E1 222 Stochastic Models and Applications

Hints for Problem Sheet 3.2

1. Let (X, Y) have joint density

$$f_{XY}(x, y) = x + y, \; 0 < x < 1, \; 0 < y < 1.$$

Find the marginal densities and the conditional densities. Find $P[X > 2Y]$. Are X, Y independent? Calculate $P[X > 0.5 \mid Y = 0.5]$ and $P[X > 0.5]$.

Hint: For marginal densities, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) \; dy = \int_0^1 (x + y) \; dy = x + 0.5, \; 0 < x < 1$$

It is easy to see that we would get $f_Y(y) = y + 0.5, \; 0 < y < 1$. Now you can write down both the conditional densities. We can conclude X, Y are not independent because the product of marginals is not equal to the joint density.

To calculate $P[X > 2Y]$,

$$P[X > 2Y] = \int_{\{(x,y):x>2y\}} f_{XY}(x, y) \; dx \; dy = \int_0^{0.5} \int_{2y}^1 (x+y) \; dx \; dy = \frac{1.25}{6}$$

The conditional density of X conditioned on Y is given by

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{x + y}{y + 0.5}, \; 0 < x, y < 1$$

Hence we have

$$P[X > 0.5 \mid Y = 0.5] = \int_{0.5}^1 f_{X|Y}(x \mid 0.5) \; dx = \int_{0.5}^1 (x + 0.5) \; dx = \frac{1.25}{2}$$

Notice that $f_{X|Y}(x \mid 0.5) = f_X(x)$ and hence we get the same value for $P[X > 0.5]$.
(Should this surprise us because we know X, Y are not independent?)

2. Let X, Y be discrete random variables taking values in $\{1, 2, \cdots, N\}$ with joint probability mass function

$$f_{XY}(x, y) = \frac{1}{Nx}, \; 1 \le y \le x \le N$$

Find the marginal mass function of X, and the conditional mass function $f_{Y|X}$.

Hint: The marginal of X is

$$f_X(x) = \sum_{y=1}^{x} \frac{1}{Nx} = \frac{1}{N}, \; x = 1, 2, \cdots, N$$

Hence the conditional mass function of Y given X is

$$f_{Y|X}(m|n) = \frac{f_{XY}(m, n)}{f_X(n)} = \frac{1}{n}, \; 1 \leq m \leq n \leq N$$

3. Let X be uniform from 0 to 1, let Y be uniform from 0 to X and let Z be uniform from 0 to Y. What is the joint density of X, Y, Z? Find the marginal densities, f_X, f_Y, f_Z and the joint density of Y, Z.

Hint: From the given verbal description we conclude

$$\begin{aligned} f_{XYZ}(x, y, z) &= f_X(x) f_{Y|X}(y, x) f_{Z|Y,X}(z|y, x) \\ &= 1 \frac{1}{x} \frac{1}{y} = \frac{1}{xy}, \quad 0 < z < y < x < 1 \end{aligned}$$

The rest of the problem is an exercise in integration. This is a good problem to test your skills in multiple integrals. The final answers are

$$f_Y(y) = -\ln(y), \; 0 < y < 1; \quad f_Z(z) = \frac{1}{2}(\ln(z))^2, \; 0 < z < 1$$

$$f_{YZ}(y, z) = -\ln(y)\frac{1}{y}, \; 0 < z < y < 1$$

4. Let X, Y be independent discrete random variables each being uniform over $\{0, 1, \cdots, N\}$. Find $P[X > Y]$, $P[X < Y]$ and $P[X = Y]$.

Hint: The event $[X < Y]$ is the mutually exclusive union of events of the type $[X = k, Y > k]$. Hence

$$P[X < Y] = \sum_{k=0}^{N} \frac{1}{N+1} \frac{N-k}{N+1} = \frac{1}{(N+1)^2}\left(N(N+1) - \frac{N(N+1)}{2}\right) = \frac{N}{2(N+1)}$$

Now, what would be $P[X > Y]$? The intuitive idea is that since X, Y are iid, essentially it is arbitrary which is called X and which is called Y. Hence, the two probabilities should be same. The calculation gives you the same answer. What would be $P[X = Y]$? You can get it either by $P[X = Y] + P[X > Y] + P[X < Y] = 1$ or by $P[X = Y] = \sum_k P[X = k, Y = k]$. You can check the final answer by yourself.

5. Let X, Y be iid random variables which are uniform over $(0,1)$. Find $P[X > Y]$ and $P[Y > X]$.

Hint:

$$P[X > Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x,y)\ dy\ dx = \int_{0}^{1} \int_{0}^{x} dy\ dx = 0.5$$

Since X, Y are iid, by intuition, we expect $P[X > Y] = P[Y > X]$. Since X, Y are continuous random variables and independent, intuitively, $P[X = Y] = 0$. Hence we expect $P[X > Y] = 0.5$.

This does not depend on the density function. Suppose, X, Y are iid cont rv with density f and df F. Then

$$P[X > Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x,y)\ dy\ dx = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(y) f(x)\ dy\ dx = \int_{-\infty}^{\infty} f(x) F(x)\ dx = 0.5$$

6. Let A, B be two events. Let I_A and I_B denote the indicator random variables of these events. Show that I_A and I_B are independent iff A and B are independent.

Answer: Note that I_A, I_B take values in $\{0, 1\}$. We have

$$P[I_A = 1, I_B = 1] = P[AB]; \quad \text{and} \quad P[I_A = 1] = P[A],\ P[I_B = 1] = P[B]$$

Hence $f_{I_A I_B}(1,1) = f_{I_A}(1) f_{I_B}(1)$ if and only if A, B are independent. Since independence of A, B implies independence of A, B^c etc, we similarly show this factorization for other combinations and thus show that I_A, I_B are independent if and only if A, B are independent.

7. Let $X, Y \in \{0, 1, \cdots, N\}$ be two discrete random variables with joint mass function given by $f_{XY}(i,j) = 1/(N+1)^2,\ 0 \leq i, j \leq N$. Are X, Y independent? Find the pmf of Z when (i). $Z = |X - Y|$, and (ii). $Z = X + Y$.

Hint: I hope you can easily show that X, Y are independent. Hence, for $Z = X + Y$ you can use the formula derived in class. That is,

$$P[X + Y = z] = \sum_k P[X = k, Y = z - k] = \sum_k P[X = k] P[Y = z - k]$$

When $Z = |X - Y|$, we have

$$P[Z = z] = \sum_k P[X = k, Y \in \{k + z,\ k - z\}]$$

Now you should be able to complete the problem.

8. Let X and Y be independent random variables each having an exponential distribution with the same value of parameter λ. Show that $Z = \min(X, Y)$ is exponential with parameter 2λ.

Hint: We have

$$P[Z > z] = P[X > z, Y > z] = e^{-\lambda z}\, e^{-\lambda z} = e^{-2\lambda z}$$

because X, Y are independent exponential random variables with the same parameter. Hence, $F_Z(z) = P[Z \leq z] = 1 - e^{-2\lambda z}$ and Z is exponential with parameter 2λ.