Unit 3 (Part-II)

Numerical Differentiation

1) Euler's Method : Initial Condition

$$\left[y(x_0) = y_0 \right]$$

$$\left\{ y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \right\}$$

$n=1$; h = Width

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

Ques) Using Euler's method, find an approximate value of y, corresponding to $x = 0.1$. Given,

$$\frac{dy}{dx} = \frac{y-x}{y+x} \; ; \; y(0) = 1$$

Let $n = 5$, $\because x_n = x_5 \Rightarrow x_5 = 0.1$

$$h = \frac{0.1 - 0}{5} = 0.02$$

$x_0 = 0 \Rightarrow y_0 = 1$

$\Rightarrow y_1 \Rightarrow x_1 = x_0 + h = 0 + 0.02 = 0.02$

$y_2 \Rightarrow x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$y_3 \Rightarrow x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

$y_4 \Rightarrow x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

$y_5 \Rightarrow x_5 = 0.1$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

$$= 1 + 0.02 \left[\frac{1-0}{1+0} \right]$$

$$= 1 + 0.02 = 1.02$$

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$= 1.02 + (0.02) \cdot \left[\frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$= 1.03923$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.03923 + 0.02 \left[\frac{1.03923 - 0.04}{1.03923 + 0.04} \right]$$

$$= 1.057747462$$

$$y_4 = y_3 + 0.02 \left[\frac{y_3 - x_3}{y_3 + x_3} \right]$$

$$= 1.075652707$$

Also, $y_5 = 1.092883709 = y(0.1)$

Ques) Using Euler's Method find y at $x=1$; $\frac{dy}{dx} = x+y$; $y(0) = 1$

Let us consider, $n=5$.

$\therefore h = \frac{1-0}{5} = 0.2$

$\Rightarrow y_0 = 1 \Rightarrow x_0 = 0$

$y_1 \Rightarrow x_1 = 0+0.2 = 0.2$

$y_2 \Rightarrow x_2 = 0.2+0.2 = 0.4$

$y_3 \Rightarrow x_3 = 0.4+0.2 = 0.6$

$y_4 \Rightarrow x_4 = 0.6+0.2 = 0.8$

$y_5 \Rightarrow x_5 = 0.8+0.2 = 1$

$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.2[0+1]$

$= 1.2$

$\Rightarrow y_2 = 1.2 + 0.2[0.2 + 1.2]$

$= 1.48$

$\Rightarrow y_3 = 1.48 + 0.2[0.4 + 1.48] = 1.856$

$\Rightarrow y_4 = 1.856 + 0.2[0.6 + 1.856]$

$y_4 = 2.3472$

$\Rightarrow y_5 = 2.3472 + 0.2[0.8 + 2.3472]$

$y_5 = 2.97664$

10.1.24

Runge Kutta Method :

consider O.D.E;

$\frac{dy}{dx} = f(x,y)$ and initial condition, $y(x_0) = y_0$

To find $y(x)$.

$K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + h/2, y_n + K_1/2)$

$K_3 = hf(x_n + h/2, y_n + K_2/2)$

$K_4 = hf(x_n + h, y_n + K_3)$

$K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$$y_{n+1} = y_n + K \quad ; \quad y_1 = y_0 + K$$

Ques) Apply Runge Kutta of fourth order. Solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0) = 1$ at $x = 0.2$ and $x = 0.4$.

Here, $x_0 = 0$ and $y_0 = 1$

$h = 0.2$

Now, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = y(x_1)$

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$y_2 = y(x_2)$

Now, $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

$K_1 = hf(x_0, y_0) = hf(0,1)$

$K_1 = 0.2\left[\frac{1-0}{1+0}\right] = 0.2$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= 0.2f(0+0.1, 1+0.1)$

$= 0.2f(0.1, 1.1)$

$= 0.2\left[\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right] = 0.19672$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= 0.2f(0+0.1, 1+0.09836)$.

$= 0.2f(0.1, 1.09836)$

$= 0.2\left[\frac{1.09836^2 - 0.1^2}{1.09836^2 + 0.1^2}\right]$

$= 0.196711$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2f(0+0.2, 1+0.196711)$

$= 0.2f(0.2, 1.196711)$

$= 0.2\left[\frac{1.196711^2 - 0.2^2}{1.196711^2 + 0.2^2}\right]$

$= 0.1891313$

$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 0.195998883$

$y_1 = y_0 + K \Rightarrow y_1 = 1 + 0.19599883 = 1.19599883$

$y_2 = y_1 + K = 1.19599883 + 0.19599883 = 1.39199766$

Picard's Method (Successive Approximation)

$$\frac{dy}{dx} = f(x,y)$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\left[\, y = \int_{x_0}^{x} f(x,y)\,dx + y_0 \,\right]$$

$$y_1 = \int_{x_0}^{x} f(x,y)\,dx + y_0$$

$$y_2 = \int_{x_0}^{x} f(x, y_1)\,dx + y_0$$

$$y_3 = \int_{x_0}^{x} f(x, y_2)\,dx + y_0$$

$$y_{n+1} = \int_{x_0}^{x} f(x, y_n)\,dx + y_0$$

Ques) Solve the differential equation, $\frac{dy}{dx} = 1 + xy$ with $x_0 = 0; y = 0$ upto third approximation.

$$y_1 = y_0 + \int f(x,y)\,dx$$

Now, $\frac{dy}{dx} = 1 + xy = f(x,y)$.

$$y_1 = \int_0^x f(x,y)\,dx + 0$$

$$= \int_0^x dx = x. \Rightarrow [y_1 = x]$$

$$y_2 = \int_0^x f(x,x)\,dx + 0.$$

$$= \int_0^x (1 + x^2)\,dx. \Rightarrow \left[y_2 = x + \frac{x^3}{3}\right]$$

$$y_3 = \int_0^x f\left(x, x + \frac{x^3}{3}\right) dx + 0.$$

$$\int_0^x \left(1 + x^2 + \frac{x^4}{3}\right) dx$$

$$= x + \frac{x^3}{3} + \frac{x^5}{15}$$

12.01.24

Numerical Solution of ODE by Taylor's Series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

If $y(x)$ is the solⁿ of O.D.E

$$\frac{dy}{dx} = f(x,y) \quad \text{---} \ (i) \quad \& \quad y_0 = y(x_0) - [\text{Given}]$$

Taylor's Series;

$$y_1 = y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots$$

$$[x - x_0 = h]$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots$$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \cdots$$

$$y_{n+1} = y_n + hy_1' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Ques) Solve $\frac{dy}{dx} = x+y$ by Taylor's method starting from $x=1, y=0$ & $x=1.2$ with $h=0.1$.

$x_0 = 1, \ y_0 = 0, \ h = 0.1$

$$\frac{dy}{dx} = x+y = f(x,y)$$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$

$$y_2 = y(x_2) = y(1.2) = ?$$

Using $y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$y_0 = 0 \Rightarrow (y_0') = (y') = (y')_{x=x_0} = (x_0 + y_0)$$

$$= 1 + y_0 = 1 + 0 = 1$$

$$(y_0'') = (y'')_{x=x_0} = [1+y']_{x=x_0} = 1 + y'(x_0) = 1 + y'(1)$$

$$y' = x + y$$

$$(y')_0 = (y')_{x=x_0} = x_0 + y_0 = 1 + 0 = 1$$

$$y_0'' = (y'')_{x=x_0} = (1 + y_1)_{x=x_0} = 1 + y'(x_0) = 1 + 1 = 2$$

$$y_0''' = (y''')_{x=x_0} = y'''(x=x_0) = 2 = y''(x=1) = 2$$

$$y_0^{iv} = (y^{iv})_{x=x_0} = y''' = 2$$

$$y_1 = 0 + (0.1) + 1 + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \cdots$$

$$y_1 = 0.1103$$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$y(x_1) = y_1 = 0.1103$

$y_1' = (y')_1 = \left(\frac{dy}{dx}\right)_{x=x_1} = (x+y)_{x=1.1} = x_1 + y_1 = 1.1$

$y_1'' = (y'')_{x=1.1} = 1 + y'(1.1) = 1 + 1.2102 = 2.2102$

$y_1''' = (y''')_{x=x_1} = (y'')_{x=1.1} = 2.2102$

$= 0.1103 + (0.1)(1.2103) + \frac{(0.1)}{2!}(2.2102) + \frac{(0.1)^3}{2!}(2.2102) + \cdots$

$\Rightarrow y_1''' = 0.24276$

Numerical differentiation:

equal interval

NFI NBI CD

$\underbrace{\text{NFI} \quad \text{NBI}}_{\text{Numerical Difference}}$

NFI:

$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$

$y(x) = y_0 + u\Delta y_0 + \frac{(u^3-u)}{2!}\Delta^2 y_0 + c \cdots$

$u = \frac{x - x_0}{h} \Rightarrow x = x_0 + uh$

Differentiating w.r.t. x,

$h\frac{dy}{dx}\frac{du}{dx} = \frac{du}{dx}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0\right]$

$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 \cdots\right]$

$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[x\,\Delta^2 y_0 + (u-1)\Delta^3 y_0 \cdots\right]$

Finite difference Method

To solve O.D.E. of 2^{nd} order diff eq^n using B.V.P.

Consider II order ODE of the form

$$f(x, y, y', y'') = 0 \quad \text{--- (1)}$$

subject to Boundary cond^n

$$\boxed{y(x_0) = y_0 \ \& \ y(x_n) = y_n}$$

Working Rule

Step (1) find $h = \frac{x_n - x_0}{n}$ by choosing n

Thus the x values x_0, x_1, x_2 —— x_n

Aim → we need to find y values at x_1, x_2 —— x_n

Step (2)

Replace the derivatives y', y'' in (1)

$\frac{dy}{dx}$ $\frac{d^2y}{dx^2}$

$$y_i' = \frac{1}{2h}\left[y_{i+1} - y_{i-1}\right]$$

$$y_i'' = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right]$$

Replace y by y_i & x by x_i in (1)

$$\boxed{f(x, y, y', y'')_{=0}} \xrightarrow{\text{turns}} \boxed{f(x_i, y_i, y_i', y_i'')=0}$$

by solving $i = 1, 2, \ldots n-1$ in (2)

a simultane system $(n-1)$ L.eq^n. Thus using known method we solve the for $y_1, y_2, \ldots y_{n-1}$.

Problem → To solve the diff eq^n $y'' = x+y$ with B.V conds

$$\boxed{y(0) = y(1) = 0}$$

$x = x_0$ $\quad$ $x = x_n$

Sol.^{n}

$$y_i'' = x_i + y_i \quad , \quad x_0 = 0, \; x_n = 1$$

$$h = \frac{x_n - x_0}{n} = \frac{1-0}{4} \qquad \underline{n=4}$$

$$\boxed{h = 0.25} = \frac{1}{4}$$

$$x_1 = x_0 + h = 0 + 0.25 = 0 + 1/4 = 1/4$$

$$x_2 = 2/4 \;, \; x_3 = 3/4 \;, \; x_4 = 1$$

Hence we need to find y values at x_1 x_2 x_3

ie we find y_1 y_2 y_3

now replace y_i'' by finite diff approx

$$y_i'' = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] \quad \text{--- i.e. (1)}$$

$$\Rightarrow \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

$i = 1,2,3$

$i=1 \Rightarrow 16[y_2 - 2y_1 + y_0] = x_1 + y_1$

$i=2 \rightarrow 16[y_3 - 2y_2 + y_1] = x_2 + y_2$ $\quad$ (2)

$i=3 \Rightarrow 16[y_4 - 2y_3 + y_2] = x_3 + y_3$

Expt. No. $y_0 = 0$ Date .... Page No. .... $y_4 = 0$

x_0	x_1	x_2	x_3	x_4
0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4} = 1$
	y_1	y_2	y_3	

$y_0 = 0, \quad y_4 = 0$
$x_0 = 0 \quad x_4 = 0$] put all in set (2)

& solving all we get

$y_1 = y(1/4) = -0.03488$

$y_2 = y(2/4) = -0.05632$

$y_3 = y(3/4) = -0.05003$

(1) Q) To solve $y''(x) = y(x)$ B.V. Cond^n $y(0) = 0$
$y(1) = 1$

(2) Q) To solve $xy'' + y = 0$, $y(1) = 1$, $y(2) = 2$, $h = 0.2$

Sol^n

x_0	x_1	x_2	x_3	x_4
1	1.25	1.50	1.75	2.0
$y_0 = 1$	y_1	y_2	y_3	$y_4 = 2$

need to find

we get three eq^n

$-39y_1 + 20y_2 = -20$
$24y_1 - 97y_2 + 24y_3 = 0$
$28y_2 - 55y_3 = -56$

$\Rightarrow AY = B$ (Cramer's method) any method

$y_1 = 1.85$
$y_2 = 1.63$
$y_3 = 1.35$

Teacher's Signature :