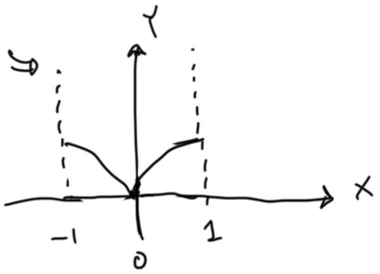
HW 4 CP220 Aditya Shirwalkar 21232

Q.1 $X \sim [-1, 1]$ then pdf of $\sqrt{|X|}$ & $-\ln|X|$

a] $Y = \sqrt{|X|} \Rightarrow$



1st getting CDF $x \in [0,1] \Rightarrow F_Y(y) = P[g(x) \le y]$

$$= P[\sqrt{X} \le y] = P[x \le y^2]$$

$$= \frac{y^2 - (-1)}{1-(-1)} = \frac{y^2+1}{2}$$

$x \in [-1, 0] \Rightarrow F_Y(y) = P[g(x) \le y] = P[\sqrt{-x} \le y]$

$$= P[x \ge -y^2]$$

$$= 1 - P[x < -y^2]$$

$$= 1 - \left(\frac{-y^2 - (-1)}{1-(-1)}\right)$$

$$= 1 - \left(\frac{1-y^2}{2}\right) = \frac{1+y^2}{2}$$

$\therefore$ pdf of $Y \Rightarrow f_Y(y) = \frac{d F_Y(y)}{dy} = y$

b] $Y = -\ln|X|$

$\therefore$ CDF of $Y \Rightarrow F_Y(y) = P[-\ln|X| \le y]$

$$= P[|X| \ge e^{-y}]$$

$x \ge 0$ $x < 0$

$= P[x \geq e^{-y}]$

$= 1 - \left(\dfrac{e^{-y}+1}{2}\right) = \dfrac{1-e^{-y}}{2}$

$= P[x \leq -e^{0}]$

$= \dfrac{-e^{-y}+1}{2}$

$\therefore$ PDF of $Y \Rightarrow f_Y(y) = \dfrac{dF_Y(y)}{dy} = \dfrac{e^{-y}}{2}$

Q.2 $Y = e^{X}$

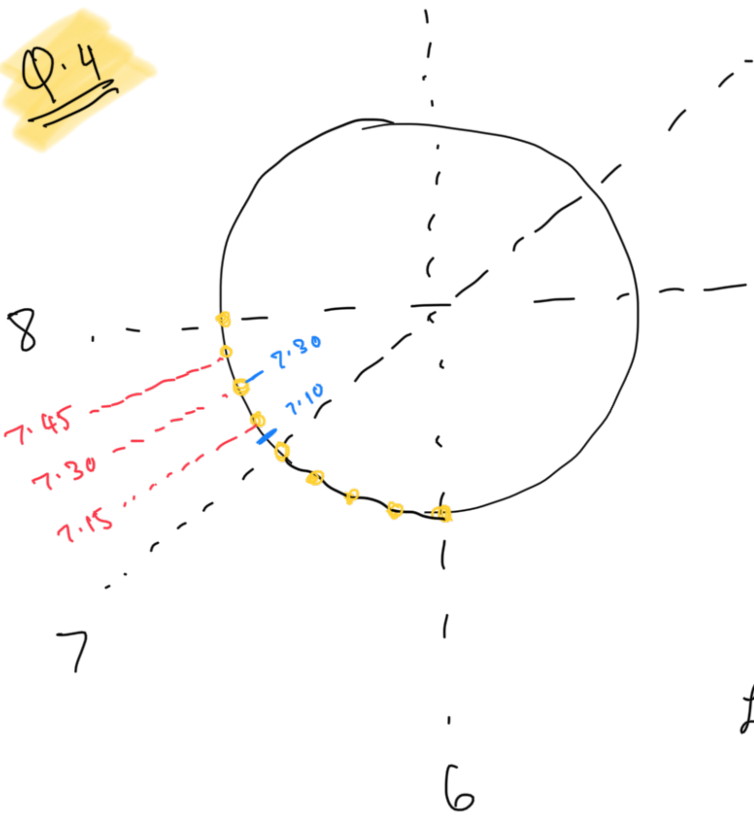
$\therefore$ cdf of $Y \Rightarrow F_Y(y) = P[e^{x} \leq y]$

$= P[x \leq \ln y]$

if uniform $F_Y(y) = \dfrac{\ln y - 0}{1-0} = \ln y$

$\therefore$ PDF of $Y \Rightarrow f_Y(y) = \dfrac{1}{y}$

Q.4



$X \sim \text{Uniform}[7.10\text{ am}, 7.30\text{ am}]$

Y time to wait till train comes

Lets calculate X as a real number in $[10, 30]$ for simlicity

$\therefore X \sim \text{Uniform}[10, 30]$

$$Y = g(x) = \begin{cases} 15 - x & ; \text{ if } x \in [10, 15] \Rightarrow \text{event } A \\ 30 - x & ; \text{ if } x \in (15, 30] \Rightarrow \text{event } B \\ 0 & ; \text{ otherwise} \Rightarrow \text{event } C \end{cases}$$

$\therefore$ CDF of Y in terms of $x = P[g(x) \le y]$

$F_Y(y)$

A:

$$= P[(15-x) \le y]$$
$$= P[x \ge 15 - y]$$
$$= 1 - P[x < 15 - y]$$
$$= 1 - \left(\frac{(15-y) - 10}{30 - 10} \right)$$
$$= 1 - \left[\frac{5-y}{20} \right]$$
$$= \frac{15+y}{20}$$

B:

$$= P[(30-x) \le y]$$
$$= P[x \ge 30 - y]$$
$$= 1 - P[x < 30 - y]$$
$$= 1 - \left(\frac{(30-y) - 10}{30 - 10} \right)$$
$$= 1 - \left(\frac{20-y}{20} \right)$$
$$= \frac{y}{20}$$

C:

$$= P[0 \le y]$$
$$= 0$$

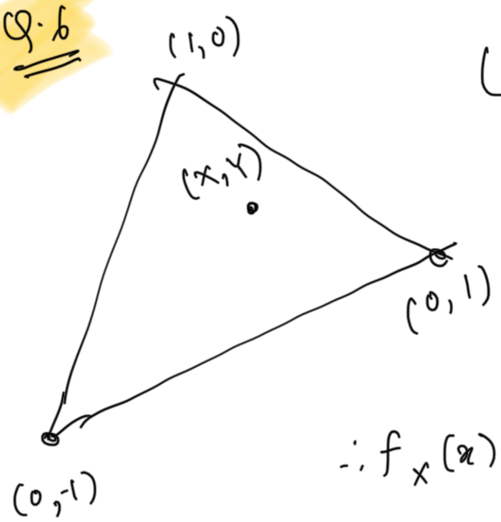
$$\therefore F_Y(y) \begin{cases} = \frac{15+y}{20} & ; x \in A \\ = \frac{y}{20} & ; x \in B \\ = 0 & ; \text{else} \end{cases}$$

$$\therefore \text{pdf of } Y \Rightarrow \frac{d F_Y(y)}{dy} \Rightarrow f_Y(y) \begin{cases} = 1/20 & ; x \in A \\ = 1/20 & ; x \in B \\ = 0 & ; \text{else} \end{cases}$$

; if $x \in A \cup B$

$$f_Y(y) \Rightarrow \begin{cases} 1/20 & , \; 0 \ldots \\ 0 & ; \; \text{else} \end{cases}$$

Q.6



$(X, Y) \sim$ Uniform within a Δle

$$\therefore f_{X,Y}(x,y) = \frac{\text{Area of } \Delta\text{le}}{\text{Area of rectangle}} = \frac{1/2 \cdot 2 \cdot 1}{2 \cdot 1} = 1/2$$

$$\therefore f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^1 1/2\, dy = \frac{1}{2}$$

$$\therefore f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_{-1}^{1} 1/2\, dy = 1$$

$$\therefore Z = |X - Y|$$

$$\therefore \text{CDF of } Z \Rightarrow P[|Z| \leq z] = P[|X-Y| \leq z]$$

$|X-Y| \geq 0$

$$= P[X - Y \leq z]$$

$$= P[X \leq z + Y]$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z+y} f_{X,Y}(x,y)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} [x]_{-1}^{z+y}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}(z + y - 1)\, dy$$

$$= \frac{1}{2}\left[zy + \frac{y^2}{2} - y \right]_0^1$$

$$= \frac{1}{2}\left(z + \frac{1}{2} - 1\right)$$

$|X-Y| < 0$

$$= P[X - Y \geq -z]$$

$$= P[X \geq Y - z]$$

$$= 1 - P[X \leq (Y - z)]$$

$$= 1 - \int_{-\infty}^{\infty} \left(\int_{-\infty}^{y-z} f_{X,Y}(x,y)\, dx \right) dy$$

$$= 1 - \int_{-\infty}^{\infty} \frac{1}{2}(y - z - 1)\, dy$$

$$= 1 - \frac{1}{2}\left(\frac{y^2}{2} - zy - y \right)_0^1$$

$$= 1 - \frac{1}{2}\left(\cancel{\frac{1}{2}} - z - \cancel{\frac{1}{2}} \right)$$

$= \frac{2z-1}{4}$

$= 1 + \frac{1}{\cdot}\left(2z + \frac{1}{2}\right)$

$= \frac{2z+5}{4}$

$\therefore$ Pdf of $Z = \begin{cases} 1/2 & ; z \in \Delta le \\ 0 & ; \text{else} \end{cases}$

Q.7

$X, Y \sim \text{Uniform}[0,1]$

$\therefore$ since independent

$f_{X,Y}(x,y) = 1$

$\therefore Z = |X - Y|$

$\therefore E[Z] = ?$

$\therefore$ CDF of $Z \Rightarrow P[\,|X-Y| \leq z\,]$