E1 222 Stochastic Models and Applications
Problem Sheet 3.5

1. Find $E[X|Y]$ when X, Y have joint density given by
$$f_{XY}(x,y) = \frac{y}{2}e^{-xy},\ x > 0,\ 1 < y < 3$$

2. Let X, Y be discrete random variables, taking non-negative integer values, with joint mass function
$$P[X = i, Y = j] = e^{-(a+bi)}\frac{(bi)^j\ a^i}{j!\ i!},\ i, j = 0, 1, \cdots$$
Find $E[Y|X]$ and $\mathrm{Cov}(X, Y)$.

3. Let X and Y be iid random variables having Poisson distribution with parameter λ. Let $Z = X + Y$. Find $E[Z|Y]$.

4. Let X and Y be independent random variables each having geometric density with parameter p. Let $Z = X + Y$. Find $E[Y|Z]$.

5. Suppose X, Y are random variables with $E[Y|X] = 1$. Show that $EXY = EX$ and $\mathrm{Var}(XY) \geq \mathrm{Var}(X)$.

6. Let Y be a continuous random variable with density
$$f_Y(y) = \frac{1}{\Gamma(0.5)}\sqrt{\frac{0.5}{y}}e^{-0.5y},\ y > 0.$$
Let the conditional density of X given Y be
$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}}\ \sqrt{y}\ e^{-0.5yx^2},\ -\infty < x < \infty,\ y > 0$$
Show that $E[X|Y] = 0$. Show that marginal of X is given by
$$f_X(x) = \frac{1}{\pi(1 + x^2)}$$
(Note that $\Gamma(0.5) = \sqrt{\pi}$). Does EX exist? Is there something useful that this example tells us regarding expectation of conditional expectation?
(Notice that $f_{X|Y}$ is Gaussian with mean zero and variance $1/\sqrt{y}$, Y is Gamma with parameters $0.5, 0.5$ and X has Cauchy distribution).

7. Define $\text{Var}[X|Y] = E\left[(X - E[X|Y])^2|Y\right]$. Show that

$$\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}[X|Y]]$$

8. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed repeatedly until the same outcome occurs k consecutive times. Let N denote the number of trials needed. Show that

$$E[N] = \frac{m^k - 1}{m - 1}$$

9. Let $X_1, X_2, \cdots$ be *iid* discrete random variables with $P[X_i = +1] = P[X_i = -1] = 0.5$. Find EX_i. Let N be a positive integer-valued random variable (which is a function of all X_i) defined as $N = \min\{k : X_k = +1\}$. Find EX_N.

10. Let $I_1, I_2, \cdots, I_n$ be independent random variables that take values 0 or 1, each with probability 0.5. Let

$$P_m(k) = P\left[\sum_{j=1}^{m} jI_j \ \leq \ k\right].$$

Show that $P_m(k) = 0.5P_{m-1}(k) + 0.5P_{m-1}(k - m)$.

11. Let X be an exponential random variable. Calculate $E[X|X > 1]$.