E1 222 Stochastic Models and Applications
Problem Sheet 3.4

1. Recall that random variables $X_1, X_2, \cdots X_n$ are said to be *exchangeable* if any permutation of them has the same joint density. Suppose X_1, X_2, X_3 are exchangeable random variables. Show that

$$E\left[\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right] = \frac{2}{3}$$

   (Hint: Is there a relation between $E\left[\frac{X_1}{X_1+X_2+X_3}\right]$ and $E\left[\frac{X_2}{X_1+X_2+X_3}\right]$?)

2. Let X, Y have joint density given by

$$f_{XY}(x, y) = 6(1 - x),\ 0 < y < x < 1$$

   Find the correlation coefficient between X and Y.

3. A coin, with probability heads being p, is tossed repeatedly till we get r heads. Let N be the number of tosses needed. Calculate EN.
   (Hint: Try to express N as a sum of geometric random variables).

4. A fair dice is rolled repeatedly till each of the numbers $1, 2, \cdots, 6$, appears atleast once. Find the expected number of rolls needed.

5. Consider the random experiment of n independent tosses of a coin whose probability of heads is p. In any outcome (of this random experiment) we say a change- over has occurred at the i^{th} toss if the result of the i^{th} toss differs from that of the $(i-1)^{th}$ toss. Let X be a random variable whose value is the number of change-overs. Find EX. (Hint: You need not find the distribution of X. Note that we are considering the same random variable as in Q1 problem sheet 2.3 except that the coin is not fair now).

Comment: Note that Q3, Q4 and Q5 represent situations where we can calclate EX without explicitly finding the distribution of X. This is often done.

6. Let X_1, X_2, X_3 be independent random variables with finite variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ respectively. Find the correlation coefficient of $X_1 - X_2$ and $X_2 + X_3$.

7. Let X and Y be random variables having mean 0, variance 1, and correlation coefficient ρ. Show that $X - \rho Y$ and Y are uncorrelated, and that $X - \rho Y$ has mean 0 and variance $1 - \rho^2$.

8. Let X, Y, Z be random variables having mean zero and variance 1. Let ρ_1, ρ_2, ρ_3 be the correlation coefficients between $X \& Y$, $Y \& Z$ and $Z \& X$, respectively. Show that

$$\rho_3 \geq \rho_1 \rho_2 - \sqrt{1 - \rho_1^2}\sqrt{1 - \rho_2^2}.$$

(Hint: Write $XZ = [\rho_1 Y + (X - \rho_1 Y)][\rho_2 Y + (Z - \rho_2 Y)]$, and then use the previous problem and Cauchy-Schwartz inequality).

9. Let X be a random variable with mass function given by

$$\begin{aligned} f_X(x) &= \frac{1}{18}, \quad x = 1, 3 \\ &= \frac{16}{18}, \quad x = 2. \end{aligned}$$

Show that there exists a δ such that $P[|X - EX| \geq \delta] = \mathrm{Var}(X)/\delta^2$. This shows that the bound given by Chebyshev inequality cannot, in general, be improved.

10. Use Chebyshev inequality to show that for any real number $K > 1$, we have

$$e^{K+1} \geq K^2$$

(Hint: Try Chebyshev inequality for an exponential rv with $\lambda = 1$)