E1 222 Stochastic Models and Applications
Hints for Problem Sheet 3.1

1. Consider the random experiment of rolling two fair dice. Let X denote the maximum of the two numbers and let Y denote the minimum of the two numbers. Calculate the joint probability mass function of X, Y and calculate the marginal mass function of X and Y from the joint mass function. Find the conditional mass function of Y given X.

Hint: This problem is very similar to the example we solved in class. The minimum has to be less than or equal to maximum and both should be between 1 and 6. For example, the event $[X = 4, Y = 3]$ is $\{(4, 3),\ (3, 4)\}$ while the event $[X = 4, Y = 4]$ would be $\{(4, 4)\}$. Now, it should be easy to argue that the joint mass function is given by

$$f_{XY}(m, n) = \begin{cases} \frac{2}{36} & 1 \leq n < m \leq 6 \\ \\ \frac{1}{36} & 1 \leq n = m \leq 6 \end{cases}$$

You should verify that this is a joint mass function. That is, show that $\sum_{m=1}^{6} \sum_{n=1}^{6} f_{XY}(m, n) = 1$.

Now you can find the marginals as we did in class. You can compute marginal of X and verify that it is the same as what you got earlier for this random variable. Let us calculate marginal of Y. For a given y, $f_{XY}(x, y)$ is non-zero only when both x, y are between 1 and 6 and when $x \geq y$. Thus, for $1 \leq k \leq 6$,

$$f_Y(k) = \sum_j f_{XY}(j, k) = \sum_{j=k+1}^{6} \frac{2}{36} + \frac{1}{36} = \frac{13 - 2k}{36}$$

(Note that the sum above is taken to be zero if $k + 1 > 6$). You can verify that this is a mass function. You can also easily verify that this is the mass function for Y which is the minimum of the two numbers on the two dice.

Now suppose you want to calculate the conditional mass function $f_{X|Y}$. Note that $f_{X|Y}(m \mid n)$ is zero if $m < n$. (Since we know that X, Y take values from 1 to 6, we need to consider only that range). We should also remember that f_{XY} is given by two different expressions, one for the case $m = n$ and one for the case $m > n$. Now you can easily show that the conditional mass function is

$$f_{X|Y}(m|n) = \begin{cases} \frac{2}{13-2n} & 1 \le n < m \le 6 \\ \\ \frac{1}{13-2n} & 1 \le n = m \le 6 \end{cases}$$

You can verify that for each (permissible) n, we have $\sum_m f_{X|Y}(m|n) = 1$. Similarly you can find the other conditional mass function.

2. Let (X, Y) have joint density

$$f_{XY}(x, y) = \frac{1}{4}[1 + xy(x^2 - y^2)], \ |x| \le 1, \ |y| \le 1.$$

Find the marginal and conditional densities. Find $P[X > 0 \mid Y = 0]$.

Hint: For the marginals we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \frac{1}{4}\int_{-1}^{1} \left(1 + x^3 y - xy^3\right)\, dy = \frac{1}{2}, \ -1 \le x \le 1$$

Similarly, you can show $f_Y(y) = 0.5, \ |y| \le 1$. Since we know marginals and joint, we can find conditional densities. We have

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{1}{2}\left(1 + xy(x^2 - y^2)\right), \ |x| \le 1, \ |y| \le 1$$

and similarly for $f_{Y|X}$. You can easily verify that, e.g., $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = 1$. Finally, we can calculate

$$P[X > 0 \mid Y = 0] = \int_{x>0} f_{X|Y}(x|0)\, dx = \int_0^1 \frac{1}{2}\, dx = \frac{1}{2}$$

3. Let $f(x, y) = e^{-x-y}, \ x > 0, y > 0$. Show that this a density function. Find the marginals and the conditional densities.

Hint: The marginal of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^{\infty} e^{-x}\, e^{-y}\, dy = e^{-x}, \ x > 0$$

Now you know all marginals and conditional densities!

4. Let F_{XY} be a joint distribution function with F_X and F_Y being the corresponding marginal distribution functions. Show that

$$1-(1-F_X(x)+1-F_Y(y)) \leq F_{XY}(x,y) \leq \min(F_X(x), F_Y(y)), \ \forall x, y$$

Answer: For any events A, B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Hence $P(AB) \leq P(A)$ and $P(AB) \leq P(B)$. This gives us $F_{XY}(x,y) \leq \min(F_X(x), F_Y(y)), \ \forall x, y$. We also have $P(A \cap B) = 1 - P(A^c \cup B^c) \geq 1 - (P(A^c) + P(B^c))$ because $P(A^c \cup B^c) \leq (P(A^c) + P(B^c))$. This gives us $1-(1-F_X(x)+1-F_Y(y)) \leq F_{XY}(x,y)$.

We can show something more. Suppose as earlier F_{XY} is a 2D distribution function. Suppose F_1 and F_2 are distribution functions satisfying

$$1-(1-F_1(x)+1-F_2(y)) \leq F_{XY}(x,y) \leq \min(F_1(x), F_2(y)), \ \forall x, y$$

Then F_1 and F_2 are the marginals from F_{XY}. By putting $y = \infty$ in the above we get $F_1(x) \leq F_{XY}(x, \infty) \leq F_1(x), \ \forall x$, showing F_1 is marginal from F_{XY}. Similarly you can show for F_2.

5. Let

$$\begin{aligned} F(x,y) &= 0, \ \text{ if } x < 0, \text{ or } \ y < 0, \text{ or } \ x + y < 1 \\ &= 1, \ \text{ otherwise} \end{aligned}$$

Show that F satisfies the following: $F(-\infty, y) = F(x, -\infty) = 0$; $F(\infty, \infty) = 1$; F is non-decreasing in each variable. Is $F(x,y)$ a distribution function? (Hint: If it were the joint distribution of two random variables, X, Y, what would be $P[1/3 < X \leq 1, \ 1/3 < Y \leq 1]$).

Hint: I hope it is straight-forward to see that $F(x, -\infty) = F(-\infty, y) = 0$ and $F(\infty, \infty) = 1$ from the definition of F. Next we want to show $F(x_1, y) \leq F(x_2, y), \ \forall x_1, x_2$ with $x_1 < x_2$ and $\forall y$. If $y < 0$ then both the quantities are zero and hence the inequality is satisfied. Similarly, if $x_1 < 0$ then $F(x_1, y) = 0$ while $F(x_2, y)$ is either 0 or 1 and hence the inequality is again satisfied. So, let us consider the only remaining case: $y > 0, x_2 > x_1 > 0$. If $x_1 + y \geq 1$ then $x_2 + y \geq 1$ and hence the inequality is satisfied because both terms are 1. If $x_1 + y < 1$ then $F(x_1, y) = 0$ and hence once again the inequality is satisfied. Similarly you can show $F(x, y_1) \leq F(x, y_2), \ \forall y_1, y_2$ with $y_1 < y_2$ and $\forall x$. We can also show that the function is right continuous in each variable because all inequalities are strict in the first line of function definition. However, this function is not a distribution function:

$$F(1,1) - F(1/3,1) - F(1,1/3) + F(1/3,1/3) = 1 - 1 - 1 + 0 < 0$$

6. Let F_1 and F_2 be two one dimensional continuous distribution functions with f_1 and f_2 being the corresponding densities. Define a function $f : \Re^2 \to \Re$ by

$$f(x,y) = f_1(x)f_2(y)\left[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)\right]$$

where α is a real number. Show that $f(x,y)$ is a two dimensional density function for all $\alpha \in (-1,1)$. Show that the two marginals of $f(x,y)$ are f_1 and f_2. What does this imply about determining the joint density from the marginals? (Note that $\int_{-\infty}^{\infty} F_1(x)f_1(x)\, dx = \frac{1}{2}$).

Hint: Since $0 \le F_1(x) \le 1$, we have $-1 \le (2F_1(x) - 1) \le 1$. Same is true of F_2. If $\alpha \in (-1,1)$ then $-1 \le \alpha(2F_1(x) - 1)(2F_2(y) - 1) \le 1$ and hence $f(x,y) \ge 0$, $\forall x, y$. Now to show it is a density we need to show

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

The first term in the integral above is

$$\int_{-\infty}^{\infty} f_1(x)\, dx \int_{-\infty}^{\infty} f_2(y)\, dy = 1$$

because both f_1 and f_2 are densities. The second term in the integral of $f(x,y)$ is zero because

$$\int_{-\infty}^{\infty} f_1(x)(2F_1(x) - 1)\, dx = 2\int_{-\infty}^{\infty} f_1(x)F_1(x)\, dx - \int_{-\infty}^{\infty} f_1(x)\, dx = 2\frac{1}{2} - 1 = 0$$

So, $f(x,y)$ is a density. Let us say it is the joint density of X, Y. Now suppose we want to find marginal of Y. Then

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = f_2(y)$$

because $\int_{-\infty}^{\infty} f_1(x)\, dx = 1$ and $\int_{-\infty}^{\infty} f_1(x)(2F_1(x) - 1)\, dx = 0$. Similarly we can show the other marginal is $f_1(x)$.

So, no matter what the value of α is, the marginals are same. But changing α changes $f(x,y)$. Thus, there are infinitely many joint densities all having the same marginals.