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M.sc. 3rd semester

Date of Assignment-03/11/2020

Date of Submission-10/11/2020

Experiment No -03

Topic- Tracing of Power curve for testing variance of a Normal Population**.**

Problem – A random sample of size 16 is drawn from a N$(5,\sigma^2)$, where σ^2 is unknown.Draw the Power curves for testing $H_0 : \sigma^2 = 3$ against

i) H_1: $\sigma^2 > 3$ ii) H_1: $\sigma^2 < 3$ iii) H_1: $\sigma^2 \neq 3$

Given that the size of the test in each of the cases is $\alpha = 0.05$

Theory and Calculation-

Using Neyman's pearson fundamental lemma, the critical region is given by-

$$W = \{\underset{\sim}{x} \ : \frac{L(\underset{\sim}{x},\theta_1)}{L(\underset{\sim}{x},\theta_0)} \geq k\}$$

Here, $f(x,\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $\quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$

$$\therefore \ \frac{L(\underset{\sim}{x},\sigma_1^2)}{L(\underset{\sim}{x},\sigma_0^2)} = \frac{\prod_{i=1}^{n} f(x_i,\mu\ ,\sigma_1^2)}{\prod_{i=1}^{n} f(x_i,\mu\ ,\sigma_0^2)} \geq k$$

$$\Rightarrow \frac{\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma_1^2}\sum_{i1}^{n}(x_i-\mu)^2}}{\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)^2}} \geq k$$

$$\Rightarrow \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)} \geq k$$

$$\Rightarrow n\log\left(\frac{\sigma_0}{\sigma_1}\right) + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right) \geq \log k$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\left(\frac{\sigma_1^2-\sigma_0^2}{\sigma_0^2\sigma_1^2}\right) \geq \log k - n\log\left(\frac{\sigma_0}{\sigma_1}\right)$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu)^2\left(\sigma_1^2-\sigma_0^2\right) \geq 2\sigma_0^2\sigma_1^2[\log k - n\log\left(\frac{\sigma_0}{\sigma_1}\right)]$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu)^2\left(\sigma_1^2-\sigma_0^2\right) \geq k' \quad \text{(say)}$$

Where,

$$k' = 2\sigma_0{}^2\sigma_1{}^2[\log k - n\log\left(\frac{\sigma_0}{\sigma_1}\right)]$$

Case 1:

When $\sigma_1{}^2 > \sigma_0{}^2$, then the C.R. is,

$$W_1 = \{\underset{\sim}{x} : \sum_{i=1}^{n}(x_i - \mu)^2 \geq k_1\}$$

Case 2:

When $\sigma_1{}^2 < \sigma_0{}^2$, then the C.R. is,

$$W_2 = \{\underset{\sim}{x} : \sum_{i=1}^{n}(x_i - \mu)^2 < k_2\}$$

(i) The C.R. for testing $H_0 : \sigma^2 = 3$ against $H_1 : \sigma^2 > 3$ is given by

$$W_1 = \{\underset{\sim}{x} : \sum_{i=1}^{16}(x_i - 5)^2 > k_1\}$$

where k_1 is a constant to be determined such that the size of the C.R. is equal to α ,

i.e., $P(x \in W_1 \mid H_0) = \alpha$

$$\Rightarrow P\{\sum_{i=1}^{16}(x_i - 5)^2 > k_1 \mid H_0\} = .05$$

$$\Rightarrow P\{\frac{\sum_{i=1}^{16}(x_i - 5)^2}{\sigma_0{}^2} > \frac{k_1}{\sigma_0{}^2}\} = .05$$

$$\Rightarrow P\{\chi_{(n)}{}^2 > \frac{k_1}{\sigma_0{}^2}\} = .05$$

$$\Rightarrow 1 - P\{\chi_{(n)}{}^2 > \frac{k_1}{\sigma_0{}^2}\} = 1 - .05$$

$$\Rightarrow P\{\chi_{(n)}{}^2 < \frac{k_1}{\sigma_0{}^2}\} = .95$$

To find the value of k_1,we use the following R-command :

a = qchisq(0.95,16,0) (0 is the non-centrality parameter)

This gives us the value $\frac{k_1}{\sigma_0{}^2} = 26.29623$

$$\therefore k_1 = 26.29623 \times \sigma_0{}^2 = 26.29623 \times 3 = 78.88869$$

Thus the C.R. is given by,

$$W_1 = \{\underset{\sim}{x} : \sum_{i=1}^{16} (x_i - 5)^2 > 78.88869\}$$

Now, the Power of the test is given by,

Power=1-β

=P{Reject H_0|H_1 is true}

$$= P(x \in W_1 \mid H_1)$$

$$= P\{\sum_{i=1}^{16} (x_i - 5)^2 > 78.88869 \mid H_1\}$$

$$= P\{\frac{\sum_{i=1}^{16} (x_i - 5)^2}{\sigma_1{}^2} > \frac{78.88869}{\sigma_1{}^2}\}$$

$$= P\{\chi_{(n)}^{2} > \frac{78.88869}{\sigma_1^{2}}\}$$

$$= 1 - P\{\chi_{(n)}^{2} < \frac{78.88869}{\sigma_1^{2}}\}$$

Where, $P\{\chi_{(n)}^{2} < \frac{78.88869}{\sigma_1^{2}}\}$ is the distribution function of the chi-square distribution with ‘n’ d.f.

Now to trace the power curve we consider different trial values of $\sigma_1^{2} > 3$ and construct the following table using R-Programming.

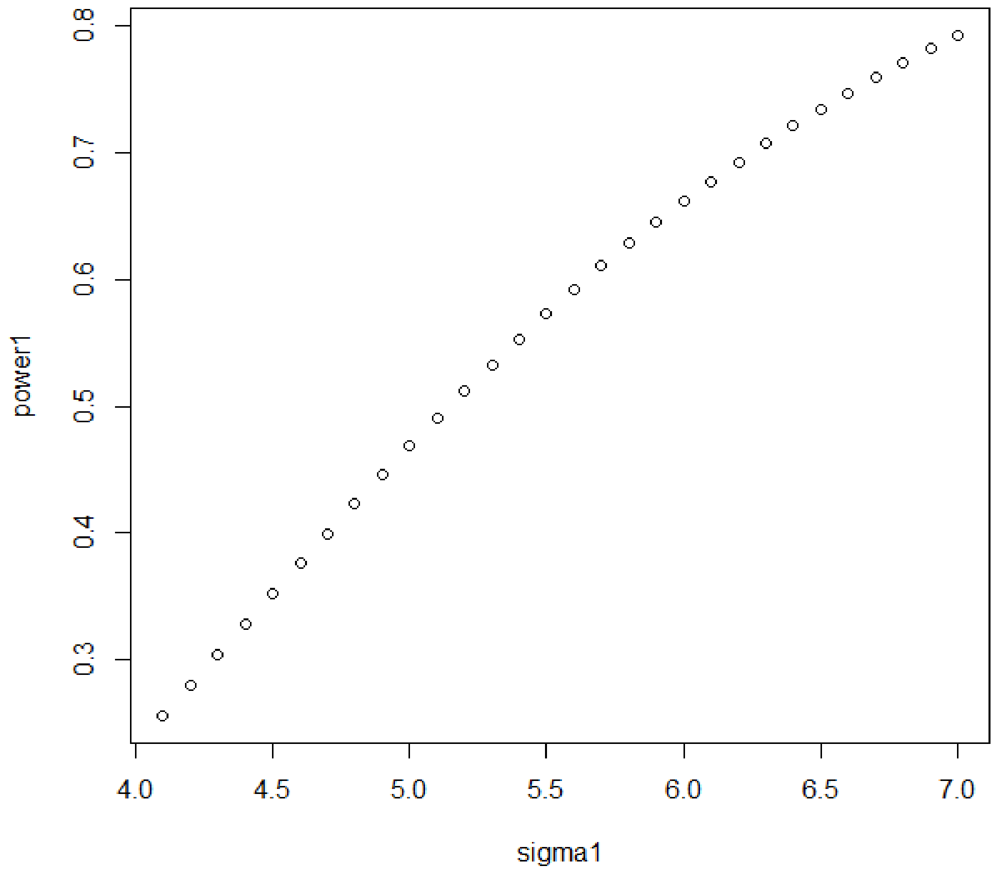
<u>Programming in R for case 1</u>

```
a = qchisq(0.95,16,0)

a

var = 3

k1 = var*a

k1

sigma1=c(3.1,3.2,3.3,3.4,3.5,3.6,3.7,3.8,3.9,4.0,4.1,4.2,4.3,4.4,4.5,4.6,4.7,4.8,4.9,5.0,5.1,5.2,5.3,
5.4,5.5,5.6,5.7,5.8,5.9,6.0)

sigma11 = k1/sigma1

power = mat.or.vec(30,1)

power1 = mat.or.vec(30,1)

for(i in 1:30){

power[i] = pchisq(sigma11[i],16,0)

power1[i] = 1-power[i]}

power1

plot(sigma1,power1)
```

TABLE: 1

Trial values of σ_1^2 (>3)	**Power** $= 1 - P\{\chi_{(n)}^2 < \frac{78.88869}{\sigma_1^2}\}$
4.1	0.2563569
4.2	0.2800755
4.3	0.3040172
4.4	0.3280564
4.5	0.3520771
4.6	0.3759738
4.7	0.3996513
4.8	0.4230254
4.9	0.4460222
5.0	0.4685780
5.1	0.4906386
5.2	0.5121589
5.3	0.5331021
5.4	0.5534392
5.5	0.5731480
5.6	0.5922128
5.7	0.6106233
5.8	0.6283744
5.9	0.6454653
6.0	0.6618989
6.1	0.6776815
6.2	0.6928221
6.3	0.7073322
6.4	0.7212251
6.5	0.7345157
6.6	0.7472204
6.7	0.7593562
6.8	0.7709412
6.9	0.7819939
7.0	0.7925329

Power curve for case 1



(ii) The C.R. for testing $H_0 : \sigma^2 = 3$ against $H_1 : \sigma^2 < 3$ is given by

$$W_2 = \{\underset{\sim}{x} : \sum_{i=1}^{16} (x_i - 5)^2 < k_2\}$$

where k_2 is a constant to be determined such that the size of the C.R. is equal to α ,

i.e., $P(x \in W_2 \mid H_0) = \alpha$

$$\Rightarrow P\{\sum_{i=1}^{16}(x_i-5)^2 < k_2 \mid H_0\} = .05$$

$$\Rightarrow P\{\frac{\sum_{i=1}^{16}(x_i-5)^2}{\sigma_0{}^2} < \frac{k_2}{\sigma_0{}^2}\} = .05$$

$$\Rightarrow P\{\chi_{(n)}{}^2 < \frac{k_2}{\sigma_0{}^2}\} = .05$$

To find the value of k_2,we use the following R-command :

a = qchisq(0.05,16,0) (0 is the non-centrality parameter)

This gives us the value $\frac{k_2}{\sigma_0{}^2} = 7.961646$

$\therefore k_2 = 7.961464 \times \sigma_0{}^2 = 7.961646 \times 3 = 23.884938$

Thus the C.R. is given by,

$$W_2 = \{\underset{\sim}{x} : \sum_{i=1}^{16}(x_i-5)^2 < 23.884938\}$$

Now, the Power of the test is given by,

Power=1-β

=P{Reject $H_0|H_1$ is true}

$= P(x \in W_2 \mid H_1)$

$$= P\{\sum_{i=1}^{16}(x_i-5)^2 < 23.884938 \mid H_1\}$$

$$= P\{\frac{\sum_{i=1}^{16}(x_i-5)^2}{\sigma_1{}^2} < \frac{23.884938}{\sigma_1{}^2}\}$$

$$= P\{\chi_{(n)}^{\ 2} < \frac{23.884938}{\sigma_1^{\ 2}}\}$$

Where, $P\{\chi_{(n)}^{\ 2} < \frac{23.884938}{\sigma_1^{\ 2}}\}$ is the distribution function of the chi-square distribution with 'n' d.f.

Now to trace the power curve we consider different trial values of $\sigma_1^{\ 2} < 3$ and construct the following table using R-Programming.

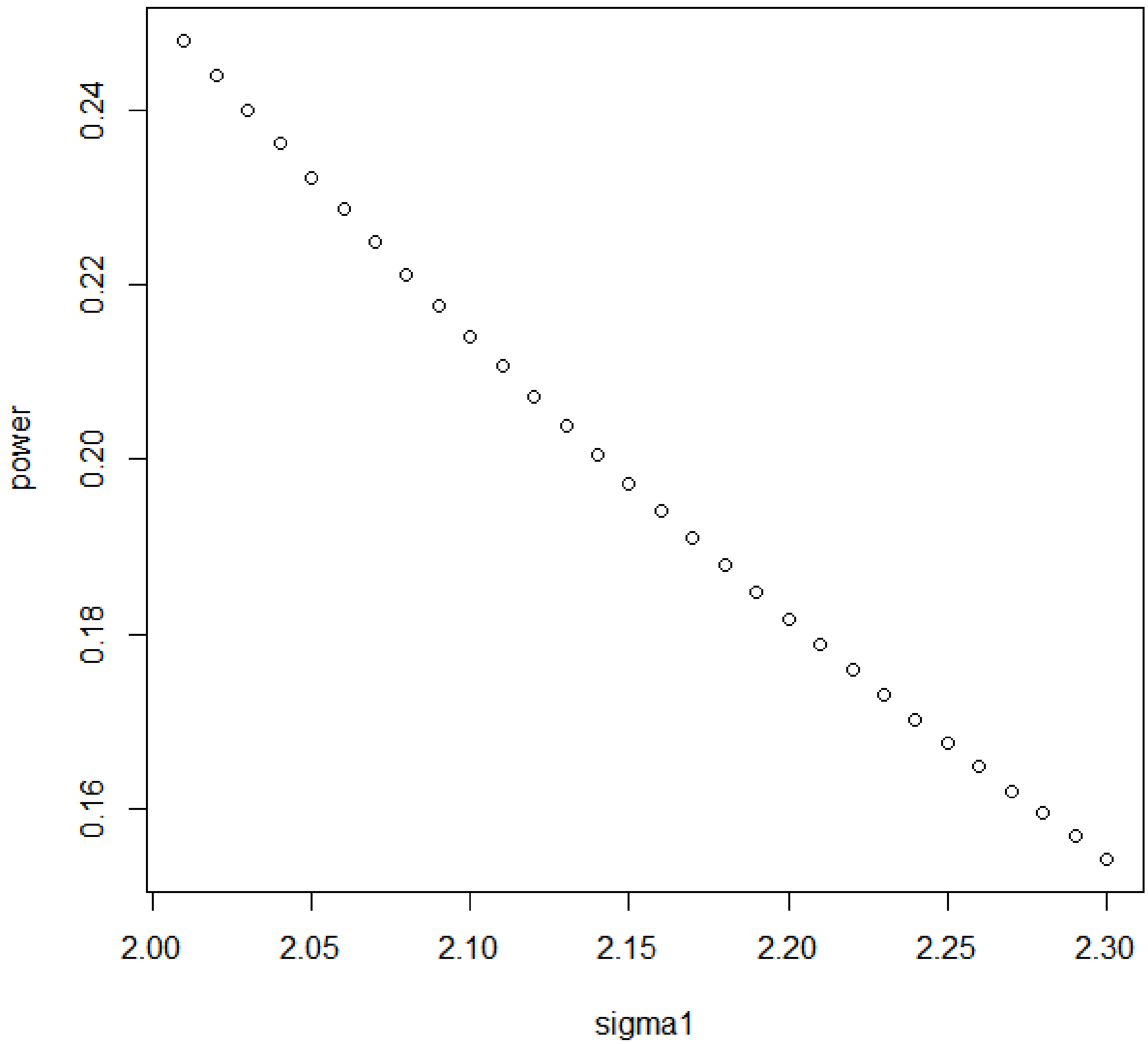
<u>Programming in R for case 2</u>

```
a = qchisq(0.05,16,0)

a

var = 3

k1 = var*a

k1

sigma1 =
c(2.01,2.02,2.03,2.04,2.05,2.06,2.07,2.08,2.09,2.10,2.11,2.12,2.13,2.14,2.15,2.16,2.17,2.18,2.19,
2.20,2.21,2.22,2.23,2.24,2.25,2.26,2.27,2.28,2.29,2.30)

sigma11 = k1/sigma1

power = mat.or.vec(30,1)

for(i in 1:30){

power[i] = pchisq(sigma11[i],16,0)}

power

plot(sigma1,power)
```

TABLE: 2

Trial values of σ_1^2 (<3)	**Power** $= P\{\chi_{(n)}^2 < \frac{23.884938}{\sigma_1^2}\}$
2.01	0.2480098
2.02	0.2440114
2.03	0.2400737
2.04	0.2361961
2.05	0.2323777
2.06	0.2286180
2.07	0.2249163
2.08	0.2212720
2.09	0.2176842
2.10	0.2141525
2.11	0.2106760
2.12	0.2072542
2.13	0.2038862
2.14	0.2005715
2.15	0.1973093
2.16	0.1940989
2.17	0.1909398
2.18	0.1878311
2.19	0.1847723
2.20	0.1817626
2.21	0.1788013
2.22	0.1758879
2.23	0.1730216
2.24	0.1702017
2.25	0.1674277
2.26	0.1646988
2.27	0.1620144
2.28	0.1593739
2.29	0.1567766
2.30	0.1542218

Power curve for case 2



(iii) The C.R. for testing $H_0:\sigma^2 = 3$ against $H_1:\sigma^2 \neq 3$ is given by

W_3={ $\underset{\sim}{x}: \sum_{i=1}^{16}(x_i - 5)^2 < k_3$ or $\sum_{i=1}^{16}(x_i - 5)^2 > k_4$ }

where k_3 and k_4 are constants to be determined such that,

$$P(x \in W_3 \mid H_0) = \alpha$$

$$\Rightarrow P\{\sum_{i=1}^{16}(x_i - 5)^2 < k_3 \text{ or } \sum_{i=1}^{16}(x_i - 5)^2 > k_4 \mid H_0\} = .05$$

$$\Rightarrow P\{\frac{\sum_{i=1}^{16}(x_i - 5)^2}{\sigma_0^{\ 2}} < \frac{k_3}{\sigma_0^{\ 2}}\} + P\{\frac{\sum_{i=1}^{16}(x_i - 5)^2}{\sigma_0^{\ 2}} > \frac{k_4}{\sigma_0^{\ 2}}\} = .05$$

Since, both are mutually exclusive

Assuming that the test is equitailed we have,

$$P\{\frac{\sum_{i=1}^{16}(x_i - 5)^2}{\sigma_0^{\ 2}} < \frac{k_3}{\sigma_0^{\ 2}}\} = .05/2 = .025 \Rightarrow P\{\chi_{(n)}^{\ \ 2} < c\} = .025$$

$$P\{\frac{\sum_{i=1}^{16}(x_i - 5)^2}{\sigma_0^{\ 2}} > \frac{k_4}{\sigma_0^{\ 2}}\} = .05/2 = .025 \Rightarrow P\{\chi_{(n)}^{\ \ 2} > d\} = .025 \Rightarrow P\{\chi_{(n)}^{\ \ 2} < d\} = .975$$

To calculate the value of c and d, we use the following R-command :

```
c = qchisq(0.025,16,0)

var = 3

k3 = var*c
```

d = qchisq(0.975,16,0)

var = 3

k4 = var*d

$\therefore k_3 = 20.72299$

$k_4 = 86.53605$

Thus the C.R. is given by,

$W_3 = \{ \underset{\sim}{x} : \sum_{i=1}^{16} (x_i - 5)^2 < 20.72299 \text{ or } \sum_{i=1}^{16} (x_i - 5)^2 > 86.53605 \}$

Now, the Power of the test is given by,

$\text{Power} = P(x \in W_3 \mid H_1)$

$$= P\{\underset{\sim}{x} : \sum_{i=1}^{16} (x_i - 5)^2 < 20.72299 \text{ or } \sum_{i=1}^{16} (x_i - 5)^2 > 86.53605 | H_1 \}$$

$$= P\{\frac{\sum_{i=1}^{16} (x_i - 5)^2}{\sigma_1^{\ 2}} < \frac{20.72299}{\sigma_1^{\ 2}}\} + P\{\frac{\sum_{i=1}^{16} (x_i - 5)^2}{\sigma_1^{\ 2}} > \frac{86.53605}{\sigma_1^{\ 2}}\}$$

$$= P\{\chi_{(n)}^{\ 2} < \frac{20.72299}{\sigma_1^{\ 2}}\} + [1 - P\{\chi_{(n)}^{\ 2} < \frac{86.53605}{\sigma_1^{\ 2}}\}]$$

Now to trace the power curve we consider different trial values of $\sigma_1^{\ 2} \neq 3$ and construct the following table using R-Programming.

Programming in R for case 3

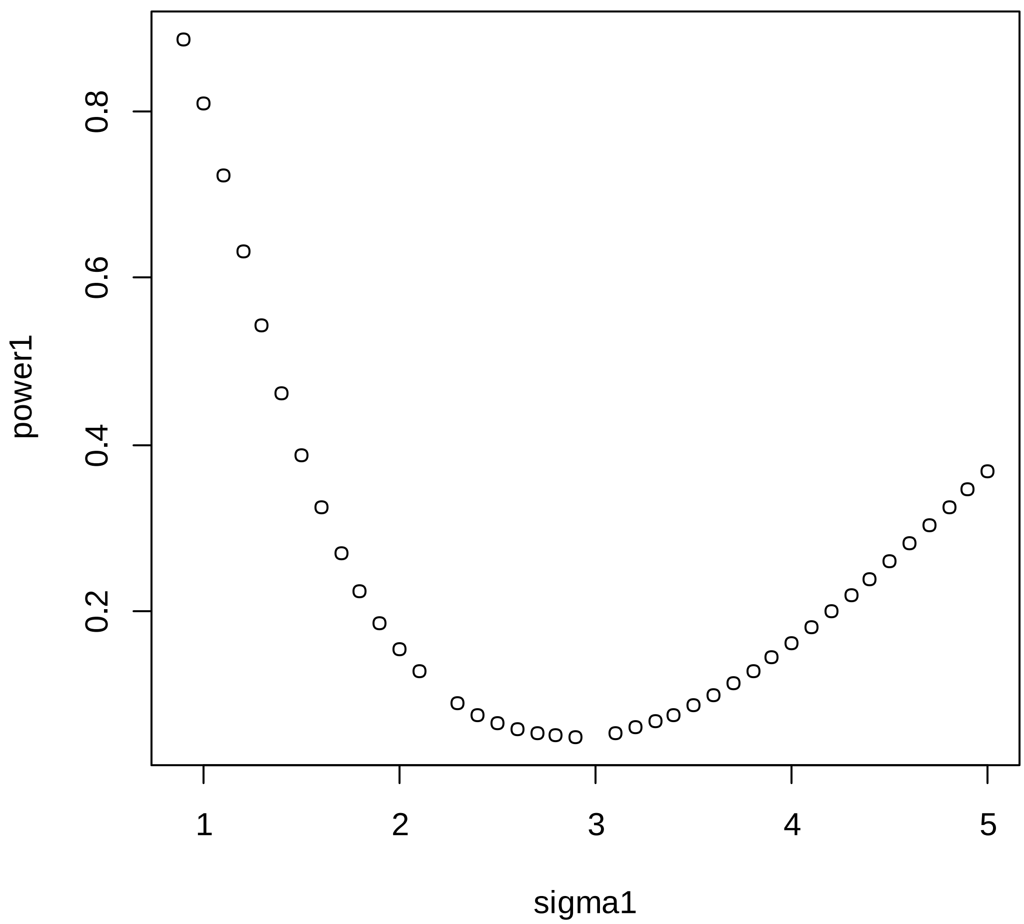
```
c = qchisq(0.025,16,0)

d = qchisq(0.975,16,0)

var = 3

k3 = var*c

k4 = var*d

k3

k4

sigma1=c(0.9,1,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9,2,2.1,2.3,2.4,2.5,2.6,2.7,2.8,2.9,3.1,3.2,3.3,3.4,
3.5,3.6,3.7,3.8,3.9,4.0,4.1,4.2,4.3,4.4,4.5,4.6,4.7,4.8,4.9,5.0)

sigma11 = k3/sigma1

sigma11

sigma22 = k4/sigma1

sigma22

power1 = mat.or.vec(40,1)

for(i in 1:40){

power1[i] = pchisq(sigma11[i],16,0)+(1-pchisq(sigma22[i],16,0))}

power1

plot(sigma1,power1)
```

TABLE: 3

Trial values of σ_1^2 ($\neq 3$)	**Power** $= P\{\chi_{(n)}^2 < \frac{20.72299}{\sigma_1^2}\} + [1 - P\{\chi_{(n)}^2 < \frac{86.53605}{\sigma_1^2}\}]$
0.9	0.88694674
1.1	0.81059418
1.2	0.72290398
1.3	0.63161004
1.4	0.54289735
1.5	0.46082501

1.6	0.38753198
1.7	0.32373148
1.8	0.26921596
1.9	0.22326209
2.0	0.18491564
2.1	0.15317390
2.2	0.12709197
2.3	0.08871020
2.4	0.07514532
2.5	0.06469934
2.6	0.05703406
2.7	0.05189660
2.8	0.04909969
2.9	0.04850352
3.1	0.05349975
3.2	0.05892191
3.3	0.06618671
3.4	0.07521037
3.5	0.08590204
3.6	0.09816245
3.7	0.11188378
3.8	0.12695044
3.9	0.14324051
4.0	0.16062749
4.1	0.17898227
4.2	0.19817508
4.3	0.21807731
4.4	0.23856317
4.5	0.25951116
4.6	0.28080528
4.7	0.30233591
4.8	0.32400060
4.9	0.34570450
5.0	0.36736067

power curve for case 3



Conclusion-

Thus we get three different power curves for testing $H_0 : \sigma^2 = 3$ against

i) H_1: σ^2 >3 ,ii) H_1: σ^2 <3 and iii) H_1: $\sigma^2 \neq 3$ respectively at the level of significance $\alpha = 0.05$

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