E1 222 Stochastic Models and Applications

Problem Sheet 3.3

(You need not submit solutions)

1. Consider a communication system. Let Y denote the bit sent by transmiter. (Y is a binary random variable). The receiver makes a measurement, X, and based on its value decides what is sent. The decision at the receiver can be represented by a function $h : \Re \to \{0,\ 1\}$. For any specific h, let $R_0(h)$ represent the set of all $x \in \Re$ for which $h(x) = 0$ and let $R_1(h)$ represent the set of $x \in \Re$ for which $h(x) = 1$. An error occurs if a wrong decision is made. Argue that the event of error is: $[h(X) = 0, Y = 1] \cup [h(X) = 1, Y = 0]$. Show that probability of error for a decision rule h is

$$\int_{R_0(h)} p_1 f_{X|Y}(x|1)dx \ + \ \int_{R_1(h)} p_0 f_{X|Y}(x|0)dx$$

where $p_i = P[Y = i]$. Now consider a h given by

$$h(x) = 1 \ \text{ if } \ f_{Y|X}(1|x) \geq f_{Y|X}(0|x)$$

(Otherwise $h(x) = 0$). Show that this h would achieve minimum probability of error.

2. Let X, Y have a joint distribution that is uniform over the quadrilateral with vertices at $(-1, 0), (1, 0), (0, -1)$ and $(0, 1)$. Find $P[X > Y]$. Are X, Y independent? (Hint: Can you decide on independence here without calculating the marginal densities?)

3. Let X, Y be *iid* uniform over $(0,\ 1)$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$. Find the density of $Z - W$.

4. Let X, Y be iid exponential random variables with mean 1. Let $Z = X + Y$ and $W = X - Y$. Find the conditional density $f_{W|Z}$

5. Let X, Y be independent Gaussian random variables with mean zero and variance unity. Define random variables D and θ by

$$D = X^2 + Y^2; \qquad \theta = \tan^{-1}(Y/X)$$

(where, by convention, we assume θ takes values in $[0, 2\pi]$; for this we first calculate $\tan^{-1}(|Y|/|X|)$ in the range $[0, \pi/2]$ and then put that angle in the appropriate quadrant based on signs of Y and X). Find the joint density of D and θ and their marginal densities. Are D and θ independent? (Hint: Note that the (D, θ) to (X, Y) mapping is invertible. Hence you can use the formula).

6. Consider the following algorithm for generating random numbers X and Y :

    1. Generate U_1 and U_2 uniform over $[0,\ 1]$.
    2. Set $X = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$ and $Y = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$.

    What would be the joint distribution of X and Y ? (Hint: Recall that when U is uniform over $[0,\ 1]$, we know that $-a\log(U)$ is exponential with parameter $1/a$ and $2\pi U$ is uniform over $[0, 2\pi]$).

7. Consider the following algorithm for generating random variables V_1 and V_2 :

    1. Generate X_1 and X_2 uniform over $[-1,\ 1]$.
    2. If $X_1^2 + X_2^2 > 1$ then go to step 1; else set $V_1 = X_1,\ V_2 = X_2$ and exit.

    What would be the joint distribution of V_1 and V_2 ?

8. Suppose we have access to a random number generator that can generate random numbers uniformly distributed over $(0, 1)$. Using the results of the previous problems, suggest a method for generating samples of X when X has Gaussian density with mean zero and variance unity.

9. Let $p_i, q_i,\ i = 1, \cdots, N$, be positive numbers such that $\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i = 1$ and $p_i \leq Cq_i,\ \forall i$ for some positive constant C. Consider the following algorithm to simulate a random variable, X:

    1. Generate a random number Y such that $P[Y = j] = q_j,\ j = 1, \cdots, N$. (That is, the mass function of Y is $f_Y(j) = q_j$).
    2. Generate U uniform over $[0,\ 1]$.
    3. Suppose the value generated for Y in step-1 is j. If $U < (p_j/Cq_j)$, then set $X = Y$ and exit; else go to step-1.

On any iteration of the above algorithm, if condition in step-3 becomes true, we say the generated Y is accepted. Find the value of $P[Y \text{ is accepted} \mid Y = j]$. Show that $P[Y \text{ is accepted}, \ Y = j] = p_j/C$. Now calculate $P[Y \text{ is accepted}]$. Use these to calculate the mass function of X.

10. Suppose X is a discrete rv taking values $\{x_1, x_2, \cdots, x_m\}$ with probabilities $p_1, \cdots p_m$. The usual method of simulating such a rv is as follows. We divide the [0, 1] interval into bins of length p_1, p_2 etc. Then we generate a rv, uniform over [0, 1] and depending on the bin it falls in, we decide on the value for X. That is, if $U \leq p_1$ we assign $X = x_1$; if $p_1 < U \leq p_1 + p_2$ then we assgn $X = x_2$ and so on.
    Suppose X is a discrete random variable taking values $1, 2, \cdots, 10$. Its mass function is: $f_X(1) = 0.08, f_X(2) = 0.13, f_X(3) = 0.07, f_X(4) = 0.15, f_X(5) = 0.1, f_X(6) = 0.06, f_X(7) = 0.11, f_X(8) = 0.1, f_X(9) = 0.1, f_X(10) = 0.1$. Can you use the result of previous problem to suggest an efficient method for simulating X.