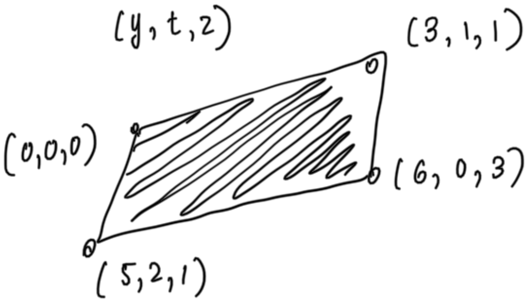
$$= A(A^TA)^{-1}A^T = P$$

Thus P is Projection Matrix

Pg 192

Q.18

$y = C + Dt + Ez$



a) $$\begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix} = C\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + D\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + E\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\underset{4\times1}{\begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}} = \underset{4\times3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} C \\ D \\ E \end{bmatrix}$$

$\therefore C = 0,$ $\quad 5 = 2D + E \Rightarrow E = 1$

$6 = 3D \Rightarrow D = 2$

$\therefore 3 = 0 + D + E = 3$ //

$\boxed{\therefore C = 0,\ D = 2,\ E = 1}$ passes a plane through the points

b) A^{+} : pseudo inverse

$$A^{+} := A(A^TA)^{-1}A^T$$

$$\therefore \hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = A^{+}b = A(A^TA)^{-1}A^T \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}$$

$\Rightarrow$ using python

$$= \begin{bmatrix} -0.12 \\ 1.46 \\ 2.02 \end{bmatrix}$$

Pg 193

Q. 27

$b = (b_1, \ldots, b_m)$ on line $a = (1, \ldots, 1)$

a) $a^T a \hat{x} = a^T b$

$$\therefore \hat{x} = (a^Ta)^{-1}a^Tb$$

$$= \left([1 \cdots 1] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} [1 \cdots 1] \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\hat{x} = \frac{(b_1 + b_2 + \cdots b_m)}{m}$$

b) $e = b - a\hat{x} = b - a\left(\frac{b_1 + \cdots b_m}{m}\right) \Rightarrow$ let $\bar{b} = \sum_i^m b_i / m$

$$= \begin{bmatrix} b_1 - \bar{b} \\ \vdots \\ b_m - \bar{b} \end{bmatrix}$$

$$\therefore \|e\|^2 = (b_1 - \bar{b})^2 + \cdots + (b_m - \bar{b})^2$$

$$\therefore \|e\| = \sqrt{(b_1 - \bar{b})^2 + \cdots + (b_m - \bar{b})^2}$$

c) $\hat{b} = 3 \qquad b = (1,2,6) \qquad e = b - a\hat{b} = (1,2,6) - (3,3,3)$

$$= (-2,-1,3)$$

$\therefore \ p = (3,3,3)$

$$\therefore \langle p, e \rangle = p^T e = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = -6 - 3 + 9 = 0$$

$$\therefore P = \frac{aa^T}{a^T a} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$