E1 222 Stochastic Models and Applications
Problem Sheet 4.3

1. Let $X(t)$ be a wide-sense stationary stochastic process with autocorrelation $R(\tau)$. Show that $\text{Prob}[|X(t+\tau)-X(t)| \geq a] < 2[R(0)-R(\tau)]/a^2$.

2. Consider a stochastic process $X(t) = e^{At}$ where A is a continuous random variable with density f_A. Express the mean $\eta(t)$ and the autocorrelation $R(t_1, t_2)$ in terms of f_A.

3. Suppose vehicles pass a certain point in a highway as a Poisson process with rate 1 per minute. Suppose 5% of the vehicles are vans. What is the probability that at least one van passes by during half an hour? Given that 10 vans passed by in an hour what is the expected number of vehicles to have passed in that hour.

4. Suppose people arrive at a bust stop in accordance with a Poisson process with rate λ. Let t be some fixed time and suppose the next bus departs at t. All people who arrive till t would get on the bus that departs at t. Let X denote the total amount of waiting time of all people who got on the bus at t. (Note that a person who arrived at $s < t$ would contribute $t - s$ to the waiting time).

   - Show that $E[X|N(t)] = N(t)\frac{t}{2}$
   - Show that $\text{Var}[X|N(t)] = N(t)\frac{t^2}{12}$
   - Using these two, calculate $\text{Var}(X)$

5. Suppose customers arrive at a single server queuing system in accordance with a Poisson process with rate λ. However an arriving customer will join the queue with probability α_n if he sees there are n people in the system. (With the remaining probability he just departs). Represent this a birth-death process (of a continuous time markov chain) and specify the birth and death rates.

6. Consider a system with two machines. The time till next failure of machine i is $\exp(\lambda_i)$, $i = 1, 2$. The repair time of machine i is $\exp(\mu_i)$. The machines act independently of each other. Model this as a four state continuous time Markov chain and calculate the infinitesimal generator of the chain.

7. Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion process. Consider a process defined by
$$V(t) = e^{-\alpha t/2}\, B\left(e^{\alpha t}\right)$$
where $\alpha > 0$ is a parameter. Find the mean and autocorrelation of $V(t)$.

8. Let $\{X(t),\ t \geq 0\}$ be a Brownian motion process with variance parameter σ^2 and drift μ. Write down the joint distribution of $X(s), X(t)$ for $s < t$. Find the variance of $aX(s) + bX(t)$ where a, b are some real constants.