E1 222 Stochastic Models and Applications
Hints for Problem Sheet 4.2

1. Let $\{X_n,\ n = 0, 1, \cdots\}$ be a Markov chain. Show that

$$\text{Prob}[X_{n+1} = z, X_{n-1} = y | X_n = x] = \text{Prob}[X_{n+1} = z | X_n = x] \ \text{Prob}[X_{n-1} = y | X_n = x]$$

(That is, conditioned on the 'present' the 'past' is conditinally independent of the 'future')

Hint: We have

$$\text{Prob}[X_{n+1} = z, X_{n-1} = y | X_n = x] = \frac{\text{Prob}[X_{n+1} = z, X_{n-1} = y, X_n = x]}{\text{Prob}[X_n = x]}$$

$$= \frac{\text{Prob}[X_{n+1} = z | X_n = x, X_{n-1} = y] \ \text{Prob}[X_n = x, X_{n-1} = y]}{\text{Prob}[X_n = x]}$$

$$= \frac{\text{Prob}[X_{n+1} = z | X_n = x] \ \text{Prob}[X_n = x, X_{n-1} = y]}{\text{Prob}[X_n = x]}$$

where the last step follows because $\{X_n\}$ is a markov chain. Now the above expression is equal to

$$\text{Prob}[X_{n+1} = z | X_n = x] \ \frac{\text{Prob}[X_n = x, X_{n-1} = y]}{\text{Prob}[X_n = x]}$$

$$= \text{Prob}[X_{n+1} = z | X_n = x] \ \text{Prob}[X_{n-1} = y | X_n = x]$$

2. Suppose we have two boxes and $2d$ balls, of which d are black and d are red. Initially d of the balls are placed in box-1 and the remaining in box-2. At each instant n, $n = 1, 2, \cdots$, a ball is chosen at random from each box and the two balls are placed in the opposite boxes. Let X_0 denote number of black balls initially in box-1 and let X_n denote number of black balls in box-1 after the exchange at n, $n = 1, 2, \cdots$. Argue that $\{X_n\}$ is a Markov chain. Find the transition probabilities of the Markov chain $\{X_n\}$. State which are transient states and which are recurrent states. Is the chain irreducible?

Hint: Given the number of black balls in box-1, we know number of black and red balls in both the boxes. Hence, we can calculate the probabilities of events involving the colors of the drawn balls. Suppose $X_n = y$. Then

X_{n+1} can be only one of $y-1, y, y+1$. The probabilities of each of these depend only on the number of black and red balls in each box and hence depend only on X_n. Thus it is a Markov chain. The state space would be $\{0, 1, \cdots, d\}$ because these are the possible values for the number of black balls.

Suppose $X_n = y$. That means there are y black balls and $d-y$ red balls in box-1. Since the remaining balls are in box-2, this means we have $d-y$ black balls and y red balls in box-2. Now, X_{n+1} would become $y+1$ if we draw a red ball from box-1 and a black ball from box-2. Thus, we can get the transition probability as below. Let C_n^1, C_n^2 denote the colors of the balls drawn from box-1 and box-2 at n.

$$\text{Prob}[X_{n+1} = y+1 | X_n = y] = \text{Prob}[C_n^1 = \text{red},\ C_n^2 = \text{black} | X_n = y] = \frac{d-y}{d}\,\frac{d-y}{d}$$

Arguing the same way we get

$$\begin{aligned} \text{Prob}[X_{n+1} &= y | X_n = y] = \frac{y}{d}\,\frac{d-y}{d} + \frac{d-y}{d}\,\frac{y}{d} \\ \text{Prob}[X_{n+1} &= y-1 | X_n = y] = \frac{y}{d}\,\frac{y}{d} \end{aligned}$$

The above three probabilities sum to one and this is to be expected because we know that from y we can only go to $y-1$or y or $y+1$. Note that the above are valid even when $y = 0$ or $y = d$

It is easy to see that the chain is irreducible. From 0 we always go to 1 and from d we always go to $d-1$. For $1 \le y \le d-1$, there is a positive probability of going from y to any of $y-1$or y or $y+1$. Hence we can go from any state to any state and hence the chain is irreducible. Since this is a finite irreducible chain, all states are recurrent.

3. Consider a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find the absorption probabilities from each of the transient states to each of the closed irreducible subsets of recurrent states.

Hint: Here the state space is $S = \{0, 1, \cdots, 5\}$. From the transition probability matrix we see that the subset $\{0, 1\}$ is closed (because we cannot go from these two to any other state) and irreducible (because we can reach 1 from 0 and 0 from 1). From state 2 we can reach state 0 and once we hit state 0 we will never go to state 2. Hence, state 2 is transient. Similarly, from state 3, we can reach state 1 and hence state 3 is also transient. The set $\{4, 5\}$ forms another closed irreducible set. Thus, from the transition probability matrix we conclude that 2 and 3 are transient and the rest are recurrent. Among the recurrent states, 0 and 1 form one closed set while 4 and 5 form another. Thus we get the decomposition of state space as

$$S_T = \{2, 3\}, \quad S_R = \{0, 1, 4, 5\} = \{0, 1\} + \{4, 5\}$$

Let $C_1 = \{0, 1\}$ and $C_2 = \{4, 5\}$. For absorption probabilities, we have

$$\rho_{C_1}(2) = P(2,0)+P(2,1)+P(2,2)\rho_{C_1}(2)+P(2,3)\rho_{C_1}(3) = 0.2+0.2+0.2\rho_{C_1}(2)+0.2\rho_{C_1}(3)$$

Similarly we get

$$\rho_{C_1}(3) = 0 + 0.3 + 0.2\rho_{C_1}(2) + 0.2\rho_{C_1}(3)$$

Solving these two linear equations, we get $\rho_{C_1}(2) = \frac{19}{30}$ and $\rho_{C_1}(3) = \frac{8}{15}$. From these we get $\rho_{C_2}(2) = \frac{11}{30}$ and $\rho_{C_2}(3) = \frac{7}{15}$.

4. Consider a Markov chain on non-negative integers having transition probabilities, $P(x, x+1) = p$ and $P(x, 0) = 1 - p$ where $0 < p < 1$. Show that the chain has unique stationary distribution and find that distribution.

Hint: Since we can go from any state x to 0 and also from any state x to $x + 1$, every state is reachable from every state. Hence the chain is irreducible. Hence, if any one state is positive recurrent then all states are positive recurrent. We can show state 0 is recurrent and hence the chain is recurrent:

$$P_0[T_0 < \infty] = \sum_{k=1}^{\infty} P_0[T_0 = k] = \sum_{k=1}^{\infty} p^{k-1}(1-p) = 1$$

We can similarly show state 0 is positive recurrent.

$$m_0 = E_0[T_0] = \sum_{k=1}^{\infty} k\ p^{k-1}(1-p) = \frac{1}{1-p} < \infty$$

where we have used our knowledge of expectation of a geometric random variable. This shows that the chain is positive recurrent. Since the chain is also irreducible, it would have a unique stationary distribution.

But we actually do not need the above calculation. We know that an irreducible chain is positive recurrent if and only if it has a stationary distribution. So, we can directly try to find a stationary distribution.

If π has to be a stationary distribution, it has to satisfy

$$\pi(y) = \sum_{x=0}^{\infty} \pi(x)P(x,y) = \pi(y-1)p,\ \forall y \geq 1$$

because, from the given transition probabilities, for $y \geq 1$, the only way we can reach y is from $y-1$. From the above, we have

$$\pi(y) = p\pi(y-1) = p^2\pi(y-2) = \cdots = \pi(0)p^y,\ \forall y \geq 1$$

Since π has to be a distribution, we have

$$1 = \sum_{y=0}^{\infty} \pi(y) = \sum_{y=0}^{\infty} p^y \pi(0) = \pi(0)\frac{1}{1-p}$$

which gives us $\pi(0) = (1-p)$. Thus, we get the stationary distribution as

$$\pi(y) = p^y(1-p),\ y = 0, 1, \cdots$$

5. A transition probability matrix is called doubly stochastic if both the rows as well as columns sum to one. Consider a finite irreducible Markov chain whose transition probability matrix is doubly stochastic. Show that the chain has a unique stationary distribution given by $\pi(y) = \frac{1}{n},\ \forall y$, where n is the number of states.

Hint: Since the transition probability matrix is doubly stochastic, we have $\sum_i P(i,j) = 1, \ \forall j$. Since the chain is irreducible and finite it has a unique stationary distribution. So, we can directly check whether $\pi(y) = \frac{1}{n}, \ \forall y$ is a stationary distribution.

$$\sum_x \pi(x) P(x,y) = \frac{1}{n} \sum_x P(x,y) = \frac{1}{n} = \pi(y), \ \forall y$$

where we have used the fact that the transition probability matrix is doubly stochastic.

6. On a road, three out of every four trucks are followed by a car while only one out of every five cars is followed by a truck. Find the ratio of trucks to cars on the road.

Hint: We can think of this as a 2-state Markov chain with state space as $\{T, C\}$. The information given in the problem translates to the following transition probabilities:

$$P(T,T) = \frac{1}{4}; \ P(T,C) = \frac{3}{4}; \ P(C,T) = \frac{1}{5}; \ P(C,C) = \frac{4}{5}$$

With this transition probability matrix, we can calculate the stationary distribution and it comes out as

$$\pi(T) = \frac{4}{19}; \ \pi(C) = \frac{15}{19}$$

These give the fraction of trucks and cars. Hence the ratio of trucks to cars is 4 to 15.

7. A professor keeps giving a sequence of exams to the class. The exams are of three types. Let q_i denote the probability that the class does well on exam of type i. It is known that $q_1 = 0.4$, $q_2 = 0.6$, $q_3 = 0.8$. If the class does well in the current exam, the next exam is equally likely to be any of the three types. If the class does badly on the current exam, then the next exam would always be of type 3. What proportion of exams are of type i, $i = 1, 2, 3$?

Hint: We can model it as a Markov chain with the state being the type of current exam. We can take the state space to be $\{1, 2, 3\}$. Suppose current state is 1. Then $q_1 = 0.4$ is the probability of the class doing well on the current exam. When the class does well in the current exam next state is equally likely to be any of the three; but when class does badly, it is always 3. So, the transition probabilities out of state 1 are:

$$P(1,1) = 0.4\frac{1}{3} = \frac{4}{30};\ P(1,2) = 0.4\frac{1}{3} = \frac{4}{30};\ P(1,3) = 0.4\frac{1}{3}+(1-0.4) = \frac{22}{30}$$

Arguing like this, we get the transition probabilities as

$$P = \begin{bmatrix} 0.4/3 & 0.4/3 & 2.2/3 \\ 0.2 & 0.2 & 0.6 \\ 0.8/3 & 0.8/3 & 1.4/3 \end{bmatrix}$$

This is a finite irreducible chain and the stationary probabilities are solutions of

$$\begin{aligned} \pi(1) &= \frac{0.4}{3}\pi(1) + 0.2\pi(2) + \frac{0.8}{3}\pi(3) \\ \pi(2) &= \frac{0.4}{3}\pi(1) + 0.2\pi(2) + \frac{0.8}{3}\pi(3) \\ \pi(3) &= \frac{2.2}{3}\pi(1) + 0.6\pi(2) + \frac{1.4}{3}\pi(3) \\ 1 &= \pi(1) + \pi(2) + \pi(3) \end{aligned}$$

Solving these, we get the stationary probabilities as $\pi(1) = \frac{2}{9}; \pi(2) = \frac{2}{9}; \pi(3) = \frac{5}{9}$. These give the fractions of the three types of exams.

8. Let Y_n be the sum of numbers obtained on n independent rolls of a fair die, $n = 1, 2, \cdots$. Find

$$\lim_{n\to\infty} P[Y_n \text{ is divisible by } 3]$$

(Hint: Think of a 3-state Markov chain where the state at n could be the remainder obtained when Y_n is divided by 3).

Hint: Let X_n denote the remainder you get when Y_n is divided by 3. Then X_n takes values in $\{0, 1, 2\}$. It is easy to see $\{X_n\}$ is a markov chain. By definition, X_{n+1} is the remainder with Y_{n+1}. The Y_{n+1} is obtained by adding the result of the next roll of dice to Y_n. So, if we know the remainder with Y_n then we know the remainders with Y_{n+1} in terms of different outcomes of the roll of dice. For example, suppose $X_n = 0$.

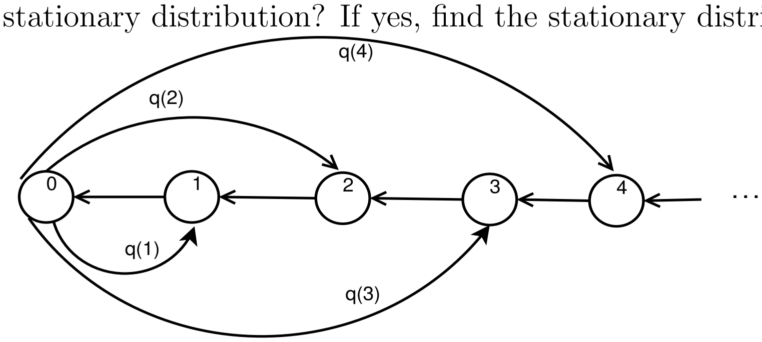
That is, Y_n is divisible by 3. Now if the next roll of dice results in 3 or 6 then $X_{n+1} = 0$, if it results in 1 or 4 then $X_{n+1} = 1$ and if it results in 2 or 5 then $X_{n+1} = 2$. Thus we have the following probabilities for transition out of state 0:

$$P(0,0) = \frac{2}{6} = \frac{1}{3};\ P(0,1) = \frac{2}{6} = \frac{1}{3};\ P(0,2) = \frac{2}{6} = \frac{1}{3};$$

Similarly we can get transitions from all other states. (For example, take $X_n = 1$. Then, if we get 1 or 4 on the dice then $X_{n+1} = 2$; if we get 2 or 5 then $X_{n+1} = 0$; if we get 3 or 6 $X_{n+1} = 1$). It is easy to see that all other transitions also have probability 1/3.

Now, for all n, Prob[Y_n is divisible by 3] = Prob[$X_n = 0$]. So, its limit as $n \to \infty$ is the stationary probability of state 0 for the Markov chain X_n. As is easy to see, the transition probability matrix of this chain is doubly stochastic. Hence, this probability is 1/3.

9. Consider the following Markov chain on state space $\{0, 1, \cdots\}$. Take $q(k) = (1-p)^{k-1}p,\ k = 1, 2, \cdots$ with $0 < p < 1$. Will this chain have a stationary distribution? If yes, find the stationary distribution.



Hint: This is an irreducible Markov chain. It would have a unique stationary distribution if and only if it is positive recurrent. We can show the chain to be positive recurrent by calculating mean return time to state 0. Alternately, we can directly show that there is a distribution that satisfies the conditions for stationary distribution.

Suppose π is a stationary distribution. Then it has to satisfy

$$\pi(j) = \sum_i \pi(i)P(i,j)$$

Taking $j = 0, 1, \cdots$, we get

$$
\begin{array}{rcll}
\pi(0) & = & \pi(1) & \\
\pi(1) & = & \pi(0)q(1) + \pi(2) & \Rightarrow \; \pi(2) = (1 - q(1))\pi(0) \\
\pi(2) & = & \pi(0)q(2) + \pi(3) & \Rightarrow \; \pi(3) = (1 - q(1) - q(2))\,\pi(0) \\
 & \vdots & & \\
\pi(k-1) & = & \pi(0)q(k-1) + \pi(k) & \Rightarrow \; \pi(k) = \left(1 - \sum_{j=1}^{k-1} q(j)\right)\pi(0) \\
 & \vdots & &
\end{array}
$$

Since $\left(1 - \sum_{j=1}^{k-1} q(j)\right) = \sum_{j=k}^{\infty} q(j)$, we get

$$
\pi(k) = \pi(0)\sum_{j=k}^{\infty} q(j) = \pi(0)\sum_{j=k}^{\infty} p(1-p)^{j-1} = \pi(0)(1-p)^{k-1}, \; k = 1, 2, \cdots
$$

This would be a stationary distribution if we can find $\pi(0)$ (with $0 < \pi(0) \leq 1$) such that $\sum_{j=0}^{\infty} \pi(j) = 1$. Thus we need

$$
\pi(0) + \pi(0)\sum_{k=1}^{\infty}(1-p)^{k-1} = \pi(0) + \pi(0)\frac{1}{p} = 1
$$

This implies $\pi(0) = \frac{p}{1+p}$. Hence the stationary distribution is

$$
\pi(0) = \frac{p}{1+p}, \quad \pi(k) = \frac{p(1-p)^{k-1}}{1+p}, \; k = 1, 2, \cdots
$$

10. Consider a 3-state Markov chain with state space $S = \{0, 1, 2\}$. Let the transition probability matrix be

$$
P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}
$$

Is the chain irreducible? Is the chain periodic and, if so, what is the period? Find the stationary distribution of the chain.

Hint: It is easily seen that we can reach any state from any other state. (We can go from 0 to 1 in two steps and 0 to 2 in one step; we can go from 1 to 0 in one step and 1 to 2 in two steps and so on). hence the chain is irreducible.

By drawing the graph corresponding to the given transition matrix we can see that $P^2(0,0) > 0$ and $P^3(0,0) > 0$. Hence the period of 0 is 1. Since the chain is irreducible, this means that the chain is aperiodic.

The stationary distribution has to satisfy

$$\begin{aligned}
\pi(0) &= \pi(1) + 0.5\pi(2) \\
\pi(1) &= 0.5\pi(2) \\
\pi(2) &= \pi(0) \\
1 &= \pi(0) + \pi(1) + \pi(2)
\end{aligned}$$

This give the stationary distribution: $\pi(0) = 0.4$, $\pi(1) = 0.2$, $\pi(2) = 0.4$.

11. Suppose that whether or not it rains today depends on whether or not it rained for the previous three days. Explain how we can set up a Markov chain model for this. Suppose that if it rained on each of the previous three days then it will rain today with probability 0.6; if it did not rain on any of the previous three days then it will rain today with probability 0.2; in all other cases the weather today would be same as that of yesterday with probability 0.5. Now find the transition probability matrix for the chain.

Hint: We can think of representing the weather on each day as a binary variable, Y_n (denoting whether or not it rains). We know Y_n is not markov because it depends on $Y_{n-1}, Y_{n-2}, Y_{n-3}$. But we can define state at n as $X_n = (Y_n, Y_{n-1}, Y_{n-2})$. Thus, we have $X_{n-1} = (Y_{n-1}, Y_{n-2}, Y_{n-3})$. It is easy to see that X_n depends only on X_{n-1} and hence $\{X_n\}$ is a Markov chain. The state here consists of a triple of binary variables and hence this is a 8-state chain. The state space would be $\{(0,0,0), (0,0,1), (0,1,0), \cdots, (1,1,1)\}$.

We can represent a general state as (b_1, b_2, b_3) where $b_i \in \{0,1\}$. Note that if $X_n = (b_1, b_2, b_3)$ then only possible next states are $X_{n+1} = (b, b_1, b_2)$ where $b \in \{0,1\}$. This is because of the way we defined X_n in terms of Y_n.

Now we can write the transition probabilities of the chain as follows (from what is given in the problem):

$$P((1,1,1),(1,1,1)) = 0.6;\ P((1,1,1),(0,1,1)) = 0.4;$$

$$P((0,0,0),(0,0,0)) = 0.8;\ P((0,0,0),(1,0,0)) = 0.2;$$

and for all other states we have

$$P((b_1,b_2,b_3),(b_1,b_1,b_2)) = 0.5;\ P((b_1,b_2,b_3),(\bar{b}_1,b_1,b_2)) = 0.5;$$

where $\bar{b}_1$ is the complement of b_1.

What we did here is a standard trick to convert processes with more than one-step memory into a Markov chain. Sometimes, the Y_n that we considered here is called a Markov Chain with memory depth 3 or a 3-step Markov chain. In practice, this may not be efficient because the size of the state space grows exponentially.