Submitted by-Aditya Gautam

Roll No-12

MSc 3rd semester

Date of Experiment-07/10/2020

Date of Submission-15/10/2020

**Experiment No-02**

**Topic-** DRAWING OF A RANDOM SAMPLE FROM A TRIVARIATE NORMAL DISTRIBUTION

**Problem-** Draw a random sample of size 15 from tri variate normal population

$N_3\left(\underset{\sim}{\mu}, \sum\right)$, where,

$$\underset{\sim}{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

$$\sum = \begin{pmatrix} 1 & 0.8 & -0.4 \\ 0.8 & 1 & -0.56 \\ -0.4 & -0.56 & 4 \end{pmatrix}$$

**Theory-**

A procedure of drawing a random sample from $N_3\left(\underset{\sim}{\mu}, \Sigma\right)$ is as follows-

Let, $\underset{\sim}{X} \sim N_3\left(\underset{\sim}{\mu}, \Sigma\right)$, where $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$

<u>Step 1</u>- We write the marginal distribution of $X_1$ which is $X_1 \sim N_1\left(\underset{\sim}{\mu_1}, \sigma_{11}\right)$

<u>Step 2</u>- We then draw a random number from U(0,1) say $R_1$.

<u>Step 3</u>- We consider the c.d.f. of $X_1$ given by

$$F(x_1) = P(X_1 \le x_1)$$

and set it equal to $R_1$ ,i.e.,

$$\begin{aligned} & F(x_1) = R_1 \\ & \Rightarrow x_1 = F^{-1}(R_1) \\ & \Rightarrow x_1 = \phi(R_1) \end{aligned}$$

Where $\phi(.)$ is the Quantile function of $N_1\left(\underset{\sim}{\mu}, \sigma_{11}\right)$

<u>Step 4</u>- The $x_1$ thus obtained is the random value of $X_1$

<u>Step 5</u>- We write down the conditional distribution of $X_2$ given $X_1 = x_1$ where

$$X_2 \mid X_1 = x_1 \sim N_1(\mu_{2.1}, \sigma_{2.1})$$

Where, $\mu_{2.1} = \mu_{X_2} \mid X_1 = x_1 = \mu_2 + \rho_{12}\sqrt{\dfrac{\sigma_{22}}{\sigma_{11}}}(x_1 - \mu_1)$ and $\sigma_{2.1} = \sqrt{\sigma_{22}(1-\rho_{12}^2)}$

<u>Step 6</u>- Since the conditional distribution of $X_2$ given $X_1 = x_1$ again univariate normal. We repeat the steps 1, 2, 3 in order to obtain $X_2$. Which is a random value of $X_2 \mid X_1 = x_1$. In the course of repeating the steps, we equate $F(x_2)$ to $R_2$ so that $x_2 = \phi_1(R_2)$ where $F_1(.)$ is the c.d.f., $\phi(.)$ is the Quantile function of $N_2\left(\mu_{2.1}, \sigma_{2.1}^2\right)$ and $R_2 \sim U(0,1)$

Step 7- After obtaining $x_1$ and $x_2$ , we now write down the conditional distribution of $X_3 \mid X_1 = x_1$ , $X_2 = x_2$. For this we partion $\underset{\sim}{X}, \underset{\sim}{\mu}$ and $\sum$ as follows-

$$\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_3 \end{pmatrix} = \begin{pmatrix} \underset{\sim}{X}^{(1)} \\ \cdots \\ \underset{\sim}{X}^{(2)} \end{pmatrix}, \quad \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \underset{\sim}{\mu}^{(1)} \\ \cdots \\ \underset{\sim}{\mu}^{(2)} \end{pmatrix}$$

$$\sum = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \vdots & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \vdots & \sigma_{23} \\ \cdots & \cdots & \vdots & \cdots \\ \sigma_{31} & \sigma_{32} & \vdots & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix}$$

Where, $\sum_{11} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ , $\sum_{12} = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix}$ , $\sum_{21} = \begin{pmatrix} \sigma_{31} & \sigma_{32} \end{pmatrix}$ , $\sum_{22} = \left(\sigma_{33}\right)$

Now , $X_3 \mid X_1 = x_1, X_2 = x_2 = \underset{\sim}{X}^{(2)} \mid \underset{\sim}{X}^{(1)} = \underset{\sim}{x}^{(1)} \sim N\left( \underset{\sim}{\mu}_{\underset{\sim}{X}^{(2)}} \mid \underset{\sim}{X}^{(1)} = \underset{\sim}{x}^{(1)}, \sigma_{\underset{\sim}{X}^{(2)}} \mid \underset{\sim}{X}^{(1)} = \underset{\sim}{x}^{(1)} \right)$

Where,

$$\underset{\sim}{\mu}_{\underset{\sim}{X}^{(2)}} \mid \underset{\sim}{X}^{(1)} = \underset{\sim}{x}^{(1)} = \mu_{3.12} = \underset{\sim}{\mu}^{(2)} + \sum_{21} \sum_{11}^{-1} \left( \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \right) = \underset{\sim}{\mu}^{(2)} + \sum_{21} \sum_{11}^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

$$\sigma_{\underset{\sim}{X}^{(2)}} \mid \underset{\sim}{X}^{(1)} = \underset{\sim}{x}^{(1)} = \sigma_{3.12} = \sqrt{\sum_{22} - \sum_{21} \sum_{11}^{-1} \sum_{12}}$$

Step 8- Since the conditional distribution of $X_3 \mid X_1 = x_1, X_2 = x_2$ in univariate normal, we repeat the steps 1, 2, 3 in order to obtain $x_3$ , a random value of $X_3$

Step 9- Step 1 to step 7 are repeated a times in case a random sample of size n is required. The triplet of n values $\left(x_1, x_2, x_3\right)$ thus obtained constitute a random sample from a trivariate normal population $N_3\left(\underset{\sim}{\mu}, \sum\right)$

**Calculation-**

The R-program for obtaining a solution to the given problem is as follows-

```
m1 = 1; m2 = 2; m3 = 3; s11 = 1; s12 = 0.8; s13 = -0.4; s22 = 1; s23 = -0.56; s33 = 2

rho12 = s12/(s11*s22)

rho12

r1 = runif(15,0,1)

x1 = mat.or.vec(15,1)

for(i in 1:15){

x1[i] = qnorm(r1[i],m1,s11)}

x1

m21 = m2+(rho12*(s22/s11)*(x1-m1))

sd21 = sqrt((s22^2)*(1-(rho12^2)))

sd21

r2 = runif(15,0,1)

x2 = mat.or.vec(15,1)

for(i in 1:15){

x2[i] = qnorm(r2[i],m21[i],sd21)}

x2

mu_2 = m3

sig_21 = array(c(s13,s23),dim=c(1,2))

sig_21

sig_11 = array(c(s11^2,s12,s12,s22^2),dim = c(2,2))

sig_11

sig_11_inv = solve(sig_11)

sig_11_inv
```

```
m3_12 = mat.or.vec(15,1)

for(i in 1:15){

m3_12[i] = mu_2+(sig_21%*%sig_11_inv%*%array(c(x1[i]-m1,x2[i]-m2),dim = c(2,1)))}

m3_12

sig_22 = s33^2

s3_12 = sqrt(sig_22-(sig_21%*%sig_11_inv%*%t(sig_21)))

s3_12

r3 = runif(15,0,1)

x3 = mat.or.vec(15,1)

for(i in 1:15){

x3[i] = qnorm(r3[i],m3_12[i],s3_12)}

x3
```

**Conclusion-**

The random sample of size 15 from the Tri variate Normal Distribution with given parameters $\underset{\sim}{\mu}$ and $\sum$ is given by-

(-0.2907752, 1.5177635, 4.1712416), ( 0.4141062, 1.2762563, 2.5851419), ( 1.3071449, 3.3168894, -0.6844437), ( 2.1878500, 2.1295468, 4.5558198), ( 1.2802354, 2.1658699, 4.7783961), ( -0.1951040, 1.1474077, -1.0225458), (1.2668237, 3.0180022, 4.8959129), (3.5035905, 3.9220359, 5.3989565), (2.5346543, 2.8554809, 0.5191732), (0.7826550, 1.6176832, 3.2203853), (-0.3998525, 0.9784032, 3.2361431), (1.1307895, 1.8270080, 3.9014278), (1.8492856, 3.0072154, 6.7756629), (1.4291538, 3.1014745, 3.6869538), (1.6860085, 2.4713094, 6.9431049)