E1 222 Stochastic Models and Applications
Hints for Problem Sheet 4.3

1. Let $X(t)$ be a wide-sense stationary stochastic process with autocorrelation $R(\tau)$. Show that $\text{Prob}[|X(t+\tau)-X(t)| \geq a] < 2[R(0)-R(\tau)]/a^2$.

Hint: Since the process is wide-sense stationary, $E[X^2(t+\tau)] = E[X^2(t)] = R(0)$ and $E[X(t+\tau)X(t)] = R(\tau)$

By Markov inequality

$$\begin{aligned}
\text{Prob}[|X(t+\tau) - X(t)| \geq a] &\leq \frac{E\left[|X(t+\tau) - X(t)|^2\right]}{a^2} \\
&= \frac{E\left[X^2(t+\tau) + X^2(t) - 2X(t)X(t+\tau)\right]}{a^2} \\
&= \frac{2(R(0) - R(\tau))}{a^2}
\end{aligned}$$

2. Consider a stochastic process $X(t) = e^{At}$ where A is a continuous random variable with density f_A. Express the mean $\eta(t)$ and the autocorrelation $R(t_1, t_2)$ in terms of f_A.

Hint:

$$\eta(t) = E[X(t)] = E[e^{At}] = \int e^{at}\, f_A(a)\, da = M_A(t)$$

where M_A is the moment generating function of A.

$$R(t_1, t_2) = E[e^{At_1} e^{At_2}] = \int e^{a(t_1+t_2)}\, f_A(a)\, da = M_A(t_1 + t_2)$$

3. Suppose vehicles pass a certain point in a highway as a Poisson process with rate 1 per minute. Suppose 5% of the vehicles are vans. What is the probability that at least one van passes by during half an hour? Given that 10 vans passed by in an hour what is the expected number of vehicles to have passed in that hour.

Hint: The vehicles process has rate 60 per hour. Since 5% are vans, they constitute a poisson process with rate 3 per hour. Probability of at least one van in half hour would be same as one minus probability of no van in half hour which is $1 - e^{-3*0.5} = 1 - e^{-1.5}$. Let N_1 represent

Vans, N_2 all other vehicles. Then, N_1 and N_2 are ind Poisson processes with rates 3 and 57 per hour. Taking unit of time as hours, we have

$$E[N_1(1) + N_2(1)|N_1(1) = 10] = 10 + E[N_2(1)] = 10 + 57$$

because N_1 and N_2 are independent.

4. Suppose people arrive at a bust stop in accordance with a Poisson process with rate λ. Let t be some fixed time and suppose the next bus departs at t. All people who arrive till t would get on the bus that departs at t. Let X denote the total amount of waiting time of all people who got on the bus at t. (Note that a person who arrived at $s < t$ would contribute $t - s$ to the waiting time).

   - Show that $E[X|N(t)] = N(t)\frac{t}{2}$
   - Show that $\mathrm{Var}[X|N(t)] = N(t)\frac{t^2}{12}$
   - Using these two, calculate $\mathrm{Var}(X)$

Hint: Suppose $N(t) = n$. Let $S_1, \cdots, S_n$ denote the time instants when the n persons arrived. Now, the waiting time of the i^{th} person would be $t - S_i$. Hence, when $N(t) = n$, we have $X = \sum_i (t - S_i) = nt - \sum_{i=1}^n S_i$. Conditioned on $N(t) = n$, we know that the joint density of $S_1, \cdots, S_n$ is same as order statistics of n iid random variables uniform over $[0, t]$. Hence, their sum would have same distribution as that of sum of n iid rv which are uniform over $[0, t]$. Hence,

$$E[X|N(t) = n] = nt - n\,\frac{t}{2} = n\,\frac{t}{2} \quad \Rightarrow \quad E[X|N(t)] = N(t)\frac{t}{2}$$

Since variance of a random variable uniform over $[0, t]$ is $t^2/12$, by the same argument we get $\mathrm{Var}[X|N(t)] = N(t)\frac{t^2}{12}$.

We have proved in one of the problem sheets that, for any random variable, X,

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

Using this for the X in this problem, we get

$$\mathrm{Var}(X) \quad = \quad E[\mathrm{Var}(X|N(t))] + \mathrm{Var}(E[X|N(t)])$$

$$
\begin{aligned}
&= E\left[N(t)\frac{t^2}{12}\right] + \mathrm{Var}\left(N(t)\frac{t}{2}\right) \\
&= \frac{t^2}{12}\,\lambda t + \frac{t^2}{4}\,\lambda t \\
&= \lambda\,\frac{t^3}{3}
\end{aligned}
$$

5. Suppose customers arrive at a single server queuing system in accordance with a Poisson process with rate λ. However an arriving customer will join the queue with probability α_n if he sees there are n people in the system. (With the remaining probability he just departs). Represent this a birth-death process (of a continuous time markov chain) and specify the birth and death rates.

Hint: This is a queuing system and would be a birth-death chain. Since it is a single server system, (assuming the service time of a customer is $\exp(\mu)$) the service rate is constant and hence in all states n, the 'death' rate is μ. That is, the rate of transition from n to $n-1$ is μ for all $n \geq 1$. Customers arrive in a Poisson process. When the system is in state n, an arriving customer is type-1 with probability α_n and only type-1 customers actually join the system. Thus the birth rate or rate of transition from n to $n+1$ is $\alpha_n \lambda$.

6. Consider a system with two machines. The time till next failure of machine i is $\exp(\lambda_i)$, $i = 1, 2$. The repair time of machine i is $\exp(\mu_i)$. The machines act independently of each other. Model this as a four state continuous time Markov chain and calculate the infinitesimal generator of the chain.

Hint: Let us represent the system state by (b_1, b_2) where b_i is state of machine i, $i = 1, 2$. We take $b_i \in \{0, 1\}$ with 0 representing that the machine is working and 1 representing that the machine is not working.

It is easy to see that from state (b_1, b_2) we can go to either $(1 - b_1, b_2)$ or $(b_1, 1 - b_2)$. Let us write the rate of transition from $(0, 0)$ to $(0, 1)$ as $q_{00,01}$ and similarly for other transitions.

Since machines act independently, we have $q_{00,01} = \lambda_2$ and $q_{00,10} = \lambda_1$.

(We can see this as follows. Let $X_i \sim \exp(\lambda_i)$. Then, when in state 00, time till next transition is $\min(X_1, X_2)$ which is $\exp(\lambda_1 + \lambda_2)$. Thus

$\nu_{00} = \lambda_1 + \lambda_2$. Now $z_{00,01}$ is the probability that machine 2 would fail before machine 1 which is same as $P[X_2 < X_1]$ which is $\frac{\lambda_2}{\lambda_1+\lambda_2}$. This gives us $q_{00,01} = \nu_{00} z_{00,01} = \lambda_2$. But we can directly infer this because the rate at which transit from 00 to 01 is the rate at which machine 2 fails).

Similarly we can get all the other entries for the infinitesimal generator matrix:

$$q_{01,00} = \mu_2, \;\; q_{01,11} = \lambda_1, \;\; q_{10,00} = \mu_1, \;\; q_{10,11} = \lambda_2, \;\; q_{11,01} = \mu_1, \;\; q_{11,10} = \mu_2$$

7. Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion process. Consider a process defined by

$$V(t) = e^{-\alpha t/2}\, B\left(e^{\alpha t}\right)$$

where $\alpha > 0$ is a parameter. Find the mean and autocorrelation of $V(t)$.

Hint: It is easily seen that $E[V(t)] = 0$ because $E[B(t)] = 0, \forall t$. Now the autocorrelation, which would be same as auto covariance, is (for $s > 0$)

$$R_V(t, t+s) = E[V(t)V(t+s)] = e^{-\alpha t/2} e^{-\alpha(t+s)/2} E\left[B\left(e^{\alpha t}\right) B\left(e^{\alpha(t+s)}\right)\right]$$

Since $\alpha, s \geq 0$, we have $e^{\alpha t} \leq e^{\alpha(t+s)}$, and hence

$$E\left[B\left(e^{\alpha t}\right) B\left(e^{\alpha(t+s)}\right)\right] = e^{\alpha t}$$

This gives us

$$R_V(t, t+s) = e^{-\alpha t/2} e^{-\alpha(t+s)/2} e^{\alpha t} = e^{-\alpha s/2}$$

8. Let $\{X(t),\ t \geq 0\}$ be a Brownian motion process with variance parameter σ^2 and drift μ. Write down the joint distribution of $X(s), X(t)$ for $s < t$. Find the variance of $aX(s) + bX(t)$ where a, b are some real constants.

Hint: We know $X(s), X(t)$ are jointly Gaussian because Brownian motion is a Gaussian process. We can write the joint density if we know the means and the covariance matrix. We know that $\mathrm{Cov}(X(s), X(t)) =$

$\sigma^2 \min(s,t)$. We know $E[X(s)] = E[X(t)] = \mu$. Let Σ_{st} be the covariance matrix. Let us assume $s < t$. Then we have

$$\Sigma_{st} = \sigma^2 \begin{bmatrix} s & s \\ s & t \end{bmatrix}$$

Now we can write down the joint distribution.

Since $X(s), X(t)$ are jointly Gaussian with covariance matrix Σ_{st}, we have

$$\mathrm{Var}(aX(s) + bX(t)) = \begin{bmatrix} a & b \end{bmatrix} \Sigma_{st} \begin{bmatrix} a \\ b \end{bmatrix} = \sigma^2(a^2 s + b^2 t + 2abs)$$