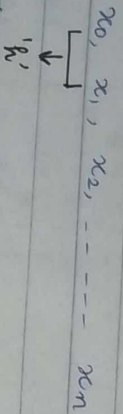
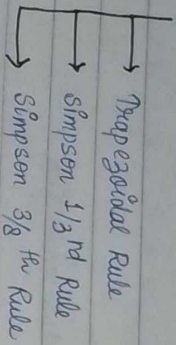


Unit : 3

Numerical Integration

$$I = \int_a^b y \, dx$$

 $n$  - equal intervals width:  $h$ 

$$x_0 = a$$

$$x_1 = a + h$$

$$x_2 = a + 2h$$

$$x_3 = a + 3h$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$x_n = a + nh$$

$$\vdots$$

$$\vdots$$

$$\therefore b - a = nh$$

(Here,  $a = x_0$ and  $b = x_n$ )

$$\Rightarrow h = \frac{b-a}{n}$$

$$I = \int_a^b f(x) \, dx$$

→ Trapezoidal Rule:

$$\int_a^b f(x) \, dx = h \left[ \frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right]$$

→ Simpson's  $\frac{1}{3}^{\text{rd}}$  Rule:

$$\int_a^b f(x) \, dx = \frac{h}{3} \left[ (y_0 + y_n) + 4(\text{Odd terms}) + 2(\text{Even Terms}) \right]$$

→ Simpson's  $\frac{3}{8}^{\text{th}}$  Rule:

$$\int_a^b f(x) \, dx = \frac{3h}{8} \left[ (y_0 + y_n) + 3(\text{Remaining}) + 2(\text{Multiples of 3}) \right]$$

Ques) Evaluate  $\int_0^1 \frac{dx}{1+x^2}$  using

a) Trapezoidal Rule

b) Simpson's  $1/3^{\text{rd}}$  Rulec) Simpson's  $3/8^{\text{th}}$  Rule.

$$dx, \quad \pi = \int_a^b f(x) dx.$$

$$\text{Here, } a=0; b=1; f(x) = \frac{1}{1+x^2}$$

NOTE: If not given, always take  $n=6$

$$\Delta t \quad n=6$$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$x_0 \rightarrow$	$x$	$f(x)$	$\frac{1}{1+x^2}$	$y$
$x_0 \rightarrow x_0 + h$	0	$f(x_0)$	$\frac{1}{1+(0)^2}$	1
$x_1 \rightarrow x_0 + 2h$	$\frac{1}{6}$	$f(x_1)$	$\frac{1}{1+(\frac{1}{6})^2}$	0.9729
$x_2 \rightarrow x_0 + 3h$	$\frac{2}{6}$	$f(x_2)$	$\frac{1}{1+(\frac{2}{6})^2}$	0.9
$x_3 \rightarrow x_0 + 4h$	$\frac{3}{6}$	$f(x_3)$	$\frac{1}{1+(\frac{3}{6})^2}$	0.8
$x_4 \rightarrow x_0 + 5h$	$\frac{4}{6}$	$f(x_4)$	$\frac{1}{1+(\frac{4}{6})^2}$	0.6923
$x_5 \rightarrow x_0 + 6h$	$\frac{5}{6}$	$f(x_5)$	$\frac{1}{1+(\frac{5}{6})^2}$	0.5901
$x_6 \rightarrow x_0 + 6h$	$\frac{6}{6}$	$f(x_6)$	$\frac{1}{1+(\frac{6}{6})^2}$	0.5

$x$	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
$y$	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$
	1	0.9729	0.9	0.8	0.6923	0.5901	0.5

Trapezoidal Rule:

$$\int_0^1 \frac{1}{1+x^2} dx = h \left[ \frac{y_0 + y_6}{2} + y_1 + y_2 + y_3 + y_4 + y_5 \right]$$

$$= \frac{1}{6} \left[ \frac{1+0.5}{2} + 0.9729 + 0.9 + 0.8 + 0.6923 + 0.5901 \right]$$

$$= 0.7842$$

Simpson's 1<sup>st</sup> Rule:-

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &= \frac{h}{3} \left[ (y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right] \\ &= \frac{1}{18} \left[ (1 + 0.5) + 4(0.9729 + 0.8 + 0.5901) + 2(0.9 + 0.6923) \right] \\ &= 0.7853 \end{aligned}$$

Simpson's 3<sup>rd</sup> Rule:-

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &= \frac{3h}{8} \left[ (y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5) \right] \\ &= \frac{3 \times \frac{1}{6}}{8} \left[ (1 + 0.5) + 2(0.8) + 3(0.9729 + 0.9 + 0.6923 + 0.5901) \right] \\ &= 0.7853 \end{aligned}$$

Waddle's Rule:-

$$\int f(x) dx = \frac{3h}{10} \left[ (y_0 + 5y_1) + (y_2 + 6y_3) + (y_4 + 5y_5) + y_6 \right]$$



Ques- Evaluate  $\int_0^1 \frac{dx}{1+x^2}$  by using Waddle's Rule and hence obtain the approximate value of Integral dividing six equal terms.

By Waddle's Rule,

$$\begin{aligned} \int f(x) dx &= \frac{3h}{10} [(y_0 + 5y_1) + (y_2 + 6y_3) + (y_4 + 5y_5 + y_6)] \\ &= \frac{3}{10} \times \frac{1}{6} [(1 + 5 \times 0.9729) + (0.9 + 6 \times 0.8) + (0.6923 + 5 \times 0.5901) + 0.5] \\ &= 0.785365 \end{aligned}$$

Now,  $\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4} = 0.7853$

## Numerical Differentiation

1) Euler's Method : Initial condition

$$[y(x_0) = y_0]$$

$$\{y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})\}$$

$$n=1; \quad h = \text{width}$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

Ques)

Using Euler's method, find an approximate value of  $y$ , corresponding to  $x = 0.1$ . Given,

$$\frac{dy}{dx} = \frac{y-x}{y+x}; \quad y(0) = 1$$

$$\text{at } n=5, \quad \therefore x_n = x_5 \Rightarrow x_5 = 0.1$$

$$h = \frac{0.1-0}{5} = 0.02$$

$$x_0 = 0 \Rightarrow y_0 = 1$$

$$\Rightarrow y_1 \Rightarrow x_1 = x_0 + h = 0 + 0.02 = 0.02$$

$$y_2 \Rightarrow x_2 = x_1 + h = 0.02 + 0.02 = 0.04$$

$$y_3 \Rightarrow x_3 = x_2 + h = 0.04 + 0.02 = 0.06$$

$$y_4 \Rightarrow x_4 = x_3 + h = 0.06 + 0.02 = 0.08$$

$$y_5 \Rightarrow x_5 = 0.1$$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

$$= 1 + 0.02 \left[ \frac{1-0}{1+0} \right]$$

$$= 1 + 0.02 = 1.02$$

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$= 1.02 + (0.02) \cdot \left[ \frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$= 1.03923$$

$$y_3 = y_2 + h \cdot f(x_2, y_2)$$

$$= 1.03923 + 0.02 \left[ \frac{1.03923 - 0.04}{1.03923 + 0.04} \right]$$

$$= 1.057747462$$

$$y_4 = y_3 + 0.02 \left[ \frac{y_3 - x_3}{y_3 + x_3} \right]$$

$$= 1.075652707$$

$$\text{Also, } y_5 = 1.092883709 = y(0.1)$$

Ques) Using Euler's Method find  $y$  at  $x=1$ ;  $\frac{dy}{dx} = x+y$ ;  $y(0)=1$

Let us consider,  $n=5$ .

$$h = \frac{1-0}{5} = 0.2$$

$$\Rightarrow y_0 = 1 \Rightarrow x_0 = 0$$

$$y_1 \Rightarrow x_1 = 0 + 0.2 = 0.2$$

$$y_2 \Rightarrow x_2 = 0.2 + 0.2 = 0.4$$

$$y_3 \Rightarrow x_3 = 0.4 + 0.2 = 0.6$$

$$y_4 \Rightarrow x_4 = 0.6 + 0.2 = 0.8$$

$$y_5 \Rightarrow x_5 = 0.8 + 0.2 = 1$$

$$\Rightarrow y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.2 [0 + 1]$$

$$= 1.2$$

$$\Rightarrow y_2 = 1.2 + 0.2 [0.2 + 1.2]$$

$$= 1.48$$

$$\Rightarrow y_3 = 1.48 + 0.2 [0.4 + 1.48] = 1.856$$

$$\Rightarrow y_4 = 1.856 + 0.2 [0.6 + 1.856]$$

$$y_4 = 2.3472$$

$$\Rightarrow y_5 = 2.3472 + 0.2 [0.8 + 2.3472]$$

$$y_5 = 2.97664$$