E1 222 Stochastic Models and Applications
Solutions to Problem Sheet 2.1

1. Let $(\Omega,\ \mathcal{F},\ P)$ be a probability space and let $A_1, A_2 \in \mathcal{F}$. Consider the following random variable:

$$\begin{aligned} X(\omega) &= -1 \quad \text{if} \quad \omega \in A_1 \\ &= +1 \quad \text{if} \quad \omega \in A_1^c A_2 \\ &= \ \ 0 \quad \text{if} \quad \omega \in A_1^c A_2^c \end{aligned}$$

What is the event $[X < 0.5]$? Find the distribution function of X.

Answer: $[X < 0.5] = [X = -1 \ \text{ or } \ X = 0] = A_1 \cup A_1^c A_2^c$

This is a discrete random variable taking three distinct values. So, we know the distribution function would have jump discontinuities at those points and we know the magnitudes of jumps. This gives us the distribution function as

$$F_X(x) = \begin{cases} 0 & \text{if } \ x < -1 \\ P(A_1) & \text{if } \ -1 \le x < 0 \\ P(A_1) + P(A_1^c A_2^c) & \text{if } \ 0 \le x < 1 \\ 1 & \text{if } \ x \ge 1 \end{cases}$$

2. Two fair dice are rolled and X is the maximum of the two numbers. Specify a reasonable probability space and then define this random variable as a function on the appropriate Ω. Derive its probability mass function.

Answer: Obvious choice is $\Omega = \{(a, b) : a, b \in \{1, 2, \cdots, 6\}\}$ and $X((a, b)) = \max(a, b)$. We can assume that each singleton event here would have probability 1/36. It is easy to see that $X \in \{1, \cdots, 6\}$. We can write the event $[X = k]$ as the mutually exclusive union of three sets as

$$[X = k] = \{(k, a) : a < k\} + \{(a, k) : a < k\} + \{(k, k)\}$$

This gives us the mass function as $f_X(k) = \frac{2k-1}{36},\ 1 \le k \le 6$.
(Verify that $\sum_k f_X(k) = 1$).
(Can we write the $[X = k]$ event as

$$[X = k] = \{(k, a) : a \le k\} + \{(a, k) : a \le k\}?$$

)

3. Consider the probability space with $\Omega = [0, 1]$ and the usual probability assignment (where probability of an interval is the length of the interval). Define X by $X(\omega) = 2\omega$ if $0 \leq \omega \leq 0.5$, and $X(\omega) = 2\omega - 0.5$ if $0.5 < \omega \leq 1$. What is the event $[X \in (0.5,\ 0.75)]$? Find the distribution function of X.

Answer: First consider the event, $[X \in (0.5,\ 0.75)] = \{\omega\ :\ 0.5 < X(\omega) < 0.75\}$. The function $X(\omega)$ is given by two different expressions in two different ranges. Hence, the above event is same as:

$$\{\omega \in [0, 0.5] : 0.5 < 2\omega < 0.75\} \cup \{\omega \in [0.5, 1.0] : 0.5 < 2\omega - 0.5 < 0.75\}$$

Thus, we can see that this event is given by

$$[X \in (0.5,\ 0.75)] = (0.25,\ 0.75/2) \cup (0.5,\ 1.25/2)$$

Actually, it is easier to see this by plotting the function $X(\omega)$ against ω. You can plot this graph, draw two lines parallel to x-axis through $y = 0.5$ and $y = 0.75$ and can ask for what all values on the x-axis the graph is between these two lines. You can get the df also by the same technique. You should plot the graph and see how you can solve it geometrically.

Consider the df. In the range $0 \leq \omega \leq 0.5$, $X(\omega)$ ranges between 0 and 1. In the range $0.5 \leq \omega \leq 1.0$, $X(\omega)$ ranges between 0.5 and 1.5 Thus, the range of values for X is 0 to 1.5.

Hence we can easily see that $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x \geq 1.5$.

For $0 \leq x < 0.5$, $[X \leq x]$ would be $\{\omega \in [0, 0.5] : 2\omega \leq x\}$. Becuase when $\omega \in [0.5, 1.0]$, we have $X(\omega) \geq 0.5$. This shows us that the event $[X \leq x]$ here would be the interval $[0,\ x/2]$ and its probability (length) is $x/2$.

When $\omega \in [0, 0.5]$, we have $X(\omega) \leq 1.0$. Hence, for $1 < x < 1.5$, the event $[X \leq x]$ is equal to $[0,\ 0.5] \cup \{\omega \in [0.5, 1.0] : 2\omega - 0.5 \leq x\}$ which is $[0,\ 0.5] \cup [0.5,\ (x + 0.5)/2]$ whose length is $(x + 0.5)/2$.

In the range, $0.5 \leq x < 1$, subintervals from both ranges would contribute to the event $[X \leq x]$. I hope you can now calculate this.

Thus the df is given by

$$
\begin{aligned}
F_X(x) &= 0 \ \text{ if } \ x < 0 \\
&= x/2 \ \text{ if } \ 0 \leq x \leq 0.5 \\
&= (2x - 0.5)/2 \ \text{ if } \ 0.5 \leq x \leq 1 \\
&= (x + 0.5)/2 \ \text{ if } \ 1 \leq x \leq 1.5 \\
&= 1 \ \text{ if } \ x \geq 1.5
\end{aligned}
$$

Actually, we can see the event $[X \leq x]$ by drawing the grpah of $X(\omega)$ Vs ω and drawing lines prallel to x-axis at different heights and seeing for what all values on the x-axis, the graph is below the line. Such a geometric method makes the concept more easier to understand.

4. Let X be a random variable with $P[X = a] = 0$. Express $P[|X| \geq a]$ in terms of the distribution function of X.

Answer:

$$
P[|X| \geq a] = 1 - P[|X| < a] = 1 - P[-a < X < a] = 1 - (P[-a < X \leq a] - P[X = a])
$$

$$
= 1 - P[-a < X \leq a] = 1 - (F_X(a) - F_X(-a))
$$

5. Let X be geometric. Calculate probabilities of the events (i). $[X \leq 10]$, (ii). $[X = 3 \ \text{ or } \ 5 \leq X \leq 7]$.

Answer:

$$
P[X \leq 10] = \sum_{k=1}^{10} f_X(k) = \sum_{k=1}^{10} p(1-p)^{k-1} = 1 - (1-p)^{10}
$$

$$
\begin{aligned}
&P[X = 3 \ \text{ or } \ 5 \leq X \leq 7] = P[X \in \{3, 5, 6, 7\}] \\
&= p\left[(1-p)^2 + (1-p)^4 + (1-p)^5 + (1-p)^6\right]
\end{aligned}
$$

6. Let X be exponential random variable. Calculate probabilities of (i). $[|X| \leq 3]$, (ii). $[X \leq 4 \ \text{ or } \ X \geq 10]$.

Answer:

$$
P[|X| \leq 3] = \int_{-3}^{3} f_X(x)\, dx = \int_{0}^{3} \lambda e^{-\lambda x}\, dx = 1 - e^{-3\lambda}
$$

$$
P[X \leq 4 \ \text{ or } \ X \geq 10] = P[X \leq 4] + P[X \geq 10] = (1 - e^{-4\lambda}) + e^{-10\lambda}
$$

7. Let X be a rv with density function

$$f(x) = cx^3, \quad \text{if} \quad 0 \leq x \leq 1.$$

($f(x)$ is zero for all other values of x). Find the value of c and the distribution function of X. Find $P[X > 0.5]$.

Answer:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 cx^3\, dx = 1 \Rightarrow c = 4$$

It is easily seen that $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 1$. For $0 \leq x \leq 1$,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \int_0^x 4x^3\, dx = x^4$$

Now

$$P[X > 0.5] = 1 - F_X(0.5) = 1 - (1/16) = 15/16$$