Local Defect Correction using Cubic Splines for Incompressible Viscous Flows

MA498 Project I

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   - Motivation
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   - Why Cubic Polynomial ?
   - Natural splines
4. A Generalized Numerical Scheme for LDC using Cubic Splines
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Problem Description

To devise a **Local Defect Correction** based approach using **Cubic Splines** for **Incompressible Viscous Flows.**

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- Infeasibility of analytical solution for many problems.
- Numerical methods are used to solve such problems.
- Some Boundary Value Problems (BVP) have rapid variations in their solution in some parts of the domain.
- A single discretization scheme will require a very fine grid overall, leading to unnecessary calculations.
- **LDC** uses different discretization schemes for coarser and finer grid, saving a lot of computations.

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Description

LDC uses two different discretization schemes.

A general problem

A boundary value problem with domain Ω, like

$$Lu = f \tag{1}$$

with boundary condition

$$Bu = g \tag{2}$$

on

$$\Gamma = \delta\Omega \tag{3}$$

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$$L u_\omega = f \tag{5}$$

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The discretization in this subdomain ω is given by

$$L^{'}_{h^{'}} u^{'}_{h^{'},\omega} = f^{'}_{h^{'},\omega} \tag{6}$$

$L^{'}$ and $u^{'}$ indicate the discretization may be different from L_h and u_h. $h^{'}$ subscript means a different step size from h.

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On the interior boundary Γ_1 we have

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The discretization of equation (7) is given by:

$$u'_{h',\omega} = \gamma u_{h,\Omega} \tag{8}$$

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Where γ is an interpolation on $u_{h,\Omega}$ in $\Gamma_{1,h'}$. In our experiments its the cubic spline interpolation

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$$f^0_{h,\Omega} = f_{h,\Omega} \tag{9}$$

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$$g^i = \gamma u^i_{h,\Omega} \tag{11}$$

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and

$$g^{i} = \gamma u^{i}_{h,\omega} \tag{13}$$

- We have sub-grids $\omega^{''}_{h} \Subset \omega^{'}_{h} \Subset \omega$
Interpolate $u^{i}_{h,\omega}$ on $\omega^{'}_{h} = \omega^{'} \epsilon \Omega_h$

$$\overline{u}_{h,\omega'} = \pi u_{h',\omega} \tag{14}$$

where π is some interpolation.

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$$d_{h,\omega^{''}} = L_h \overline{u}_{h,\omega'} - f_{h,\Omega} \tag{15}$$

A general algorithm(...continued)

- Compute the defect in $\omega_h^{"}$

$$d_{h,\omega^{"}} = L_h \overline{u}_{h,\omega'} - f_{h,\Omega} \tag{15}$$

- Now for the next iteration, define in the coarser grid

$$f_{h,\Omega}^{i+1} = f_{h,\Omega}^{i} + \chi_{\omega^{"}} d_{h,w^{"}} \tag{16}$$

where $\chi_{\omega^{"}}$ is the characterization function for $\omega^{"}$

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- LDC based discretization scheme use two grids.
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IMPROVED BOUNDARY INFORMATION! :)

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- Acquire the boundary value of common points from coarser grid.

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- High Degree polynomials can oscillate erratically because of a minor fluctuation over a small portion.
- Divide approximation interval into sub-intervals.
- Given n point(s) we approximate $n-1$ piecewise cubic polynomial(s).

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Why Cubic Polynomial ?

Smoothness!

Theorem

If S is the natural cubic spline function that interpolates a twice-continuously differentiable function f at knots $a = t_0 < t_1 < \ldots < t_n = b$ then

$$\int_a^b [S''(x)]^2 dx \leq \int_a^b [f''(x)]^2 dx$$

Constraints

Definition

Given a function f defined on $[a, b]$ and set of n+1 points such that

$$a < x_0 < x_1 < x_2 ... < x_n < b$$

A cubic spline interpolant S for the function f satisfies the following conditions:

1. $S(x)$ is a cubic polynomial, denoted $S_j(x)$ for every jth interval $[x_j, x_{j+1}]$, $j = 0, 1, 2, ..., n-1$
2. $S_j(x_{j+1}) = S_j(x_{j+1})$ for all $j = 0, 1, 2, ..., n-2$
3. $S_j^{'}(x_{j+1}) = S_j^{'}(x_{j+1})$ for all $j = 0, 1, 2, ..., n-2$
4. $S_j^{''}(x_{j+1}) = S_j^{''}(x_{j+1})$ for all $j = 0, 1, 2, ..., n-2$
5. One of the following boundary conditions is satisfied:
    1. $S^{''}(x_0) = S^{''}(x_n)$ (free boundary)
    2. $S^{'}(x_0) = f^{'}(x_0)$ and $S^{'}(x_n) = f^{'}(x_n) S^{'}(x_n)$ (clamped boundary)

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Natural Splines

The general form of cubic spline interpolant

$$a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \tag{17}$$

The following theorem proves that for any given set of n points, there will always exist a natural spline interpolant.

Theorem

If a function f is defined at nodes $x_0, x_1, ..., x_n$ such that $a < x_0 < x_1 ... < x_n < b$, then there exists a unique natural spline interpolant S for the given function passing through the points $(x_0, f(x_0)), (x_1, f(x_1))...(x_n, f(x_n))$ satisfying the natural boundary condition $S^{''}(a) = 0$ and $S^{''}(b) = 0$.

Proof by using **Levy Desplanques** theorem for dominant matrices.

A General Numerical Scheme for LDC using Cubic Splines

Steps

1. The coarser grid is solved first(till convergence or for some fixed number of iterations) by initializing it with proper boundary conditions (and zero values for the interior points).

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4. The finer grid is then solved using a similar scheme as used in the coarser grid in step 1.

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3. The remaining boundary points are then interpolated using cubic spline for the virtual boundaries.
4. The finer grid is then solved using a similar scheme as used in the coarser grid in step 1.
5. The common points are now used to update the coarser grid and we move back to step 1. The process is repeated until we reach below tolerance on comparing solution on common points for both the grids.

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Problem description and Governing Equations

The following PDE governs the steady-state temperature T(x,y):

$$T_{xx} + T_{yy} = 0 \tag{18}$$

subject to the following constraints:

$$0 < x < 4$$
$$0 < y < 1$$

and the following boundary conditions:

$$T(x,0) = 200x(x-4)$$
$$T(x,1) = T(0,y) = T(4,y) = 0$$

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Discretization

To obtain the FDE we replace T_{xx} and T_{yy} by their respective second-order centered-difference approximation at grid point (i, j) :

$$\overline{T}_{xx}(i,j) = \frac{\overline{T}_{i+1,j} - 2\overline{T}_{i,j} + \overline{T}_{i-1,j}}{\Delta x^2} + O(\Delta x^2) \tag{19}$$

$$\overline{T}_{yy}(i,j) = \frac{\overline{T}_{i,j+1} - 2\overline{T}_{i,j} + \overline{T}_{i,j-1}}{\Delta y^2} + O(\Delta y^2) \tag{20}$$

Now putting (19) and (20) in (18) we get:

$$\frac{\overline{T}_{i+1,j} - 2\overline{T}_{i,j} + \overline{T}_{i-1,j}}{\Delta x^2} + \frac{\overline{T}_{i,j+1} - 2\overline{T}_{i,j} + \overline{T}_{i,j-1}}{\Delta y^2} + E_{i,j} = 0 \tag{21}$$

with the following truncation error:

$$E_{i,j} = O(\Delta x^2) + O(\Delta y^2)$$

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LDC Grid for numerical simulation

The global domain was discretized using a coarser of dimension 41x11 points(step size of .1 on both axes) and a finer grid was imposed in the lower right corner of dimension 41x11 points(step size of .05 on both axes)

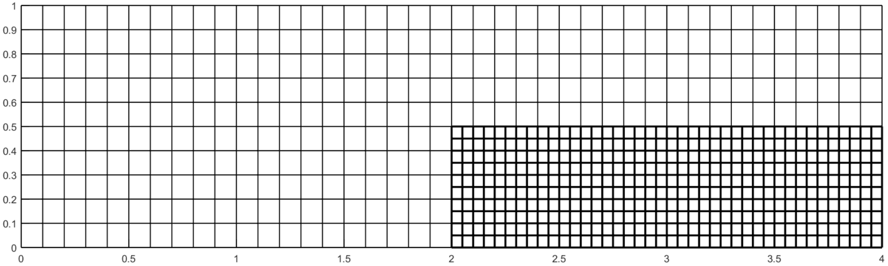

Figure 1: Grid structure for steady state heat conduction

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Observations

Going back and forth achieving convergence for the coarser grid and for the finer grid gave encouraging results.

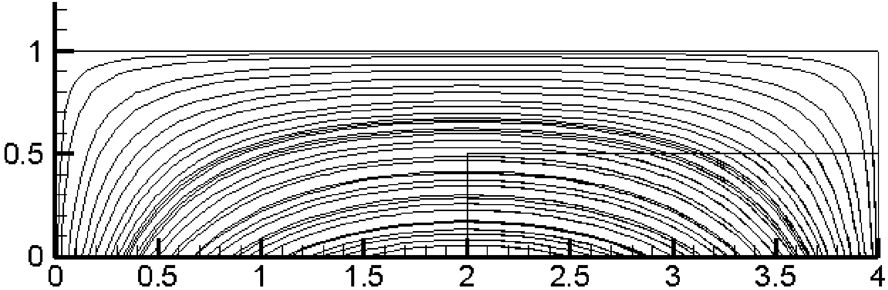

Figure 2: Contour plot of the whole domain consisting of the coarser and finer grids.

Observations(...continued)

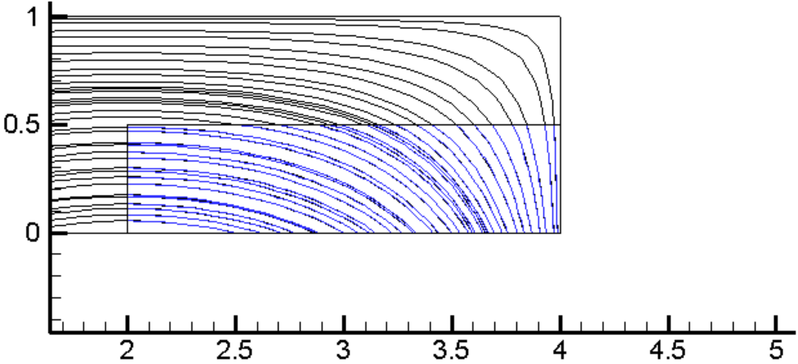

Figure 3: Contour plot of the lower right region

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2D Navier-Stokes equations in a square cavity

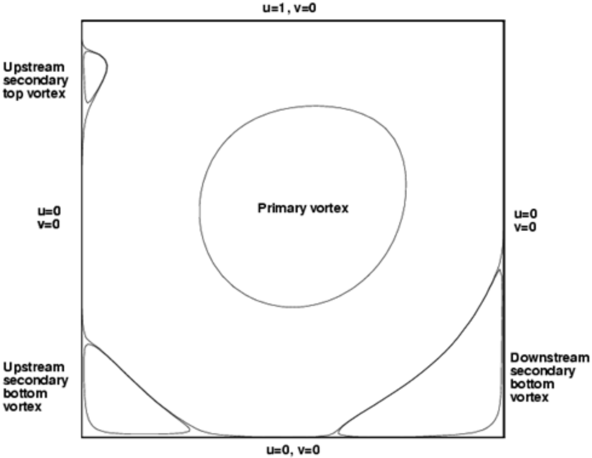

Figure 4: Lid Driven Square Cavity.

Variation in flow complexity with change in *Re*

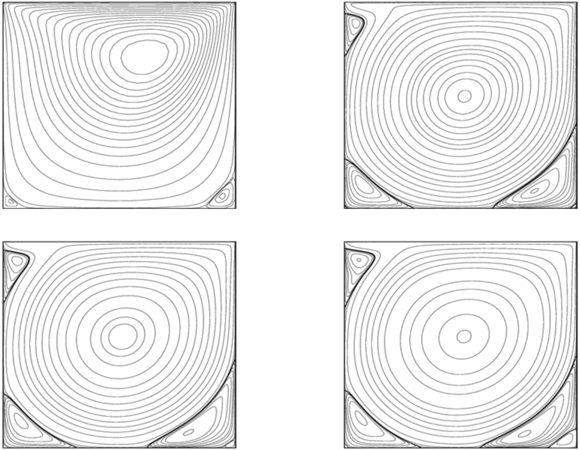

Figure 5: Flow for different values of *Re*.

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Governing Equations

The governing equations for the present flow configuration are the two-dimensional (2D) incompressible Navier-Stokes (NS) equations. The 2D NS equations in the **non-dimensional** primitive-variables form are given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{22}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{23}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 v = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{24}$$

Non-Dimensionality and Variable Reduction

For a steady 2D flow, it is possible to have an equivalent non-dimensional system of two equations known as the streamfunction-vorticity (or $\psi - \omega$) form of the NS equations:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega \tag{25}$$

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \frac{1}{Re}\nabla^2\omega = \frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) \tag{26}$$

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Discretization

- Numerically solving the 2D Navier-Stokes equations in a square cavity:.

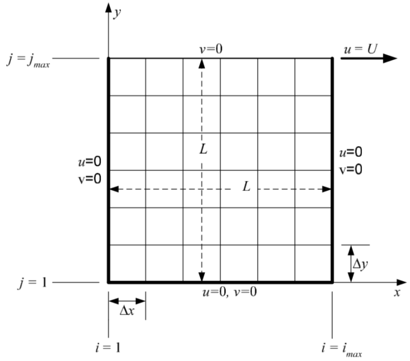

Figure 6: Lid Driven Square Cavity with an uniform grid for the whole domain, $Re = \frac{LU}{\nu}$.

Discretizing ψ

We use second-order accurate central differencing for all the space derivatives.The discretized ψ-equation at point (i, j) is written as:

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2} = -\omega_{i,j} \tag{27}$$

We introduce the notation $\beta = \frac{\Delta x}{\Delta y}$. Upon rearrangement of equation (4.6) we shall get

$$\psi_{i,j} = \frac{\psi_{i+1,j} + \psi_{i-1,j} + \beta^2(\psi_{i,j+1} + \psi_{i,j-1}) + (\Delta x)^2\omega_{i.j}}{2(1+\beta^2)} \tag{28}$$

Discretizing ω

The discretized form of the ω equation at (i, j) point is

$$u_{i,j}\frac{\omega_{i+1,j}-\omega_{i-1,j}}{2\Delta x}+v_{i,j}\frac{\omega_{i,j+1}-\omega_{i,j-1}}{2\Delta y}=\frac{1}{Re}\left(\frac{\omega_{i+1,j}-2\omega_{i,j}+\omega_{i-1,j}}{(\Delta x)^2}+\frac{\omega_{i,j+1}-2\omega_{i,j}+\omega_{i,j-1}}{(\Delta y)^2}\right) \quad (29)$$

Upon rearrangement after multiplying on both sides by $(\Delta x)^2$ we get

$$\omega_{i,j}=\frac{\omega_{i+1,j}+\omega_{i-1,j}+\beta^2(\omega_{i,j+1}+\omega_{i,j-1})-\frac{1}{2}\Delta x Re(\omega_{i+1,j}-\omega_{i-1,j})u_{i,j}-\frac{1}{2}\Delta x\beta Re(\omega_{i,j+1}-\omega_{i,j-1})v_{i,j}}{2(1+\beta^2)} \quad (30)$$

Boundary Conditions

Take $\psi = 0$ on all the walls.

$$\text{Left wall: } \omega_{1,j} = -\frac{2\psi_{2,j}}{(\Delta x)^2} \qquad 2 \leq j \leq j_{max-1} \tag{31}$$

$$\text{Right wall: } \omega_{imax,j} = -\frac{2\psi_{imax-1,j}}{(\Delta x)^2} \qquad 2 \leq j \leq j_{max-1} \tag{32}$$

$$\text{Bottom wall} \omega_{i,1} = -\frac{2\psi_{i,2}}{(\Delta y)^2} \qquad 2 \leq i \leq i_{max-1} \tag{33}$$

$$\text{Top wall} \omega_{i,jmax} = -\frac{2\psi_{i,jmax-1} + 2\Delta y}{(\Delta y)^2} \qquad 2 \leq i \leq i_{max-1} \tag{34}$$

These conditions were obtained by coupling with (28) and (30) the conditions of zero wall-tangential velocity at the side and bottom walls and a tangential velocity of U at the top wall.

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LDC Grid for Numerical Simulation

The following grid structure was used to apply local defect correction to the lower left corner of the global domain.

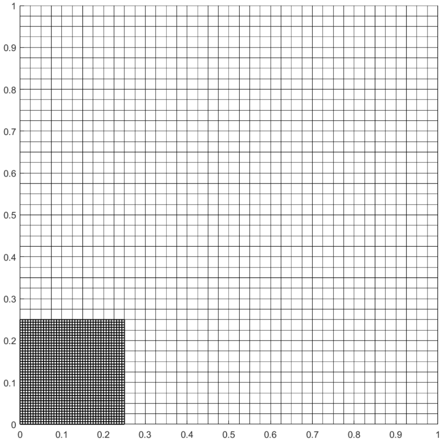

Figure 7: Grid structure for lid driven cavity

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Coarser Grid

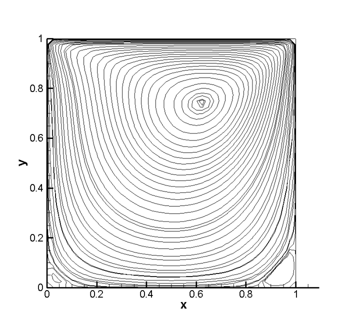

Figure 8: Contour plot of the global domain.

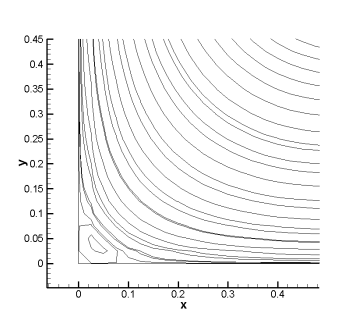

Figure 9: Contour plot of the lower left region

Observation:

Using a uniform coarser grid global domain does not give us good results in high gradient regions

Virtual BC $\simeq$ Physical BC

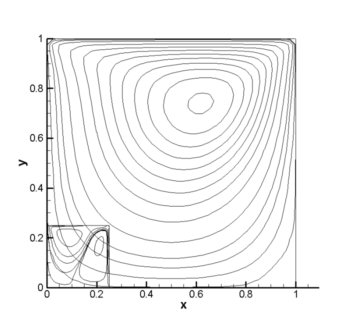

Figure 10: Contour plot of the whole domain.

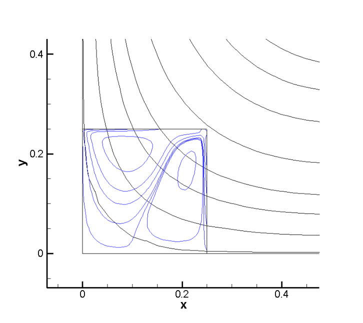

Figure 11: Contour plot of the lower right region.

Observation:

Expected results! This gives us a solution where the finer grid now corresponds to a new global domain with actual physical boundaries.

Fixed ψ and ω on boundaries

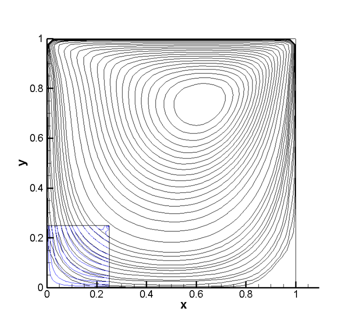

Figure 12: Contour plot of the whole domain.

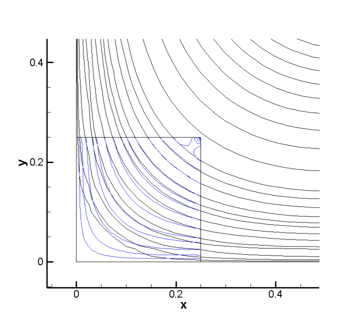

Figure 13: Contour plot of the lower right region.

Observation:

Since there is no refinement in the boundary values of the both ψ & ω, the solution resembles the overall solution but the error is significantly high.

Fixed ψ and varying ω

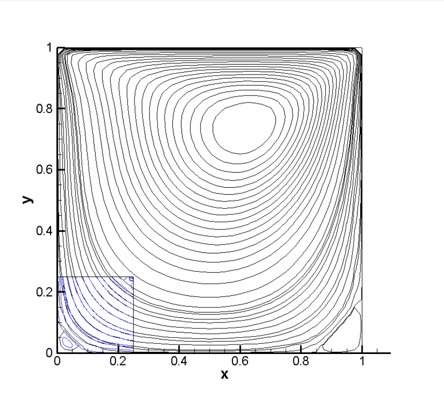

Figure 14: Contour plot of the whole domain.

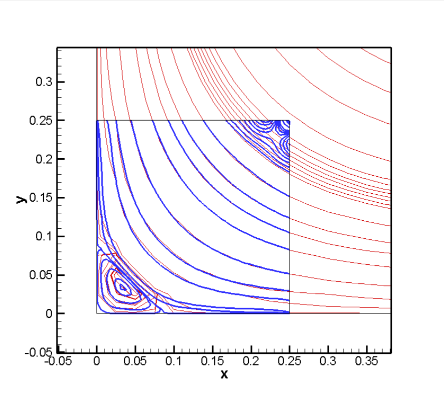

Figure 15: Contour plot of the lower right region.

Fixed ψ and varying ω (...continued)

Observation I:

When solving for finer grid, we use the discretization for ω and we update ω every time by using the new *refined* values from the interior of the finer grid and old *fixed* values from the coarser grid. This significantly improved the solution in the region of interest matching with the set benchmarks (will be discussed later).

Observation II:

There are some 'unwanted vortices' in the upper right corner. The reason of which could be ascribed to error which exists due to cubic spline interpolation since the top right corner's result are affected values from both the top and the right boundaries (both are virtual in our case).

Moreover, the error prevails due to the interpolation in the values of ψ which are not refined over the course of the solution. Doing this, will further improve the quality of numerical solution.

BL_1 comparision

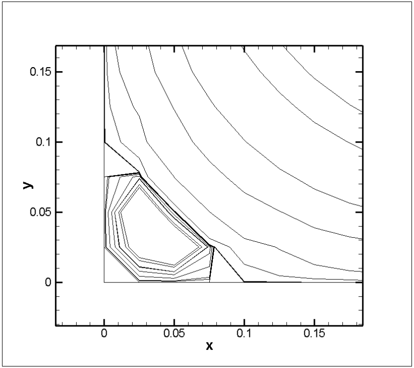

Figure 16: First BL (BL_1) from coarser grid .

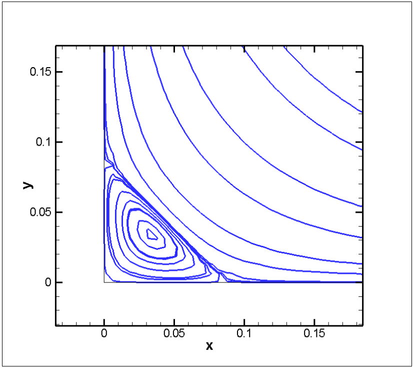

Figure 17: First BL (BL_1) from finer grid.

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Comparing with Benchmark

Parameter	Ghia's benchmark	Finer grid	Coarser grid
ψ_{max}	1.74877 $\times 10^{-6}$	1.733538 $\times 10^{-6}$	2.488781 $\times 10^{-6}$
$\omega_{v.c.}$	-1.55509 $\times 10^{-2}$	-1.547339 $\times 10^{-2}$	-1.68443 $\times 10^{-2}$
(x, y)	0.0313,0.0391	0.03125,0.03125	0.03578,0.03729
H_L	0.0781	0.07892	0.7652
V_L	0.0781	0.07976	0.8078

Grid Point Comparison

The solution mentioned in the paper uses a single 129 $\times$ 129 (16641) global grid whereas we evaluate by using two 41 $\times$ 41 grids (3262).

Immediate Improvement Strategy

- Tuning hyper parameters for over and under relaxation.
- Trying different grid structures and sizes.
- Updating ψ as well.

Future Work

- **Improved interpolation scheme** for the virtual boundaries and **better approximation for nodal points** in using cubic spline(s).

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- Deriving and experimenting with **Higher Order Compact schemes.**

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- Using **efficient solvers** which are specific to the problem. Currently, basic iterative solver, namely Gauss-Seidel, was used in all the experiments.

Future Work

- **Improved interpolation scheme** for the virtual boundaries and **better approximation for nodal points** in using cubic spline(s).
- Deriving and experimenting with **Higher Order Compact schemes.**
- Using **efficient solvers** which are specific to the problem. Currently, basic iterative solver, namely Gauss-Seidel, was used in all the experiments.
- **Parallel computing** for faster back and forth computation. Also, computation on two disjoint finer grids.
- **Error analysis** and **convergence analysis**

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