E1 222 Stochastic Models and Applications
Hints for Problem Sheet 3.4

1. Recall that random variables $X_1, X_2, \cdots X_n$ are said to be *exchangeable* if any permutation of them has the same joint density. Suppose X_1, X_2, X_3 are exchangeable random variables. Show that

$$E\left[\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right] = \frac{2}{3}$$

(Hint: Is there a relation between $E\left[\frac{X_1}{X_1+X_2+X_3}\right]$ and $E\left[\frac{X_2}{X_1+X_2+X_3}\right]$?)

Hint: I hope the hint in the problem was sufficient. Here is an extended hint.

Suppose X, Y are exchangeable. We have

$$E\left[\frac{X}{X+Y}\right] = \int\int \frac{a}{a+b}\, f_{XY}(a,b)\, da\, db \quad \text{and} \quad E\left[\frac{Y}{X+Y}\right] = \int\int \frac{b}{a+b}\, f_{XY}(a,b)\, da\, db$$

By changing variables in the second integral as a to b and b to a, and noting that $f_{XY}(a,b) = f_{XY}(b,a)$ because X, Y are exchangeable, we conclude that both the above expectations are equal. Also, we easily see that the sum of the two expectations above is 1.

Now I hope you can easily solve the given problem.

2. Let X, Y have joint density given by

$$f_{XY}(x,y) = 6(1-x),\ 0 < y < x < 1$$

Find the correlation coefficient between X and Y.

Hint: You can easily find the two marginals:

$$\begin{aligned} f_X(x) &= \int_0^x 6(1-x)\, dy = 6x(1-x),\ 0 < x < 1 \\ f_Y(y) &= \int_y^1 6(1-x)\, dx = 3(1-y)^2,\ 0 < y < 1 \end{aligned}$$

Now we can calculate that

$$EX = \frac{1}{2},\ \ \mathrm{Var}(X) = \frac{1}{20},\ \ EY = \frac{1}{4},\ \ \mathrm{Var}(Y) = \frac{3}{80}$$

either directly by integration or by using the moments of beta density. We can also calculate

$$E[XY] = \int_0^1 \int_0^x xy\, 6(1-x)\, dy\, dx = \frac{3}{20}$$

where we can again use the beta density expression to simplify calculation. With all this, we can now get $\rho_{XY} = 1/\sqrt{3}$.

3. A coin, with probability of heads being p, is tossed repeatedly till we get r heads. Let N be the number of tosses needed. Calculate EN. (Hint: Try to express N as a sum of geometric random variables).

Hint: We are waiting for r heads. So, we first wait for the first head and then for the second head and so on. Hence $N = X_1 + X_2 + \cdots X_r$ where each X_i is a geometric rv with parameter, p. Hence $EN = r/p$

4. A fair dice is rolled repeatedly till each of the numbers $1, 2, \cdots, 6$, appears atleast once. Find the expected number of rolls needed.

Hint: To start with we are waiting for any of the six numbers (which happens on the first roll!). Then we are waiting for any of the remaining 5 numbers. Once we get that, we are waiting for any of the remaining 4 numbers. So, if X is the number of rolls needed, it can be expressed as a sum of geometric random variables. I hope you can now see that the answer is

$$EX = \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$$

5. Consider the random experiment of n independent tosses of a coin whose probability of heads is p. In any outcome (of this random experiment) we say a change- over has occurred at the i^{th} toss if the result of the i^{th} toss differs from that of the $(i-1)^{th}$ toss. Let X be a random variable whose value is the number of change-overs. Find EX. (Hint: You need not find the distribution of X. Note that we are considering the same random variable as in Q1 problem sheet 2.3 except that the coin is not fair now).

Hint: You can define indicator random variables, $Y_1, \cdots, Y_{n-1}$ where Y_i is 1 if there is a changeover at toss number $i+1$. For change over at toss j, we need either HT or TH on tosses $j-1$ and j and hence

$EY_i = 2p(1-p), \forall i$. Since X is sum of Y_i's, you can now find $EX = 2p(1-p)(n-1)$
(Are Y_i independent?)

Comment: Note that Q3, Q4 and Q5 represent situations where we can calculate EX without explicitly finding the distribution of X. This is often done.

6. Let X_1, X_2, X_3 be independent random variables with finite variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ respectively. Find the correlation coefficient of $X_1 - X_2$ and $X_2 + X_3$.

Answer: One can show this through straight-forward algebra:

$$\begin{aligned}
\mathrm{Cov}(X_1 - X_2, X_2 + X_3) &= E[(X_1 - X_2)(X_2 + X_3)] - E[(X_1 - X_2)]E[(X_2 + X_3)] \\
= E[X_1X_2 + X_1X_3 - X_2^2 - X_2X_3] &- [EX_1EX_2 + EX_1EX_3 - (EX_2)^2 - EX_2EX_3] \\
&= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) - \mathrm{Var}(X_2) - \mathrm{Cov}(X_2, X_3) \\
&= -\sigma_2^2
\end{aligned}$$

where we used the fact that X_1, X_2, X_3 are independent and hence uncorrelated.
Since the random variables are uncorrelated, $\mathrm{Var}(X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = \sigma_1^2 + \sigma_2^2$. Similarly $\mathrm{Var}(X_2 + X_3) = \sigma_2^2 + \sigma_3^2$. Now you can calculate the correlation coefficient.

Comment: By its definition, covariance satisfies: $\mathrm{cov}(kX, Y) = k\,\mathrm{cov}(X, Y)$, where k is a real constant, $\mathrm{cov}(X, Y) = \mathrm{cov}(Y, X)$ and $\mathrm{cov}(X, Y + Z) = \mathrm{cov}(X, Y) + \mathrm{cov}(X, Z)$. (Covariance is like an inner product). This can be used to directly deduce $\mathrm{Cov}(X_1 - X_2, X_2 + X_3) = \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) - \mathrm{Cov}(X_2, X_2) - \mathrm{Cov}(X_2, X_3) = -\sigma_2^2$, because these random variables are uncorrelated.

7. Let X and Y be random variables having mean 0, variance 1, and correlation coefficient ρ. Show that $X - \rho Y$ and Y are uncorrelated, and that $X - \rho Y$ has mean 0 and variance $1 - \rho^2$.

Hint: $E[X - \rho Y] = 0$ because $EX = EY = 0$. Hence, variance of $X - \rho Y$ is $E[X - \rho Y]^2 = 1 - \rho^2$ because $EX^2 = EY^2 = 1$ and $EXY = \rho$. (Since means are zero and variances 1, $\mathrm{Cov}(X, Y) = \rho_{XY} = EXY$). For uncorrelatedness, $E[(X - \rho Y)Y] = EXY - \rho EY^2 = 0$

Comment: As we discussed in class, we can think of all mean-zero random variables to be vectors in a vector space with covariance as the inner product. $\rho_{XY}\ Y$ can be thought of as projection of X on Y and hence the 'residual', $X - \rho Y$ is 'orthogonal' to Y.

8. Let X, Y, Z be random variables having mean zero and variance 1. Let ρ_1, ρ_2, ρ_3 be the correlation coefficients between $X\&Y$, $Y\&Z$ and $Z\&X$, respectively. Show that

$$\rho_3 \geq \rho_1\rho_2 - \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}.$$

(Hint: Write $XZ = [\rho_1 Y + (X - \rho_1 Y)][\rho_2 Y + (Z - \rho_2 Y)]$, and then use the previous problem and Cauchy-Schwartz inequality).

Answer: Since all random variable have mean zero and variance 1, $\rho_1 = EXY$, $\rho_2 = EYZ$ and $\rho_3 = EXZ$. Using the hint, we have

$$\rho_3 = EXZ = \rho_1\rho_2 E[Y^2] + \rho_1 E[Y(Z-\rho_2 Y)] + \rho_2 E[Y(X-\rho_1 Y)] + E[(X-\rho_1 Y)(Z-\rho_2 Y)]$$

From the previous problem we know Y and $(Z - \rho_2 Y)$ are uncorrelated. Hence, $E[Y(Z - \rho_2 Y)] = EYE(Z - \rho_2 Y) = 0$ because $EY = 0$. Similarly, $E[Y(X - \rho_1 Y)] = 0$. We also have $EY^2 = 1$. Hence we get

$$\rho_3 = EXZ = \rho_1\rho_2 + E[(X - \rho_1 Y)(Z - \rho_2 Y)]$$

By Cauchy-Schwartz inequality (and results of previous problem)

$$|E[(X-\rho_1 Y)(Z-\rho_2 Y)]| \leq \sqrt{\mathrm{Var}(X - \rho_1 Y)\mathrm{Var}(Z - \rho_2 Y)} = \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}$$

Hence

$$-|E[(X - \rho_1 Y)(Z - \rho_2 Y)]| \geq -\sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}$$

Thus, we get

$$\begin{aligned} \rho_3 &= \rho_1\rho_2 + E[(X - \rho_1 Y)(Z - \rho_2 Y)] \\ &\geq \rho_1\rho_2 - |E[(X - \rho_1 Y)(Z - \rho_2 Y)]| \\ &\geq \rho_1\rho_2 - \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2} \end{aligned}$$

9. Let X be a random variable with mass function given by

$$\begin{aligned} f_X(x) &= \frac{1}{18}, \quad x = 1, 3 \\ &= \frac{16}{18}, \quad x = 2. \end{aligned}$$

Show that there exists a δ such that $P[|X - EX| \geq \delta] = \mathrm{Var}(X)/\delta^2$. This shows that the bound given by Chebyshev inequality cannot, in general, be improved.

Hint: Take $\delta = 1$ and calculate $P[|X - EX| \geq \delta]$ and $\mathrm{Var}(X)$.

10. Use Chebyshev inequality to show that for any real number $K > 1$, we have

$$e^{K+1} \geq K^2$$

(Hint: Try Chebyshev inequality for an exponential rv with $\lambda = 1$)

Hint If X is a continuous rv having exponential density with parameter $\lambda = 1$, then, by Chebyshev inequality,

$$e^{-(K+1)} = P[X > K + 1] \leq P[|X - 1| > K] \leq \frac{1}{K^2}$$

Now I hope the solution is clear. (Note that for an exponential rv with parameter 1, both mean and variance are 1).