E1 222 Stochastic Models and Applications
Hints for Problem Sheet 2.3

1. Consider the random experiment of five independent tosses of a fair coin. In any outcome (of this random experiment) we say a change-over has occurred at the i^{th} toss if the result of the i^{th} toss differs from that of the $(i-1)^{th}$ toss. Let X be a random variable whose value is the number of change-overs. For example, if the outcome of the random experiment is $HTTHH$ then the value of X would be 2. Note that the minimum value of X is 0 (e.g., when the outcome is $HHHHH$) and the maximum value of X is 4 (e.g., when the outcome is $HTHTH$). Find the probability mass function of X. Generalize this to the case of n tosses.

Hint: From toss number 2 to n, at each toss, we may or may not have a change-over. The idea is to ask whether we can consider these as repeated independent 'trials'. At toss number 2, a change-over means either a HT or a TH on first two tosses. Hence probability of a change-over is $0.25 + 0.25 = 0.5$. A little thought should show you this is the probability of chage-over at any toss. Next we ask: is the event of change-over at toss 2 independent of that at toss 3? The intersection of these two events is a change-over at both toss 2 and toss 3. That would mean the outcomes of first three tosses should be either THT or HTH. Hence the probability of a change-over at toss 2 and toss 3 is 0.25 which shows these two are independent. Using this idea you should be able to show that change-over at different tosses are independent events. Now I think the problem is easy to solve. (The case of $n = 5$ is given only so that you may want to check your intuition by brute-force enumeration).

2. Let p be a number such that $0 < p < 1$ and let U be a random variable distributed uniformly over (0, 1). Let X be a random variable defined by

$$X = \mathrm{Int}\left(\frac{\log(1-U)}{\log(1-p)}\right) + 1$$

where $\mathrm{Int}(x)$ is the largest integer smaller than or equal to x. Find the distribution of X.

Hint: For any real number, a, $\text{Int}(a)$ is an integer. Hence, X is a discrete rv. Since $0 < U < 1$ and since $0 < p < 1$, both $\log(1-U)$ and $\log(1-p)$ are negative. Thus X takes only positive integer values. Also, note that for any real number, a, $\text{Int}(a) = k$ iff $k \leq a < k+1$. Hence, for any integer $k \geq 1$,

$$P[X = k] = P\left[k - 1 \leq \frac{\log(1-U)}{\log(1-p)} < k\right]$$

The RHS above is probability of an event involving the random variable U and we know distribution of U. Now simple algebra would allow you to show that X is geometric with parameter p.

3. Let F be the distribution function of a random variable that takes values in $\{0, 1, 2, \cdots\}$. Consider the following procedure (written as a pseudo-code) for determining the value of a random variable X

   1 Generate Z uniform over $[0,\ 1]$.

   2 Set k=0.

   3 *while* $Z > F(k)$ set $k = k + 1$

   4 Set $X = k$ and exit

   What would be the distribution function of X? What is the expected number of steps spent in the while loop?

Hint: We start with $k = 0$ and keep incrementing k as long as $Z > F(k)$. Hence, when we come out of the while loop, the k is such that $Z \leq F(k)$ and $Z > F(k-1)$. Hence $P[X = k]$ is same as $P[F(k-1) < Z \leq F(k)]$. Now I hope you can easily argue that the rv X would have F as its df.

The event of going through the while loop k times has the same probability as that of the event $[X = k]$. Hence expected number of passes through the while loop is same as EX.

I hope you can see that this method can be used to simulate any discrete rv starting with a uniform random number generator.

Can you think of a modification that can improve the expected time complexity of the method?

4. The price of some commodity is Rs. 2 per gram this week. Next week the price would be either Rs.1 per gm or Rs. 4 per gram, each with probability 0.5. You have a capital of Rs.1000. What would be your strategy if (i) you want to maximize expected amount of money with you (next week), (ii) you want to maximize the expected quantity of the commodity with you.

Hint : We have to first decide what we mean by 'strategy'. Given what is stated in the problem, it is reasonable to consider only the following class of strategies. This week we will buy x grams of the commodity and keep the remaining money. Next week, if we want commodity we will buy whatever we get with the money we have; if we want money, we will sell the commodity we have at the prevailing price. So, a strategy is just fixing a value for x. Given that it is selling at Rs.2 per gram this week and that we have Rs. 1000, we know $0 \leq x \leq 500$. If we buy x grams, we currently have x grams of commidity and $1000 - 2x$ rupees. Expected amount of commodity next week would be $[x + (0.5(1000 - 2x) + 0.5((1000 - 2x)/4)]$. Similarly the expected amount of money would be $[1000 - 2x + (0.5(x) + 0.5(4x))]$. Now you can complete the calculation and show that if we want to maximize the (expected) quantity of commodity then we should buy all our commodity at next week's prices. On the other hand, if we want to maximize our (expected) money, we should convert all our money into commodity this week and sell it next week.

5. Children from a school went to a picnic in four buses. Different buses carried different number of students. Define two random variables, X, Y, as follows. We select one of the four drivers at random and X is the number of students in the bus driven by that driver. We select a student at random and Y is the number of students in the bus in which the selected student travelled. Can you say whether $EX > EY$ or $EY > EX$ (or the information given is not sufficient to decide which of EX, EY is greater)?

Hint : Let n_1, n_2, n_3, n_4 be the numbers of children in the four buses. Both X, Y are discrete rv's with $X, Y \in \{n_1, n_2, n_3, n_4\}$. X takes the four values with equal probability. The probability of Y taking value n_1 is $(n_1 / \sum_i n_i)$. Thus Y takes higher values with higher probability. Hence

EY would be higher. You can actually write the expressions for EX and EY and show this algebraically.

6. For a continuous random variable, X, the real number a that satisfies $\int_{-\infty}^{a} f_X(x)\,dx = 0.5$ is called the median of X. Show that for a continuous random variable, X, the number x_0 that minimizes $E[\,|X - x_0|\,]$ is the median of X.

Hint: Split the integral of $E|X - x_0|$ into two parts one for $x \leq x_0$ and the other for $x > x_0$ and thus get rid of absolute value inside the integral. You can now simplify this so that the needed expectation can be shown to be: $x_0(2F_X(x_0) - 1) + \int_{x_0}^{\infty} x f_X(x)\,dx \;-\; \int_{-\infty}^{x_0} x f_X(x)\,dx$. Now you need to find the value of x_0 for which this expression is minimized. You can differentiate it with respect to x_0. You may need the Liebnitz formula for differentiating an integral:

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(x,t)\,dt = f(x,g(x))\frac{d}{dx}g(x) - f(x,h(x))\frac{d}{dx}h(x) + \int_{h(x)}^{g(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

7. Suppose X is a discrete random variable taking positive integer values. Assume $E|X| < \infty$. Show that

$$E[X] = \sum_{k=0}^{\infty} P[X > k].$$

Hint: We have $EX = P[X = 1] + 2 * P[X = 2] + 3 * P[X = 3] + \cdots$. We can rewrite the sum as $(P[X = 1) + (P[X = 2] + P[X = 2]) + (P[X = 3] + P[X = 3] + P[X = 3]) + \cdots$. This can be rearranged to $(P[X = 1] + P[X = 2] + P[X = 3] + \cdots) + (P[X = 2] + P[X = 3] + \cdots) + \cdots$. I hope now you can see how to solve the problem. All such rearrangements are permitted because the expectation is finite and hence the infinite sum is absolutely convergent.

8. Let X be a continuous random variable with $E|X|^k < \infty$ for some $k > 0$. Then show that $n^k P[|X| > n] \to 0$ as $n \to \infty$. (Hint: Write the expectation integral of $|X|^k$ as two parts one for $|x| \leq n$ and the other for $|x| > n$. Since the integral is finite, argue that the second part goes to zero. Then try and bound the second integral in terms of $P[|X| > n]$).

Hint: We follow the hint as given in the problem.

$$E|X|^k = \int_{-\infty}^{\infty} |x|^k \; f_X(x) \; dx \; = \; \int_{|x|\le n} |x|^k \; f_X(x) \; dx + \int_{|x|>n} |x|^k \; f_X(x) \; dx, \; \forall n$$

We know $\lim_{n\to\infty} \int_{|x|\le n} |x|^k \; f_X(x) \; dx = E[|X|^k]$. (Remember we are given $E|X|^k < \infty$). Hence, $\lim_{n\to\infty} \int_{|x|>n} |x|^k \; f_X(x) \; dx = 0$.

Now in the range of integration of the second integral, $|x|^k > n^k$. Hence

$$0 \le n^k \int_{|x|>n} f_X(x) \; dx \le \int_{|x|>n} |x|^k \; f_X(x) \; dx$$

Now you can easily show that $\lim_{n\to\infty} n^k P[|X| > n] \to 0$.

9. Let X be a nonnegative continuous random variable and suppose EX exists. Show that

$$EX = \int_0^{\infty} (1 - F(x)) \; dx$$

(Hint: Integrate by parts and use the previous problem).

Hint: (Since X is a non-negative rv, by saying expectation exists we actually mean $EX < \infty$). We first note that

$$\int_0^{\infty} (1 - F(x)) \; dx = \lim_{n\to\infty} \int_0^n (1 - F(x)) \; dx$$

Now we get

$$\int_0^n (1 - F(x)) \; dx = n - \int_0^n F(x) \; dx$$

$$\int_0^n F(x) \; dx = xF(x)|_0^n - \int_0^n x \; f(x) \; dx$$

$$\text{Hence} \quad \int_0^n (1 - F(x)) \; dx = n(1 - F(n)) + \int_0^n x \; f(x) \; dx$$

Now, taking limit as $n \to \infty$ (and realizing that $n(1-F(n)) = nP[X > n]$ and that X is non-negative) you can complete this problem by using the result in the previous problem.

10. Consider the following density function (called Beta density)

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},\ 0 \le x \le 1.$$

where $\Gamma(\cdot)$ is the gamma function and $a, b \ge 1$ are parameters. Show that this is a density as follows. By definition of gamma function, we have

$$\Gamma(a)\Gamma(b) = \int_0^\infty x^{a-1}e^{-x}\,dx \int_0^\infty y^{b-1}e^{-y}\,dy$$

First bring the integral over y inside the integral over x. Now in the inner integral change the variable from y to t using $t = y + x$. Now change the order of the x and t integrals so that the x integral becomes the inner integral. Now, in the inner integral change the variable from x to s using $x = ts$. The final expression you get can then be used to show that the above $f(x)$ is a density.

11. Suppose an experiment can result in one of r possible outcomes and the i^{th} outcome has probability p_i, $i = 1, 2, \cdots, r$. (Note that $\sum_{i=1}^{r} p_i = 1$). Suppose we have n independent repetitions of this experiment. Argue that the probability that the first outcome occurs x_1 times, the second x_2 times and so on, is

$$\frac{n!}{x_1! x_2! \cdots x_r!}\, p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$$

where $x_1 + x_2 + \cdots + x_r = n$. This is known as the mutinomial distribution. What would this be if $n = 2$?

12. A coin having probability p of coming up heads is successively tossed till the r^{th} head appears. (p and r are parameters). Let X denote the number of tosses needed. Find the mass function of X. (Hint: To calculate $P[X = n]$, think of how many heads are allowed in the first $n-1$ tosses).

Hint: If r^{th} head has to appear on the n^{th} toss, then, the first $n-1$ tosses should have $r-1$ heads and the n^{th} toss should be a head. Hence

$$P[X = n] = {}^{n-1}C_{r-1}\, p^{r-1}(1-p)^{n-r}p$$

13. Consider a random variable X with the mass function

$$f(x) = {}^{(\alpha+x-1)}C_x \, p^\alpha (1-p)^x, \quad x = 0, 1, \cdots$$

where $\alpha > 0$. Is this related to the X in the previous problem? This is known as the negative binomial distribution. The motivation for the name can be seen as follows. For any positive real number α and a nonnegative integer x we have

$$\begin{aligned} {}^{-\alpha}C_x &= \frac{-\alpha(-\alpha-1)\cdots(-\alpha-x+1)}{x!} \\ &= \frac{(-1)^x(\alpha)(\alpha+1)\cdots(\alpha+x-1)}{x!} \\ &= {}^{(\alpha+x-1)}C_x \, (-1)^x \end{aligned}$$

Thus ${}^{(\alpha+x-1)}C_x \, p^\alpha(1-p)^x = {}^{-\alpha}C_x p^\alpha (-1)^x (1-p)^x$. Thus our distribution can be seen to involve binomial coefficients for negative index and hence the name. What will this distribution be for $\alpha = 1$?

14. The binomial distribution can be approximated by the Poisson distribution for large n. Consider a binomial distribution with parameters n and p. Since, the expectation is np, if we want an approximation as n tends to infinity we need to ensure that the expectation is finite. So, let us write p_n as the probability of success when we consider n trials and let us assume that as $n \to \infty$, $np_n \to \lambda$. Noting that, as $n \to \infty$, we have (i). $(1 - \frac{\lambda}{n})^n \to e^{-\lambda}$, (ii). $(1 - \frac{\lambda}{n})^{-k} \to 1$, (iii). $(n(n-1)\cdots(n-k+1))/(n^k) \to 1$, show that

$$\lim_{n\to\infty} {}^nC_k (p_n)^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

15. This problem is about calculating the mode of binomial distribution. That is, if we consider n tosses of a coin with probability of heads as p, we want to know what is the most probable number of heads. Let X be binomial with parameters, n, p. Then, as k goes from 0 to n, $P[X = k]$ first increases monotonically and then decreases monotonically. You can show this as follows. Derive a condition on k to satisfy $P[X = k] < P[X = k+1]$ and similarly for $P[X = k] > P[X = k+1]$. Using these, show the following.

a. If $(n+1)p$ is an integer then $P[X = k]$ attains its maximum value at $(n+1)p - 1$ or at $(n+1)p$

b. If $(n+1)p$ is not an integer then $P[X = k]$ attains its maximum value when k satisfies $(n+1)p - 1 < k < (n+1)p$.