14/08/24 # Signals And Systems 11:00AM

Evaluation: (relative grading)		
	Attendance	10%
	Quiz	16%
	Assignments	15%
	mid sem	24%
	end sem	35%

yoooooooooooo :fire emoji:

Book: Signals & System 2nd edition by Alan V. oppenheimer

Lecture : 1 — 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ GROUP : consists of ....

- A set G
- A rule / binary operation "$*$"
  - a. associative
    $$x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$$
  - b. There exists an element "e" called the identity of group G such that
    $$e*x = x*e = x \quad \forall x \in G$$
  - c. $\forall x \in G, \exists\ x^{-1}$ such that
    $$x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$$
  - d. if $x*y = y*x \quad \forall x,y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

a

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abllion Group

eg: set of continuous time periodic signals with time period T with "$*$" = "$+$"

⇒ FIELD : consists of the following

- A set F
- Two binary operations "$+$" and "$\cdot$" such that ....
  - $(F,+)$ is an abllian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - left distributive
      $$x\cdot(y+z) = xy + xz \quad \forall x,y,z \in F$$
    - Right distributive
      $$(x+y)\cdot z = xz + yz \quad \forall x,y,z \in F$$

eg: F = Real Numbers $\mathbb{R}$

* Vector Space : A set V with a map...

- '$+$' : $V \times V \to V$
  $(v_1, v_2) \to (v_1 + v_2)$ called vector addition
- '$\cdot$' : $F \times V \to V$
  $(a, v_1) \to (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a\cdot(v_1+v_2) = av_1 + av_2$
- $(a_1+a_2)\cdot v = a_1 v + a_2 v$
- $(\alpha\beta)v = \alpha(\beta v) = \beta(\alpha v)$
- $1\cdot v = v$
- $a\cdot 0 = 0$
- $0\cdot v = 0$
- if $v \neq 0$, then $a\cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \to R$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X,d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM: let V be a F-vector space.

A map $\|\cdot\| : V \to R$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a|\,\|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space m

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

Lecture : 2

16/08/24 : 9:30Am

* Inner Product :

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v,v \rangle \geq 0$ and $\langle v,v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ " — " : complex conjugate operation
- it is linear in the first co-ordinate
  $$\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$$
  $\forall\ v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $$\langle v, a_1 w_1 + a_2 w_2 \rangle = \bar{a}_1 \langle v, w_1 \rangle + \bar{a}_2 \langle v, w_2 \rangle$$
  $\forall\ w_1, w_2, v \in V$ and $a_1, a_2 \in F$

measures cosine similarity
" norm(x) · norm(y) · cosθ

eg: dot product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner Product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot \bar{y}_i$$

complex conjugate

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \qquad \text{implies } a_i = 0\ \forall i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

weighted linear combination of vector V

if $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ is a basis for $V_n(F)$, then $\forall\ v_i \in V\ \&\ a_i \in F$

$$V = \sum_{i=1}^{n} a_i e_i$$

$V = a_1 e_1 + a_2 e_2 + \dots a_n e_n$

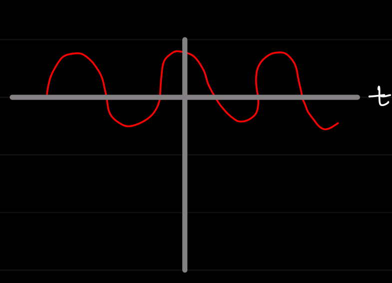
* ORTHOGONAL & ORTHONORMAL Basis

A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

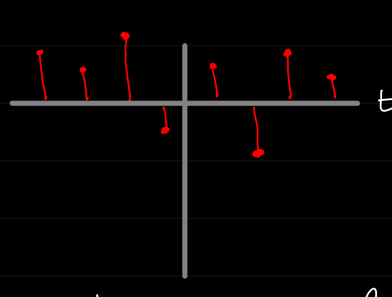
$$v_i \neq 0 \quad \forall i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall i \& j$$

SIGNALS (CTS & DTS)

- Continuous time signal



- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t)\ \forall t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t)\ \forall t$
  eg: sine function

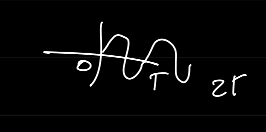
signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal :

$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

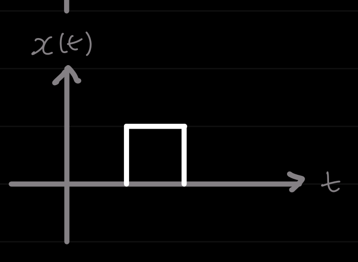


* PERIODIC : if $x(t) = x(t+T)\ \forall t$
then T is the time period

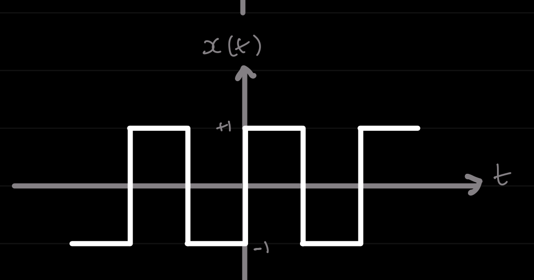
The smallest time period T defined for this property is called the fundamental time period

eg:

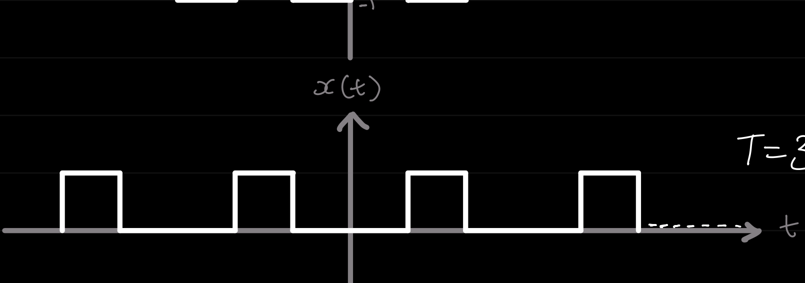
Not periodic



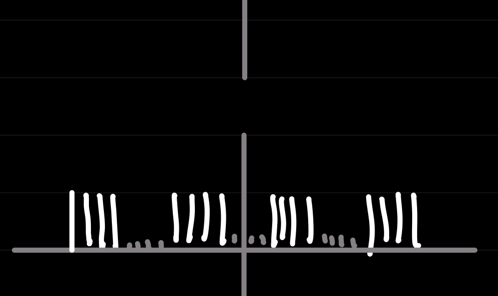
$T \to \infty$
Aperiodic



Periodic with $T=2$



$T=3$



$T=8$

classification { Probabilistic
Dataministic

* ENERGY OF SIGNAL

Continuous Time signal :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time signal

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

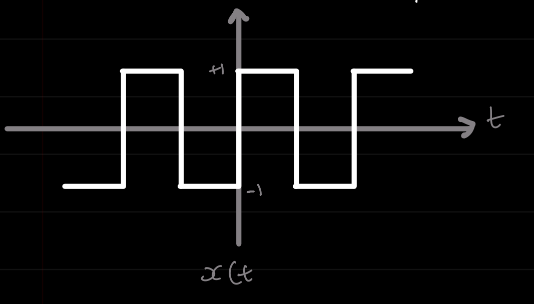
Note: Periodic Signals are power signals ✓
Aperiodic signals are not power signals ✗

$T = \infty$
so, power = not defined
hence, Aperiodic signals are not power signals

avg. energy in
" a time duration

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$T=2$



compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{x=-N}^{N} |x[n]|^2$$

* Lecture : 3

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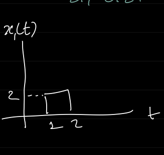
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{2}^{2} |2|^2 dt = 4\int_{0}^{2} dt$$

$$= 4 < \infty$$



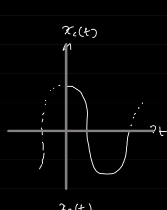
↑ energy signal

$x_1(t) \equiv$ energy signal,
continuous signal,
aperiodic,
deterministic

$x_1(t) \neq x_1(t+T) \Rightarrow$ aperiodic signal

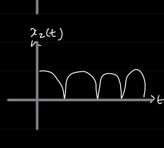
$x_2(t) =$ continuous signal,
periodic signal,
← deterministic
power signal

we know its value at every point t
↓
not probabilistic



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

after mod ⇒



$$E = \lim_{T_0 \to \infty} \underbrace{\int_{-T_0}^{T_0} |x(t)|^2 dt}_{\infty}$$

Since $E \to \infty$ and not $E < \infty$, this signal is not an energy signal

$$P = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} |x(t)|^2 dt$$

intuition $\left\{ \approx \frac{\text{infinite block}}{\text{infinit block}} = \text{finite} \right.$

$\boxed{0 < P < \infty} \Rightarrow$ Power Signal ✓

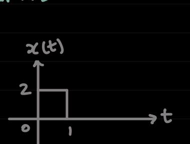
* TRANSFORMATION OF SIGNAL

① Translation

energy of signal doesn't change

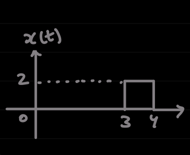
(a) DELAY

$y(t) = x(t - t_0)$



let $t_0 = 3$

$y(t) = x(t-3)$



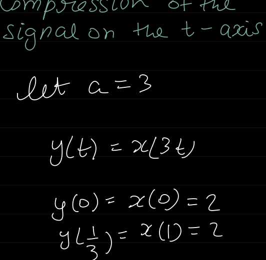
$y(0) = x(-3) = 0$
$y(3) = x(0) = 2$
$y(4) = x(1) = 2$
$y(5) = x(2) = 0$

} verification of transformation

(b) ADVANCE A SIGNAL

$y(t) = x(t + t_0)$

let $t_0 = 3$

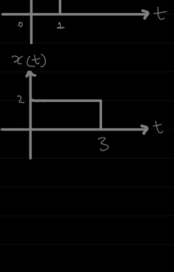


$y(-3) = x(0) = 2$
$y(-2) = x(1) = 2$
$y(-1) = x(2) = 0$
verified

energy of scale changes in accordance with area under curve

② Scaling

$y(t) = x(at)$

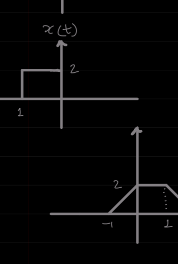


(a) $a > 1$: compression of the signal on the t-axis

let $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$
$y(\frac{2}{3}) = x(2) = 0$



(b) $0 < a < 1$: Dilation on t-axis

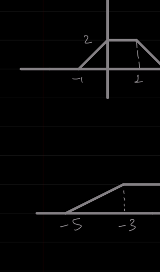
let $a = \frac{1}{3}$

$y(t) = x(at) = x(t/3)$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

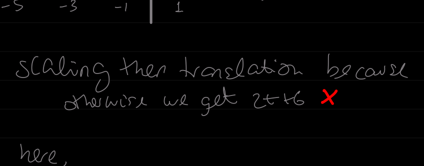
$y(6) = x(2) = 0$



ⓒ SPECIAL CASE : $a = -1$

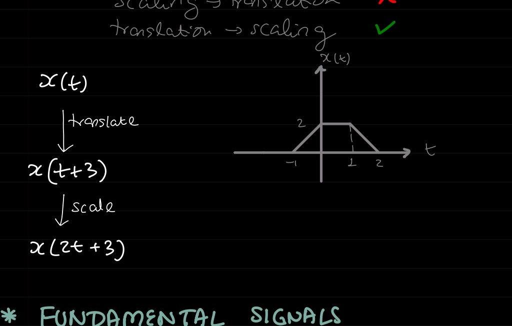
$y(t) = x(-t)$

flipped on the t-axis
≡ time reversal



eg) $y(t) = x(2t+3)$

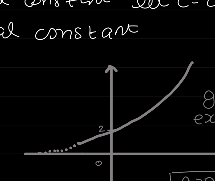
translation + scaling



scaling then translation because otherwise we get $2t+6$ ✗

here,
scaling → translation ✗
translation → scaling ✓

$x(t)$
↓ translate
$x(t+3)$
↓ scale
$x(2t+3)$



* FUNDAMENTAL SIGNALS

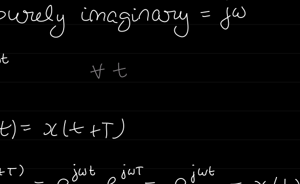
- **Continuous Time Signals**

$$x(t) = ce^{at}$$

- CASE : 1 — $c \equiv$ real constant, let $c = 2$
  $a \equiv$ real constant

$x(t) = 2e^{at}$

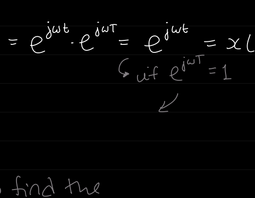
① let $a = 1$



growing exponential

$\boxed{a > 0}$

② let $a = -1$



$\boxed{a < 0}$

③ let $a = 0$

- CASE : 2 — $c \equiv$ real constant
  $a \equiv$ purely imaginary $= j\omega$

$x(t) = e^{j\omega t} \quad \forall t$

periodic if $x(t) = x(t+T)$

$$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t} \cdot e^{j\omega T} = e^{j\omega t} = x(t)$$

↳ if $e^{j\omega T} = 1$

$e^{j\omega T} = 1$ if
$\omega T = 2\pi K$

$e^{j2\pi K}$

since we need to find the fundamental time period (smallest interval), let $K = 1$

So, $\omega T = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$ fundamental time period

$$E = \int_{-\infty}^{\infty} |e^{j\omega t}|^2 dt = \int_{-\infty}^{\infty} 1\, dt = \infty$$

- CASE : 3 — $C \equiv$ complex number
  $a \equiv$ real number

$$x(t) = |C| e^{j\theta} \cdot e^{at} = |C| e^{at + j\theta}$$

e^{at} will define if it is growing / decaying exponential

we can also say, $|C|e^{j\theta} = c\cos(\theta) + c \cdot j\sin(\theta)$

$Re\{x(t)\} = |C| \cdot \cos\theta \cdot e^{at}$

$Im\{x(t)\} = |C| \cdot \sin\theta \cdot e^{at}$

- CASE : 4 — $C \equiv$ complex number $= |C|e^{j\theta}$
  $a \equiv$ complex number $= r + j\omega$

$$x(t) = ce^{at} = |C|e^{j\theta} \cdot e^{(r+j\omega)t}$$

$$= \underbrace{|C|e^{rt}}_{\text{envelope with either increasing or decaying exponential}} \cdot \underbrace{e^{j(\omega t + \theta)}}_{\cos(\omega t+\theta) + j\sin(\omega t + \theta)}$$

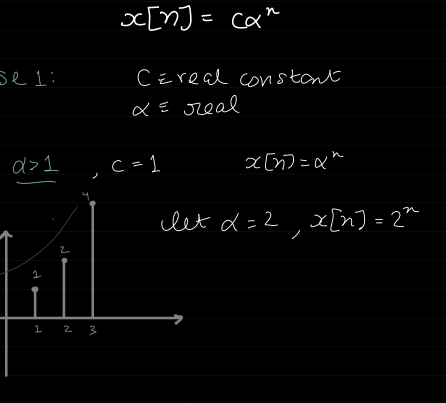
cosinosoidal signal — sinosoidal signal

$Re\{x(t)\} = |C|e^{rt} \cdot \cos(\omega t + \theta)$

$Im\{x(t)\} = |C|e^{rt} \cdot \sin(\omega t + \theta)$

let $r = 2$, $|C| = 1.5$

$Re\{x(t)\} = \underbrace{1.5e^{2t}}\underbrace{\cos(\omega t + \theta)}$



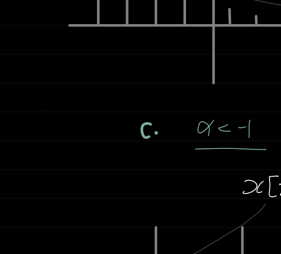
- **Discrete Time Signals**

$$x[n] = c\alpha^n$$

- Case 1: $C \equiv$ real constant
  $\alpha \equiv$ real

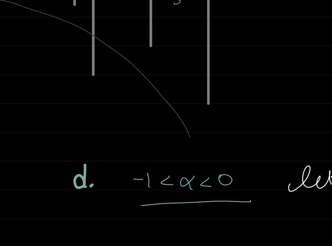
a. $\underline{\alpha > 1}$, $c = 1$ — $x[n] = \alpha^n$

let $\alpha = 2$, $x[n] = 2^n$



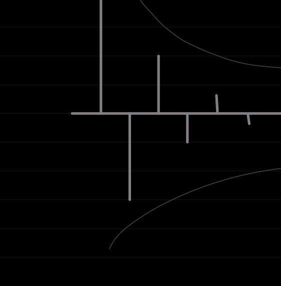
b. $\underline{\alpha < 1}$ — let $\alpha = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$



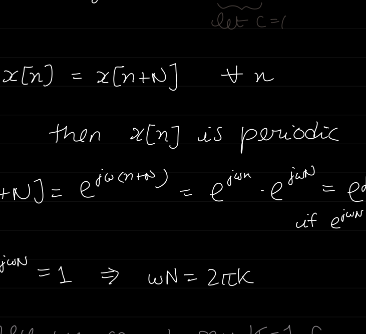
c. $\underline{\alpha < -1}$ — let $\alpha = -2$

$x[n] = (-2)^n$



d. $\underline{-1 < \alpha < 0}$ — let $\alpha = \frac{-1}{2}$

$x[n] = (\frac{-1}{2})^n$



- Case : 2 — $C \equiv$ real
  $\alpha = e^{j\omega}$

$x[n] = c\alpha^n = \underbrace{e^{j\omega n}}_{\text{let } C=1}$

if $x[n] = x[n+N] \quad \forall n$

then $x[n]$ is periodic

$$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N} = e^{j\omega n}$$

if $e^{j\omega N} = 1$

$e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi K$

here we cannot say $K = 1$ for smallest interval necessarily because we want N to be an integer

$N = \frac{2\pi K}{\omega}$ ≡ has to be integer

$\frac{N}{K} = \frac{2\pi}{\omega}$ ≡ has to be rational number

Note: ω has to be a multiple of π

* Lecture: 4 28/08/24

- for continuous time signals
  → frequency is unique $(0 \to \infty)$

- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

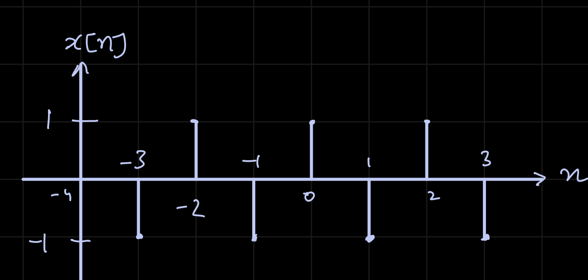
DTS

$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j \cdot 2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

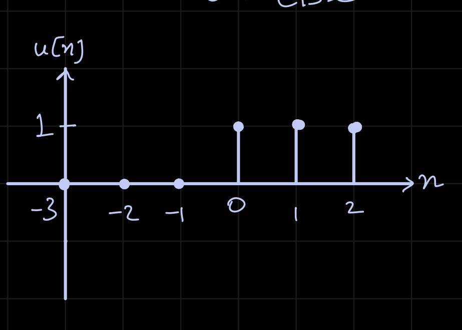
let $\omega_0 = \pi \Rightarrow e^{j \cdot \pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

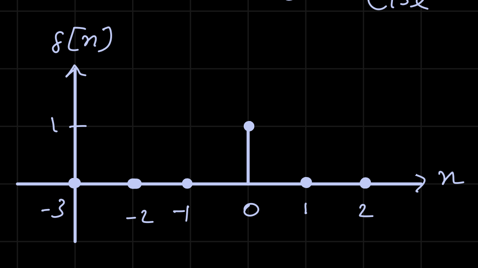
unit step signal $u[n]$

$$u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : \text{else} \end{cases}$$



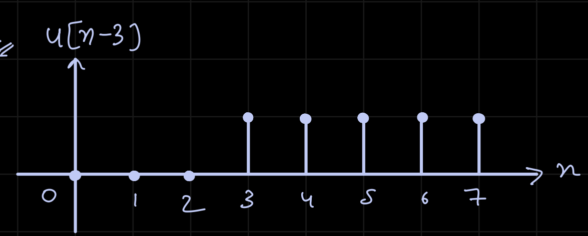
unit impulse function $\delta[n]$

$$\delta[n] = \begin{cases} 1 : n = 0 \\ 0 : \text{else} \end{cases}$$



$1 : n-3 \geq 0$
$n \geq 3$

note:



conversion btw $u[\,]$ and $\delta[\,]$

$$\boxed{\delta[n] = u[n] - u[n-1] \qquad u[n] = \sum_{k=-\infty}^{n} \delta[k]}$$

note: $u[n] = 0 \quad \forall\, n < 0$

so, $u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$

$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$

$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$

$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1$?2?

- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when $n - k = 0$ i.e. $k = -2$
but k is never negative since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n - k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n - k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underset{0}{\delta[1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[-1]} + \dots = \underline{1} \checkmark$$

- **SIFTING / SAMPLING**

$x[n] \cdot \delta[n] = ?$ $\quad x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

we can also say $x[n]\,\delta[n] = x[0]\,\delta[n]$

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

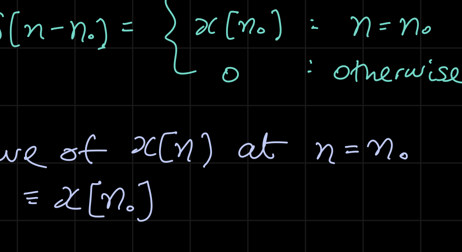
eg) $x[n] \cdot \delta[n - n_0] = ?$ $\quad x[n] \cdot \delta[n - n_0] = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- unit step function

$$u(t) = \begin{cases} 1 : t \geq 0 \\ 0 : \text{else} \end{cases}$$



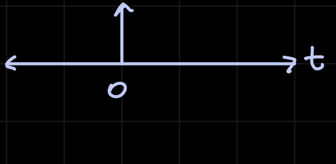
OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function $\equiv$ singularity function

$\delta(t) = 0 \quad : t \neq 0$

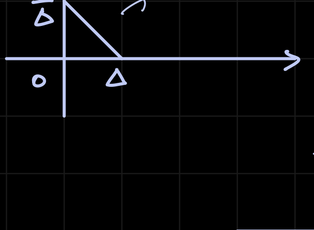
and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$



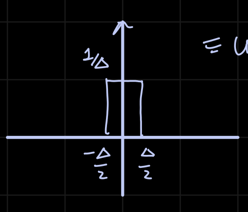
$\delta_\Delta(t)$:

infinitesimally small

$$\delta_\Delta(t) = \begin{cases} 0 : t \geq \Delta \ \&\ t < 0 \\ \frac{1}{\Delta} : \text{otherwise} \end{cases}$$



area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
$\equiv$ unit imp ulse func ✓



$\equiv$ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \int_{0}^{\infty} \delta(t - \tau)\, d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

* Tutorial : 1 30/08/24

(7) $\text{even}\{x[t]\} = \dfrac{x(t)+x(-t)}{2}$

$\text{odd}\{x[t]\} = \dfrac{x(t)-x(-t)}{2}$

8) (a) $x_1[n] = \sin(0.02\pi n)$

$\omega = 0.02\pi$

$T = \dfrac{2\pi}{\omega} =$

(3) $p(x,y) = a^2x_1y_1 + abx_1y_2 + b^2x_2y_2 + ab\,x_2y_1$

$A = \begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

$|A| = a^2b^2 - a^2b^2 = 0$

atleast one eigenvalue is zero

So, since not all eigenvalues are +ve

(d) for no values of a, b

eigenvalues are values of λ such that

$|A - \lambda I| = 0$

eigenvalues / eigenvectors

Q1) $B = \{V_1, V_2, \dots V_n\}$

B is span of V

$\bar{x} \in V$

$\bar{x} = K_1V_1 + K_2V_2 + \dots K_nV_n$

$-\ \bar{x} = C_1V_1 + C_2V_2 + \dots C_nV_n$

$0 = (K_1-C_1)V_1 + (K_2-C_2)V_2 + \dots (K_n-C_n)V_n$

since $\bar{V_i} \neq 0$, $(K_i - C_i)$ need to be 0

i.e. $K_i = C_i$

$\dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\times10^{-2}\pi} = 100 \in T$

(8) $x_1[n] = \sin(0.02\pi n)$

Periodic ✓

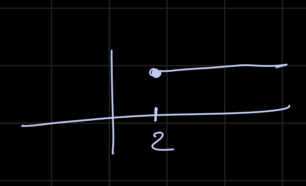
(11) $y[n] = \sum_{n=-5}^{5} \sin(2n)\,\delta[n+7]$

but range is $-5 \to 5$, so ans = 0

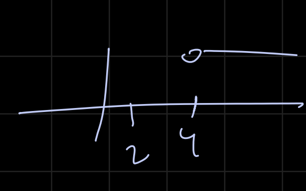
(12) $x(t) = u(t-2) - u(t-4)$

$u(t) \Rightarrow t \geq 0 : 1$

$u(t-2) \Rightarrow t \geq 2 : 1$



$u(t-4) \Rightarrow t \geq 4 : 1$



$\Rightarrow$

but since $\delta(t) = 1$ at $t = 0$

So, ans = 0

$f = \dfrac{0.3}{2\pi}$ rational / irrational ?

(8) (b) $x_2[n] = \cos\left(\dfrac{\pi}{2} + 0.3n\right)$

$\sin = \cos\left(\dfrac{\pi}{2} - \theta\right)$

(c) $x_3[n] = 1 - e^{j2\pi n/5} + e^{j4\pi n/7}$

note: $e^{j\theta} = \cos\theta + j\sin\theta$

$\Rightarrow 1 - \cos\left(\dfrac{2\pi n}{5}\right) - j\sin\left(\dfrac{2\pi n}{5}\right)$

$+ \cos\left(\dfrac{4\pi n}{7}\right) + j\sin\left(\dfrac{4\pi n}{7}\right)$

5 5 7/2 7/2

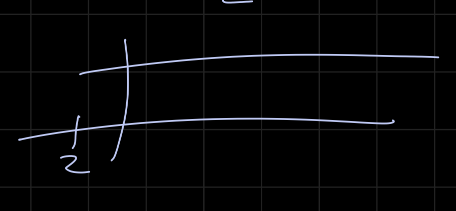
$\Rightarrow \text{LCM}\left(5, \dfrac{7}{2}\right) \Rightarrow$

(a) $\dfrac{x(t)+x(-t)}{2} = 0$

$x(t) = -x(-t)$

$\forall t$

(b) $e^{-5t}u(t+2)$



$e^{-5t}u(t+2) = -[e^{5t}u(-t+2)]$

$(e^{-5t})_{(-2,\infty)} + (e^{5t})_{(-\infty,2)}$

$(e^{-5t})_{(-2,2)} + (e^{5t})_{(-2,2)}$

(5) $P = tu(t)$

$\lim_{T\to\infty} \dfrac{1}{2T}\int_{-T}^{T} |t\,u(t)|^2 dt$

$\lim_{T\to\infty} \dfrac{1}{2T}\int_0^T t^2 dt$

$\lim_{T\to\infty} \dfrac{1}{2T}\left[\dfrac{t^3}{3}\right]_0^T$

$\lim_{T\to\infty} \dfrac{1}{2T} \times \dfrac{T^3}{3} \Rightarrow \dfrac{T^2}{6}$

$\to \infty$

neither energy nor power signal

(9) (c) $\dfrac{u[n] - u[n-4] + u[-n] - u[-n-4]}{2} = 0$

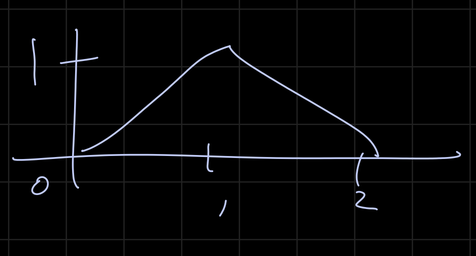
$u[n] + u[-n] = u[n-4] + u[-n-4]$

$1 : n \geq 0$ $\quad 1 : n \leq 0$ $\quad n \geq 4$ $\quad n \leq 4$

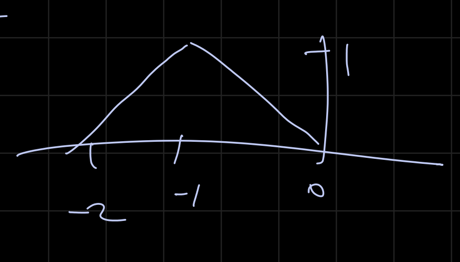
$-3 < n < 3$

$\boxed{|n| < 3}$ ✓

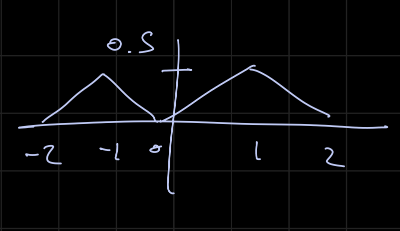
(7) $x(t) =$



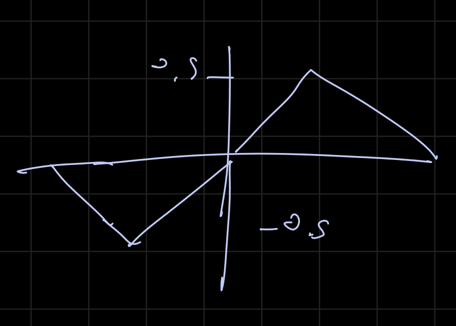
$x(-t) =$



$\text{even} = \dfrac{x(t)+x(-t)}{2} =$

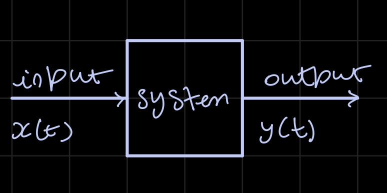


odd =



* Lecture: 5

⇒ Systems and their properties



$x[n] \rightarrow y[n]$

① System: with memory or without memory

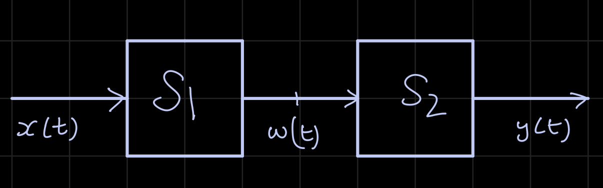
$S_1: \; y(t) = x(t)$ no memory
$S_2: \; y(t) = x^2(t)$ no memory
$S_3: \; y(t) = x(t-2)$ memory ✓ requires past value
$S_4: \; y(t) = \frac{1}{2}\{x(t) + x(t-1)\}$ mem ✓

output depends on just present inputs → no memory required

output needs to store previous outputs → requires memory

$S_5: \; y(t) = x(t+2)$: requires memory

• Invertible / Non-invertible Systems



$y(t) = x(t)$ iff $S_2 = \text{inverse}(S_1)$

ⓐ let $S_1: \; w(t) = 2x(t)$
$S_2: \; y(t) = \frac{1}{2} w(t) = \frac{1}{2} \times 2x(t) = x(t)$

S_1 is invertible and S_2 is the inverse system

ⓑ let $S_1: \; w[n] = \sum_{k=-\infty}^{n} x[n]$

then $S_2: \; y[n] = w[n] - w[n-1]$ invertible ✓

$= \{x[-\infty] + \dots x[n-1] + x[n]\} - \{x[-\infty] + \dots x[n-1]\}$

$= x[n]$

ⓒ let $S_1: \; w[n] = x[n] - x[n-1]$

$S_2: \; y[n] = \sum_{k=-\infty}^{n} w[k] = x[n]$ invertible ✓

$x[-\infty] - x[-\infty-1] + \dots x[1] - x[2] + x[2] - x[3] + x[3] - x[4] + \dots - x[n-1] + x[n-1] - x[n]$

ⓔ $S_1: \; w(t) = x^2(t)$
$S_2: \; y(t) = \sqrt{w(t)}$
} S_1 is not invertible

* System is invertible if and only if distinct inputs lead to distinct outputs

(very casual)

• Causal Systems

$S_1: \; y(t) = x(t - t_0)$: t_0 is a +ve real number

$S_2: \; y[n] = x[n+3]$

System output is depending on past system output

$S_1 \equiv$ causal system
$S_2 \equiv$ non-causal system

Causality: unless there is no cause, there should be no effect.

OR unless there is a non-zero input, there should not be non-zero output

OR a causal system depends on current and past input

eg) causal system: $y[n] = \sum_{k=-\infty}^{n} x[k]$

image processing: $S_4: \; y[n,m] = \frac{1}{5}\{x[n,m] + x[n-1,m] + x[n+1,m] + x[n,m-1] + x[n,m+1]\}$

non-causal system

• Stability System

A bounded input $\equiv |x(t)| < M_x < \infty$
finite value $\forall t$

not on the duration (y input) → but on the amplitude (x axis) ↓

OR $|x[n]| < M_x < \infty$

This bounded input is fed to a system. If the system generates an output which is also bounded, we call it a stable system.

eg) $x(t) = u(t)$; $x[n] = u[n]$
$[0,1]$

unbounded: $x(t) = tu(t)$ OR $x[n] = tu[n]$

if $|y(t)| < M_y < \infty \; \forall t$, for bounded input,
↳ finite number

then the system is stable

eg) $S_1: \; y[n] = \frac{1}{3}\{x[n] - x[n-1] - x[n+1]\}$

$|y[n]| = |\frac{1}{3}x[n] - \frac{1}{3}x[n-1] - \frac{1}{3}x[n+1]|$

since $|x[n]| < M_x < \infty \; \forall n$

and the sum & difference of 3 finite terms is finite, so,

$|y[n]| < M_y < \infty \; \forall n$

$S_3: \; y[n] = nx[n]$

$|x[n]| < M_x < \infty \; \forall n$
bounded input assumed

$\Rightarrow |y[n]| = |nx[n]| \rightarrow \infty$ as $n \rightarrow \infty$

unstable system

$S_4: \; y(t) = x(2t)$

no manipulation on the y axis
2t means manipulation on the x-axis

• Time Invariant System

if $x(t) \xrightarrow{\text{system}} y(t)$

then $x(t - t_0) \rightarrow y(t - t_0)$

and $x(t + t_0) \rightarrow y(t + t_0)$

system output does not change if there is a change in time

if $x_1(t) \rightarrow y_1(t)$
and $x_2(t) \rightarrow y_2(t)$
and $x_2(t) = x_1(t - t_0)$

then $y_2(t)$ must be equal to $y_1(t - t_0)$

if it is a time invariant system

eg) $S: \; y(t) = x(2t)$

⇒ let $t_0 = 4$: $y(t - t_0) = x(2t - 8)$

⇒ time variant system

$x_1(t) \rightarrow y_1(t) = x_1(2t)$
$x_2(t) \rightarrow y_2(t) = x_2(2t)$

$y_1(t - t_0) = x_1(2t - 2t_0)$
first scaling : $x_1(2t)$
$\downarrow t \rightarrow t - t_0$
then delay : $x_1(2(t - t_0))$

$y_2(t - t_0) = x(2t - t_0)$
first delay : $x(t - t_0)$
↓
then scaling : $x(2t - t_0)$

$S_2: \; y[n] = nx[n]$

$x_1[n] \rightarrow y_1[n] = nx_1[n]$
↓ delayed output
$y_1[n - n_0] = (n - n_0)x[n - n_0]$

$x_2[n] \rightarrow y_2[n] = nx_2[n]$

let $x_2[n] = x_1[n - n_0]$ be delayed input

$y_2[n] = (n)x[n - n_0]$

since $y_1[n] \neq y_2[n]$, system is time varying

note: 1st system had delayed output and we gave delayed input

• Linearity

: A system is linear if the properties of additivity & homogeneity holds true

$x_1(t) \rightarrow y_1(t)$
$x_2(t) \rightarrow y_2(t)$

⇒ $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
then system is linear

eg) $y(t) = 2x(t) + 3$

$x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$

$x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$

$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) = 2(x_1 + x_2) + 2(3)$

but $y(x_1 + x_2) = 2(x_1 + x_2) + 3$

Since, $y_1(t) + y_2(t) \neq y(x_1 + x_2)$

not a linear system

• A system which is linear and time invariant $\equiv$ LTI
↓
linear & time invariant system

* Lecture: SNS 6 4/9/24

Q) $x[n] = \cos(\frac{\pi}{8} n^2)$

periodic if $x(n) = x[n+N] \quad \forall n$

$\cos(\frac{\pi}{8}(n^2 + N^2 + 2nN)) = \cos(\frac{\pi}{8} n^2)$

$\cos(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{4} nN) = \cos(\frac{\pi}{8} n^2)$

we know $\cos(2\pi\theta + K) = \cos\theta \quad \forall K \in \mathbb{Z}$

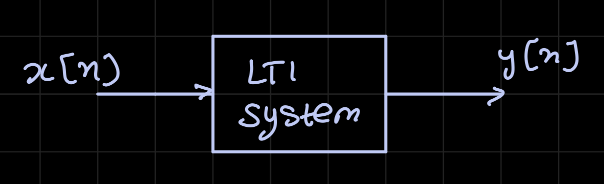
$\frac{\pi}{8} N^2 \Rightarrow$ can it be a multiple of 2π?

if $N = 4 \Rightarrow 2\pi = K2\pi$ ✓ ($K=1$)
true for $N=8$ as well
but not for $N=2$

$\frac{\pi}{4} nN \to K2\pi$?
$N=4 \Rightarrow \pi 4 n \to$ ✗
$N=8 \Rightarrow$ ✓ $2\pi n N$ = ✓

---

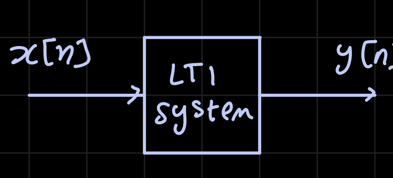
* Discrete Time Convolution



$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K]$

weighted linear
combination of delayed signals

* Impulse Response of an LTI System



$\delta[n] \to y[n] = h[n]$
= Impulse response
of the LTI system

$\delta[n-1] \to h[n-1]$
$x[1]\,\delta[n-1] \to x[1]\,h[n-1]$

$x(n) = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K] \longrightarrow \sum_{K=-\infty}^{\infty} x[K]\,h[n-K] = y[n]$

discrete time
convolution of
$x[n] \times h[n] = y[n]$

Valid for LTI systems only

convolution $\to$ linear & commutative

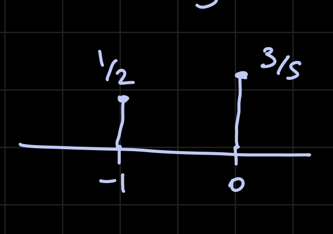
$x[n] \times h[n] = h[n] \times x[n]$

$(x_1[n] + x_2[n])\,h[n] = x_1[n]\,h[n] + x_2[n]\,h[n]$

eg) given an LTI system whose impulse response is

$h[n] = \begin{cases} 1 & n = 0,1,2 \\ 0 & \end{cases}$

and $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$



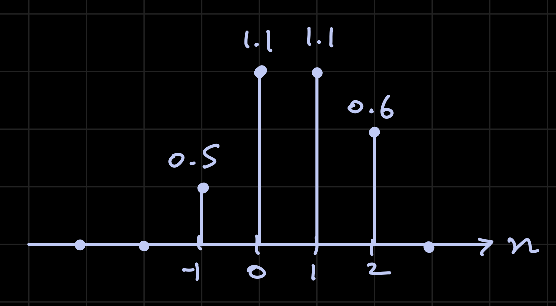
compute $y[n]$

Ans) $y[n] = h[n] * x[n]$

$y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1]$

because for $x[n] \to y[n]$,
$\delta[n] \to h[n]$

$y[n] = \frac{3}{5}[\delta(n) + \delta(n-1) + \delta(n-2)]$
$+ \frac{1}{2}[\delta[n+1] + \delta[n] + \delta(n-1)]$



eg) LTI system (← given)

$h[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$

$x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

Ans) $y(n) = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$

OR $\equiv \sum_{-\infty}^{\infty} h[K]\,x[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} x[n-K] = \sum_{K=0}^{\infty} \alpha^{(n-K)} u[n-K]$

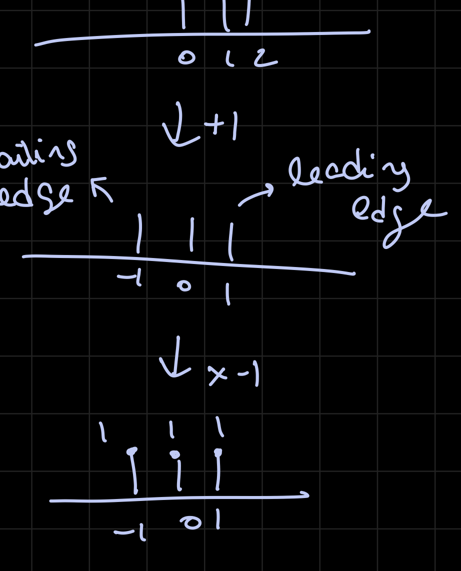
$\Rightarrow \sum_{K=0}^{\infty} \frac{\alpha^n}{\alpha^K} u[n-K]$ (1 : $n \geq K$)

$y(n) = \sum_{K=0}^{n} \frac{\alpha^n}{\alpha^K}$ ✓

$\frac{a(1-r^n)}{1-r}$

$\times -1$: $-K \xrightarrow{+1} -K+1$

$K \to (1-K)$
$\downarrow +1$
$K+1$
$\times -1$: $-K+1$



* LECTURE: 7

06/09/24

Note: $y[n] \rightarrow$ function $\equiv$ variable
$y[10] \rightarrow$ scalar $\equiv$ constant

$h(n) \equiv$ characteristic impulse response of the system

$$y[n] = x[n]\,h[n]$$

An LTI system is uniquely charaterized by its $h[n]$ - hence two systems are same if they are LTI and their $h[n]$ are same

bonus question} $\int_{-\infty}^{t} \cos(\tau)\, x(\tau)\, d\tau$

let $x(\tau) = \cos(\tau)$

$$\Rightarrow \int_{-\infty}^{t} \cos^2(\tau)\, d\tau \Rightarrow \int_{-\infty}^{t} \frac{1+\cos(2\tau)}{2}\, d\tau$$

$$\Rightarrow \left[\frac{t}{2} + \underbrace{\overbrace{\frac{\sin(2t)}{4}}^{\in[-1,1]}}_{\text{bounded}}\right]_{-\infty}^{t} \Rightarrow$$

becomes unbounded as $t \rightarrow \infty$ ($\equiv$ unstable)

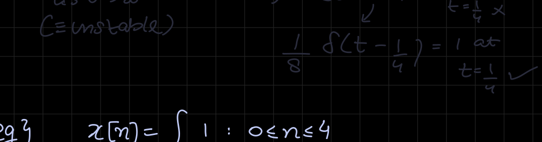
Q1) $\delta(8t-2) = 1$ at $t = \frac{1}{4}$ ✗

$\frac{1}{8}\delta(t - \frac{1}{4}) = 1$ at $t = \frac{1}{4}$ ✓

eg} $x[n] = \begin{cases} 1 : & 0 \le n \le 4 \\ 0 : & \text{else} \end{cases}$

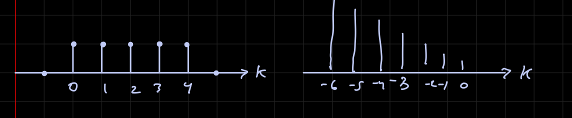
$x[n] = u[n] - u[n-4]$

$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \quad \& \ \alpha > 1 \\ 0 & \text{else} \end{cases}$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$y[0] = \sum_{-\infty}^{\infty} x[k]\,h(-k)$$



only common point : $k=0$

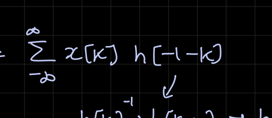
$$y[n] = x[0]\,h[-0] = h[n] = \alpha^0 = \underline{1}$$

$$y[1] = \sum_{-\infty}^{\infty} x[k]\,h[-1-k]$$

$h(k) \xrightarrow{-1} h(k-1) \rightarrow h(-k-1)$

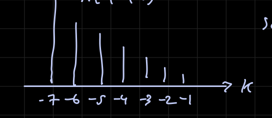
shifting → scaling → reversing

✗



$h[1-k] = 1$ at $1-k = 0 \Rightarrow k = 1$

but we do scaling then translation here



Check: $h[1-k] = 1$ at $1-k=0$ because $\alpha^k = \alpha^0 = 1$

$k=1$

$\Rightarrow y(k) = x(k)\,h[1-k]$

now, we shifted $h[-k]$ 1 unit to the right and we have overlap with $x(k)$ on $k = 0,1$

Similarly for $x[k-n]$, we have overlap with $k = 0, 1, \ldots, n-1, n$

404: Brain not found
2nd attempt initiated ...

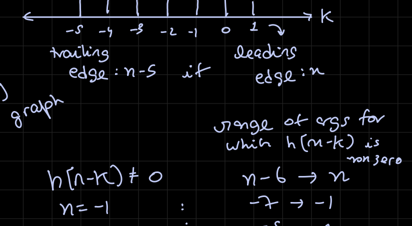
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$x[k]$

we are not shifting $x(k)$ in this case but we could

$h[-k]$ $(n=0)$

$h[1-k]$ $(n=1)$



verification: $h(0)$ at $1-k=0$, $k=1$ $\equiv$ same as $h[-0]$ in previous graph (argument is zero)

trailing edge: $n-5$ if leading edge: n

range of args for which $h(n-k)$ is non zero

$h[n-k] \neq 0$		
		$n-6 \rightarrow n$
$n=-1$	:	$-7 \rightarrow -1$
$n=1$	:	$-5 \rightarrow 1$
$n=-2$	:	$-8 \rightarrow -2$

① when $n<0$, there is no overlap with $x[n]$ hence $y[n] = 0$

② $n=11$: 5 to 11
$n=10$: 4 to 10

So, no overlap with $x(n)$ when $n>10$

hence, $y[n] = 0$: $n<0$ and $n>10$

③ $0 \le n < 4$: $y[n] = \sum_{k=0}^{n} x[k]\,h[n-k]$

overlap	n	range
$k=0$	0	$-6 \rightarrow 0$
$k=0,1$	1	$-5 \rightarrow 1$
$k=0,1,2$	2	$-4 \rightarrow 2$
$k=0,1,2,3$	3	$-3 \rightarrow 3$
$k=0,1,2,3,4$	4	$-2 \rightarrow 4$
$k=0,1,2,3,4$	5	$-1 \rightarrow 5$
$k=0,1,2,3,4$	6	$0 \rightarrow 6$
$k=1,2,3,4$	7	$1 \rightarrow 7$
$k=2,3,4$	8	$2 \rightarrow 8$
$k=3,4$	9	$3 \rightarrow 9$
$k=4$	10	$4 \rightarrow 10$

leading edge of h left $x(n)$

So, $y[n]$ is defined for $0 \le n \le 10$

② $0 \le n \le 4 \Rightarrow y[n] = \sum_{k=0}^{n} x[k]\,h[n-k]$

③ $4 < n \le 6 \Rightarrow y[n] = \sum_{k=0}^{4} x(k)\,h[n-k]$

④ $6 < n \le 10 \Rightarrow y[n] = \sum_{k=n-6}^{4} x[k]\,h[n-k]$

overlap: $1,2,3,4 \rightarrow 2,3,4 \rightarrow 3,4 \rightarrow 4$
n: $7 \rightarrow 8 \rightarrow 9 \rightarrow 10$

final $\rightarrow$ $y[n] = \sum_{k=0}^{n} \alpha^{n-k}$: $0 \le n \le 4$
$+ \sum_{k=0}^{4} \alpha^{n-k}$: $4 < n \le 6$
$+ \sum_{k=n-6}^{4} \alpha^{n-k}$: $6 < n \le 10$

convolution for CTS = DTS

$$\sum \alpha^{n-k} = \alpha^n \sum \frac{1}{\alpha^k} = \alpha^n \{GP\}$$

$\frac{a(1-r^n)}{1-r}$

$\boxed{CTS}$

$x(t) \rightarrow \boxed{LTI} \rightarrow y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

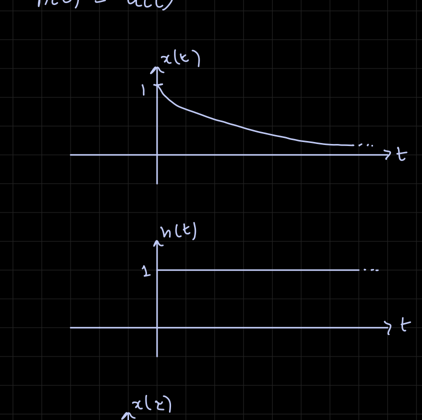
useful when $h(t) = u(t)$

$y(t) = x(t) * h(t)$

convolution operation

eg) $x(t) = e^{-at}u(t)$: $a>0$

$h(t) = u(t)$



leading edge: $\tau = 0$

no overlap when we shift $h(-\tau)$ to left, there will be no overlap and hence that integral $= 0$ ✗

shift to right ✓

① $t<0 \Rightarrow y(t) = 0$

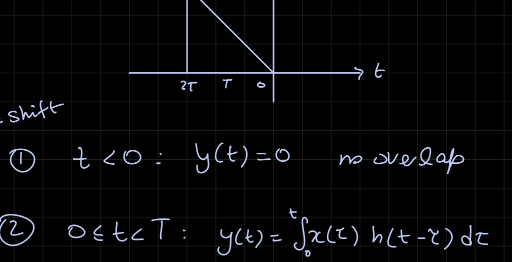
② $t>0 \Rightarrow y(t) = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$ ✓

integral $-\frac{e^{-at}}{a}$

starts at 0 then saturates as $t \rightarrow \infty$

eg} $x(t) = \begin{cases} 1 & : 0 < t < T \\ 0 & : \text{else} \end{cases}$

$h(t) = \begin{cases} t & : 0 < t < 2T \\ 0 & : \text{else} \end{cases}$



t = shift

① $t<0$: $y(t) = 0$ no overlap

② $0 \le t < T$: $y(t) = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$

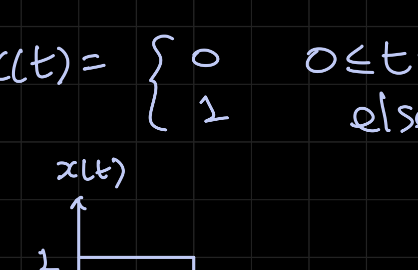
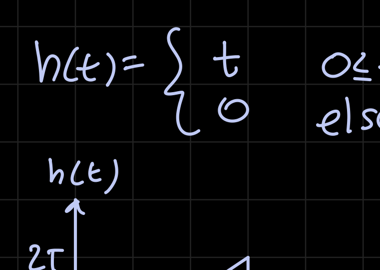
③ $T \le t \le 2T$: $y(t) = \int_{0}^{T} x(\tau)\,h(t-\tau)\,d\tau$

④ $2T < t \le 3T$ $y(t) = \int_{t-2T}^{T} x(\tau)\,h(t-\tau)\,d\tau$

* Lecture : 8 11/09/24

⇒ convolution

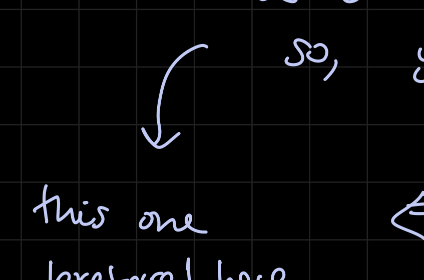
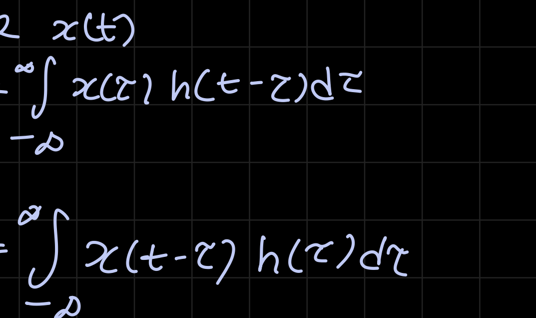
$x(t) = \begin{cases} 0 & 0 \le t \le T \\ 1 & \text{else} \end{cases}$ $h(t) = \begin{cases} t & 0 \le t \le 2t \\ 0 & \text{else} \end{cases}$

easier to reverse $x(t)$

so, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

this one preferred here ⇐ $= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$

at $y(0)$ ⇒ overlap = none

so, $y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

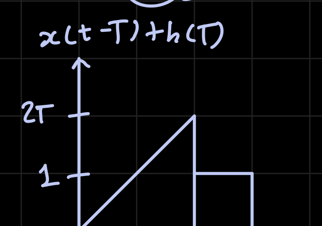
moving forward, $y(1) \dots y(n)$
if we shift $x(-\tau)$ to left
i.e. $t<0$, it will never overlap

So, we move to right,
$t>0$
when plotting $x(t-\tau)$

case ① } $t<0$

$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

case ② } $t>3t$ again no overlap

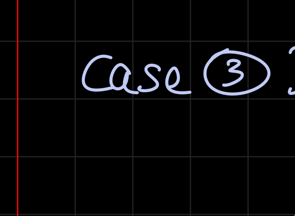


$y(t) = 0$

no non-zero overlap

Case ③ } $0<t<T$

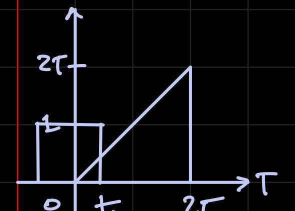
$y(t) = \int_0^t h(\tau)\, d\tau = \int_0^t \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t$

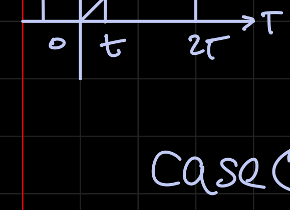


$= \frac{t^2}{2}$

Case ④ : $T \le t \le 2T$

$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{-T^2}{2} + tT$





case ⑤ } $2T \le t \le 3T$

$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$

$= \frac{4T^2}{2} - \frac{(t-T)^2}{2}$

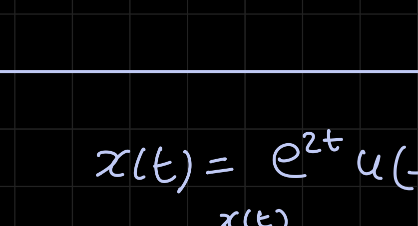
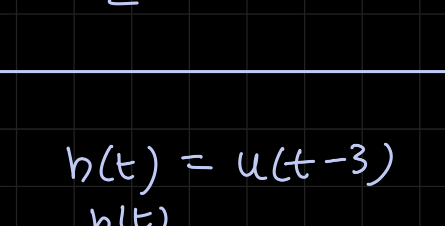
$= \frac{4T^2 - t^2 - T^2 + 2Tt}{2}$

$= \frac{3T^2 + 2tT - t^2}{2}$

---

eg } $x(t) = e^{2t} u(-t)$ $h(t) = u(t-3)$

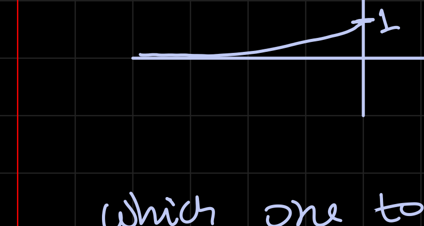
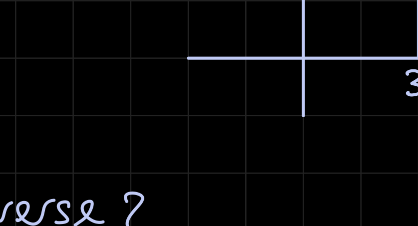
$y(t) = \int x(\tau)\, h(t-\tau)\, d\tau$
$= \int x(t-\tau)\, h(\tau)\, dt$

Which one to reverse?
we can reverse
both but which one is easier?

$t \ge 3$

$h(-\tau)$

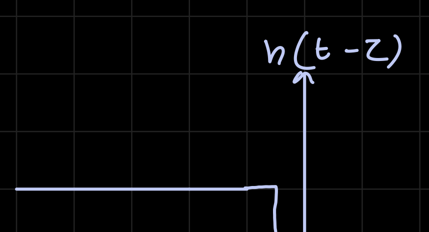



case 1: $t \le 3$ in $h(t-\tau)$

moves to right if $t>0$
left if $t<0$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{t-3} x(\tau)\, d\tau$



$\int_{-\infty}^{t-3} e^{2\tau}\, d\tau$

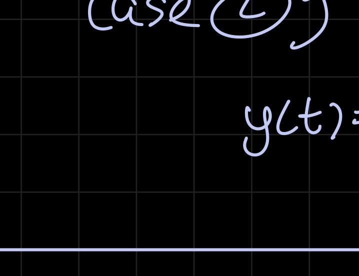
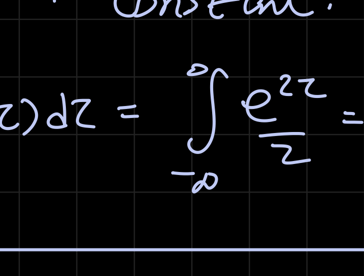
⇒ $\left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t-6}}{2} = \frac{e^{2t}}{2e^6}$

case ② } $t>3$: constant: $e^{2\tau}$

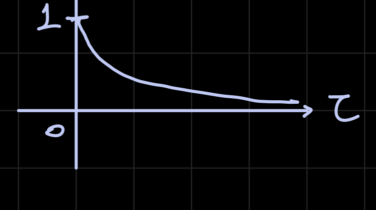
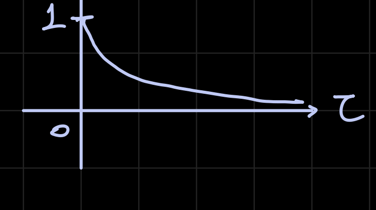
$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \int_{-\infty}^{0} \frac{e^{2\tau}}{2} = \frac{1}{2}$ ✓

---

eg } $x(t) = e^{at} u(-t)$: $a>0$
$h(t) = e^{-at} u(t)$

flip any here as similar

Shifting by t Ranges: $t>0$, $t\le 0$
OR $t\ge 0$, $t<0$

Case ① : $t=0$

$y(0) = \int_0^{\infty} x(-\tau)\, h(\tau) = \int_0^{\infty} e^{-2a\tau}\, d\tau$

⇒ $\frac{e^{-2a\tau}}{-2a}\Big|_0^{\infty}$

⇒ $0 + \frac{1}{2a} = \frac{1}{2a}$

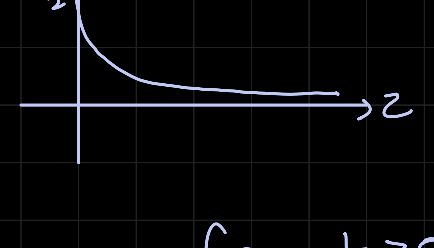
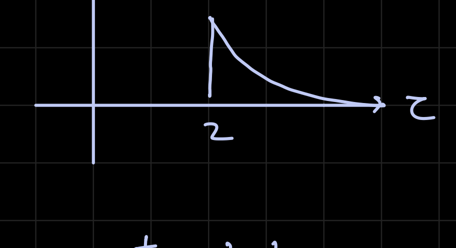
case ③: $t<0$

$y(t) = \int_0^{\tau - t} x(t-\tau)\, h(\tau)\, d\tau$

$t - \tau \to \infty$

$= \int_0^{\tau - t}$

left edge of $x(-\tau)$ will always be equal to t

for $t \ge 0$: $t \to \infty$

for $t<0$: $0 \to \infty$

Case ① : $t \ge 0$ $t \to \infty$

$y(t) = \int_t^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$\int_t^{\infty} e^{-at}\, e^{a(t-\tau)}\, d\tau$

case ② : $t<0$ $0 \to \infty$

$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_0^{\infty} e^{-at} \cdot e^{a(t-\tau)}\, d\tau$

* SYSTEM PROPERTIES

given an LTI system and

$h[n] = \begin{cases} 1 & : n=0,1 \\ 0 & : \text{else} \end{cases}$

defaults: input : $x[\,]$
output : $y[\,]$
impulse response : $h[\,]$

S: $y[n] = ?$ ⇒ System output = ?

apply convolution

$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

from the given defn, $h[n] = \delta[n] + \delta[n-1]$

⇒ $h[k] = \delta[k] + \delta[k-1]$

so, $y[n] = \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1])\, x[n-k]$

$= \sum_{-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{-\infty}^{\infty} \delta[k-1]\, x[n-k]$

$= x[n] + x[n-1]$

eg } find impulse response of systems

S_2 : $y[n] = (x[n] + x[n-1])^2$

S_3 : $y[n] = \max\{x[n], x[n-1]\}$

we know: $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

There is only one LTI system
that gives the same impulse response

these systems S_2, S_3 give same response
because they are not LTI

* Lecture: 9

13/09/24

• SYSTEM PROPERTIES

② (i) Commutativity

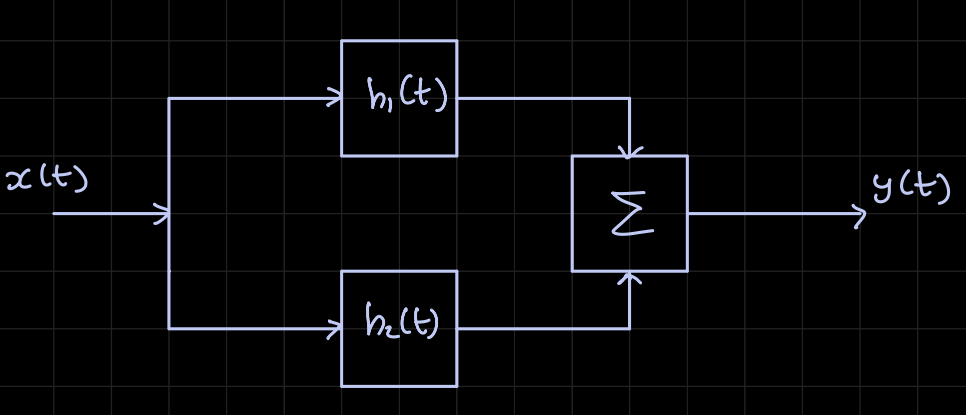
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Prove

(ii) Convolution distributes over addition

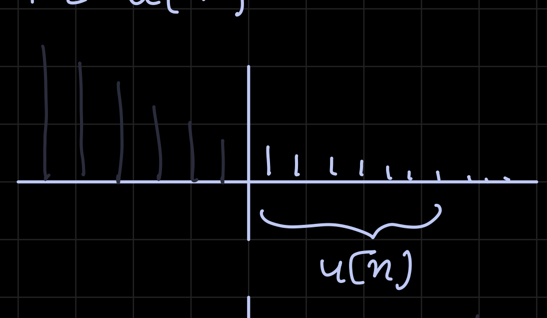
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



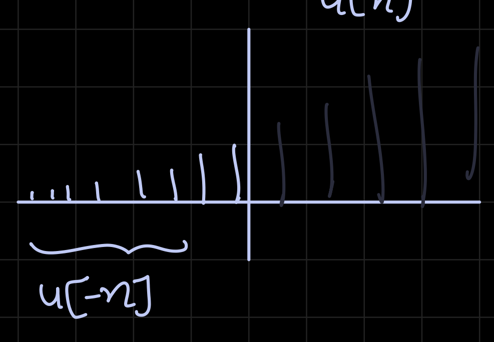
holds true for both CTS and DTS

eg) $x[n] = (\frac{1}{2})^n u[n] + 2^n u[-n]$

$x_1[n] = 2^n u[n]$



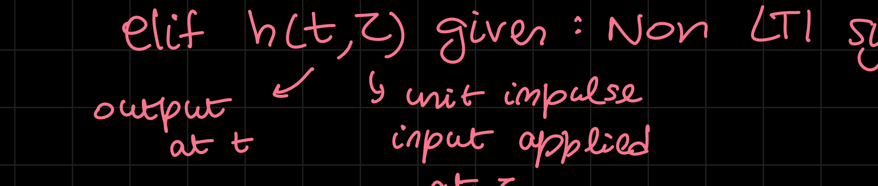
$x_2[n] = 2^n u[-n]$



easier: $y[n] = x_1[n] * h[n] + x_2[n] * h[n]$

(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x_1(t) * h_1(t)] * h_2(t)$$



* only LTI

for LTI systems, in cascading, order of these systems can be swapped

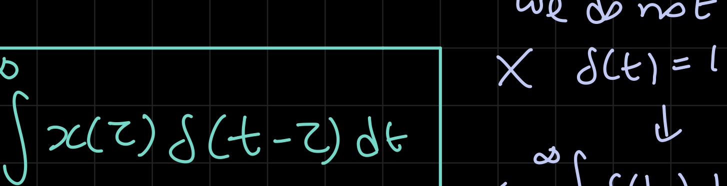
③ Invertibility :

Note: if $h(t)$ given: LTI system ✓

elif $h(t,\tau)$ given : Non LTI system ✓

output at t

unit impulse input applied at τ



eg) if $h_1(t) * h_2(t) = \delta(t)$ continuous delta(t)

$$y(t) = x(t) * (h_1(t) * h_2(t))$$

$$= x(t)\,\delta(t) \quad \} \text{ operational definition of } \delta(t)$$

$$= x(t)$$

we do not write

$\times$ $\delta(t) = 1$ at $t=0$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, dt}$$

✓ $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

not a scalar signal?

can we have a signal with ∞ value at a single point?

Discrete Time

$$x[n] = \sum_{-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$= x[n] * \delta[n]$$

for this we can say $\delta[n] = 1 : n = 0$

④ Causality

claim: if a system is causal,

$h(t) = 0$ for $t<0$ (CTS)
$h[n] = 0$ for $n<0$ (DTS)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

because x cannot have future values so $k>0$

hence, for causal systems,

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

OR $$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

if $h[n] = 0$ at $n<0$

integrator : S if $h(n) = u(n)$ CTS
accumulator : S if $h[n] = u[n]$ DTS

⑤ memory / memoryless system

$S_1 : y(t) = Kx(t) \Rightarrow h(t) = K\delta(t)$
$S_2 : y[n] = Kx[n] \Rightarrow h[n] = K\delta[n]$

⑥ stability (BIBO Stability Criteria)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

let input signal is bounded and $|x(t)| < M_x \ \forall t$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau) d\tau| \le M_x \times (\text{some finite number}) < \infty \ \forall t$$

if the system is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau) d\tau| < \infty$$

DTS: $|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

if the system holds absolute summability,

then the system is stable

Note: $h[n] = u[n]$, $h(t) = u(t)$ } LTI, unstable, causal, with memory

but still we use these for integrators & accumulators because we use clipper and clamper since we know the upper and lower bound

Identity : system with $h(t) = \delta(t)$

* Constant Coefficient difference eqⁿ for causal LTI systems

$$\sum_{k=0}^{N} b_k\, y[n-k] = \sum_{k=0}^{M} a_k\, x[n-k]$$

a_k, b_k: not functions of time ∴ constant coeff

① $N \equiv$ order of the system

② The system is at rest if :

if, $x[n] = 0 : n < n_0$
then, $y[n] = 0 : n < n_0$
$y[n-1] = 0 : n < n_0$
$\vdots$
$y[n-N] = 0 : n < n_0$

} auxilliary conditions

note: $N > 0$

MA

→ case : 1 $\Rightarrow N = 0$

$$b_0 y[n] = a_0 x(n) + a_1 x[n-1] + \ldots a_m x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_1} x[n-1] + \ldots \frac{a_m}{b_m} x[n-M]$$

causal system ✓ $\equiv$ moving average system → weighted sum

$n \to n-1 \to n-2 \to n-3 \cdots$

system is called finite impulse response system

$\equiv$ FIR system

IIR / AR

→ case 2: $N = 1$ and $M = 0$ (simplicity)

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] - \alpha_1 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\alpha_1 = \frac{b_1}{b_0}$

if $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$h[0] = \alpha_0$
$h[1] = -\alpha_1\alpha_0$
$h[2] = +\alpha_1^2\alpha_0$
$\vdots$

} FIR? $\times$

Infinite impulse response system ✓ (IIR)

* Auto Regressive System

$\equiv$ AR system

output depends on regresses of its previous/past inputs

ARMA

→ Case : 3 $N > 0$, $M > 0$

* Lecture : 10

18/09/24

⇒ constant coeff differential eq^n representation of causal LTI systems

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k x(t)}{dt^k}$$

$N \equiv$ order of the system

let $N = 2$ & $M = 3$

$$b_2 \frac{d^2 y(t)}{dt^2} + b_1 \frac{dy(t)}{dt} + b_0 y(t) =$$
$$a_0 x(t) + a_1 \frac{d\, x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a^3 \frac{d^3 x(t)}{dt^3}$$

Initial condition:

if $x(t) = 0 \quad \forall \; t < t_0$

then $\frac{d^k y(t)}{dt^k} = 0 \quad \forall \; t < t_0$

$\forall \; k \in [0, N]$

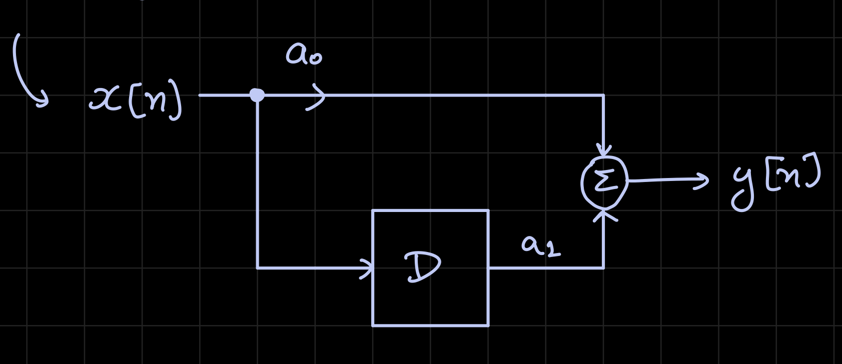
* Block diagram representation

useful for DTS

let $a_0 x[n] \equiv$ scalar multiplier

then $x[n-1] \equiv$ delay element

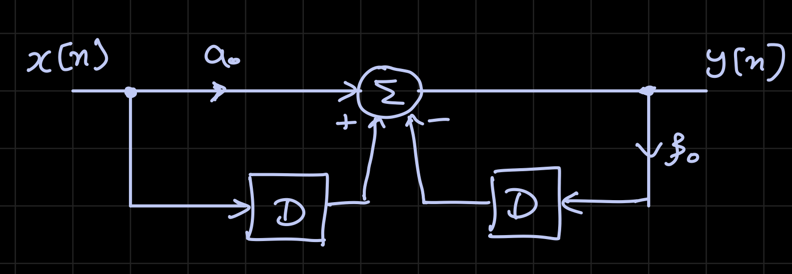
let S_1: $y[n] = a_0 x[n] + a_1 x[n-1]$



S_2: $b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$
$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\beta_0 = \frac{b_1}{b_0}$



* FOURIER SERIES

we know, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

and $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)$

$x[k]$ are scalars / constants
we can write them as a_k

$$x[n] = \sum_{k=-\infty}^{\infty} a_k\,\delta[n-k]$$

$\boxed{CT}$ $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Synthesis equation $\left\{ x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \right.$

weighted sum of frequencies

finite basis

$\phi_k(t) = e^{jk\omega_0 t}$

(freq. of oscillation)

weighted linear combination of vectors

$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

$\omega_0 = \frac{2\pi}{T}$

fundamental time period of $x(t)$

$f = \omega_0$ rad/s
freq ≠ -ve
Time Period = T
counter clockwise $\phi_1(t) = e^{j\omega_0 t}$ $\quad\quad \phi_2(t) = e^{2j\omega_0 t}$

completes 2 cycles in one time period T
freq = $2\omega_0$ rad/s

Time Period = T
clockwise $\phi_{-1}(t) = e^{-j\omega_0 t}$ $\quad\quad \phi_{-2}(t) = e^{-2j\omega_0 t}$

freq ≠ -ve
$f = \omega_0$ rad/s

$\phi_0(t) = e^{j0} = e^0 = 1 = \cos 0 + j\sin 0$ ✓

DC component of signal
no fluctuation

can we write $\phi_2(t)$ in terms of $\phi_1(t)$?

No

they are independent of each other

inner product
$$\langle \phi_k(t), \phi_L(t) \rangle_T = \frac{1}{T}\int_0^T \phi_k(t)\,\phi_L^*(t)\,dt$$
$$= \frac{1}{T}\int_0^T e^{jk\omega_0 t} \cdot e^{-jL\omega_0 t}\,dt$$

integers; let = m

$$= \frac{1}{T}\int_0^T e^{j(k-L)\omega_0 t}\,dt$$
$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt$$
$$= \begin{cases} 0 & : k \neq L \\ 1 & : k = L \end{cases}$$

defining the inner product

$$\langle \phi_k(t), \phi_L(t) \rangle_T = \frac{1}{T}\int_T \phi_k(t)\,\phi_L^*(t)\,dt$$

(continuous time periodic signals)

⇒ **Synthesis:** $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

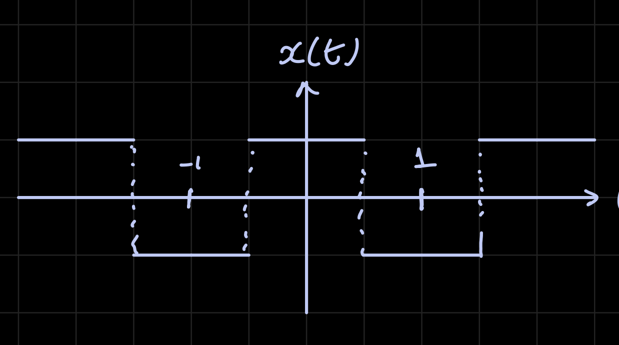
$\{\phi_k(t)\}$ = span a vector space from an orthonormal basis

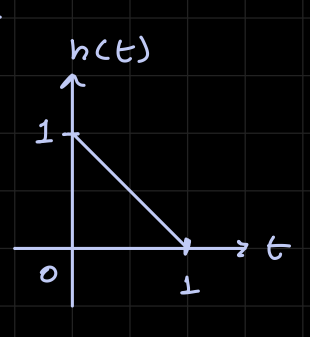
⇒ **Analysis:** $a_k = \langle x(t), \phi_k(t) \rangle_T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_L(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_L(t) \right\rangle_T$$
$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\langle \phi_k(t), \phi_L(t) \rangle_T}_{\delta[k-L]}$$

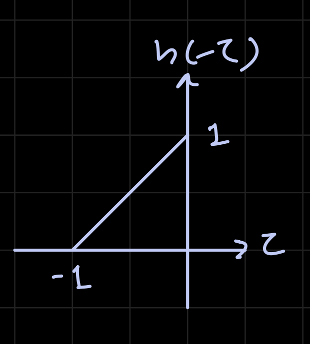
eg3



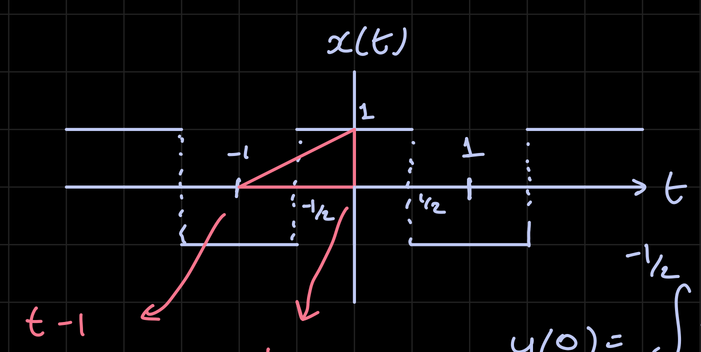


convolute $x(t)$ & $h(t)$

Ans3 $\quad h(\tau) = 1 - \tau$
$\quad\quad\; h(-\tau) = 1 + \tau$ reflect



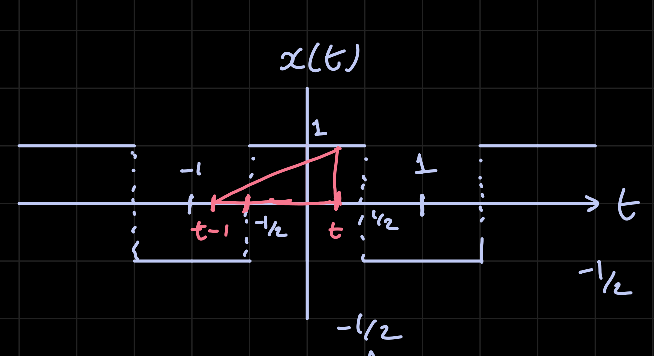
overlap



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(0) = \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

case : 1 : $\quad 0 \leq t \leq \frac{1}{2}$



$$y(t) = \int_{t-1}^{-1/2} (1+\tau+t)\,d\tau + \int_{-1/2}^{t} (1+\tau+t)\,d\tau$$

case : 2 ⇒

* Lecture: 11

⇒ Continuous time periodic signals FOURIER SERIES

$e^{jk\omega_0 t} \to e^{j2k\omega_0 t}$

time period is decreasing

constant → pure imaginary exponential (no real part) (in power)

synthesis equation
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$ rad/s, $f_0 = \frac{1}{T}$ per sec or Hz

eg: $x(t) = \underbrace{3}_{(DC)\ 3} + \underbrace{2e^{j\omega_0 t}}_{\text{amplitude } 2} + \underbrace{e^{j2\omega_0 t}}_{1}$ synthesis equation

$= 3 + 2(\cos\omega_0 t + j\sin\omega_0 t) + \cos 2\omega_0 t + j\sin 2\omega_0 t$

will keep oscillating till ∞ and pattern holds and since it is not decaying ⇒ periodic

$= \sum_{k=0}^{2} a_k e^{jk\omega_0 t}$

$a_0 = 3$, $a_1 = 2$, $a_2 = 1$ } in this case

all other a_k ∀ $k \neq 0,1,2$ are zero

analysis equation

$a_k = \langle x(t), \phi_k(t) \rangle_T$ — 2nd coordinate → hence we take conjugate

$= \frac{1}{T}\int_T x(t)\phi_k^*(t)\,dt$

$= \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt$

$\{\phi_k(t)\}_{k\in\mathbb{Z}} \equiv$ orthogonal basis $\Rightarrow \int_T \phi_m(t)\phi_k^*(t)\,dt = T\delta[m-k]$

$$a_k = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m\left(\int_T \phi_m(t)\phi_k^*(t)\,dt\right)\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T e^{j(m-k)\omega_0 t}\,dt\right]$$

$$= \frac{1}{\cancel{T}}\sum_{m=-\infty}^{\infty} a_m \cdot \cancel{T} \cdot \delta[m-k] = a_k = \underline{LHS}$$

$\phi_1(t), \phi_{-1}(t) \Rightarrow$ 1st harmonic

$\phi_2(t), \phi_{-2}(t) \Rightarrow$ 2nd harmonic

and so on...

* Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

* Analysis: $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt$

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

lets assume $x(t) = e^{st}$ where $s = j\omega_0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{st}\cdot e^{-s\tau}\,d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \;\} \text{ let } H(s)$$

constant

$$\boxed{y(t) = e^{st}\cdot H(s)}$$

e^{st} = act as eigenfunctions of an LTI system
$H(s)$ = corresponding eigenvalue $A\bar{v} = \lambda\bar{v}$

$$e^{st} \longrightarrow \boxed{LTI} \longrightarrow e^{st}\cdot H(s)$$

↳ here the LTI system is acting as a linear operator

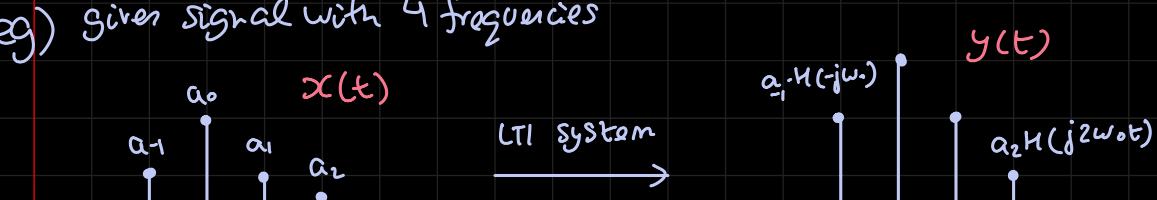
if i feed an LTI system with a continuous signal with some frequencies, then, it outputs a signal with same frequencies but with different amplitudes of the frequencies

output ≡ **Line Spectrum** of $y(t)$

existing frequencies can disappear in output but a different frequency will not get added up

eg: set $H(-j\omega_0) = 0$
so, $a_{-1}\cdot H(-j\omega_0) = 0$
this freq. will disappear in output

eg) given signal with 4 frequencies



freq domain representation

eg) S: $y(t) = x(t-2)$ and $x(t) = e^{j2t}$

Ans) only delaying system observed from eqn

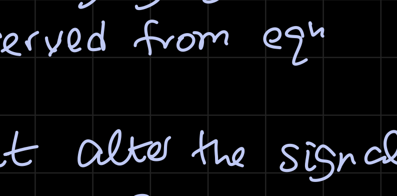
will it alter the signal frequencies? Intuitively, No

$x(t) = e^{j2t}$ ↳ $\omega_0 = 2$ rad/s

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$ $\boxed{T = \pi}$

We have only one freq. component in the line spectrum of $x(t)$

$x(t) = \underset{\text{amplitude}}{1}\cdot e^{j2t}$

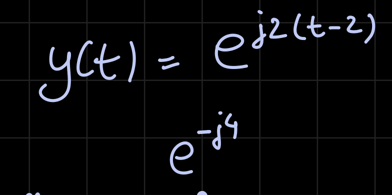


$$y(t) = e^{j2(t-2)} = e^{j2t}e^{-j4} = e^{j2t}\cdot H(j2)$$

$\boxed{H(j2) = e^{-j4}}$

note: in $H(s)$, $s = j\omega$, so, $H(j2) = e^{-j4}$

line spectrum of $y(t)$



no more freq. will be added

same freq. comes out with altered amplitude

---

eg} What is $h(t)$? $h(t) = \delta(t-2)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau = \int_{-\infty}^{\infty}\delta(\tau-2)e^{-j2\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty}\delta(\tau-2)e^{-j2\cdot 2}\,d\tau \Rightarrow H(j2) = e^{2}$$

NOTE: $x(t) * \delta(t) = x(t)$

$x(t) * \delta(t-t_0) = x(t-t_0)$

shifts the signal by t_0 units

---

eg} $x(t) = \sin(\omega_0 t)$

find and plot the line spectrum of $x(t)$

$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$
$e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$

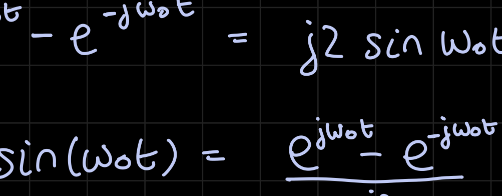
$e^{j\omega_0 t} - e^{-j\omega_0 t} = j2\sin\omega_0 t$

so, $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2} = \frac{-j}{2}\cdot e^{j\omega_0 t} + \frac{j}{2}e^{-j\omega_0 t}$

2 frequencies with amplitude $\frac{-j}{2}$ and $\frac{j}{2}$

we have form $a_k e^{jk\omega_0 t}$

So we have 2 frequencies at $k = 1$ and -1



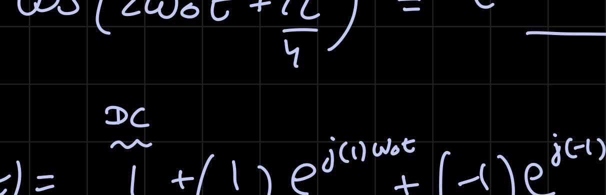
eg} $x(t) = 1 + \sin(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

Ans} $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$\cos\left(2\omega_0 t + \frac{\pi}{4}\right) = \frac{e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{j(-2\omega_0 t + \frac{\pi}{4})}}{2}$

$$x(t) = \underbrace{1}_{DC} + \left(\frac{1}{2j}\right)e^{j(1)\omega_0 t} + \left(\frac{-1}{2j}\right)e^{j(-1)\omega_0 t} +$$
$$\left(\frac{1}{2}\right)e^{j2\omega_0 t}\cdot e^{j\frac{\pi}{4}} + \left(\frac{1}{2}\right)e^{j(-2)\omega_0 t}e^{j\frac{\pi}{4}}$$

line spectrum of $x(t)$

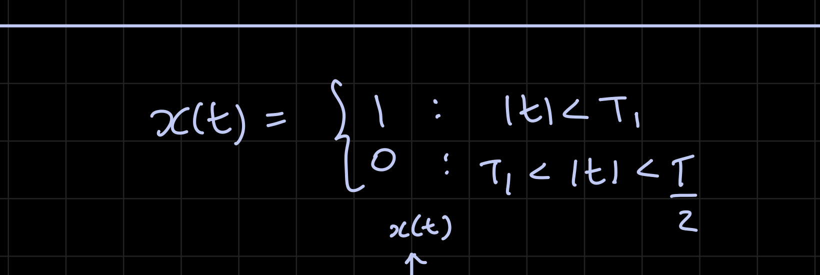


5 frequencies

if the question asks for absolute value line spectrum, then just write the mod of the complex amplitudes

---

eg} $x(t) = \begin{cases} 1 & : |t| < T_1 \\ 0 & : T_1 < |t| < \frac{T}{2} \end{cases}$



① if $T \to \infty$, signal will become aperiodic

② There should be a DC term, so $a_0 \neq 0$

HW

$$a_0 = \frac{1}{T}\int_T x(t)e^{0}\,dt = \frac{1}{T}\int_T x(t)\,dt$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} = \underline{a_0} = \frac{2T_1}{4T_1} = \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{1}{T}\left[-\frac{e^{-jk\omega_0 t}}{jk\omega_0}\right]_{-T_1}^{T_1}$$

euler

$a_k = \frac{1}{k\pi}\sin(k\omega_0 T_1)$: $k \neq 0$

$a_k = \frac{1}{k\pi}\sin\left(k\frac{2\pi}{4T_1}T_1\right)$

$a_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$

line spectrum has becomes dense
more frequency points created

* lecture: 12 — 25/09/24

• LINEARITY

• Prove the linearity property of signals

if $x(t) \overset{\text{fourier series coefficients}}{\rightleftharpoons} a_k$

and $y(t) \rightleftharpoons b_k$

$x(t)$ and $y(t)$ are periodic with fundamental time period $= T$

prove $z(t) = \alpha x(t) \beta y(t) \rightleftharpoons \alpha a_k + \beta b_k$

and $z(t)$ = same fundamental time period T

sol^n)

let $z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ where $\omega_0 = \frac{2\pi}{T}$

$$C_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T (\alpha x(t) + \beta y(t)) e^{-jk\omega_0 t} dt$$

$$= \alpha \underbrace{\frac{1}{T}\int_T x(t) e^{jk\omega_0 t} dt}_{a_k} + \beta \cdot \underbrace{\frac{1}{T}\int_T y(t) e^{jk\omega_0 t} dt}_{b_k}$$

$$= \alpha a_k + \beta b_k$$ hence proved

• TIME SHIFTING

$$x(t) \xrightarrow[T]{F.S.} a_k$$

find fourier series coefficients for $x(t - \underset{\text{constant}}{t_0})$

$$z(t) = x(t - t_0) \xrightarrow{F.S.} C_k$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

$$C_k = \frac{1}{T}\int_T x(t - t_0) e^{-jk\omega_0 t} dt$$

let $t - t_0 = z$

$dt = dz$

let limits: $-\frac{T}{2}$ to $\frac{T}{2}$

now: $-\frac{T}{2} - t_0$ to $\frac{T}{2} - t_0$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}-t_0}^{\frac{T}{2}-t_0} x(z) e^{-jk\omega_0 (t_0 + z)} dz$$

$$C_k = e^{-jk\omega_0 t_0} \underbrace{\frac{1}{T}\int_T x(z) e^{-jk\omega_0 z} dz}_{a_k}$$

$$C_k = a_k \cdot e^{-jk\omega_0 t_0}$$

There is a shift in frequency but the magnitude remains the same

• TIME REVERSAL

$$x(t) \xrightarrow[T]{FS} a_k$$

$$z(t) = x(-t) \xrightarrow[T]{FS} C_k = ?$$

note: same time period

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(-t) e^{jk\omega_0 t} dt$$

let $z = -t$ and $-\frac{T}{2} \to \frac{T}{2}$

$dz = -dt$ to $\frac{T}{2} \to -\frac{T}{2}$

$$C_k = \frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} x(z) e^{jk\omega_0 z} \underset{\text{reversed the limits instead of } -dz}{-dz}$$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(z) e^{jk\omega_0 t} dz$$

$$C_{-k} = \frac{1}{T}\int_T x(z) e^{-jk\omega_0 t} dz$$

$$C_{-k} = a_k$$

$$\boxed{C_k = a_{-k}}$$

no change in frequencies only change in magnitude

• COMPLEX CONJUGATE

$x(t) \xrightarrow[T]{F.S.} a_k$ time period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$z(t) = x^*(t) \xrightarrow{F.S.} ?$$

$$z(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$\downarrow k \to -k$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$\Rightarrow$ so,

$$z(t) = x^*(t) \xrightarrow[T]{F.S.} a_{-k}^*$$

pattern changes but timeperiod & ω_0 : same

• **CASE: 1** let $x(t)$ is real

$$x(t) = x^*(t)$$

$$\downarrow \qquad \downarrow$$

$$a_k = a_{-k}^*$$

so, $\boxed{a_k^* = a_{-k}}$ magnitude will be symmetric about y-axis

eg: $a_0^* = a_0$, $a_1^* = a_{-1}$, $a_2^* = a_{-2}$ } provided $x(t)$ is real

• **CASE: 2** let $x(t)$ is real and even

real $\to a_k^* = a_{-k}$; even $\to x(t) = x(-t) \to a_k = a_{-k}$

$\boxed{a_k^* = a_k}$ $x(t)$ is real and even $\downarrow$ a_k are also real and even

$\Rightarrow a_k$ are even and real

• **CASE: 3** let $x(t)$ is real and odd

real $\to a_k^* = a_{-k}$; odd $\to x(t) = -x(-t)$

$\boxed{a_k^* = -a_k}$ let $a_k = \alpha + j\beta \Rightarrow a_k^* = \alpha - j\beta$

but $a_k^* = -a_k$ only when real part doesn't exist in a_k

$\Rightarrow a_k$ are purely complex

• TIME SCALING

$$x(t) \xrightarrow[T]{F.S.} a_k$$

$$z(t) = x(\alpha t) \xrightarrow[\frac{T}{\alpha}]{F.S.} C_k = ?$$: $\alpha \neq -1$ (time reversal done already)

$T \to \frac{T}{\alpha}$

$\omega_0 \to \alpha\omega_0$

$$z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\alpha\omega_0 t}$$

$$C_k = \frac{1}{T/\alpha}\int_T x(\alpha t) e^{-jk\alpha\omega_0 t} dt$$

let $z = \alpha t$

$dz = \alpha dt$

and $-\frac{T}{2} \to \frac{T}{2}$ to $-\frac{T}{2\alpha} \to \frac{T}{2\alpha}$

$$C_k = \frac{1}{\alpha} \cdot \frac{1}{T/\alpha} \cdot \int_{-T/2\alpha}^{T/2\alpha} x(z) e^{-jk\omega_0 z} dz = a_k$$

Coefficients are same fundamental time period and fundamental freq. changes

• multiplication

$$x(t) \xrightarrow[T]{FS} a_k$$

$$y(t) \xrightarrow[T]{FS} b_k$$

$$z(t) = x(t) \cdot y(t) \xrightarrow[T]{FS} C_k = ?$$

$$C_k = \frac{1}{T}\int_T x(t) y(t) e^{-jk\omega_0 t} dt$$

$$x(t) = \sum_{l=-\infty}^{\infty} a_l e^{jk\omega_0 t} dt$$

$$y(t) = \sum_{m=-\infty}^{\infty} a_m e^{jk\omega_0 t} dt$$

$$C_k = \frac{1}{T}\int_T \sum_l \sum_m a_l b_m e^{j(l+m)\omega_0 t} \cdot e^{-jk\omega_0 t} dt$$

$$C_k = \sum_l \sum_m a_l b_m \underbrace{\left(\frac{1}{T}\int_T e^{-j(k-l-m)\omega_0 t} dt\right)}_{=1}$$

when $k - l - m = 0$

$k = l + m$

OR

$l = k - m$

$$C_k = \sum_m a_{k-m} b_m$$

$$= \sum_l a_l b_{k-l}$$

multiplication in the time domain leads to convolution in the frequency domain and VICE VERSA

• PARSEVALS RELATION

$$x(t) \xrightarrow[T]{FS} a_k$$

$$\underbrace{\frac{1}{T}\int |x(t)|^2 dt}_{\text{Avg. Power in time domain}} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

avg. power of a signal is partitioned in its Fourier coefficients

* DISCRETE TIME FOURIER SERIES

given a discrete time periodic signal $x[n]$

SYNTHESIS

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$ $\omega_0 = \frac{2\pi}{N}$, fundamental time period $= N$

$k = \langle N \rangle$: same contiguous time period

$0\omega_0 \to 1\omega_0 \to 2\omega_0 \to (N-1)\omega_0 \to N\omega_0$

$0 \to \frac{2\pi}{N} \to \frac{4\pi}{N} \to \frac{(N-1)2\pi}{N} \to 2\pi$

frequencies are also periodic

$$x[n] = \sum_{k=\langle N \rangle} a_k \Phi_k[n]$$

where $\Phi_k[n] = e^{jk\omega_0 n}$

$$\Phi_{k+N}[n] = \Phi_k[n] = e^{jk\omega_0 n}$$

$$\langle x[n], \Phi_l[n] \rangle_N = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jl\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N \rangle}\left(\sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}\right) e^{-jl\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N \rangle}\sum_{k=\langle N \rangle} a_k e^{j(k-l)\omega_0 n}$$

$$= \sum_{k=\langle N \rangle} a_k \underbrace{\left(\frac{1}{N}\sum_{n=\langle N \rangle} e^{j(k-l)\omega_0 n}\right)}_{=1}$$

when $k - l = 0 \Rightarrow k = l$

$$\boxed{a_l = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] \Phi_l^*[n]}$$