E1 222 Stochastic Models and Applications
Hints for Problem Sheet 3.5

1. Find $E[X|Y]$ when X, Y have joint density given by

$$f_{XY}(x,y) = \frac{y}{2} e^{-xy}, \;\; x > 0, \;\; 1 < y < 3$$

Hint: We can write the joint density as $\left(\frac{1}{2}\right) \; ye^{-xy}$. From this I hope it is easy to see that $f_Y(y) = \frac{1}{2}$ and $f_{X|Y}(x|y) = ye^{-xy}$ for the given ranges of the variables in the problem. (What we mean is that $f_Y(y) = \frac{1}{2}, 1 < y < 3$, is a density function and $f_{X|Y}(x|y) = ye^{-xy}, x > 0$, is a density in x for every y in the range $1 < y < 3$).

So, conditional density of X given $Y = y$ is exponential with parameter y. Hence, $E[X|Y] = \frac{1}{Y}$.

You can also get this by first finding f_Y by integrating joint density and then finding conditional density and so on. But that would take more effort.

2. Let X, Y be discrete random variables, taking non-negative integer values, with joint mass function

$$P[X = i, Y = j] = e^{-(a+bi)} \frac{(bi)^j \; a^i}{j! \; i!}, \;\; i, j = 0, 1, \cdots$$

Find $E[Y|X]$ and $\text{Cov}(X, Y)$.

Hint: I hope you can once again see a convenient factorization of joint mass function

$$f_{XY}(i,j) = \left(e^{-a} \frac{a^i}{i!}\right) \; \left(e^{-bi} \frac{(bi)^j}{j!}\right) \;\; i, j = 0, 1, \cdots$$

Thus we can easily see that marginal of X is Poisson with parameter a and the conditional mass function of Y conditioned on $X = i$ is Poisson with parameter bi.

We can get the same by finding marginal of X (which you need because you need to calculate $E[Y|X]$). If we sum the joint over j, we see that $\sum_j e^{-bi} \frac{(bi)^j}{j!} = 1$ for all i and hence see that marginal of X is Poisson with parameter a. Now, using $f_{Y|X} = f_{XY}/f_X$, we can see what the conditional of Y given X is.

Once we get the conditional mass fn of Y given X and notice its Poisson structure, we can easily show that $E[Y|X] = bX$.

To calculate covariance, we need EX, EY and $E[XY]$. While we can get them by appropriate expectation sums, we can get them very easily by looking at the structure of the joint density.

Given marginal of X is Poisson, we get $EX = a$.

Now, $EY = E[E[Y|X]] = E[bX] = ba$.

Similarly, $E[XY] = E[E[XY|X]] = E[XE[Y|X]] = E[X \;\; bX] = bE[X^2] = b(a + a^2)$

Now we get, $\mathrm{Cov}(X, Y) = E[XY] - EXEY = b(a + a^2) - aab = ab$.

3. Let X and Y be iid random variables having Poisson distribution with parameter λ. Let $Z = X + Y$. Find $E[Z|Y]$.

Hint:

$$E[Z|Y] = E[X + Y|Y] = E[X|Y] + E[Y|Y] = E[X] + Y = \lambda + Y$$

4. Let X and Y be independent random variables each having geometric density with parameter p. Let $Z = X + Y$. Find $E[Y|Z]$.

Hint:

$$Z = E[Z|Z] = E[X + Y|Z] = E[X|Z] + E[Y|Z]$$

Since X, Y are iid we should have $E[X|Z] = E[Y|Z]$. So, each of them should be equal to $Z/2$.

You can verify this by direct calculation. $f_{Y|Z}(y|z) = P[Y = y|Z = z] = P[Y = y, Z = z]/P[Z = z]$. You know how to find this conditional mass function from your mid-term test. Now simple algebra would allow you to verify the above answer.

5. Suppose X, Y are random variables with $E[Y|X] = 1$. Show that $EXY = EX$ and $\mathrm{Var}(XY) \geq \mathrm{Var}(X)$.

Hint:

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]] = EX$$

because we are given $E[Y|X] = 1$.

We have (using $E[XY] = EX$)

$$\mathrm{Var}(XY) = E[X^2Y^2] - (E[XY])^2 = E[X^2Y^2] - (EX)^2$$

Hence all we need to show is that $E[X^2Y^2] > E[X^2]$.

By jensen's inequality, $E[Y^2|X] \geq (E[Y|X])^2 = 1$.

This gives, $E[X^2Y^2] = E[E[X^2Y^2|X]] = E[X^2 \; E[Y^2|X]] \geq E[X^2]$ because $E[Y^2|X] \geq 1$.

6. Let Y be a continuous random variable with density

$$f_Y(y) = \frac{1}{\Gamma(0.5)}\sqrt{\frac{0.5}{y}}e^{-0.5y}, \; y > 0.$$

Let the conditional density of X given Y be

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}}\sqrt{y}\, e^{-0.5yx^2}, \; -\infty < x < \infty, \; y > 0$$

Show that $E[X|Y] = 0$. Show that marginal of X is given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

(Note that $\Gamma(0.5) = \sqrt{\pi}$). Does EX exist? Is there something useful that this example tells us regarding expectation of conditional expectation?
(Notice that $f_{X|Y}$ is Gaussian with mean zero and variance $1/\sqrt{y}$, Y is Gamma with parameters $0.5, 0.5$ and X has Cauchy distribution).

Comment No hint needed here. If you multiply $f_{X|Y}$ and f_Y you get a simple expression which is easily integrated over y to get f_X.

This is an example where $E[E[X|Y]] = E[X]$ does not hold because one of the expectations does not exist.

7. Define $\mathrm{Var}[X|Y] = E\left[(X - E[X|Y])^2|Y\right]$. Show that

$$\mathrm{Var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}[X|Y]]$$

Hint: $\mathrm{Var}[X|Y] = E\left[(X - E[X|Y])^2|Y\right] = E\left[(X^2 + (E[X|Y])^2 - 2X(E[X|Y]) \mid Y\right] = E[X^2|Y] + (E[X|Y])^2 - 2(E[X|Y])E[X|Y] = E[X^2|Y] - (E[X|Y])^2$
where we used the fact that $E[X|Y]$ is a function of Y.

Hence, $E[\mathrm{Var}[X|Y]] = E[X^2] - E\left[(E[X|Y])^2\right]$.

We have
$\mathrm{Var}(E[X|Y]) = E[(E[X|Y])^2] - (E[E[X|Y]])^2 = E[(E[X|Y])^2] - (EX)^2$.
I hope now you can complete the solution.

8. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed repeatedly until the same outcome occurs k consecutive times. Let N denote the number of trials needed. Show that

$$E[N] = \frac{m^k - 1}{m - 1}$$

Hint: This is a slight variation of the problem done in class where N_k is number of tosses to get m consecutive heads and we want EN_k. We conditioned on N_{k-1}. Suppose $N_{k-1} = n$. If next toss is head you are done; otherwise you wasted n plus one tosses and have to start all over again.

Now suppose we wanted N_k which is number of tosses needed to get either k consecutive heads or k consecutive tails and assume coin is fair. Once again condition on N_{k-1}. Suppose you know $N_{k-1} = n$. If the next toss results in the same outcome as all these, we are done. Otherwise we wasted n tosses. Note that you have not wasted the $n + 1$ toss. because you can start counting another run from that toss onwards.

Hope the hint is enough for you to get an idea on how to solve this problem. Look at the recurrence relation we derived in class for finding expected number of tosses for k consecutive heads and see how you can modify it to take care of what is said above.

9. Let $X_1, X_2, \cdots$ be *iid* discrete random variables with $P[X_i = +1] = P[X_i = -1] = 0.5$. Find EX_i. Let N be a positive integer-valued

random variable (which is a function of all X_i) defined as $N = \min\{k : X_k = +1\}$. Find EX_N.

Hint: What is the value of $E[X_i|N = i]$? By definition of N, if $N = k$ then $X_k = +1$. Hence, $P[X_k = +1|N = k] = 1$. Hence, $E[X_N] = E[E[X_N|N]] = \sum_i E[X_N|N = i]P[N = i] = \sum_i E[X_i|N = i]P[N = i] = \sum_i (+1)P[N = i] = 1$.

But you should easily see this intuitively. N is the first index i such that $X_i = +1$. Hence the rv corresponding to index N has to take value 1 only!

10. Let $I_1, I_2, \cdots, I_n$ be independent random variables that take values 0 or 1, each with probability 0.5. Let

$$P_m(k) = P\left[\sum_{j=1}^{m} jI_j \;\leq\; k\right].$$

Show that $P_m(k) = 0.5P_{m-1}(k) + 0.5P_{m-1}(k - m)$.

Hint: We once again use the conditioning argument.

$$P\left[\sum_{j=1}^{m} jI_j \;\leq\; k\right] = \sum_{j=0}^{1} P\left[\sum_{j=1}^{m} jI_j \;\leq\; k \mid I_m = j\right] \; P[I_m = j]$$

Writing $\sum_{j=1}^{m} jI_j = \sum_{j=1}^{m-1} jI_j + mI_m$ and noting that I_j are independent, we get

$$P\left[\sum_{j=1}^{m} jI_j \;\leq\; k \mid I_m = 0\right] = P\left[\sum_{j=1}^{m-1} jI_j \;\leq\; k\right]$$

$$P\left[\sum_{j=1}^{m} jI_j \;\leq\; k \mid I_m = 1\right] = P\left[\sum_{j=1}^{m-1} jI_j \;+ m \;\leq\; k\right]$$

Now you should be able to complete the solution

11. Let X be an exponential random variable. Calculate $E[X|X > 1]$.

Hint: If the expression is confusing to you, you can rewrite it as follows. Suppose Z is the rv which is indicator of the event $[X > 1]$. Then what you want is $E[X|Z = 1]$.

For this we essentially need the conditional density of X given Z at $Z = 1$. We can write the conditional distribution as

$$\begin{aligned} P[X \leq x | X > 1] &= \frac{P[X \leq x, X > 1]}{P[X > 1]} \\ &= \begin{cases} 0 & \text{if } \ x \leq 1 \\ \frac{F_X(x) - F_X(1)}{1 - F_X(1)} & \text{if } \ x \geq 1 \end{cases} \end{aligned}$$

Hence the corresponding density is (using the rv Z so that the notation is easier)

$$f_{X|Z}(x|1) = \begin{cases} 0 & \text{if } \ x \leq 1 \\ \frac{f_X(x)}{1 - F_X(1)} & \text{if } \ x \geq 1 \end{cases}$$

Now we get (remembering that $1 - F_X(1) = e^{-\lambda}$)

$$E[X|X > 1] = \frac{1}{e^{-\lambda}} \int_1^{\infty} x \ \lambda e^{-\lambda x} \ dx = 1 + \frac{1}{\lambda}$$

The answer is intuitively clear given the memoryless property of exponential rv.