E1 222 Stochastic Models and Applications

Problem Sheet 3.1

(You need not submit the solutions)

1. Consider the random experiment of rolling two fair dice. Let X denote the maximum of the two numbers and let Y denote the minimum of the two numbers. Calculate the joint probability mass function of X, Y and calculate the marginal mass function of X and Y from the joint mass function. Find the conditional mass function of Y given X.

2. Let (X, Y) have joint density
$$f_{XY}(x, y) = \frac{1}{4}[1 + xy(x^2 - y^2)], \ |x| \leq 1, \ |y| \leq 1.$$
Find the marginal and conditional densities. Find $P[X > 0 \mid Y = 0]$.

3. Let $f(x, y) = e^{-x-y}, \ x > 0, y > 0$. Show that this a density function. Find the marginals and the conditional densities.

4. Let F_{XY} be a joint distribution function with F_X and F_Y being the corresponding marginal distribution functions. Show that
$$1 - (1 - F_X(x) + 1 - F_Y(y)) \leq F_{XY}(x, y) \leq \min(F_X(x), F_Y(y)), \ \forall x, y$$

5. Let
$$\begin{aligned} F(x, y) &= 0, \ \text{ if } x < 0, \text{ or } \ y < 0, \text{ or } \ x + y < 1 \\ &= 1, \ \text{ otherwise} \end{aligned}$$
Show that F satisfies the following: $F(-\infty, y) = F(x, -\infty) = 0$; $F(\infty, \infty) = 1$; F is non-decreasing in each variable. Is $F(x, y)$ a distribution function? (Hint: If it were the joint distri- bution of two random variables, X, Y, what would be $P[1/3 < X \leq 1, \ 1/3 < Y \leq 1]$).

6. Let F_1 and F_2 be two one dimensional continuous distribution functions with f_1 and f_2 being the corresponding densities. Define a function $f : \Re^2 \rightarrow \Re$ by
$$f(x, y) = f_1(x) f_2(y) \left[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)\right]$$
where α is a real number. Show that $f(x, y)$ is a two dimensional density function for all $\alpha \in (-1, 1)$. Show that the two marginals of $f(x, y)$ are f_1 and f_2. What does this imply about determining the joint density from the marginals? (Note that $\int_{-\infty}^{\infty} F_1(x) f_1(x) \, dx = \frac{1}{2}$).