E1 222 Stochastic Models and Applications

Problem Sheet 4.1

(You need not submit solutions)

1. Given $P[X_n = 0] = 1 - n^{-2}$, $P[X_n = e^n] = n^{-2}$. Show that X_n converge almost surely but not in r^{th} mean.

2. Given $P[X_n = 0] = 1 - 1/n$, $P[X_n = n^{1/2r}] = 1/n$, X_n are independent. Show that $E|X_n|^r \to 0$ but the sequence does not converge to zero almost surely.

3. Let $\Omega = [0,\ 1]$ and let P be the usual length measure. Let $X_n = n^{0.25} I_{[0,\ 1/n]}$, $n = 1, 2, \cdots$, where I_A denotes indicator of event A. Does the sequence converge in (i) probability, (ii) r^{th} mean for some r?

4. Let $X_1, X_2, \cdots$, be random variables with distributions

$$\begin{aligned} F_{X_n}(x) &= 0 \quad \text{if} \quad x < -n \\ &= \tfrac{x+n}{2n} \quad \text{if} \quad -n \le x \le n \\ &= 1 \quad \text{if} \quad x \ge n \end{aligned}$$

   Does $\{X_n\}$ converge in distribution?

5. Consider a Probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{1, 2, \cdots\}$, $\mathcal{F}$ is the power set of Ω and $P(\{i\}) = q_i,\ \forall i$. Note that we would have $q_i \ge 0, \forall i$ and $\sum_i q_i = 1$. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables defined on this space given by

$$\begin{aligned} X_n(\omega) &= 1 \ \text{ if } \ n \le \omega \\ &= 0 \ \text{ otherwise} \end{aligned}$$

   Does the sequence converge in (i) Probability, (ii) almost surely.

6. Let $X_1, X_2, \cdots$ be iid Gaussian random variables with mean zero and variance unity. Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Let F_n be the distribution function of $\bar{X}_n$. Find Lim F_n. Is this a distribution function?

7. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables with X_n being uniform over the set $\{\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$. Does the sequence $\{X_n\}$ converge in distribution?

8. Let $\{X_n\}$ be a sequence of random variables converging in distribution to a continuous random variable X. Let a_n be a sequence of positive numbers such that $a_n \to \infty$ as $n \to \infty$. Show that X_n/a_n converges to zero in probability.

9. Let $X_1, X_2, \cdots$ be independent normally distributed random variables having mean zero and variance σ^2.

   (a). What is the mean and variance of X_1^2?

   (b). How should $P[X_1^2 + X_2^2 + \cdots + X_n^2 \leq x]$ be approximated in terms of standard normal distribution?

   (c). Suppose $\sigma^2 = 1$. Find (approximately) $P[80 \leq X_1^2 + \cdots + X_{100}^2 \leq 120]$.

   (d). Find c such that (approximately) $P[100 - c \leq X_1^2 + \cdots + X_{100}^2 \leq 100 + c] = 0.95$.

10. Candidates A and B are contesting an election and 55% of the electorate favour B. What is the (approximate) probability that in a sample of size 100 atleast one-half of the people sampled favour A.

11. A university has 300 vacancies for research students. Since not all students offered admission would accept, the university sends out offers of admission to 400 students. By past experience the university knows that only 70% of students offered admission would accept the offer. Calculate the approximate probability that more than 300 students would accept the offer of admission.

12. A fair coin is tossed until 100 heads appear. Find (approximately) the probability that atleast 230 tosses will be necessary.