Roll No-12

M.sc. 3rd semester

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Experiment No -11

Topic-- Fitting of Auto Regression (AR) series to time series data.

Problem –- Fit a auto regressive series of order 2 from the following data.

$$u_{t+2} + a u_{t+1} + b u_t = e_{t+2}$$

t	u_t
1	7
2	6
3	-6
4	-4
5	3
6	-4
7	-5
8	-8
9	10
10	10
11	6
12	-4
13	-4
14	-7
15	-2
16	6
17	17
18	24
19	13
20	1
21	1

Also find the period of the auto regressive series of order 2.

Theory and Calculation-

Fitting of AR(2)

The auto regressive series of order k is given by –

$$u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + ... + \phi_k u_{t-k} + \in_t \;\; ; \; where \; \in_t \sim N(0,\sigma^2)$$

Now, taking the deviation of the observations u_t from their mean $\bar{u}$ and replacing the u_t's with these deviations, the series becomes

$$u'_t = \phi_1 u'_{t-1} + \phi_2 u'_{t-2} + ... + \phi_k u'_{t-k} + \in_t$$

When, k=2, the AR series of order k reduces to

$$u'_t = \phi_1 u'_{t-1} + \phi_2 u'_{t-2} + \in_t \quad \text{-------(1)}$$

Multiplying eq^n by u'_{t-k} , we get

$$u'_t u'_{t-k} = \phi_1 u'_{t-1} u'_{t-k} + \phi_2 u'_{t-2} u'_{t-k} + \in_t u'_{t-k} \quad \text{---------(2)}$$

Taking expectation on both sides of equation (2) we get-

$$E(u'_t u'_{t-k}) = \phi_1 E(u'_{t-1} u'_{t-k}) + \phi_2 E(u'_{t-2} u'_{t-k}) + E(\in_t u'_{t-k})$$

$$c_k = \phi_1 c_{k-1} + \phi_2 c_{k-2} \quad \text{-------(3)}$$

Where, c_k represents the auto-covariance function at lag k. Dividing both sides by $c_0 = E(u_t{}^2)$, we get from (3)

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad \text{--------(4)}$$

Where ρ_k represents the autocorrelation at lag k.

[When k=3, the AR reduces to

$$u'_t = \phi_1 u'_{t-1} + \phi_2 u'_{t-2} + \phi_3 u'_{t-3} + \in_t \quad \text{------(5)}$$

Multiplying both sides of eqn (5) by u'_{t-k} and taking expectation, we get—

$$E(u'_t u'_{t-k}) = \phi_1 E(u'_{t-1} u'_{t-k}) + \phi_2 E(u'_{t-2} u'_{t-k}) + \phi_3 E(u'_{t-3} u'_{t-k}) + E(\in_t u'_{t-k})$$

$$c_k = \phi_1 c_{k-1} + \phi_2 c_{k-2} + \phi_3 c_{k-3} \quad \text{-----(6)}$$

Dividing both sides of eq^n (6) by c_0 , we get—

$$\frac{c_k}{c_0} = \phi_1 \frac{c_{k-1}}{c_0} + \phi_2 \frac{c_{k-2}}{c_0} + \phi_3 \frac{c_{k-3}}{c_0}$$

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \phi_3 \rho_{k-3} \text{ ------(7)} \qquad]$$

In practice, ρ_k is determined by the sample counter part of ρ_k i.e. r_k . Putting k = 1 and k = 2 in eq^n (4) we get-

$$\begin{aligned}
&\rho_1 = \phi_1 \rho_{1-1} + \phi_2 \rho_{1-2} \\
=> &\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1} \\
=> &\rho_1 = \phi_1 + \phi_2 \rho_1 \qquad \textit{Since} \quad \rho_0 = 1 \quad \& \quad \rho_{-1} = \rho_1
\end{aligned}$$

And

$$\begin{aligned}
&\rho_2 = \phi_1 \rho_{2-1} + \phi_2 \rho_{2-2} \\
=> &\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0 \\
=> &\rho_2 = \phi_1 \rho_1 + \phi_2 \qquad \textit{Since} \quad \rho_0 = 1
\end{aligned}$$

Replacing ρ_1 & ρ_2 by r_1 and r_2 respectively, we get -

$$r_1 = \phi_1 + \phi_2 r_1$$

$$r_2 = \phi_1 r_1 + \phi_2$$

$$=> \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 & r_1 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$=> \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & r_1 \\ r_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \quad \text{-------(8)}$$

Solving (8) , we get the estimate of the coefficients ϕ_i ‘s . Therefore the fitted AR series of order 2 is-

$$\widehat{u}_t' = \widehat{\phi}_1 \widehat{u}_{t-1}' + \widehat{\phi}_2 \widehat{u}_{t-2}'$$

The period of the fitted second order AR series is given by-

$$\text{period } = \frac{2\pi}{\theta} \quad , \quad where \quad \theta = \cos\left(\frac{\widehat{\phi}_1}{2\sqrt{|\widehat{\phi}_2|}} \right)$$

Finally, we have -

$$\widehat{u}_t' = \widehat{\phi}_1 \widehat{u}_{t-1}' + \widehat{\phi}_2 \widehat{u}_{t-2}'$$

$$\therefore \quad \widehat{u}_t = \widehat{u}_t' + \bar{u}$$

Calculation-

we construct the following table-

t	ut
1	0
2	0
3	4.66925
4	-4.96459
5	2.265444
6	7.23029
7	-1.89507
8	0.499542
9	-1.56221
10	14.96881
11	6.647786
12	3.282412
13	-3.28191

14	1.340886
15	-1.18315
16	4.41041
17	8.829762
18	14.38631
19	15.19064
20	2.699908
21	-2.31114

Programming in R

```
library(readxl)

df_1 = read_excel("Autocorrelation_3.xlsx")

View(df_1)

ut=c(7,6,-6,-4,3,-4,-5,-8,10,10,6,-4,-4,-7,-2,6,17,24,13,1,1)

Mean=mean(ut)

Mean

N=length(ut)

N

#Calculation of sample auto-covariance and auto-correlation at lag1 and lag2

c0=var(ut)*((N-1)/N)

c0

c_k=mat.or.vec(2,1)

r_k=mat.or.vec(2,1)

for(i in 1:2){

  c_k[i]=(sum((ut[1:(N-i)]-Mean)*(ut[(1+i):N]-Mean)))/N

  r_k[i]=c_k[i]/c0

}
```

```
c_k

r_k

#auto-covariance at lag1 and lag2

cov1=c_k[1]

cov2=c_k[2]

#auto-correlation at lag1 and lag2

r1=r_k[1]

r2=r_k[2]

#Calculation of coefficients phai1 and phai2

A=array(c(r1,r2),dim=c(2,1))

B=array(c(1,r1,r1,1),dim=c(2,2))

coeff=solve(B)%*%A

coeff

phai_1=coeff[1,1]

phai_1

phai_2=coeff[2,1]

phai_2

#Calculation of period of AR(2) process

theta=acos(phai_1/(2*sqrt(abs(phai_2))))

theta

pi=22/7

period=(2*pi)/theta

period

#Here, in R propgram pi represents 22/7 value

#Calculation of 1st and 2nd term of our fitted AR(2) model
```

```
first_term=phai_1*ut[2:20]

second_term=phai_2*ut[1:19]

ut_des_est=first_term+second_term

ut_des_est

ut_est=ut_des_est+Mean

ut_est

df_a = append(0,ut_est)

df_2 = append(0,df_a)

View(df_2)

Data = cbind(df_1,df_2)

Data

#using ggplot to plot the graph

library(ggplot2)

ggp = ggplot(NULL, mapping = aes(t)) +

    geom_point(data = Data, mapping = aes(y=ut), col = "black") + geom_line(data = Data,
mapping = aes(y=ut), col = "orange", size = 1) +

    geom_point(data = Data, mapping = aes(y=df_2), col = "blue") + geom_line(data = Data,
mapping = aes(y=df_2), col = "green", size = 1) +

 labs(

  title = paste("Plotting of observed ut and estimated ut against t"),

  subtitle = paste("orange_line=observed ut and green_line=estimated ut"),

  caption = "Data from Model",

  x = "t",

  y = "observed_ut_and_estimated_ut"

  )

 ggp
```

Graph plotting

We plot the graph of observed and estimated values for the second order series considering t along X-axis and u_t & $\hat{u}_t$ along Y-axis by Using ggplot2-

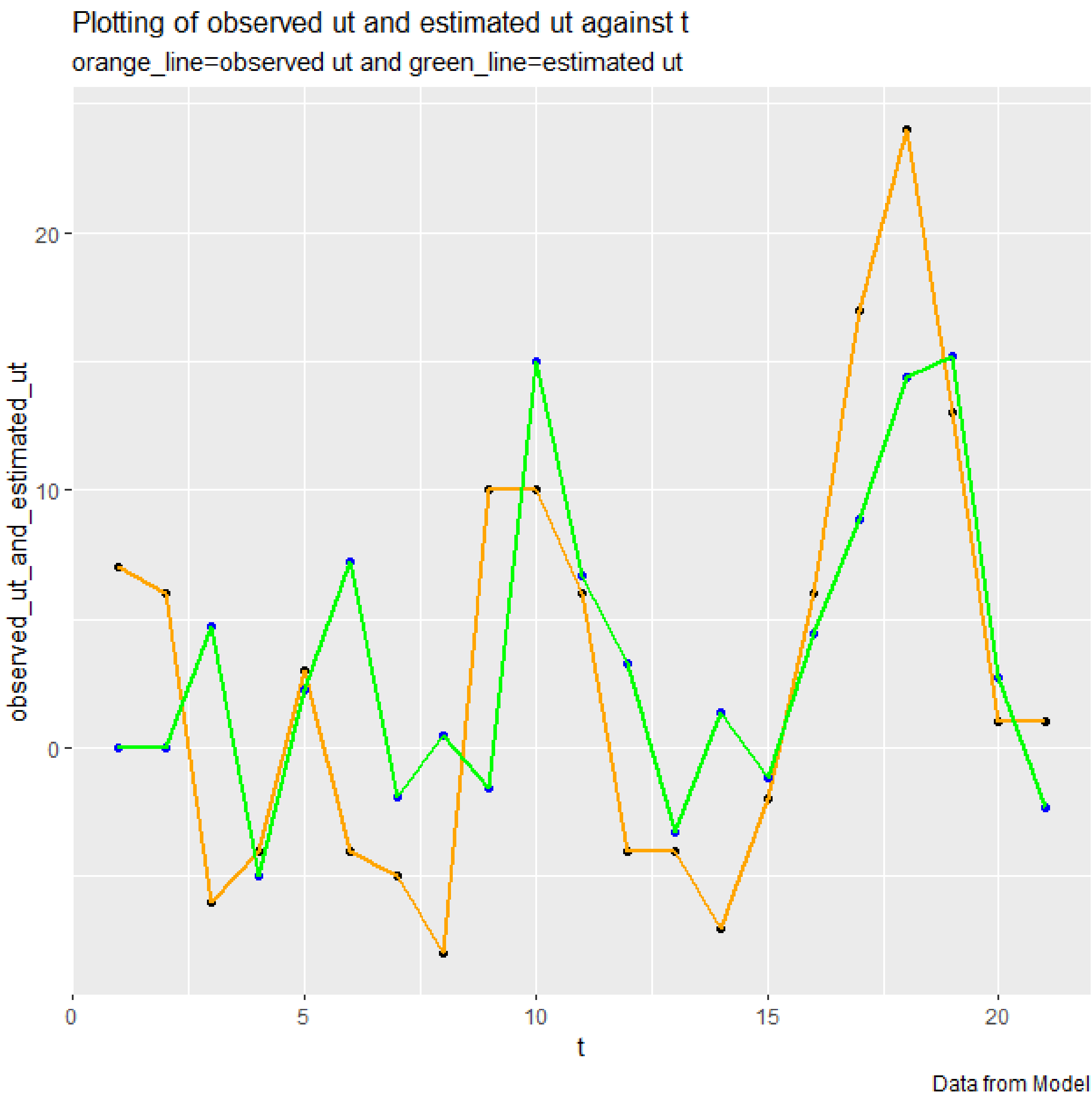
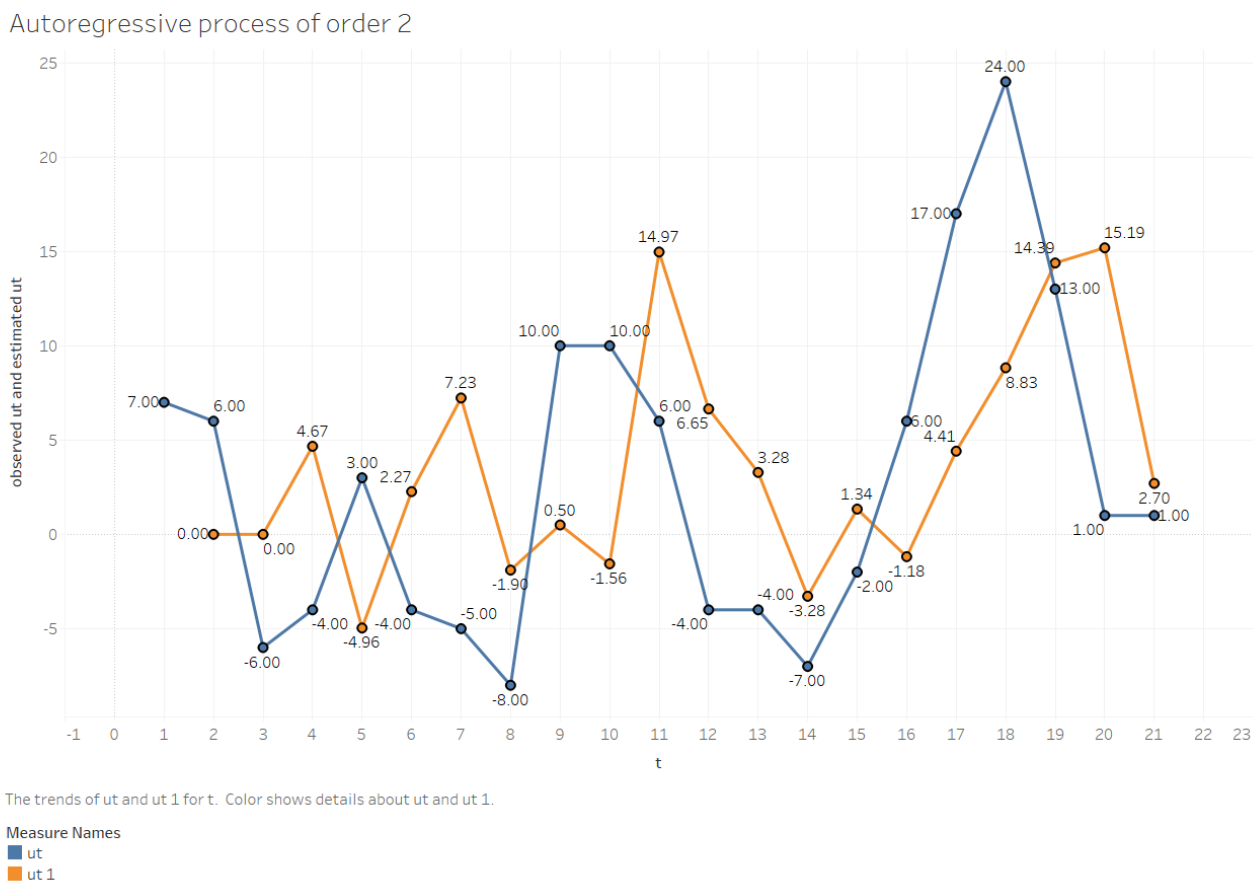


Tableau-



The trends of ut and ut 1 for t. Color shows details about ut and ut 1.

Conclusion-

The fitted AR(2) is-

$$\widehat{u}_t = 0.8413435\, u_{t-1} - 0.4622792\, u_{t-2}$$

Hence, The period of the fitted AR series of order 2 is 6.955614.