# Answers to Problem Sheet 1.1

1. A fair coin is tossed n times. What is the probability that the difference between the number of heads and the number of tails is equal to $n-3$?

Answer: If we get $k$ heads out of $n$ tosses then the difference between heads and tails is $k-(n-k)=2k-n$. Thus the difference can only take the values: $n, n-2, n-4, \cdots$ and hence can never be equal to $n-3$. Hence the probability is zero.

2. Suppose we want to find mutually exclusive events, $A_1, \cdots, A_n$ such that $P(A_i) \geq 0.1$, $\forall i$. What is the maximum possible value for $n$.

Answer: For mutually exclusive events we have $P(\cup_i A_i) = \sum_i P(A_i)$ and probability of any event should be less than or equal to 1. Hence, maximum possible value for $n$ is 10

3. State whether the following sets are finite, countably infinite or uncountably infinite.

   (i). The sample space for the situation when each of a million people vote for one of ten candidates.

   (ii). All rational numbers between 0 and 1.

   (iii). All irrational numbers between 0 and 1.

   (iv). All real numbers between 0 and 1.

Answer: (i). Finite, (ii) countably infinite, (iii) uncountably infinite, (iv). uncountably infinite

4. A standard deck of playing cards is made into four piles. Let $E_i$ denote the event that the $i^{th}$ pile contains exactly one ace. Find $P(E_1E_2E_3E_4)$.

Answer: For any events we have

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3)$$

It is easily seen that $P(E_1) = \frac{{}^{48}C_{12}\ {}^{4}C_1}{{}^{52}C_{13}}$, $P(E_2|E_1) = \frac{{}^{36}C_{12}\ {}^{3}C_1}{{}^{39}C_{13}}$ and so on. This gives the final answer as $\frac{(48!)(13!)^4(4!)}{(52!)(12!)^4}$.
(You can also directly solve this by noting that the number of possible ways to make the four piles as needed is $\frac{48!}{(12!)^4}4!$ and total number of ways of making the four piles is $\frac{52!}{(13!)^4}$).

5. A chord is drawn at random in the unit circle. What is the probability that the length of the chord is greater than the side of the inscribed equilateral triangle. (The side of an equilateral triangle inscribed inside a unit circle is $\sqrt{3}$).
(The solution to the problem depends on how we define what is meant by a random chord. One possibility is as follows. We can think of choosing a random chord to be same as that of choosing a point inside the circle. This is because any given point inside the circle can be uniquely corresponded to a chord: join the point to the center of the circle and then draw a line through the point perpendicular to the line joining the point to the center. The resulting chord is what we can uniquely associate with the point. Using this idea, calculate the above probability.)

Answer: From the explanation given above, we can take sample space to be all points inside a circle with center as origin and radius 1: $\Omega = \{(x, y) \in \Re^2 \ : \ x^2 + y^2 < 1\}$. Simple geometric calculation shows that a point has to be at a distance less than 0.5 from the center for it to correspond to a chord of length greater than $\sqrt{3}$. This means the event of interest is a circle of radius $0.5 - A = \{(x, y) \in \Re^2 \ : \ x^2 + y^2 < 0.5\}$. The probability is the ratio of areas of $A$ and $\Omega$ which is 0.25.

Comment: There can be an alternate view on what constitutes a random chord. Since we are only interested in lengths of chords we can fix one end of the chord to an arbitrary point on the circle and choose the other point randomly on the circle. In this view, we take $\Omega = \{(x, y) \in \Re^2 \ : \ x^2 + y^2 = 1\}$. It is easy to see that the other point should be on an arc of the circle constituting a third of the circumference for the chord to have length greater than $\sqrt{3}$. This view gives us the answer as 1/3. Of course, the question of which is the **correct** probability is meaningless as far as Probability Theory is concerned.

6. Consider a student answering a multiple-choice question. The student knows the answer with probability $p$, $0 < p < 1$. If the student knows the answer, the student marks the correct answer with probability 0.99. When the student does not know the answer, the student guesses and hence the probability of marking the correct answer is $\frac{1}{k}$, where $k$ is the number of choices. Calculate the probability that the student knows the answer given that the student marked the correct answer.

Answer: Let $K$ denote the event that the student knows the answer and let $M$ denote the event that the student marks the correct answer. Then

$$\begin{aligned} P(K|M) &= \frac{P(M|K)P(K)}{P(M|K)P(K) + P(M|K^c)P(K^c)} \\ &= \frac{0.99p}{0.99p + (1-p)\frac{1}{k}} \end{aligned}$$

As is easy to see, as $k \to \infty$, this probability goes to 1. But what is more interesting is to look at the probability for a fixed $k$, say, $k = 4$ and different values of $p$. Take $p = 0.1$ and $p = 0.9$ and see what is the value of the probability in each case.

7. State whether the following sequences of sets are monotone.

   (i). $A_k = [0,\ 1 + \frac{(-1)^k}{k}],\ k = 1, 2, \cdots$

   (ii). $A_k = [1/k,\ 1],\ k = 1, 2, \cdots$

Answer: (i). not monotone, (ii). monotone

Comment: Note that for the sequence $A_n$ to be, e.g., monotone increasing we need to have $A_n \subset A_{n+1}$ **for all** $n$.

8. Let $A_k = (-1/k,\ 1],\ k = 1, 2, \cdots$. Let $B = \cap_{k=1}^{\infty} A_k$. For any $x < 0$, show that there is a $K$ such that $x \notin A_K$. For any $x$, such that $0 < x < 1$, show that $x \in A_k$, $\forall k$. Now determine what $B$ is.

Answer: $B = [0, 1]$

Let $x = -\epsilon$ with $1 \geq \epsilon > 0$. (Note that if $x < -1$ then anyway $x$ does not belong to $A_k$ for all $k$). Take $K > 1/\epsilon$. Then $\epsilon > 1/K$ and hence $-\epsilon < -1/K$. Then $-\epsilon \notin A_k$, $\forall k \geq K$. So, no negative number belongs to $B$. The point 0 is in $A_k$ for all $k$. It is easy to see that $[0,\ 1] \subset A_k$, $\forall k$. Hence $B = [0, 1]$

9. Let $A_k = [1/k,\ 1]$, $k = 1, 2, \cdots$. Let $B = \cup_{k=1}^{\infty} A_k$. For any $0 < x < 1$, show that there is a $K$ such that $x \in A_K$. Now determine what $B$ is.

Answer: $B = (0, 1]$

You can argue as in the above problem to show what is asked. Also, note that the point 0 does not belong to $A_k$ for all $k$. Hence $B = (0, 1]$