HW 1

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Q. 2.2 (2)

$\dot{x}_1 = 2x_1 - x_1x_2$ & $\dot{x}_2 = 2x_1^2 - x_2$

$\therefore \dot{x}_1 = 0 = 2x_1 - x_1x_2 \Rightarrow x_1(2 - x_2) = 0 \longrightarrow x_1(1 - x_1^2) = 0$

$\dot{x}_2 = 0 = 2x_1^2 - x_2 \Rightarrow 2x_1^2 = x_2$

This gives a set of eq. pts as follows,

$(x_1, x_2) = \{ (0,0), (1,2), (-1,2) \}$

$\therefore$ when $x_2 = 0$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_1^2 \end{bmatrix}$

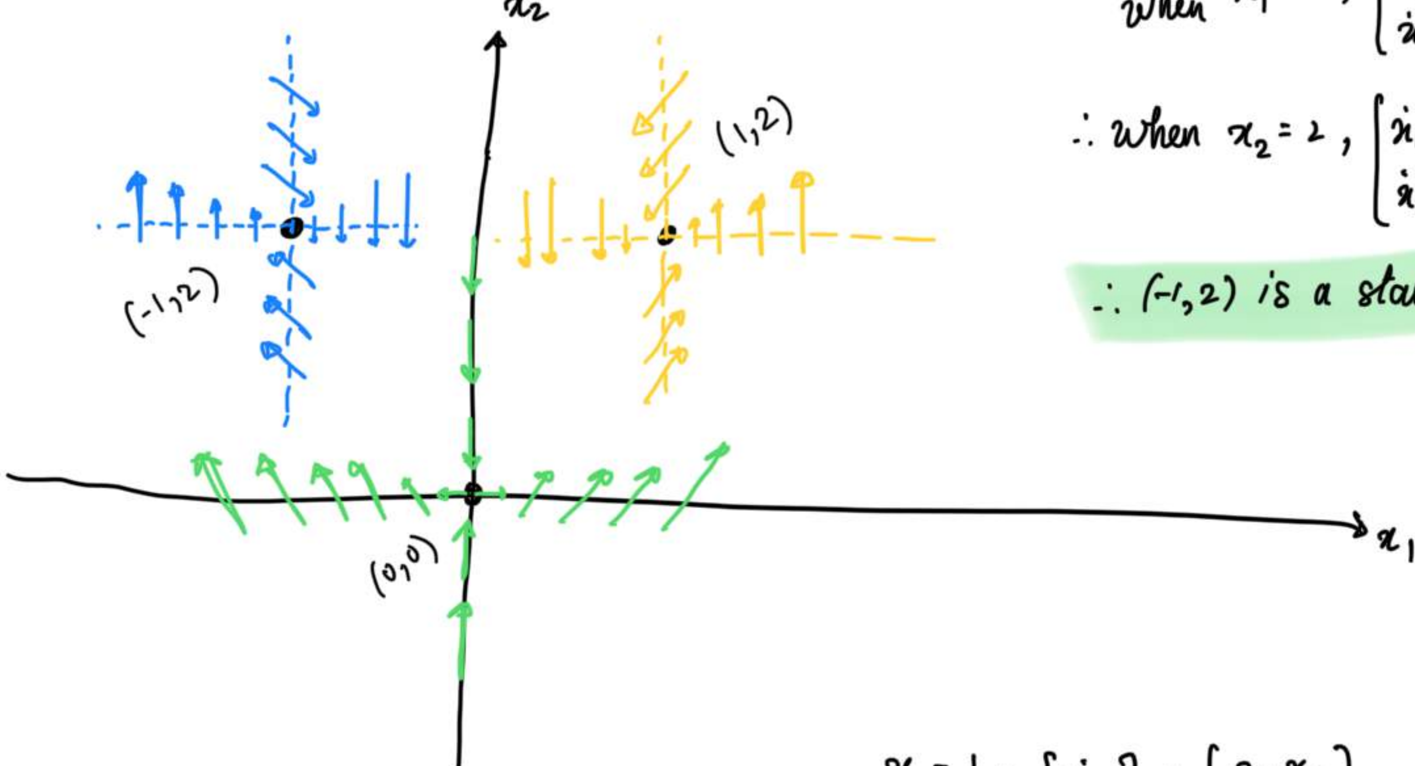
$x_1 = 0$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_2 \end{bmatrix}$

Thus a saddle node type nature can be seen at $(0,0)$

when $x_1 = -1$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - 2 \\ 2 - x_2 \end{bmatrix}$

$\therefore$ when $x_2 = 2$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2(x_1^2 - 1) \end{bmatrix}$

$\therefore (-1,2)$ is a stable node



$x_1 = 1$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 - x_2 \\ 2 - x_2 \end{bmatrix}$

Again the trajectories asymptotically converge at $(1,2)$ Hence stable node

Q. 2.12

$\therefore$ Dynamics eq^{n}s ;

① $\dot{x}_1 = \lambda \cos^2(\pi x_1/2) \left[x_2 - \dfrac{4\tan(\pi x_1/2)}{\lambda\pi} \right]$

② $\dot{x}_2 = \lambda \cos^2(\pi x_2/2) \left[x_1 - \dfrac{4\tan(\pi x_2/2)}{\lambda\pi} \right]$

& $x_i \in (-1, 1)$ is given

can never become zero

I have derived these in the python code in LateX format

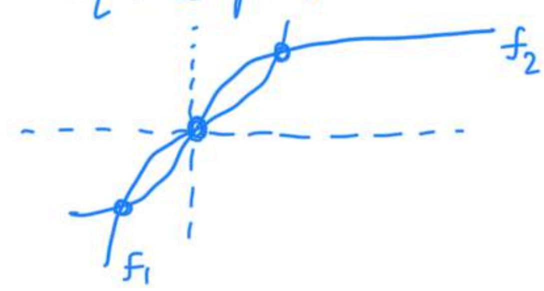
(a) $x_2 = \frac{4\tan(\pi x_1/2)}{\lambda\pi}$ & $x_1 = \frac{4\tan(\pi x_2/2)}{\lambda\pi}$

$\therefore$ (0,0) satisfies this eq^n & is a saddle node (By seeing the phase portrait in Python)

$\therefore x_1 = \frac{4}{\lambda\pi}\tan\left(\frac{\pi}{2}\times\frac{4\tan(\pi x_1/2)}{\lambda\pi}\right) = \frac{4}{\lambda\pi}\tan\left(\frac{2\tan(\pi x_1/2)}{\lambda}\right)$

$$\underbrace{\operatorname{atan}\left(\frac{\lambda\pi x_1}{4}\right)}_{f_1} = \underbrace{\frac{2}{\lambda}\tan\left(\frac{\pi x_1}{2}\right)}_{f_2}$$

intersection of the curves f_1 & f_2 will give the required points



$\therefore$ as λ varies, if $\lambda \le 2$ there is only one solution i.e the trivial sol^n $x_1 = 0$

if $\lambda > 2$ then there exists 3 sol^{n}s out of which one is trivial

$\therefore$ if $\lambda > 2$ then 3 equillibrium points & the non trivial ones are stable nodes as seen in the phase portrait in python

Q: 2.16

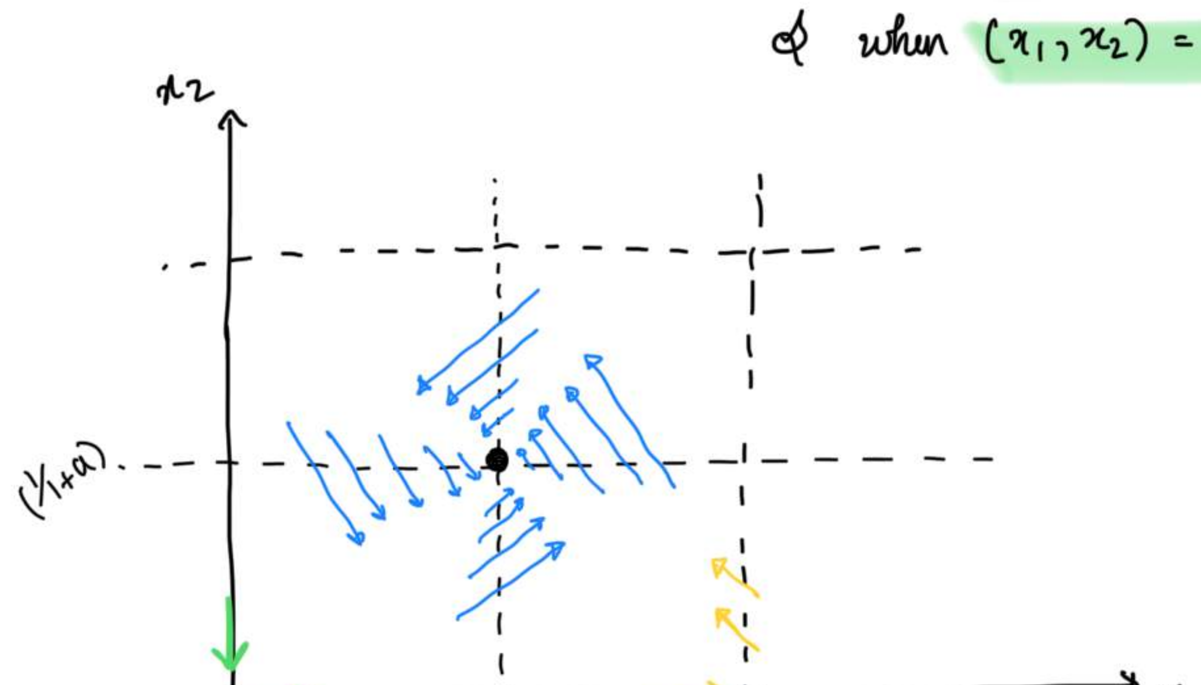
$\dot{x}_1 = x_1(1 - x_1 - a x_2)$

$\dot{x}_2 = b x_2(x_1 - x_2)$

(a) $\therefore x_1(1 - x_1 - a x_2) = 0$ $\Rightarrow$ when $(x_1, x_2) = (0,0)$

$\therefore b x_2(x_1 - x_2) = 0$ $\quad (x_1, x_2) = (1, 0)$

& when $(x_1, x_2) = \left(\frac{1}{1+a}, \frac{1}{1+a}\right)$



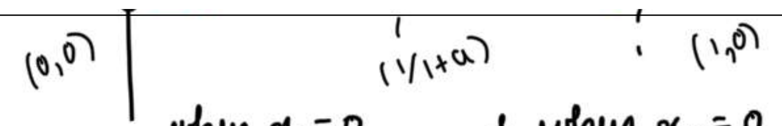
when $x_1 = 1/1+a$; $\dot{x}_1 = \frac{a}{1+a}\left(\frac{1}{1+a} - x_2\right)$

$\dot{x}_2 = b x_2\left(\frac{1}{1+a} - x_2\right)$

when $x_2 = 1/1+a$; $\dot{x}_1 = x_1\left(\frac{1}{1+a} - x_1\right)$

$\dot{x}_2 = \frac{b}{1+a}\left(x_1 - \frac{1}{1+a}\right)$

$\therefore \left(\frac{1}{1+a}, \frac{1}{1+a}\right)$ is a stable node

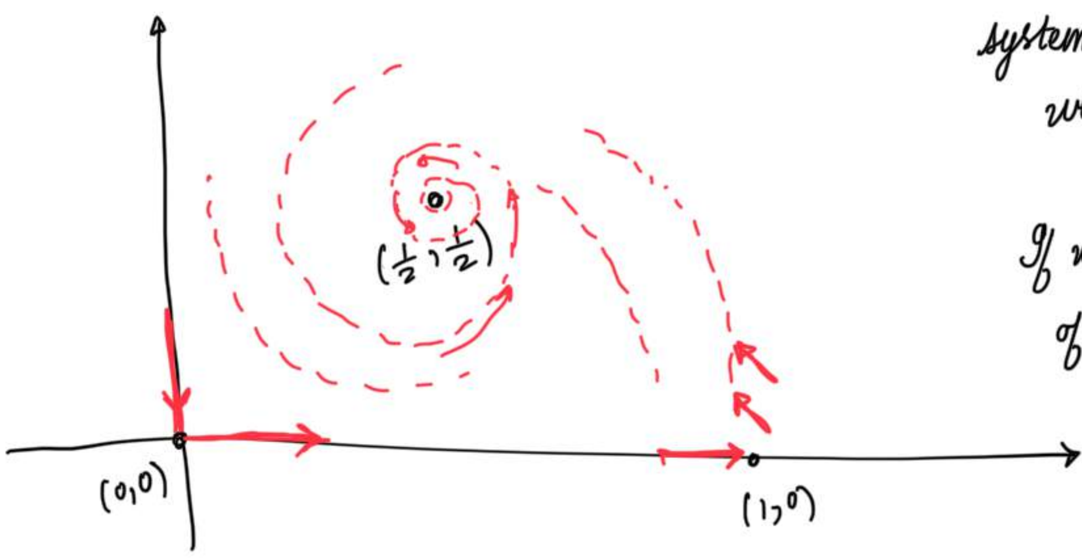


when $x_1 = 0$, $\dot{x}_1 = 0$, $\dot{x}_2 = -b x_2^2$ & when $x_2 = 0$ $\dot{x}_1 = x_1(1-x_1)$ $\dot{x}_2 = 0$ & when $x_1 = 1$ $\dot{x}_1 = 0$ $\dot{x}_2 = b x_2(1-x_2)$

$\therefore$ (1,0) is a saddle node

$\therefore$ (0,0) is a saddle node

(b) When $a = 1$, $b = 0.5$



There is 1 stable node in this system. & depending on where we start the population will stabilize

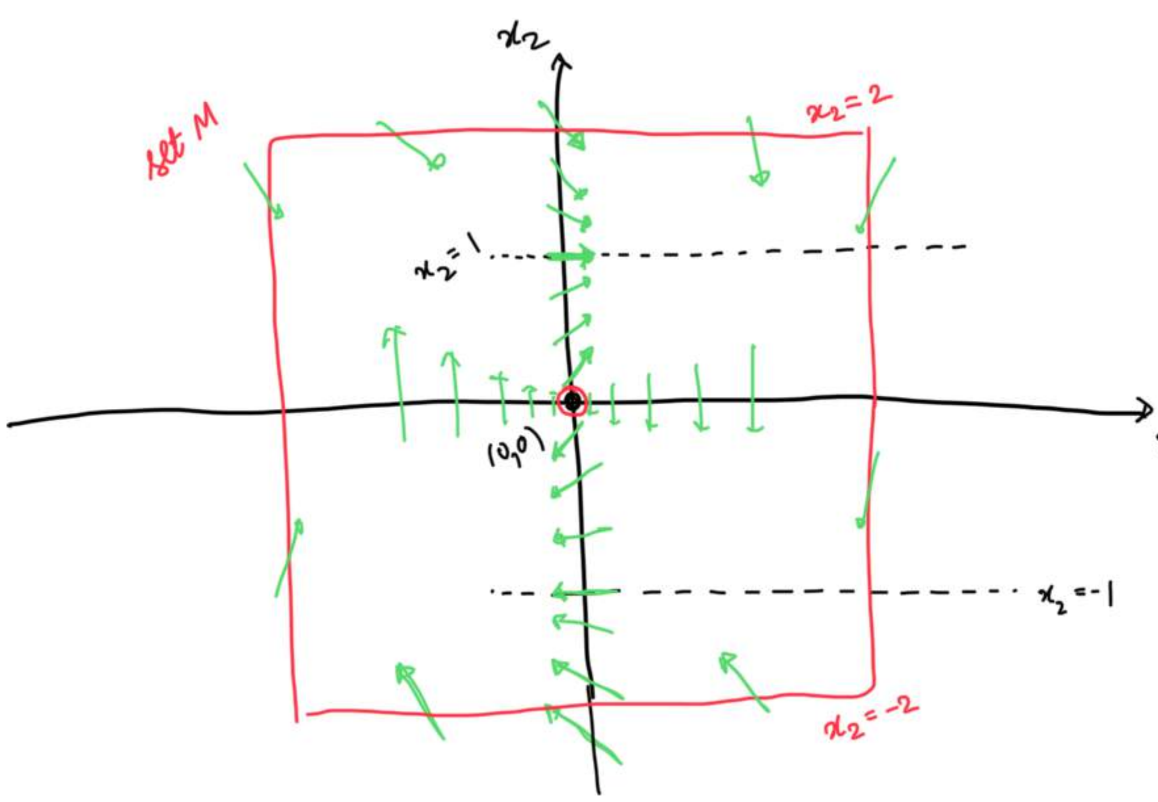
If we start with 0 population of prey or predator, then the other will die as well

Q: 2.17 (2)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2(2 - 3x_1^2 - 2x_2^2)$$

$\therefore$ Equilibrium points $\Rightarrow$ $x_2 = 0$

$$-x_1 + x_2(2 - 3x_1^2 - 2x_2^2) = 0 \Rightarrow x_1 = 0$$



when $x_2 = 0$; $\dot{x}_1 = 0$
$\dot{x}_2 = -x_1$

$x_1 = 0$; $\dot{x}_1 = x_2$
$\dot{x}_2 = 2x_2(1 - x_2^2)$

$\therefore$ (0,0) is an unstable node

Hence trajectories come out

Since the vector fields change the direction at $x_2 = \pm 1$, I am choosing a set

$$M = \{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 2,\ |x_2| \leq 2,\ x_1 \neq 0,\ x_2 \neq 0 \}$$

Since M is closed & bounded set, in which trajectories that start in M, stay in M & there are no equill. points, by Poincare Bendixon Theorem There must be a periodic orbit in M

See python code for this periodic orbit

Q: 2.17 (4)

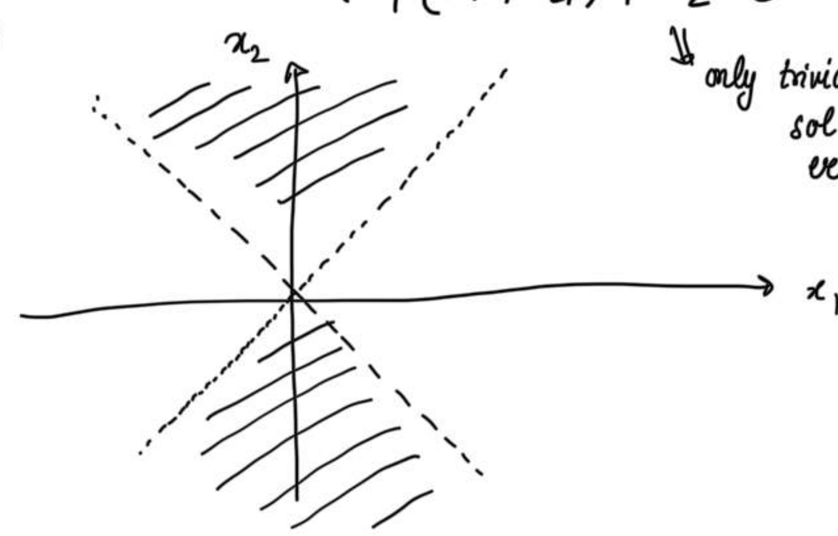
$$\dot{x}_1 = x_1 + x_2 - x_1 h(x) \qquad ; \quad h(x) = \max\{|x_1|, |x_2|\}$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2 h(x)$$

$\therefore$ Eq. pts $\Rightarrow$
$$x_1(1 - h(x)) + x_2 = 0 \quad —①$$
$$-2x_1(1 + h(x)) + x_2 = 0 \quad —②$$

$\therefore (0,0)$ is definitely an eq. pt. for other non trivial sol^{n}s

if $|x_1| < |x_2|$
$\hookrightarrow$ $x_1(1 - |x_2|) + x_2 = 0$
$-2x_1(1 + |x_2|) + x_2 = 0$
$\Downarrow$ only trivial sol^n exists

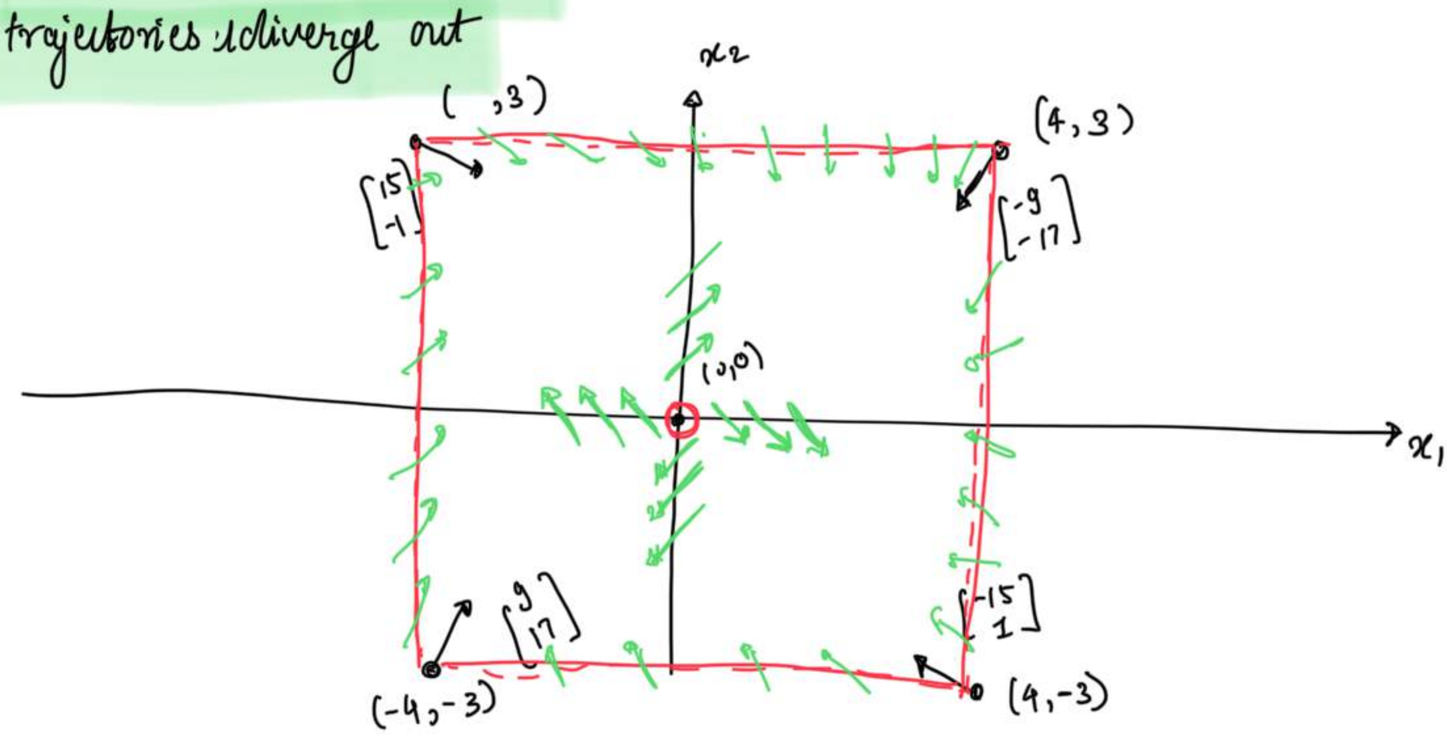
similarly we can show this for $|x_2| > |x_1|$



when $x_2 = 0$, $\dot{x}_1 = x_1 - x_1^2$
$\dot{x}_2 = -2x_1$

when $x_1 = 0$, $\dot{x}_1 = x_2$
$\dot{x}_2 = x_2 - x_2^2$

$\therefore (0,0)$ is an unstable node
$\therefore$ trajectories diverge out



$\therefore$ Let $M = \{(x_1, x_2) \mid |x_1| < 4,\ |x_2| < 3,\ x_1 \neq,\ x_2 \neq 0\}$ & since M is closed & bounded & no eq. pts are contained in M & no trajectories that start in M come out of M, by using Poincare Bendixson Criteria, A periodic orbit must exist inside M

Q: 2.20(2)

$$\dot{x}_1 = -x_1 + x_1^3 + x_1 x_2^2$$
$$\dot{x}_2 = -x_2 + x_2^3 + x_1^2 x_2$$

$\therefore$ Eq. pts $\begin{cases} -x_1 + x_1^3 + x_1 x_2^2 = 0 \\ -x_2 + x_2^3 + x_1^2 x_2 = 0 \end{cases}$

$\Rightarrow \begin{cases} x_1 (x_1^2 + x_2^2 - 1) = 0 \\ x_2 (x_1^2 + x_2^2 - 1) = 0 \end{cases}$

$\therefore x_1 = 0$ (or) $x_1^2 + x_2^2 = 1$

$\therefore x_2 = 0$ (or) $x_1^2 + x_2^2 = 1$

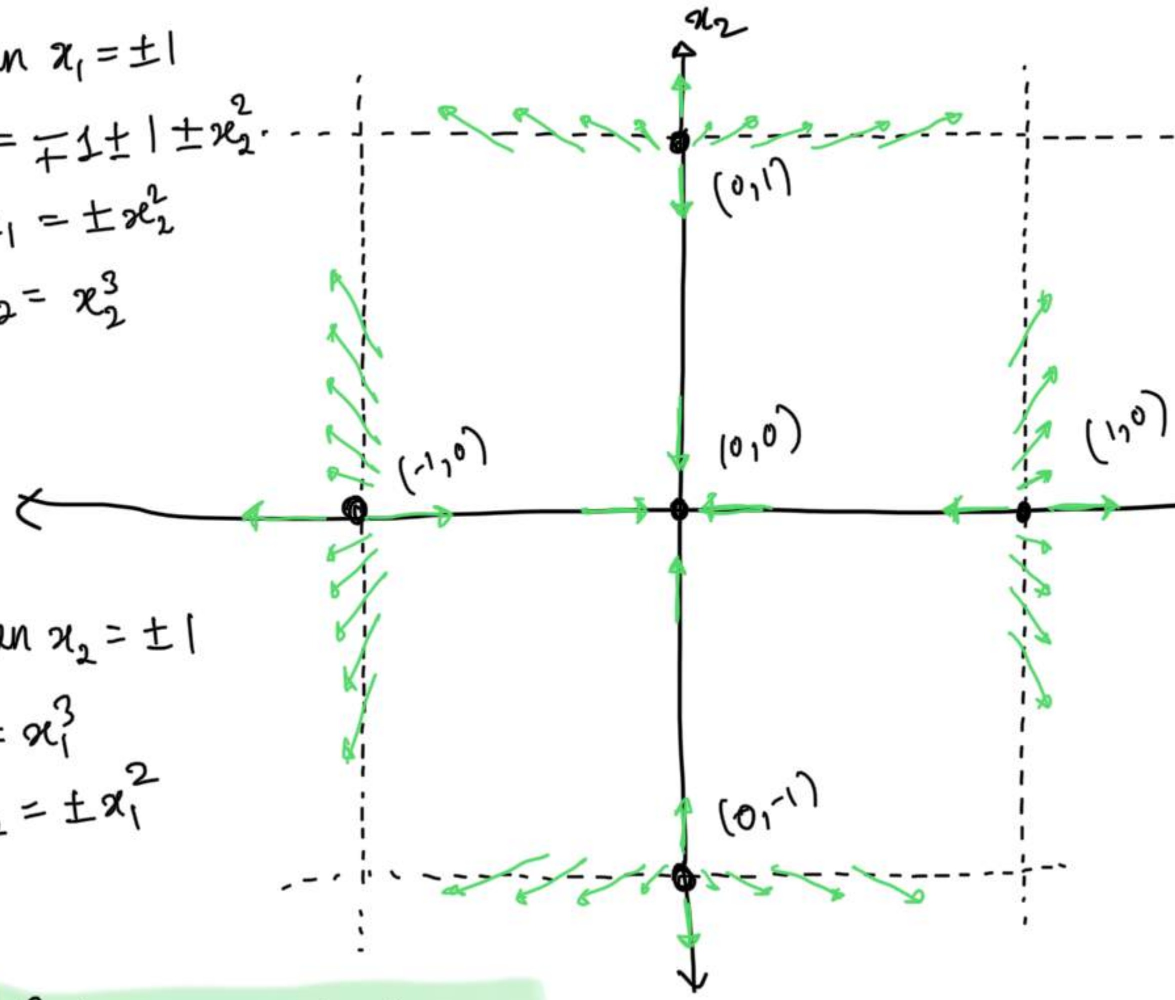
when $x_1 = \pm 1$

$\dot{x}_1 = \mp 1 \pm 1 \pm x_2^2$

$\begin{cases} \dot{x}_1 = \pm x_2^2 \\ \dot{x}_2 = x_2^3 \end{cases}$

when $x_2 = \pm 1$

$\begin{cases} \dot{x}_1 = x_1^3 \\ \dot{x}_2 = \pm x_1^2 \end{cases}$



when $x_1 = 0$

$\dot{x}_1 = 0$

$\dot{x}_2 = -x_2 + x_2^3$

when $x_2 = 0$

$\dot{x}_1 = -x_1 + x_1^3$

$\dot{x}_2 = 0$

$\therefore (0,0)$ is a stable node

Rest are unstable nodes

There exits no set M that satisfies Poincare Bendixon Theorem inside region $\textcircled{D} \in [-1,1] \times [1,1]$ - because all trajectories go out or converge to $(0,0)$

&

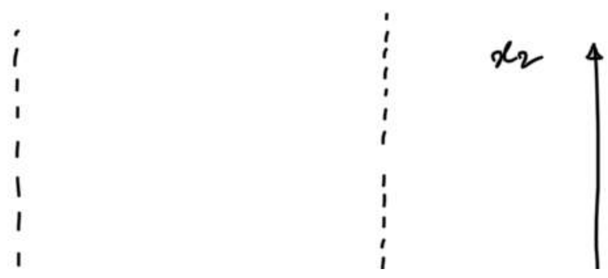
similarly outside $\textcircled{D}$ all trajectories diverge outside, Hence there exist no limit cycles

Q: 2.21

$$\dot{x}_1 = -x_1 + x_2 (x_1 + a) - b \qquad a, b, c > 0$$
$$\dot{x}_2 = -c x_1 (x_1 + a) \qquad \& \; b > a$$

$$D = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 < -a \;\; \& \;\; x_2 < \frac{x_1 + b}{x_1 + a} \right\}$$

(a)

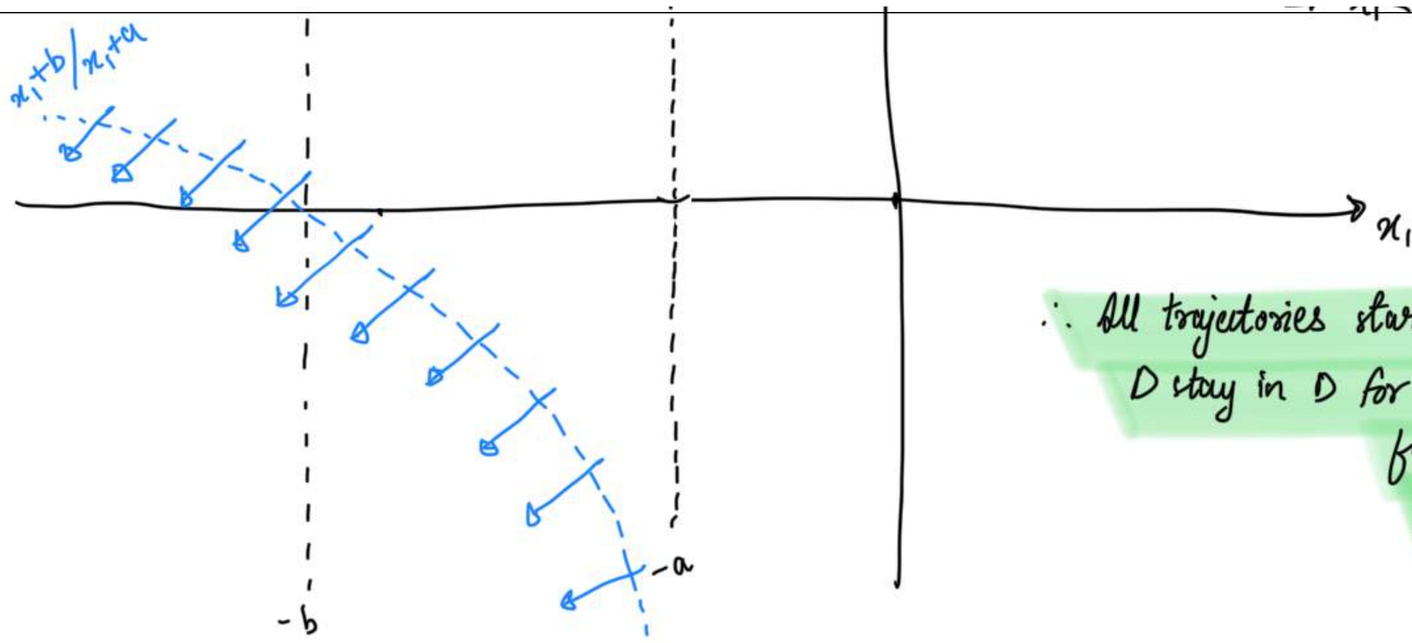


when $x_1 < -a$ & $x_2 < \dfrac{x_1 + b}{x_1 + a}$

$\dot{x}_2 < 0$

& $\dfrac{\dot{x}_1 + x_1 + b}{(x_1 + a)} < \dfrac{x_1 + b}{(x_1 + a)}$

$\Rightarrow \dot{x}_1 < 0$



$\therefore$ All trajectories starting in D stay in D for all future time

(b) $\therefore$ Eq. pts $\dot{x}_1 = 0 = -x_1 + x_2(x_1+a) - b$

$\dot{x}_2 = 0 = -cx_1(x_1+a)$

$\therefore x_1 = 0$ (or) $(x_1+a) = 0$

$\Downarrow$ $\quad$ $\hookrightarrow$ then $a-b=0$ which is not possible

$x_2 = \frac{b}{a}$ $\quad$ $\therefore$ Only one eq. pt. $(0, b/a)$

Now even though we have a region D satisfying (a) & that no equillibrium point exists in D,

D is not bounded , Hence Poincare Bendexison criterion theorem doesn't held & no periodic orbit can exist within D