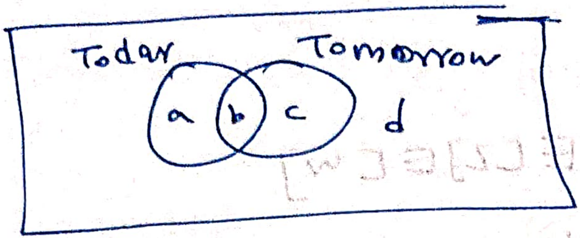
Q1)



$(a+b+c+d=1)$

Given: $a+b=0.6$

$b+c=0.5$

$d=0.3$

$a+b+c=1-d=0.7$

$c=0.1,\ b=0.4,\ a=0.2$

a) $1-d=0.7=70\%$.

b) $b=0.4=40\%$.

c) $a=0.2=20\%$.

d) $a+c=0.3=30\%$.

Q2) Sample space (Ω) =

$$\left\{\begin{array}{llllll}
(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & \boxed{(2,6)} \\
(3,1), & (3,2), & (3,3), & (3,4), & \boxed{(3,5)}, & (3,6) \\
(4,1), & (4,2), & (4,3), & \boxed{(4,4)}, & (4,5), & (4,6) \\
(5,1), & (5,2), & \boxed{(5,3)}, & (5,4), & (5,5), & (5,6) \\
(6,1), & \boxed{(6,2)}, & (6,3), & (6,4), & (6,5), & (6,6)
\end{array}\right\}$$

$P(X_1+X_2=8) = \dfrac{5}{36}$

Q3: P(all are girls | one child picked randomly is a girl)

$\underset{A}{\text{all are girls}}$, $\underset{B}{\text{one child picked randomly is a girl}}$

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

$P(A) = \left(\frac{1}{2}\right)^n,\quad P(B|A) = 1,\quad P(B) = 1/2$

$$P(A|B) = \frac{\left(\frac{1}{2}\right)^n}{\frac{1}{2}} = \left(\frac{1}{2}\right)^{n-1}$$

4) 1) $F_X(x) = p F_d(x) + (1-p) F_c(x)$

2) $f_X(x) = p f_d(x) + (1-p) f_c(x)$

3) $E[X] = p E[X_d] + (1-p) E[X_c]$

$E[X^2] = p E[X_d^2] + (1-p) E[X_c^2]$

4) $Var(X) = p E(X_d^2) + (1-p) E(X_c^2) - p^2 E^2(X_d) - (1-p)^2 E^2(X_c) + 2p(1-p) E(X_c) E(X_d)$

$Var(X) = p E(X_d^2) + (1-p) E(X_c^2) - p^2 E^2(X_d) - (1-p)^2 E^2(X_c) + p(1-p) [(E(X_c) - E(X_d))^2 - (E(X_c))^2 - (E(X_d))^2]$

$= p E(X_d^2) + (1-p) E(X_c^2) - p^2 E^2(X_d) - (1-p)^2 E^2(X_c) + p(1-p) (E(X_c) - E(X_d))^2 - (p-p^2)(E(X_c))^2 - (p-p^2)(E(X_d))^2$

$= p(E(X_d^2) - (E(X_d))^2) + (1-p)(E(X_c^2) - (E(X_c))^2) + p(1-p)(E(X_c) - E(X_d))^2$

$$\mathrm{Var}(X) = p\,\mathrm{Var}(X_d) + (1-p)\,\mathrm{Var}(X_c) + p(1-p)\left(E(X_c) - E(X_d)\right)^2$$

Q5:

$Cov(Z, W) = E[ZW] - E[Z]E[W]$

$X, Y \rightarrow$ independent random variables $\sim N(0,1)$

a) $Z = 1 + X + XY^2, \quad W = 1 + X$

$Cov(Z, W) = E(1 + X^2 + 2X + XY^2 + X^2Y^2)$
$\qquad - E((1+X)E((1+X+XY^2))$

$\mu = 0, \ \sigma^2 = 1$

$E[X] = E[Y] = 0, \ E[X^2] - (E[X])^2 = 1$

$E[X^2] = 1 = E[Y^2]$

$Cov(Z, W) = (1 + 1 + 2\cdot 0 + 0\cdot 1 + 1\cdot 1) - (1+0)(1+0+0\cdot 1)$

$= 3 - 1 = 2$

6) No.

Company A — $p = 0.2$

B — $p = 0.2$

C — $p = 0.2$

D — $p = 0.2$

$P(\text{atleast one offer}) = 1 - P(\text{no offers})$

$= 1 - (1 - 0.2)^4$

$= 1 - 0.8^4 = 1 - 0.4096$

$= 0.5904$

$= 59.04\%$.

7) $P(X > 120) = 1 - \sum_{i=1}^{120} P(X = i)$

$p = 0.1$

$$P(X>120) = 1 - \sum_{i=1}^{120} {}^{1000}C_i \, (0.1)^i (0.9)^{1000-i}$$

$$= 1 - \left({}^{1000}C_1 (0.1)^1 (0.9)^{999} + {}^{1000}C_2 (0.1)^2 (0.9)^{998} \cdots + {}^{1000}C_{120} (0.1)^{120} (0.9)^{880}\right)$$

8) $\underline{X_i} \Rightarrow 0 \to \frac{1}{4}$

$1 \to \frac{1}{2}$

$2 \to \frac{1}{4}$

$$E[X_i] = \frac{1}{2} + \frac{1}{2} = 1$$

$$E[X_i^2] = \cancel{\frac{1+2}{4+8}} \; \frac{1}{2} + 1 = \frac{3}{2}$$

$$Y = X_1 + X_2 \cdots + X_{64}$$

$$\cancel{E[Y] = 64 \times 1 = 64}$$

$$\cancel{E[Y^2] = \frac{3}{2} \times 64 = 96}$$

$$\cancel{\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 =}$$

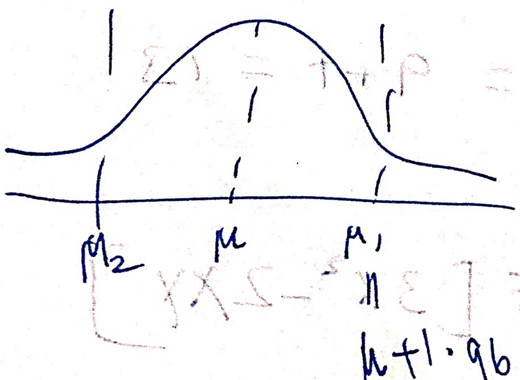
$$\mathrm{Var}(Y) = \sum_{i=1}^{64} \mathrm{Var}(X_i)$$

$$= \sum E[X_i^2] - (E[X_i])^2$$

$$= \frac{3}{2} \times 64 - 64 = 32$$

$$\sigma(Y) = \sqrt{32} \simeq 4\sqrt{2} \simeq 5.64$$

$$\mu(Y) = \sum \mu(X_i) = 64 \times 1 = 64$$



$$64 + 1.96 \times 5.64 \simeq \boxed{75.05}$$

9) $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1, & \rho \\ \rho, & 1 \end{pmatrix}\right)$

$\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} Cov(X,X), & cov(X,Y) \\ cov(Y,X), & cov(Y,X) \end{pmatrix}\right)$

$\mu_x = 0, \mu_y = 0$

2) $Var(X) = Cov(X,X) = 1$

$Var(Y) = Cov(Y,Y) = 1$

3) $cov(X,Y) = \rho$

10) $E[X|Y=y] = E[X] + \frac{cov(X,Y)}{Var(Y)}(y - E[Y])$

$Var(X|Y=y) = Var(X) - \frac{cov^2(X,Y)}{Var(Y)}$

$E[X|Y=y] = 1 + \frac{1}{3}(y-2) = \frac{y+1}{3}$

$Var(X|Y=y) = 4 - \frac{1}{3} = \frac{11}{3}$

$X|Y=y \sim N\left(\frac{y+1}{3}, \frac{11}{3}\right)$

11) $Z = aX + bY \rightarrow Z$ is also $\sim N$ when $X, Y \sim N$

$E[Z] = aE[X] + bE[Y]$

$Var(Z) = a^2 var(X) + b^2 var(Y) + 2ab\, cov(X,Y)$

$a = 3, b = -2, E[X] = E[Y] = 0, var(X) = 1, var(Y) = 4,$

$cov(X,Y) = 1$

$E[Z] = 0$

$Var(Z) = 9 \times 1 + 4 \times 4 - 12 \times 1 = 9 + 4 = 13$

$Z \sim N(0, 13)$

$cov(Z,X) = E[(Z)(X)] = E[3X^2 - 2XY]$

$\text{Cov}(Z, X) = 3E[X^2] + 2E[XY]$

$\text{Cov}(X,Y) = E[XY] = 1 \quad (E[X] = E[Y] = 0)$

$\text{cov}(Z,X) = 3(1+0) - 2 = 1$

$\text{Corr}(Z,X) = \dfrac{1}{\sigma_Z \sigma_X}$

$\sigma_Z = \sqrt{\text{Var}(Z)} = \sqrt{13}$

$\sigma_X = \sqrt{\text{Var}(X)} = 1$

$\text{Corr}(Z,X) = \dfrac{1}{\sqrt{13}}$

12) $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 5 \end{pmatrix} \right)$

$E[X \mid Y=y, Z=z] = \Sigma_{X,(YZ)} \left(\Sigma_{(YZ),(YZ)} \right)^{-1} \begin{pmatrix} y \\ z \end{pmatrix}$ [From internet]

$\Sigma_{X(YZ)} = \begin{pmatrix} 2 & 1 \end{pmatrix}$

$\Sigma_{(YZ),(YZ)} = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$

$\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}^{-1} = \dfrac{1}{14} \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{1}{14} \\ \frac{1}{14} & \frac{3}{14} \end{bmatrix}$

$E[X \mid \cancel{y} Y=y, Z=z] = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{14} & \frac{1}{14} \\ \frac{1}{14} & \frac{3}{14} \end{bmatrix}_{2\times 2} \begin{bmatrix} y \\ z \end{bmatrix}_{2\times 1}$

$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{5y+z}{14} \\ \frac{y+3z}{14} \end{bmatrix}$

$= \dfrac{10y + 2z + y + 3z}{14} = \dfrac{11y + 5z}{14}$

$$\mathrm{var}(X \mid Y=y, Z=z) = \mathrm{var}(X) - \Sigma_{X,(YZ)} \left(\Sigma_{(YZ),(YZ)}\right)^{-1} \Sigma_{(YZ),X}$$

$$\Sigma_{(YZ),X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathrm{var}(X) = 4$$

$$\mathrm{var}(X \mid Y=y, Z=z) = 4 - \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{14} & -\frac{1}{14} \\ -\frac{1}{14} & \frac{3}{14} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 4 - \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{11}{14} \\ -\frac{5}{14} \end{bmatrix}$$

$$= 4 - \frac{27}{14} = \frac{29}{14}$$

$$X \mid Y=y, Z=z \sim N\left(\frac{11y + 5z}{14}, \frac{29}{14}\right)$$