E1 222 Stochastic Models and Applications
Solutions to Problem Sheet 1.2

1. There are three chests each having two drawers. Chest 1 has a gold coin in each drawer while chest 2 has a silver coin in each drawer. Chest 3 has a gold coin in one drawer and a silver coin in the other drawer. A chest is chosen at random and one of its drawers, chosen at random, is opened. It is found to contain a gold coin. What is the probability that the other drawer has (i). a gold coin, (ii). a silver coin?

Answer: Note that each outcome of this random experiment consists of (the choice of) a chest and a drawer. So, we can take Ω as the following six element set.

$$\Omega = \{(c1, d1), (c1, d2), \cdots, (c3, d2)\}$$

We assume all these six outcomes are equally likely. Let A be the event of selected drawer having G. Let B be the event of the non-selected drawer in the selected chest having G. What we want is $P(B|A)$. We see that these events (as subsets of Ω) are:

$$A = \{(c1, d1), (c1, d2), (c3, d1)\}; \quad B = \{(c1, d1), (c1, d2), (c3, d2)\}$$

Hence we have

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(\{(c1, d1), (c1, d2)\})}{P(\{(c1, d1), (c1, d2), (c3, d1)\})} = \frac{2/6}{3/6} = \frac{2}{3}$$

Hence the probability that the other drawer (which is the non-selected drawer in the selected chest) having a S given selected one has G is 1/3.

We can get the probabilities of A, B by total probability rule too. (Let ci denote event of choosing chest ci, $i = 1, 2, 3$; this is an abuse of notation!).

$$P(A) = P(A|c1)P(c1)+P(A|c2)P(c2)+P(A|c3)P(c3) = 1\,\frac{1}{3} \;+\; 0\,\frac{1}{3} \;+\; \frac{1}{2}\,\frac{1}{3} = \frac{1}{2}$$

Similarly we get $P(B) = 0.5$. For $P(AB)$, note that $P(AB|c1) = 1$ and $P(AB|c2) = P(AB|c3) = 0$.

We can also say the following: given that A has occurred, effectively Ω is reduced to $\{(c1, d1), (c1, d2), (c3, d1)\}$ and hence the the other drawer having a G has probability 2/3. As I told you in class, while this is good for calculation, we should realize that conditional probability is defined for all events in the original sample space.

One may wonder why the probability is not 0.5. After all, the other drawer has either a G or a S. But since we are choosing a chest and a drawer, there are three outcomes for finding a G in the opened drawer and in two of them the other drawer has a G.

2. A box contains coupons labelled $1, 2, 3, \cdots, n$. Two coupons are drawn from the box with replacement. Let a, b denote the numbers on the two coupons. Find the probability that one of a, b divides the other.

Answer: Given a fixed number a which is between 1 and n, how many multiples of a are there between 1 and n? Given a, the multiples of a are $k * a$, $k = 1, 2, \cdots$. (Note a is a multiple of a). Hence, there are $\lfloor \frac{n}{a} \rfloor$ multiples of a between 1 and n. (Here $\lfloor x \rfloor$ is the floor function, that is, it is the greatest integer smaller than or equal to x). For example, there are three multiples of 6 between 1 and 20. Thus given a specific number i, $1 \leq i \leq n$, if we randomly select a number between 1 and n, the probability of it being a multiple of i is $\frac{1}{n} \lfloor \frac{n}{i} \rfloor$.

In this problem, for one of the two numbers to divide the other, we need either the first to be a multiple of the second or the other way.

Let A be the event that the second number is a multiple of the first and is not equal to the first. (Note that here the second number is strictly greater than the first). Then

$$\begin{aligned} P(A) &= \sum_{i=1}^{n} P(\text{1st is } i) P(\text{2nd is multiple of } i \text{ and greater} | \text{1st is i}) \\ &= \sum_{i=1}^{n} \frac{1}{n} \left(\frac{\lfloor \frac{n}{i} \rfloor - 1}{n} \right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \frac{1}{n} \end{aligned}$$

where we used the fact that the second number is independent of the first because we are sampling with replacement. Now let B be the event that the first number is a multiple of second and is not equal to the second. Easy to see that $P(B) = P(A)$. Let C be the event that both numbers are same. Then $P(C) = (1/n)$. Also, these three are disjoint. Their union is the event we want. hence, the required probability is

$$\frac{2}{n^2}\sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor - \frac{1}{n}$$

3. A fair dice is rolled repeatedly till we get at least one 5 and one 6. What is the probability that we need n rolls?

Answer: If it takes n rolls what can we say about the outcome of n^{th} roll and about the outcomes of the first $n-1$ rolls? We should get either a 5 or a 6 in the n^{th} roll. If we stop with a 5 (that is 5 on n^{th} roll), the first $(n-1)$ rolls should have at least one 6 and no 5. Similarly for the case of stopping with a 6. Note that the minimum n is 2.

Let A be the event that first $(n-1)$ have at least one 6 and no 5 and n^{th} is a 5. Let B be the event that first $(n-1)$ have at least one 5 and no 6 and n^{th} is a 6. These are disjoint and what we want is probability of their union.

$$\begin{aligned}
P(A) &= \sum_{k=1}^{n-1} {}^{n-1}C_k \left(\frac{1}{6}\right)^k \left(\frac{4}{6}\right)^{n-1-k} \frac{1}{6} \\
&= \frac{1}{6}\left(\sum_{k=0}^{n-1} {}^{n-1}C_k \left(\frac{1}{6}\right)^k \left(\frac{4}{6}\right)^{n-1-k} - \left(\frac{4}{6}\right)^{n-1}\right) \\
&= \frac{1}{6}\left(\left(\frac{5}{6}\right)^{n-1} - \left(\frac{4}{6}\right)^{n-1}\right)
\end{aligned}$$

It is easy to see that $P(B)$ would also be the same. Hence the required probability is

$$\frac{2}{6}\left(\left(\frac{5}{6}\right)^{n-1} - \left(\frac{4}{6}\right)^{n-1}\right)$$

To verify the answer we must have the above summing to 1 over all n.

$$\sum_{n=2}^{\infty}\frac{2}{6}\left(\left(\frac{5}{6}\right)^{n-1} - \left(\frac{4}{6}\right)^{n-1}\right) = \frac{2}{6}\left(\frac{5/6}{1/6} - \frac{4/6}{2/6}\right) = 1$$

4. Suppose E and F are mutually exclusive events of a random experiment. This random experiment is repeated till either E or F occurs. Show that the probability that E occurs before F is $P(E)/(P(E)+P(F))$.

Answer: Intuitively, we can say the question is asking the following. In the random experiment, given that $E \cup F$ occurred what is the probability that E occurred? That conditional probability is the answer.

To derive this formally. Let us first calculate the probability that exactly n repetitions are required and E occurred before F. That means in the first $n-1$ repetitions neither E nor F occurred and in the n^{th} one E occurred. The probability of this is

$$(1 - P(E \cup F))^{n-1} P(E)$$

For different n, these are disjoint and what we need is the union over all n. hence the required probability is

$$\sum_{n=1}^{\infty} (1 - P(E \cup F))^{n-1} P(E) = \frac{P(E)}{P(E \cup F)} = \frac{P(E)}{P(E) + P(F)}$$

This is a useful general formula.

5. Suppose n men put all their hats together in a heap and then each man selects a hat at random. Show that the probability that none of the n men selects his own hat is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \frac{(-1)^n}{n!}$$

Answer: Here the possible outcomes are all possible permutations of the numbers 1 to n and there are $n!$ of them. We assume all are equally likely.

Let A_i denote the event of i^{th} man getting his own hat. Then

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}, \ \forall i$$

The event $\cup_{i=1}^{n} A_i$ is the event of some man or the other getting his own hat. Hence the probability we want is $1 - P\left(\cup_{i=1}^{n} A_i\right)$.

To calculate $P\left(\cup_{i=1}^{n} A_i\right)$, we can use the general formula for probability of union of events. for this we need probabilities of intersections of subcollections of these events. We have

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{2!}\frac{(n-2)!2!}{n!} = \frac{1}{2!}\frac{1}{{}^nC_2}$$

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{3!}\frac{1}{{}^nC_3}$$

and so on.

The general formula for probability of union of events is

$$\begin{aligned} P(U_{i=1}^{n} A_i) &= \sum_i P(A_i) - \sum_i \sum_{j>i} P(A_i \cap A_j) \\ &+ \sum_i \sum_{j>i} \sum_{k>j} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\cap_i A_i) \end{aligned}$$

Note that there are ${}^nC_2$ terms represented by the two summations in the second term, ${}^nC_3$ terms represented by the three summations in the third term and so on. Now, simple algebra will complete the problem.

6. Suppose there is a TV show like this. There are three closed doors. The host of the show assures that there is a car behind one of the doors and goats behind the other two. The contestant has to choose a door and she/he gets the car if the car is behind the door chosen by the contestant. After the contestant chooses a door, the host opens one of the other two doors and shows that there is a goat behind it and then offers the contestant a chance to change her/his choice to the remaining door. Should the contestant change her/his choice?

Comment: This is a famous problem that has given rise to a lot of fierce debates some years ago. There are many such 'puzzles' possible in probability and people keep debating them. In this course we do not discuss many such puzzles. I included this one only because I thought you should experience at least one puzzle! No need to spend too much time on this.

Answer: We can assume that the contestant picks one of the three doors at random. That is, we think that the probability of car being there behind the door chosen by the contestant is 1/3. Now the question is whether the Show-host also chooses a door at random. First let us suppose it is so.

To keep things simple let us assume contestant chooses door-1 and host opens door-2. The randomness comes by the arrangement behind the doors being random. Thus we can take $\Omega = \{CGG,\ GCG,\ GGC\}$ (with C for car and G for goat). Let A be the event that there is a car behind the door chosen by contestant. Then $A = \{CGG\}$ and $P(A) = 1/3$. Let B be the event that there is a goat behind the door opened by host. Then $B = \{CGG,\ GGC\}$. Now it is easy to see $A \cap B = A$ and thus $P(A|B) = \frac{1/3}{2/3} = \frac{1}{2}$. Thus, given that there is a goat behind the door opened by host the probability that there is a car behind the door of the contestant is 0.5 and hence we may argue that there is no particular reason for the contestant to change the choice.

But all this assumes that the host chooses door at random. Under the above model, probability that there is a car behind the door opened by host is 1/3. But our experience (or the 'gut feeling') is that it never so happens that there is a car behind the door open by the host in the show. After all, if that happens the show is not interesting!

So, let us say the host knows what is behind the doors and the action of the host is purposeful and not random. The host would open a door behind which there is a goat. Then what can we say (from the point of view of probability theory)?

We will still assume that the initial choice of the contestant is random. So, two times out of three there would be a goat behind the door chosen by the contestant. But now the actions of host are not random. Thus, there is no meaning in conditioning on the 'event' that the opened door has goat. Even before the host opens the door we know there would be a goat. So, the question now simply is what is the probability that there is a car behind the **unopened** door. Note that the unopened door now is not one randomly chosen between the other two. It has a very special relation with the event of what is behind the door chosen by the contestant. Given that there is a goat behind the door chosen by the contestant the probability that there is a car behind the unopened door is 1. And given that there is a car behind the door of contestant the probability of car behind the unopened door is zero. This gives us the (unconditional) probability of car behind unopened door as same as probability of goat behind the door of the contestant, which we know is 2/3. (I leave it to you to formally show this using total probability rule). Hence it is better to switch to the unopened door.

We do not really need any probability for this. Here is a simple intuitive argument for me if I am the contestant. If there is a goat behind the door chosen by me then there is a car behind the unopened door. And 2 out of three times there is a goat behind the door chosen by me.

Many people find it counter-intuitive that the better strategy is for the contestant to *always* change. We argue: after all, every thing is random; so the fact that there is a goat behind door-2 does not make there being a car behind door-1 any more or less likely than there being a goat behind door-1. But the point is that everything is not random. The action of host is purposeful. The fact that the host chose not to open door-3 conveys the extra information.

I hope all of you also see that all this analysis is useless for any individual contestant who plays the game only once. If the same contestant would play this game many many times then the strategy of changing would make it profitable in the long run.

7. Suppose there are three special dice, A, B, C which have the following numbers on their six faces:

   A: 1, 1, 6, 6, 8, 8

   B: 2, 2, 4, 4, 9, 9

   C: 3, 3, 5, 5, 7, 7

   The dice are fair in the sense that each of the faces have the same probability of coming up.
   (i). Suppose we roll dice A and B. What is the probability that the number that comes up on A is less than the one that comes up on B?
   (ii)Suppose your friend, with whom you go out for dinner often, offers you the following. At the end of each dinner, you choose any one of the three dice that you want. She/He would then choose one of the two dice that are remaining. Then both of you roll your respective dice. Whoever gets the smaller number would pay for the dinner. Would you take the offer?

Answer: Let us denote by $A < B$, the event that the number of A would be less than that on B. Then

$$P(A < B) = P(A \text{ shows } 1) + P(A \text{ shows 6 or 8 and } B \text{ shows } 9) = \frac{2}{6} + \frac{4}{6}\frac{2}{6} = \frac{5}{9} > 0.5$$

Similarly, we can show that $P(B < C) > 0.5$ and $P(C < A) > 0.5$. Thus, no matter which dice you pick up, your fried has choice to pick up a dice that gives higher number with probability greater than 0.5. (Of course, whether or not you accept the offer depends on whether or not you feel happy to pay for a larger fraction of the dinners!)

8. Consider a communication system. The transmitter sends one of two waveforms. One waveform represents the symbol 0 and the other represents the symbol 1. Due to the noise in the channel, the receiver cannot say with certainty what was sent. The receiver is designed so that, after sensing signal coming out of the noisy channel, it puts out one of the three symbols: a, b, c. The following statistical parameters of the system are determined (either through modeling or experimentation):

$$P[a|1] = 0.6,\ P[b|1] = 0.2,\ P[c|1] = 0.2$$

$$P[a|0] = 0.3,\ P[b|0] = 0.4,\ P[c|0] = 0.3$$

Here, $p[a|0]$ denotes the probability of the receiver putting out symbol a when the symbol transmitted is 0 and similarly for all others. The transmitter sends the two symbols with probabilities: $P[0] = 0.4$ and $P[1] = 0.6$. Find $P[1|a]$ and $P[0|a]$. When receiver puts out a what should we conclude about the symbol sent? We would like to build a decision device that will observe the receiver output (that is, a, b, or c) and decide whether a 0 was sent or a 1 was sent. An error occurs if the decision device says 1 when a 0 was sent or vice versa. Find a decision rule that minimizes the probability of error. What is the resulting (minimum) probability of error?

Answer: Using Bayes rule, we get

$$P[1|a] = \frac{P[a|1]P[1]}{P[a|1]P[1] + P[a|0]P[0]} = \frac{0.6 * 0.6}{0.6 * 0.6 + 0.3 * 0.4} = \frac{36}{48} = \frac{3}{4}$$

Hence, $P[0|a] = 0.25$.

Similarly, we can calculate that $P[1|b] = \frac{3}{7}$, $P[0|b] = \frac{4}{7}$ and $P[1|c] = P[0|c] = 0.5$.

Thus, intuitively, when receiver puts out a we should conclude 1, if it puts out b we should conclude 0 and when it puts out c we can choose anything. We can actually prove that this is optimal in terms of minimizing probability of error.

Any decision rule is just a function that maps $\{a, b, c\}$ to $\{0, 1\}$. Consider any function h like that. As a notation, if $h(a) = 1$ then $\bar{h}(a) = 0$. Note that $h(a), \bar{h}(a) \in \{0, 1\}$.

We want to calculate the probability of error for any such decision rule.

Let e denote the event of h making an error. Then, $P(e|a)$ is the probability that the rule h makes an error conditioned on receiver outputting a. When a is received, h would conclude that the bit sent is $h(a)$. Hence, it would make an error if $h(a)$ is not the bit sent. Hence $P(e|a) = P(\bar{h}(a)|a)$. Now, we have

$$\begin{aligned} P(e) &= P(e|a)P(a) + P(e|b)P(b) + P(e|c)P(c) \\ &= P(\bar{h}(a)|a)P(a) + P(\bar{h}(b)|b)P(b) + P(\bar{h}(c)|c)P(c) \end{aligned}$$

In the above expression, if we change h the only things that change are $P(\bar{h}(a)|a)$. For example, if $h(a) = 1$ then the first term above would contain $P(0|a)$ and if $h(a) = 0$ it would contain $P(1|a)$.

Hence, the optimal h should satisfy $P(\bar{h}(a)|a) < P(h(a)|a)$ and similarly for the others. Thus, if $P(1|a) > P(0|a)$ then optimal h should have $h(a) = 1$. Thus, the optimal h here would have $h(a) = 1$ and $h(b) = 0$. It does not matter what $h(c)$ is. So, optimal decision rule here is not unique.

9. There is a component manufacturing facility where 5% of the products may be faulty. The factory wants to pack the components into boxes so that it can guarantee that 99% of the boxes have at least 100 good components. What is the minimum number of components they should put into each box?

Answer: We are given that the probability a component is faulty is 0.05. We assume that a component being faulty is independent of another being faulty. Thus, if we have n components in a box then the probability of there being exactly k faulty components is given by

$$^{n}C_k(0.05)^k(0.95)^{n-k}$$

So, if a box contains $100+n$ components, then the probability that at most n components are faulty is given by

$$\sum_{k=0}^{n} {}^{100+n}C_k(0.05)^k(0.95)^{100+n-k}$$

So, given any random box with $100+n$ components, the probability that there are at least 100 good components in the box is given by the above expression.

We want 99% of boxes to have at least 100 good components. Hence, we want to satisfy

$$\sum_{k=0}^{n} {}^{100+n}C_k(0.05)^k(0.95)^{100+n-k} \geq 0.99$$

What this says is the following. Suppose I randomly pack $100+n$ components into boxes. Then, if I pick a box at random then the probability that that box contains at least 100 good components is at least 0.99. In this sense, the quality guarantee we are giving is a probabilistic guarantee.

So, to solve the problem, we need to find a value of n to satisfy the above. Of course, we want to find the least n to satisfy that.

Analytically solving the above is difficult. But we can solve it numerically. We can keep calculating LHS for $n = 1, 2, \cdots$ till we find an n that satisfies it.