13.12.2023

Unit : 3

Numerical Integration

$I = \int_a^b y\,dx$
- → Trapezoidal Rule
- → Simpson $1/3^{rd}$ Rule
- → Simpson $3/8^{th}$ Rule

$\underbrace{\underbrace{x_0, \; x_1}_{'h'}, \; x_2, \; \cdots \; x_n}_{}$

n - equal Intervals Width : 'h'

$x_0 = a$ $y_0 = f(x_0)$

$x_1 = a + h$ $y_1 = f(x_1)$

$x_2 = a + 2h$ $y_2 = f(x_2)$

$x_3 = a + 3h$ $y_3 = f(x_3)$

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$x_n = a + nh$ $y_n = f(x_n)$

$\therefore \quad b - a = nh$ (Here, $a = x_0$ and $b = x_n$)

$\Rightarrow h = \frac{b-a}{n}$

$I = \int_a^b f(x)\,dx$

→ **Trapezoidal Rule :**

$$\int_a^b f(x)\,dx = h\left[\frac{y_0 + y_n}{2} + y_1 + y_2 + \cdots + y_{n-1}\right]$$

→ **Simpson $\frac{1}{3}^{rd}$ Rule :**

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\overbrace{(\text{Odd terms})}^{y_1+y_3+y_5+\cdots} + 2\underbrace{(\text{Even Terms})}_{y_2+y_4+y_6+\cdots}\right]$$

→ **Simpson $\frac{3}{8}^{th}$ Rule :**

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3\overbrace{(\text{Remaining})}^{y_1+y_2+y_4+y_5+\cdots} + 2\underbrace{(\text{Multiple of } 3)}_{y_3+y_6+y_9+\cdots}\right]$$

Ques) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using

a) Trapezoidal Rule
b) Simpson $1/3^{rd}$ Rule
c) Simpson $3/8^{th}$ Rule.

As, $I = \int_a^b f(x)\,dx$.

Here, $a=0$; $b=1$; $f(x) = \frac{1}{1+x^2}$

NOTE : If not given, always take $n=6$

Let $n=6$

$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

		x	$f(x)$	$1/(1+x^2)$	y
	x_0	0	$f(x_0)$	$1/(1+0^2)$	1
$x_1 \rightarrow$	x_0+h	1/6	$f(x_1)$	$1/(1+1/6^2)$	0.9729
$x_2 \rightarrow$	x_0+2h	2/6	$f(x_2)$	$1/(1+2/6^2)$	0.9
$x_3 \rightarrow$	x_0+3h	3/6	$f(x_3)$	$1/(1+3/6^2)$	0.8
$x_4 \rightarrow$	x_0+4h	4/6	$f(x_4)$	$1/(1+4/6^2)$	0.6923
$x_5 \rightarrow$	x_0+5h	5/6	$f(x_5)$	$1/(1+5/6^2)$	0.5901
$x_6 \rightarrow$	x_0+6h	6/6	$f(x_6)$	$1/(1+6/6^2)$	0.5

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1	0.9729	0.9	0.8	0.6923	0.5901	0.5

15.12.23

Trapezoidal Rule :

$$\int_0^1 \frac{1}{1+x^2}\,dx = h\left[\frac{y_0+y_6}{2} + y_1+y_2+y_3+y_4+y_5\right]$$

$$= \frac{1}{6}\left[\frac{1+0.5}{2} + 0.9729 + 0.9 + 0.8 + 0.6923 + 0.5901\right]$$

$$= 0.7842$$

Simpson $\frac{1}{3}$ rd Rule :-

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[(1+0.5) + 4(0.9729+0.8+0.5901) + 2(0.9+0.6923\right.$$

$$= 0.7853$$

Simpson $\frac{3}{8}$ th Rule :-

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8} \times \frac{1}{6}\left[(1+0.5) + 2(0.8) + 3(0.9729+0.9+0.6923+0.5901)\right]$$

$$= 0.7853$$

Waddle's Rule :-

$$\int f(x)\,dx = \frac{3h}{10}\left[(y_0+5y_1) + (y_2+6y_3) + (y_4+5y_5) + y_6 \ldots\right]$$

Ques- Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Waddle's Rule and hen[ce]

obtain the approximate value of Integral dividing six equal terms.

By Waddle's Rule,

$$\int f(x)\,dx = \frac{3h}{10}\left[(y_0 + 5y_1) + (y_2 + 6y_3) + (y_4 + 5y_5 + y_6\right]$$

$$= \frac{3}{10} \times \frac{1}{6}\left[(1 + 5\times 0.9729) + (0.9 + 6\times 0.8) + (0.6923 + 5\times 0.5901) + 0.5\right]$$

$$= 0.785365$$

Now, $\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4} = 0.7853$

09.01.24 Unit 3 (Part-II)

Numerical Differentiation

1) **Euler's Method :** Initial Condition

$$\left[y(x_0) = y_0 \right]$$

$$\left\{ y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}) \right\}$$

$\underline{n=1}$; h = Width

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

Ques) Using Euler's method, find an approximate value of y, corresponding to $x = 0.1$. Given,

$$\frac{dy}{dx} = \frac{y-x}{y+x} \; ; \; y(0) = 1$$

Let $n = 5$, $\because x_n = x_5 \Rightarrow x_5 = 0.1$

$$h = \frac{0.1 - 0}{5} = 0.02$$

$x_0 = 0 \Rightarrow y_0 = 1$

$\Rightarrow y_1 \Rightarrow x_1 = x_0 + h = 0 + 0.02 = 0.02$

$y_2 \Rightarrow x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$y_3 \Rightarrow x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

$y_4 \Rightarrow x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

$y_5 \Rightarrow x_5 = 0.1$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

$$= 1 + 0.02 \left[\frac{1-0}{1+0} \right]$$

$$= 1 + 0.02 = 1.02$$

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$= 1.02 + (0.02) \cdot \left[\frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$= 1.03923$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 1.03923 + 0.02 \left[\frac{1.03923 - 0.04}{1.03923 + 0.04} \right]$$

$$= 1.057747462$$

$$y_4 = y_3 + 0.02 \left[\frac{y_3 - x_3}{y_3 + x_3} \right]$$

$$= 1.075652707$$

Also, $y_5 = 1.092883709 = y(0.1)$

Ques) Using Euler's Method find y at $x=1$; $\frac{dy}{dx} = x+y$; $y(0) = 1$

Let us consider, $n=5$.

$\therefore h = \frac{1-0}{5} = 0.2$

$\Rightarrow y_0 = 1 \Rightarrow x_0 = 0$

$y_1 \Rightarrow x_1 = 0+0.2 = 0.2$

$y_2 \Rightarrow x_2 = 0.2+0.2 = 0.4$

$y_3 \Rightarrow x_3 = 0.4+0.2 = 0.6$

$y_4 \Rightarrow x_4 = 0.6+0.2 = 0.8$

$y_5 \Rightarrow x_5 = 0.8+0.2 = 1$

$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.2[0+1]$

$= 1.2$

$\Rightarrow y_2 = 1.2 + 0.2[0.2 + 1.2]$
$= 1.48$

$\Rightarrow y_3 = 1.48 + 0.2[0.4 + 1.48] = 1.856$

$\Rightarrow y_4 = 1.856 + 0.2[0.6 + 1.856]$

$y_4 = 2.3472$

$\Rightarrow y_5 = 2.3472 + 0.2[0.8 + 2.3472]$

$y_5 = 2.97664$