14/08/24

Signals And Systems

11:00AM

Evaluation : (relative grading)

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Attendance	10%
Quiz	16%
Assignments	15%
mid sem	24%
end sem	35%

Book: Signals & System 2nd edition by Alan V. oppenheimer

LECTURE : 1 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ **GROUP** : consists of ....

- A set G
- A rule / binary operation "*"
  - a. associative
    $x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$
  - b. There exists an element "e" called the identity of group G such that
    $e*x = x*e = x \quad \forall x \in G$
  - c. $\forall x \in G, \exists x^{-1}$ such that
    $x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$
  - d. if $x*y = y*x \quad \forall x,y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abelian Group

eg: set of continuous time periodic signals with time period T with "*" = "+"

⇒ **FIELD** : consists of the following

- A set F
- Two binary operations "+" and "·" such that ....
  - $(F,+)$ is an abelian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - Left distributive
      $x\cdot(y+z) = xy + xz \quad \forall x,y,z \in F$
    - Right distributive
      $(x+y)\cdot z = xz + yz \quad \forall x,y,z \in F$

eg: F = Real Numbers $\mathbb{R}$

* Vector Space : A set V with a map...

- '+' : $V \times V \to V$
  $(v_1, v_2) \to (v_1 + v_2)$ called vector addition
- '·' : $F \times V \to V$
  $(a, v_1) \to (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a\cdot(v_1+v_2) = av_1 + av_2$
- $(a_1+a_2)\cdot v = a_1 v + a_2 v$
- $(\alpha\beta)v = \alpha(\beta v) = \beta(\alpha v)$
- $1\cdot v = v$
- $a\cdot 0 = 0$
- $0\cdot v = 0$
- if $v \neq 0$, then $a\cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \to \mathbb{R}$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X,d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM : let V be a F-vector space.

A map $\|\cdot\| : V \to \mathbb{R}$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a|\,\|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

Lecture : 2

16/08/24 : 9:30am

* Inner Product :

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ ... complex conjugate operation
- it is linear in the first co-ordinate
  $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$
  $\forall\, v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $\langle v, a_1 w_1 + a_2 w_2 \rangle = \overline{a_1} \langle v, w_1 \rangle + \overline{a_2} \langle v, w_2 \rangle$
  $\forall\, w_1, w_2, v \in V$ and $a_1, a_2 \in F$

eg: dot product
(numerical case similarly complex) $\min(\{x\}) \geq 0$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner product
(complex conjugate)

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot \overline{y_i}$$

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \quad \text{implies } a_i = 0 \ \forall\, i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

(maximal linearly independent set of vectors in V)

if $\begin{cases} e_1 \\ e_2 \\ \vdots \\ e_n \end{cases}$ is a basis for $V_n(F)$, then $\forall\, v \in V$ & $a \in F$

$$v = a_1 e_1 + a_2 e_2 + \dots a_n e_n \qquad V = \sum_{i=1}^{n} a_i e_i$$

* ORTHOGONAL & ORTHONORMAL BASIS

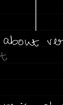
A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

$$v_i \neq 0 \quad \forall\, i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall\, i \,\&\, j$$

SIGNALS (133 & 203)

- continuous time signal

- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t) \ \forall\, t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t) \ \forall\, t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal :

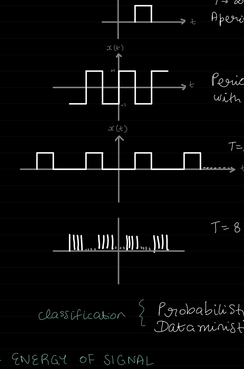
$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

* PERIODIC : if $x(t) = x(t+T) \ \forall\, t$ then T is the time period

The smallest time period T defined for this property is called the fundamental time period

eg:



classification { Probabilistic / Deterministic

* ENERGY OF SIGNAL

Continuous Time Signal :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time Signal

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

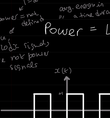
Note: Periodic signals are power signals ✓
Aperiodic signals are not power signals ✗

$T \to \infty$ ← as power is not defined here

avg energy in a time period

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Aperiodic signals are not power signals



compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

★ Lecture 3 [illegible]

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= 4 + 4 \quad \uparrow \text{energy signal}$$

$x_1(t)$ = energy signal, constant signal, periodic signal, aperiodic, power signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$\infty$$

Since $E \to \infty$ and $\text{not } E = \infty$, that signal is not an energy signal.

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$\text{attention} \quad \begin{cases} = \text{finite value} \\ = \text{finite value} \end{cases}$$

$0 < P < \infty$ ⇒ Power Signal ✓

★ TRANSFORMATION OF SIGNAL

① Translation

(a) DELAY

$y(t) = x(t-4)$

like $x_0 = 3$

$y(t) = 2(t-2)$

$y(0) = 2(-3) = 0$
$y(3) = 2(1) = 2$
$y(4) = 2(2) = 4$
$y(5) = 2(3) = 6$

(b) ADVANCE A SIGNAL

$y(t) = 2(t+4)$

like $x_0 = 3$

$y(-1) = 2(3) = 2$
$y(-2) = 2(1) = 2$
$y(-3) = 2(1) = 0$
$y(-4) = 0$

② Scaling

$y(t) = x(2t)$

(a) $a > 1$: Compression of the signal on the t-axis

like $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$
$y(\frac{2}{3}) = x(2) = 0$

(b) $0 < a < 1$: Dilation on t-axis

like $a = \frac{1}{3}$

$y(t) = x(\frac{1}{3}t) = x(\frac{t}{3})$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

$y(6) = x(2) = 0$

(c) DEFLECT $y(t) = x(-t)$

$y(t) = x(-t)$

Reflect on the t-axis a line passed.

$a = -1$

$y(3) = x(-3)$

$y(3) = x(3) = 0$

$y(-3) = x(3) = 0$

$y(t) = x(t) = 0$

$y(t) = x(2t) = 0$ ✗ $y(t) = x(2t)$ ✗ execution ✗ scaling

Scaling then translation, because answer of left $2t \neq t$ ✗

Not, scaling → translation → reflection ✓

$x(t)$
translate
$x(t+3)$
scale
$x(3t+3)$

★ FUNDAMENTAL SIGNALS

• COMPLEXES TIME SIGNALS

$x(t) = Ce^{at}$

• CASE 1 : C real constant and a real constant

$x(t) = Ce^{at}$

① $a = 0$

② let $a > 1$

③ let $a < 1$

④ let $a = 0$

• CASE 2 : C real constant a + purely imaginary $= j\omega$

$x(t) = Ce^{j\omega t}$ $\forall t$

periodic if $x(t) = x(t+T)$

$x(t+T) = Ce^{j\omega(t+T)} = Ce^{j\omega t} e^{j\omega T} = x(t)$

$e^{j\omega T} = 1$ $\omega T = 2\pi k$

Step we need to find the fundamental time period (smallest period) for $k = 1$

So, $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$ fundamental time period

$C = \frac{1}{2}(1e^{j\omega t} + 1e^{-j\omega t}) \quad \frac{1}{2}j\omega = \omega$

• CASE 3 : C + complex number a + real number

$x(t) = 1C1e^{j\theta} e^{at} = 1C1e^{j\theta + at}$

$e^{j\theta}$ will stay it is purely imaginary exponential

we can write $x(t) = 1C1e^{at} = \cos(\omega t) + j\sin(\omega t)$

$Re[x(t)] = 1C1 \cos(\omega t)$

$Im[x(t)] = 1C1 \sin(\omega t)$

• CASE 4 : C + complex number $= 1C1e^{j\theta}$ a complex number $= r + j\omega$

$x(t) = 1C1e^{j\theta} = 1C1e^{(r+j\omega)t}$

$= 1C1e^{rt} \underbrace{e^{j\omega t}}_{\text{purely imaginary exponential}}$

$Re[x(t)] = 1C1e^{rt} \cos(\omega t + \theta)$

$Im[x(t)] = 1C1e^{rt} \sin(\omega t + \theta)$

let $r = 2$, $1C1 = 1.5$

$Re[x(t)] = 1.5e^{2t} \cos(\omega t + \theta)$

• Discrete Time Signals

$x[n] = Ca^n$

• CASE 1 : C + real constant a + real

a. $a > 1$, $C = 1$ $x[n] = a^n$

let $a = 2$, $x[n] = 2^n$

b. $0 < a < 1$ let $a = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$

c. $a < -1$ let $a = -2$

$x[n] = (-2)^n$

d. $-1 < a < 0$ let $a = -\frac{1}{2}$

$x[n] = (-\frac{1}{2})^n$

• CASE 2 : C + real a + $e^{j\theta}$

$x[n] = Ca^n = C e^{j\theta n}$

if $x[n] = x[n+N]$ $\forall n$

then $x[n]$ is periodic

$x[n+N] = C e^{j\theta(n+N)} = C e^{j\theta n} e^{j\theta N} = x[n]$

$e^{j\theta N} = 1$ $\Rightarrow$ $\omega N = 2\pi k$

Note we cannot say $N = 1$ for smallest common exponent because we need N to be an integer

$N = \frac{2\pi k}{\omega}$ ← k is an integer

$\frac{N}{k} = \frac{2\pi}{\omega}$ ← $\frac{N}{k}$ is a rational number

ω must be π times a multiple of 2

* Lecture: 4

28/08/24

- for continuous time signals
  → frequency is unique $(0 \to \infty)$

- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

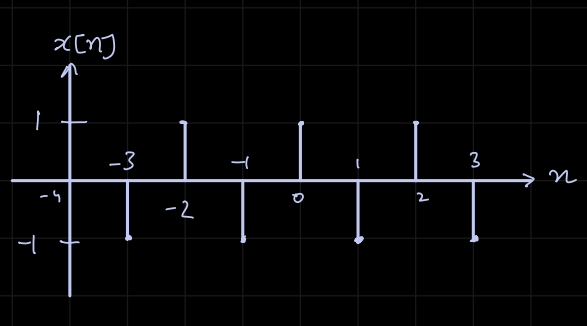
DTS

$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

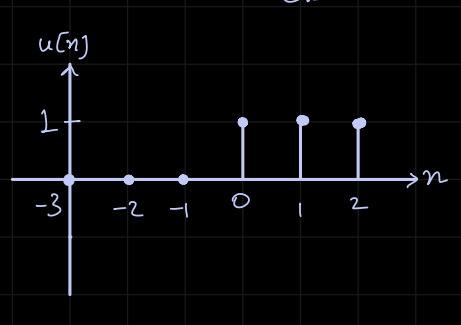
let $\omega_0 = \pi \Rightarrow e^{j\pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

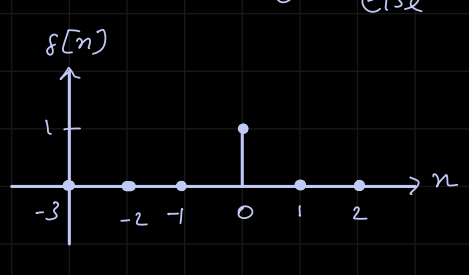
unit step signal
$u[n]$

$$u[n] = \begin{cases} 1 & : n \geq 0 \\ 0 & : \text{else} \end{cases}$$



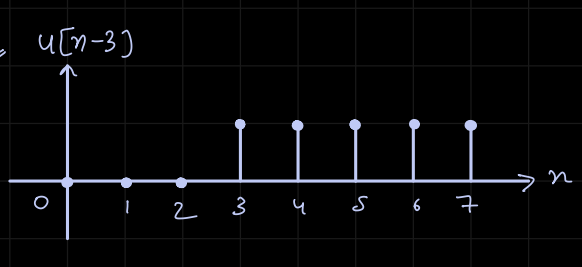
unit impulse function
$\delta[n]$

$$\delta[n] = \begin{cases} 1 & : n = 0 \\ 0 & : \text{else} \end{cases}$$



note:

$1 : n-3 \geq 0$
$n \geq 3$



conversion btw $u[\,]$ and $\delta[\,]$

$$\boxed{\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^{n} \delta[k] \end{aligned}}$$

note: $u[n] = 0 \quad \forall\, n < 0$

So,
$$u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$$
$$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$$
$$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$$
$$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1$$

- $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when
$n - k = 0$ i.e. $k = -2$
but k is never negative
since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n - k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n - k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underset{0}{\delta[1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[-1]} + \dots$$
$$= \underline{\underline{1}} \checkmark$$

- **SIFTING / SAMPLING**

$x[n] \cdot \delta[n] = ?$

$$x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$$

we can also say $x[n]\delta[n] = x[0]\,\delta[n]$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

eg) $x[n] \cdot \delta[n - n_0] = ?$

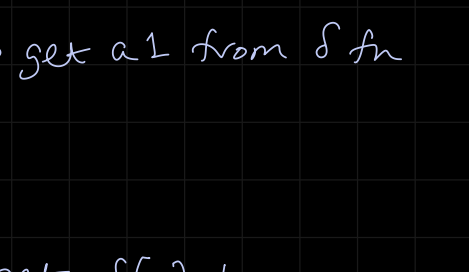
$$x[n] \cdot \delta(n - n_0) = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- unit step function

$$u(t) = \begin{cases} 1 & : t \geq 0 \\ 0 & : \text{else} \end{cases}$$



OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function ≡ singularity function ≡

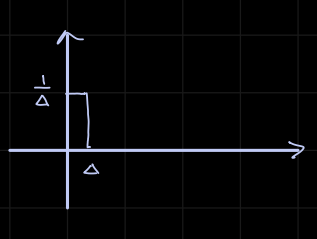
$$\delta(t) = 0 \quad : t \neq 0$$

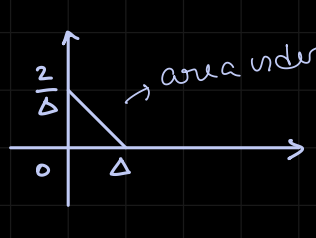
and $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$\delta_\Delta(t)$:

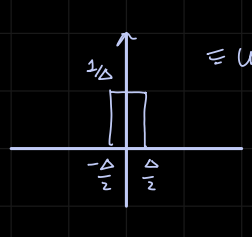
$$\delta_\Delta(t) = \begin{cases} 0 & : t \geq \Delta \ \&\ t < 0 \\ \frac{1}{\Delta} & : \text{otherwise} \end{cases}$$

infinitesimally small





area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
≡ unit imp use func ✓



≡ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t - \tau)\,d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

* Tutorial: 1 30/08/24

(7) even $\{x(t)\} = \dfrac{x(t)+x(-t)}{2}$

odd $\{x(t)\} = \dfrac{x(t)-x(-t)}{2}$

8) (a) $x[n] = \sin(0.02\pi n)$

$\omega = 0.02\pi$

$T = \frac{2\pi}{\omega} =$

(b) $p(x,y) = a^2x_1y_1 + abx_1y_2 + b^2x_2y_2 +$ $ab\ x_2y_1$

$$A = \begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$$

$|A| = a^2b^2 - a^2b^2 = 0$

atleast one eigenvalue is zero

So, sina not all eigenvalues are +ve

(c) for no values of a, b

eigenvalues are values of λ such that

$|A - \lambda I| = 0$

eigenvalues / eigenvectors

8(7) $B = \{v_1, v_2, \dots v_n\}$

B is span of V

$z \in V$

$z = k_1v_1 + k_2v_2 + \dots k_nv_n$

$\bar{x} = c_1v_1 + c_2v_2 + \dots c_nv_n$

$0 = (k_1 - c_1)v_1 + (k_2 - c_2)v_2 + \dots (k_n - c_n)v_n$

Since $\bar{v}_i \neq 0$, $(k_i - c_i)$ need to be 0

i.e. $k_i = c_i$

(8) $x_1[n] = \sin(0.02\pi n)$ $\quad \frac{2\pi}{\omega} = \frac{2\pi}{0.02\pi} = 100 \in \mathbb{Z}$

Periodic ✓

(11) $y[n] = \sum_{n=-5}^{5} \sin(n)\ \delta[n+7]$

but $\sin(n)$ is $-5 \to 5$, so ans = 0

(14) $x(t) = u(t-2) - u(t-4)$

$u(t) \to t > 0$:1

$u(t-2) \to t \geq 2$:1

$u(t-4) \to t \geq 4$:1

but since $\delta(t) = 1$ at $t = 0$

So, ans = 0

(2) (b) $x_1[n] = \cos\left(\frac{\pi}{2} + 0.5n\right)$ $\quad f = \frac{0.5}{2\pi}$ irrational / irrational ?

$\sin = \cos\left(\frac{\pi}{2} - \omega\right)$

(1) $x_1(n) = 1 - e^{j2\pi n/5} + e^{j4\pi n/7}$

note: $e^{j\theta} = \cos\theta + j\sin\theta$

$\Rightarrow 1 - \cos\left(\frac{2\pi n}{5}\right) - j\sin\left(\frac{2\pi n}{5}\right)$

$+ \cos\left(\frac{4\pi n}{7}\right) + j\sin\left(\frac{4\pi n}{7}\right)$

$\Rightarrow \text{LCM}\left(5, \frac{7}{2}\right) \Rightarrow$

(a) $\dfrac{x(t) + x(-t)}{2} = 0$

$x(t) = -x(-t)$

$\forall t$

(b) $e^{-5t}u(t+2)$

$e^{-5t}u(t+2) = -\left[e^{5t}u(-t+2)\right]$

$\left(e^{-5t}\right)_{(t,\omega)} + \left(e^{5t}\right)_{(\omega,2)}$

$\left(e^{-5t}\right)_{(t,\omega)} + \left(e^{5\omega}\right)_{(t,\omega)}$

(5) $P = tu(t)$

$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (t\,u(t))^2 dt$

$\lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} t^2 dt$

$\lim_{T\to\infty} \frac{1}{2T}\left[\frac{t^3}{3}\right]_0^T$

$\lim_{T\to\infty} \frac{1}{2T} \times \frac{T^3}{3} \Rightarrow \frac{T^2}{6}$

$\Rightarrow \infty$

neither energy nor power signal

(9) (c) $\dfrac{u[n] - u[n-7] + u[-n] - u[-n-4]}{2} = 0$

$u[n] + u[-n] = u[n-4] + u[-n-4]$

$2 \geq n \geq 0$ $\quad 1 \geq n \geq 0$ $\quad n \geq 4$ $\quad n \geq 4$

$-2 \leq n \leq 3$

$\boxed{|n| \leq 3}$ ✓

(7) $x(t) =$

$x(-t) =$

$\text{even} = \dfrac{x(t) + x(-t)}{2} =$

odd =

* Lecture : 5

⊙ Systems and their properties

① System with memory or without memory

$x(n) \rightarrow y(n)$

$S_1 : y(n) = 2(n)$ no memory

$S_2 : y(n) = 2^n(n)$ no memory

$S_3 : y(n) = x(n-2)$ memory (depends on past input)

$S_4 : y(n) = \frac{1}{3}\{x(n) + x(n-1)\}$ memory

output depends on past-present inputs + no memory required

output needs to store previous output + requires memory

$S_5 : y(t) = z(t+2)$: requires memory

• Invertible / Non-invertible Systems

$y(n) = x(n)$ iff S_2 = inverse$\{S_1\}$

① let $S_1 : w(n) = 2x(n)$

$S_2 : y(n) = \frac{1}{2} w(n) = \frac{1}{2} \times 2x(n) = x(n)$

S_1 is invertible and S_2 is the inverse system

② let $S_1 : w(n) = \sum_{k=-\infty}^{n} x[n]$

then $S_2 : y[n] = w[n] - w[n-1]$ invertible ✓

$= \{x[-\infty] + \dots x[n-1]\} - \{x[-\infty] + \dots x[n-2]\}$

$= x[n]$

③ let $S_1 : w[n] = x(n) - x[n-1]$

$S_2 : y[n] = \sum_{k=-\infty}^{n} w[k] = x[n]$ invertible ✓

$x(n) - x(n-1) + \dots x(-1) - x(-2) + x(-2) - x(-3) + \dots - x(-\infty) + x(-\infty)$

$x(n) - x(n-1) + \dots x(-1) - x(-2) + x(-2) - x(-3) + \dots - x(-\infty) + x(-\infty)$

$x(n) - x(n-1)$

④ $S_1 : w(n) = x^2(t)$
$S_2 : y(n) = \sqrt{w(n)}$
} S_1 is not invertible

* System is invertible if and only if distinct inputs lead to distinct outputs

(one input)

• Causal Systems

$S_1 : y(n) = x(t-t_0)$: t_0 is a +ve real number

$S_2 : y[n] = x[n+3]$

System output at depending on past system output

S_1 = causal system
S_2 = non-causal system

Causality : unless there is a non-zero input, there should be no effect.

OR unless there is a non-zero input, there should not be non-zero output

OR a causal system depends on current and past input

eg) Causal system : $y(n) = \sum_{k=-\infty}^{n} x[k]$

average processing : $S_1 : y[n,m] = \frac{1}{2}\{x(n,m) + x(n-1,m) + x(n,m-1) + x(n-1,m-1)\}$ non-causal system

$x(n,m)$

• Stability System

A bounded input = $|x(n)| < M_x < \infty$ (finite value ✓)

not on the average (output) → or and t

but on the amplitude OR $|x(n)| < M_x < \infty$

This bounded input is fed to a system. If the system generates an output which is also bounded, we call it a stable system.

eg) $x(t) = u(t)$; $x[n] = u[n]$

$[0,1]$

unbounded : $x(t) = tu(t)$ or $x[n] = tu[n]$

if $|y(t)| < M_y < \infty \forall t$, for bounded finite number input,

then the system is stable

eg) $S_1 : y(n) = \frac{1}{3}\{x(n) - x(n-1) - x(n+1)\}$

$|y[n]| = \left|\frac{1}{3}x(n) - \frac{1}{3}x(n-1) - \frac{1}{3}x(n+1)\right|$

since $|x[n]| < M_x < \infty \forall n$

and the sum & difference of 3 finite terms is finite, so,

$|y[n]| < M_y < \infty \quad \forall n$

$S_3 : y[n] = nx[n]$

$|x[n]| < M_x < \infty \forall n$

bounded input stored

$\Rightarrow |y[n]| = |nx[n]| \rightarrow \infty$ as $n \rightarrow \infty$

unstable system

$S_4 : y(t) = z(2t)$

no manipulation on the years
2t means manipulation on the x-axis

• Time Invariant System

if $x(t) \xrightarrow{system} y(t)$

then $x(t-t_0) \rightarrow y(t-t_0)$

and $x(t+t_0) \rightarrow y(t+t_0)$

System output does not change if there is a change in time

if $x_1(t) \rightarrow y_1(t)$
and $x_2(t) \rightarrow y_2(t)$
and $x_3(t) = x_1(t-t_0)$

then $y_3(t)$ must be equal to $y(t-t_0)$

if it is a time invariant system

eg) $S : y(t) = z(2t)$

$\Rightarrow$ let $t_0 = 4$: $y(t-4) = z(2t-8)$

$\Rightarrow$ time variant system

$x_1(t) \rightarrow y_1(t) = z(2t)$
$x_2(t) \rightarrow y_2(t) = z(2(t-t_0))$

$y_2(t-t_0) = z(2(t-t_0))$
first delay : $x(t-t_0)$
then scaling : $x(2(t-t_0))$

$y_2(t-t_0) = z(2t-4)$
first scaling : $x(2t)$
then delay : $x(2(t-t_0))$

$S_2 : y[n] = nx[n]$

$x_1[n] \rightarrow y_1[n] = nx_1[n]$

let $x_2[n] = x_1[n-n_0]$
(w/ delayed input)

$y_2[n] = x[n-n_0]$

since $y_1[n] \neq y_2[n]$, system is time varying

note: 1st system had delayed output and we gave delayed input

• Linearity

A system is linear if the properties of additivity & homogeneity holds true

$x_1(t) \rightarrow y_1(t)$
$x_2(t) \rightarrow y_2(t)$

$\Rightarrow ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

then system is linear

eg) $y(t) = 2x(t) + 3$

$x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$

$x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$

$x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t) = 2(x_1 + x_2) + 2(3)$

but $y(x_1 + x_2) = 2(x_1 + x_2) + 3$

since, $y_1(t) + y_2(t) \neq y(x_1 + x_2)$

not a linear system

* A system which is linear and time invariant = LTI

linear & time invariant system

* Lecture: SNS 6 — 4/9/24

q) $x[n] = \cos\left(\frac{\pi}{8} n^2\right)$

periodic if $x[n] = x[n+N] \ \forall n$

$$\cos\left(\frac{\pi}{8}(n^2 + N^2 + 2nN)\right) = \cos\left(\frac{\pi}{8} n^2\right)$$

$$\cos\left(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{4} nN\right) = \cos\left(\frac{\pi}{8} n^2\right)$$

we know $\cos(2\pi\theta + k) = \cos\theta \ \forall k \in \mathbb{Z}$

$\frac{\pi}{8} N^2 \Rightarrow$ can it be a multiple of 2π?

if $N=4 \Rightarrow 2\pi = k2\pi$ ✓ ($k=1$)

true for $N=8$ as well

but not for $N=2$

$\frac{\pi}{4} nN \to k2\pi$?

$N=4 \Rightarrow \pi 4n \to \times$

$N=8 \Rightarrow$ ✓ $2\pi nN =$ ✓

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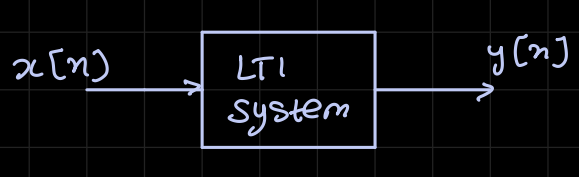
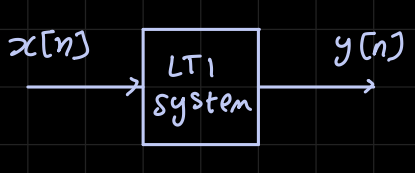
* Discrete Time convolution



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

weighted linear combination of delayed signals

* Impulse Response of an LTI System



$\delta[n] \to y[n] = h[n]$

≡ Impulse response of the LTI system

$\delta[n-1] \to h[n-1]$

$x[1]\,\delta[n-1] \to x[1]\,h[n-1]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = y[n]$$

discrete time convolution of $x[n] \times h[n] = y[n]$

Valid for LTI systems only

convolution → linear & commutative

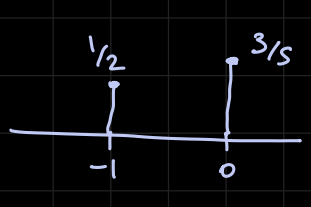
$$x[n] \times h[n] = h[n] \times x[n]$$

$$(x_1[n] + x_2[n])\,h[n] = x_1[n]\,h[n] + x_2[n]\,h[n]$$

eg) given an LTI system whose impulse response is

$$h[n] = \begin{cases} 1 & n = 0,1,2 \\ 0 & \end{cases}$$

and $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$



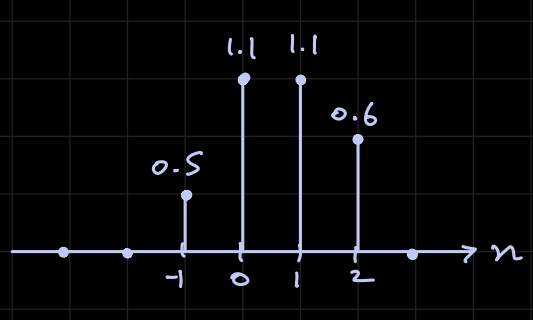
Compute $y[n]$

Ans) $y[n] = h[n] * x[n]$

$y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1]$

because for $x[n] \to y[n]$, $\delta[n] \to h[n]$

$$y[n] = \frac{3}{5}\left[\delta(n) + \delta(n-1) + \delta(n-2)\right] + \frac{1}{2}\left[\delta[n+1] + \delta[n] + \delta[n-1]\right]$$



eg) LTI System (← given)

$$h[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$$

$x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

Ans) $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

OR ≡ $\sum_{-\infty}^{\infty} h[k]\,x[n-k]$

$$\Rightarrow \sum_{k=0}^{\infty} x[n-k] = \sum_{k=0}^{\infty} \alpha^{(n-k)} u[n-k]$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{\alpha^n}{\alpha^k} u[n-k] \quad (1 : n \geq k)$$

$$y[n] = \sum_{k=0}^{n} \frac{\alpha^n}{\alpha^k} \ ✓ \qquad \frac{a(1-r^n)}{1-r}$$

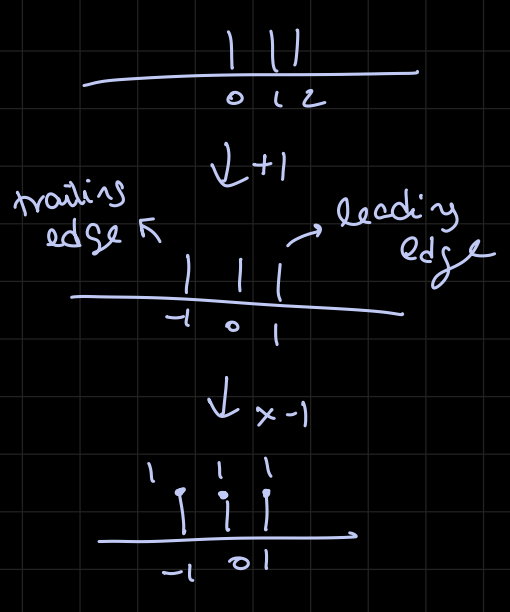
$x-1$: $-k \xrightarrow{+1} -k+1$

$k \to (1-k)$

$\downarrow +1$

$k+1$

$x-1$: $-k+1$



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Note: $y(n) = \text{output}$, $x(n) = \text{input}$, $y(t) = \text{output}$, $x(t) = \text{input}$

For a continuous input output system of the form

$$y(t) = x(t) * h(t)$$

for CT system, it is widely characterized by its $h(t)$. Here, two systems are said alike, each LTI and their $h(t)$ are same.

Extra question: $\int_{-\infty}^{\infty} |x(t)|^2 \, |x(2t)|^2 \, dt$

Ans $x(t) = \cos(t)$

$$= \int_{-\infty}^{\infty} |\cos(t)|^2 \, |\cos 2t|^2 \, dt = \int_{-\infty}^{\infty} \left|\frac{1+\cos 2t}{2}\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{4} + \frac{\cos(2t)}{2}\right) dt$$

Because integrated $\cos 2t = 0$ (symmetric)

$$= \int_{-\infty}^{\infty} \frac{1}{4} \, dt = \infty$$

eg) $x[n] = \begin{cases} 1 : & 0 \le n \le 4 \\ 0 : & \text{else} \end{cases}$

$x[n] = u[n] - u[n-4]$

$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \quad \alpha > 2 \\ 0 & \text{else} \end{cases}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \, h[n-k]$$

$$y[0] = \sum_{-\infty}^{\infty} x[k] \, h[-k]$$

only common point: $k=0$

$$y[0] = x[0] \, h[0] = h[0] = \alpha^0 = 1$$

but we do scaling then translation here

$h[0]^2 = h[k_1] \rightarrow h[-k_1]$

Flipping & Scaling

$h[-k] = 1$ at $-k = 0 \Rightarrow k = 0$

but we do scaling then translation here

Check: $h[-k] = 1$ at $-k+4 = 0$ because $\alpha^0 = 1$

$k = 4$

$\Rightarrow y[4] = x[k] \, h[-k]$

here, we started $h[-k]$ 1 with to the origin. So we have overlap with $x[k]$ on $k = 0, 1$

Similarly for $x[n-1]$, we have overlap with $k = 0, 1 \ldots\ldots n, n-1$

you can find the 2nd example yourself...

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \, h[n-k]$$

we use the same flipping $x[k]$ & $h[-k]$ but we scale

Verification:
$h[n] \text{ at } -k+n$
$k=1$
$\Rightarrow 1 - \frac{1}{2} = 0$
$\Rightarrow h\left(-\frac{1}{2}\right)$

Change of origin for which $(n-k) = 0$

(original)
$-2 \le n \le 0$

$h[n-k] \neq 0$: $n - k \rightarrow n$
$n - 4 \rightarrow -2$
$n - 1 \rightarrow 0$

$n - 6 \rightarrow -2$
$n - 0 \rightarrow 0$

① when $n=0$, there is no overlap with $x[n]$ hence $y[n] = 0$

② $n = 1$: 5 to 11
$n = 10$: 4 to 10

So, no overlap with $x[n]$ when $n > 10$

hence, $y[n] = 0$: $n < 0$ and $n > 10$

③ $0 \le n \le 4$: $y[n] = \sum_{k=0}^{n} x[k] \, h[n-k]$

overlap	n	range
$k=0$	0	$-6 \rightarrow 0$
$k=1$	1	$-5 \rightarrow 1$
$k=0,1,2$	2	$-4 \rightarrow 2$
$k=0,1,2,3$	3	$-3 \rightarrow 3$
$k=0,1,2,3,4$	4	$-2 \rightarrow 4$
$k=0,1,2,3$	5	$0 \rightarrow 5$
$k=1,2,3,4$	6	$1 \rightarrow 6$
$k=2,3,4$	7	$2 \rightarrow 7$
$k=3,4$	8	$3 \rightarrow 8$
$k=4$	9	$4 \rightarrow 9$

$\Rightarrow y[n]$ is defined for $0 \le n \le 10$

④ $0 \le n \le 4 \Rightarrow y[n] = \sum_{k=0}^{n} x[k] \, h[n-k]$

⑤ $4 \le n \le 6 \Rightarrow y[n] = \sum_{k=n-4}^{n} x[k] \, h[n-k]$

⑥ $6 \le n \le 10 \Rightarrow y[n] = \sum_{k=n-4}^{4} x[k] \, h[n-k]$

final $\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{-k}$: $0 \le n \le 4$

$+ \sum_{k=n-4}^{4} \alpha^{-k}$: $4 \le n \le 6$

$+ \sum_{k=n-4}^{4} \alpha^{-k}$: $6 \le n \le 10$

Convolution for CT = DT

$\sum \alpha^{-k} = \alpha^{k} \sum \frac{1}{\alpha^{k}} \rightarrow \alpha^{n} \left(\frac{1}{1-\frac{1}{\alpha}} \right)$

CTS

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \, h(t-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \, x(t-\tau) \, d\tau$$

useful when $u(t) * u(t)$

$y(t) = x(t) * h(t)$

convolution operation

eg) $x(t) = e^{-at} u(t)$: $a > 0$

$h(t) = u(t)$

leading edge $= 0$

no overlap when we shift $h(-\tau)$ on left, there will be no overlap and hence result comes out $= 0$

shift so range ✓

① $t < 0 \Rightarrow y(t) = 0$

② $t > 0 \Rightarrow y(t) = \int_0^t x(\tau) \, h(t-\tau) \, d\tau$

integral $= \frac{-e^{-at}}{a}$

starts at 0
has zero at ∞

eg) $x(t) = \begin{cases} 1 & : 0 < t < T \\ 0 & : \text{else} \end{cases}$

$h(t) = \begin{cases} t & : 0 < t < 2T \\ 0 & : \text{else} \end{cases}$

no shift

① $t < 0$: $y(t) = 0$ no overlap

② $0 \le t \le T$: $y(t) = \int_0^t x(\tau) \, h(t-\tau) \, d\tau$

③ $T \le t \le 2T$: $y(t) = \int_0^T x(\tau) \, h(t-\tau) \, d\tau$

④ $2T \le t \le 3T$: $y(t) = \int_{t-2T}^{T} x(\tau) \, h(t-\tau) \, d\tau$

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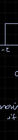
→ convolution

$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{else} \end{cases}$ $h(t) = \begin{cases} t & 0 \le t \le T \\ 0 & \text{else} \end{cases}$

easier to reverse $x(t)$

so, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

this our prefered way $\Leftarrow$ $y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$



ok $y(0) \Rightarrow$ overlap = want

so, $y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(\tau - 0)\, d\tau = 0$

moving forward, $y(1) \ldots y(n)$

i.e. we shift $x(-\tau)$ to left

i.e. $t<0$, it will never overlap

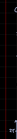
so, we must imagine $t \ge 0$

when plotting $x(t-\tau)$

case ①: $t<0$

$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

case ②: $t \ge 3T$ again no overlap



$y(t) = 0$

no non-zero overlap

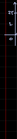
case ③: $0 < t < T$

$y(t) = \int_{0}^{t} h(\tau)\, d\tau = \int_{0}^{t} \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t$

$= \frac{t^2}{2}$

case ④: $T \le t \le 2T$

$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{t^2}{2} - \frac{(t-T)^2}{2}$



$= \frac{\cancel{t^2} - \cancel{t^2} + 2tT - T^2}{2}$

$= \frac{2tT - T^2}{2}$

case ⑤: $2T \le t \le 3T$

$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$

$= \frac{4T^2 - (t-T)^2}{2}$

$= \frac{4T^2 - t^2 + 2tT - T^2}{2}$

$= \frac{3T^2 + 2tT - t^2}{2}$

---

eg: $x(t) = e^{2t}\, u(-t)$ $h(t) = u(t-3)$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$

which one to reverse?

we can reverse both but which one is easier?

$t \ge 3$

case 2: $t \le 3$ in $h(t-\tau)$

moves to right if $t>0$

left if $t<0$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{t-3} x(\tau)\, d\tau$

$\int_{-\infty}^{t-3} e^{2\tau}\, d\tau$

$= \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t-6}}{2}$

$\left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t}}{2e^6}$

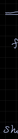
case ②: $t \ge 3$: constant: e^{2t}

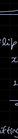
$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2}$ ✓

---

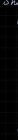
eg: $x(t) = e^{2t}\, u(-t)$: $a>0$

$h(t) = e^{-at}\, u(t)$





flip any one as similar



Shifting by t Ranges: $t>0$ $t<0$

case ①: $t<0$ case ②: $t>0$

$y(0) = \int_{0}^{\infty} x(\tau)\, h(-\tau) = \int_{0}^{\infty} e^{-2\tau}\, d\tau$

$\Rightarrow \frac{e^{-2\tau}}{-2a}\Big|_0^{\infty}$

$\Rightarrow 0 + \frac{1}{2a} = \frac{1}{2a}$

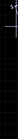
case ②: $t<0$

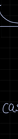
$y(t) = \int_{0}^{t-0} x(t-\tau)\, h(\tau)\, d\tau$

$t-\tau \Rightarrow$

$= \int_{0}^{t-0}$

left edge of $x(-\tau)$ will always be equal to t





for $t>0$: $t \to \infty$

for $t<0$: $0 \to \infty$

case ①: $t \ge 0$ $t \to \infty$

$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_{0}^{\infty} e^{-a\tau}\, e^{2(t-\tau)}\, d\tau$

case ②: $t<0$ $0 \to \infty$

$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_{0}^{\infty} e^{-a\tau}\, e^{2(t-\tau)}\, d\tau$

* SYSTEM PROPERTIES

given an LTI system and

$h[n] = \begin{cases} 1 & n=0, \\ 0 & \text{else} \end{cases}$

determine: input : $x[]$

output: $y[]$

impulse: $h[]$

response.

S: $y[n] = ?$ → System output = ?

apply convolution

$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

from the given def^n, $h[n] = \delta[n] + \delta[n-1]$

$\Rightarrow h[k] = \delta[k] + \delta[k-1]$

so, $y[n] = \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1])\, x[n-k]$

$= \sum_{-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{-\infty}^{\infty} \delta[k-1]\, x[n-k]$

$= x[n] + x[n-1]$

eg: find impulse response of systems

S_2: $y[n] = (x[n] + x[n-1])^2$

S_3: $y[n] = \max\{x[n], x[n-1]\}$

we know: $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

There is only one LTI system that give the same impulse response

Note system S_2, S_3 give same impulse response because they are not LTI

* lecture: 9 13/01/24

• SYSTEM PROPERTIES

① (i) Commutativity

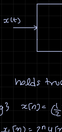
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Prove

(ii) Convolution distributes over addition

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

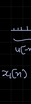


holds true for both CTS and DTS

eg: $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$

$x_1[n] = 2^n u[n]$

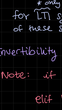
$x_2[n] = 2^n u[-n]$



easier: $y[n] = x_1[n] * h[n] + x_2[n] * h[n]$

(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



if only LTI

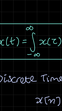
for LTI systems in cascading, order of these systems can be swapped

③ Invertibility:

Note: if $h(t)$ gives: LTI system ✓

Elif $h(t, \tau)$ gives: Non LTI system ✓

output at t — ↳ next impulse input applied at τ



eg: if $h_1(t) * h_2(t) = \delta(t)$ } contiguous deltas(t)

$$y(t) = x(t) * (h_1(t) * h_2(t)) \quad \text{— } \delta_{t_0}$$

$$= x(t) * \delta(t) \quad \text{} \}\ \text{operational definition of } \delta(t)$$

$$= x(t)$$

we do not want $\times$ $\delta(t) = 1$ at $t=0$

$\Downarrow$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau}$$ ✓

} not a scalar signal?

Em ya know, what signal with a value we a single point?

Discrete Time

$$x[n] = \sum_{-\infty}^{\infty} x[k]\, \delta[n-k]$$

$$= x[n] * \delta[n]$$

for this we can say $\delta[n] = 1 : n=0$

④ Causality

Claim: if a system is causal,

$h(t) = 0$ for $t<0$ (CTS)

$h[n] = 0$ for $n<0$ (DTS)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$ because x count is not dating values to $k<0$

hence, for causal systems,

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

OR $$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

} with $h[n] = 0$ at $n<0$

integrator: S if $h(n) = u(n)$ CTS

accumulator: S if $h[n] = u[n]$ DTS

⑤ memory / memoryless system

S_1: $y(t) = Kx(t) \Rightarrow h(t) = K\delta(t)$

S_2: $y[n] = Kx[n] \Rightarrow h[n] = K\delta[n]$

⑥ Stability (BIBO Stability Criteria)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

let input signal is bounded and $|x(t)| < M_x \ \forall t$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right|$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \le M_x \times \left(\text{some finite value, say } K\right)$$

$< \infty$ $\forall t$

if the system is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

DTS: $|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

if the system holds absolute summability,

then the system is stable

Note: $h[n] = u[n]$ } LTI, unstable, $h(t) = u(t)$ } causal, with memory

but still we use these for integrators & accumulators because we use clipper and clamper since we know the upper and lower bounds

Identity: System with $h(t) = \delta(t)$

* Constant Coefficient Difference eq for causal LTI systems

$$\sum_{k=0}^{N} b_k\, y[n-k] = \sum_{k=0}^{M} a_k\, x[n-k]$$

a_0, b_0: not dependent on time, & cannot exist

① N = order of the system

② The system is at rest if:

if $x[n] = 0 : n<n_0$

then, $y[n] = 0 : n<n_0$

$y[n-1] = 0 : n<n_0$

$y[n-N] = 0 : n<n_0$

} causality conditions

note: $N>0$

MA

→ case: 1 ⇒ $N=0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots a_M x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_1} x[n-1] + \dots \frac{a_M}{b_M} x[n-M]$$

causal system ✓ ≡ moving average system

System is called finite impulse response system

≡ FIR System

IIR/AR

→ case 2: $N=1$ and $M=0$ (simplicity)

$$b_0 y[n-1] + b_1 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0\, x[n] - \alpha_1 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\alpha_1 = \frac{b_1}{b_0}$

if $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$h[0] = \alpha_0$

$h[1] = -\alpha_1 \alpha_0$

$h[2] = +\alpha_1^2 \alpha_0$

$\vdots$

} FIR? ✗

Infinite impulse response system ✓ (IIR)

* Auto Regressive System

≡ AR system

output depends on regresses of its previous/past inputs

ARMA

→ case: 3 $N>0$, $M>0$

* Lecture : 10 18/09/24

⇒ Constant coeff differential eqn representation of causal LTI systems

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k x(t)}{dt^k}$$

N = order of the system

let N = 2 & M = 3

$$b_0 \frac{d^2 y(t)}{dt^2} + b_1 \frac{dy(t)}{dt} + b_0 y(t) =$$

$$a_0 x(t) + a_1 \frac{d\,x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a^3 \frac{d^3 x(t)}{dt^3}$$

Initial condition :

if $x(t) = 0 \quad \forall \quad t < t_0$

then $\frac{d^k}{dt^k} y(t) = 0 \quad \forall \quad t < t_0$

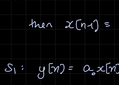
$\forall \quad k \in [0, N]$

* Block diagram representation
Useful for DTS

let $a_0 x[n]$ = scalar multiplier

then $x[n-1]$ = delay element

let S_1 : $y[n] = a_0 x[n] + a_1 x[n-1]$



S_2 : $b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$

and $\beta_0 = \frac{b_1}{b_0}$

* FOURIER SERIES

we know, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

and $x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau)$

x[k] are scalars / coefficients
we can write them as a_k

$$x[n] = \sum_{k=-\infty}^{\infty} a_k\,\delta[n-k]$$

CT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

weighted sum of frequencies
finite basis
$\phi_k(t) = e^{jk\omega_0 t}$
(deg. of oscillation)

weighted sum of vectors
combination of vectors

$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$

$\omega_0 = \frac{2\pi}{T}$

↳ fundamental time period of x(t)

f is vector
freq of f is
time period T
Cosine component
time period T
frequency
freq = 2πk / T
time period T
Sin deg. of osc
time period T
non freq component
$\phi_1(t) = e^{j\omega_0 t}$ $\quad \phi_2(t) = e^{-j\omega_0 t}$
freq of osc
fr. vector
vector
$\phi_1(t) = e^{-j\omega_0 t}$ $\quad \phi_2(t) = e^{-2j\omega_0 t}$

$\phi_0(t) = e^{j0} = e^0 \cdot 1 = \cos 0 + j\sin 0$ ✓

↳ DC component of signal
no fluctuation

can we write $\phi_2(t)$ in terms of $\phi_1(t)$?

No

they are independent of each other

inner product $\langle \phi_k(t), \phi_l(t) \rangle_T = \frac{1}{T}\int_0^T \phi_k(t)\,\phi_l^*(t)\,dt$

$$= \frac{1}{T}\int_0^T e^{jk\omega_0 t} \cdot e^{-jl\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T e^{j(k-l)\omega_0 t}\,dt$$

integral: $k\omega = m$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt$$

$$= \begin{cases} 0 & : k \neq l \\ 1 & : k = l \end{cases}$$

defining the inner product

$$\langle \phi_k(t), \phi_l(t) \rangle_T = \frac{1}{T}\int \phi_k(t)\,\phi_l^*(t)\,dt$$

(continuous time periodic signals)

⇒ Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$\{\phi_k(t)\}$ = span a vector space from an orthonormal basis

⇒ Analysis : $a_k = \langle x(t), \phi_k(t) \rangle_T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,\phi_k(t)$$

$$\langle x(t), \phi_l(t) \rangle_T = \langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_l(t) \rangle_T$$

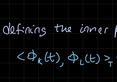
$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\langle \phi_k(t), \phi_l(t) \rangle_T}_{\delta[k-l]}$$

eg:

convolve x(t) & h(t)

firstly $h(\tau) = 1 - \tau$
$h(-\tau) = 1 + \tau$ ↻ reflect

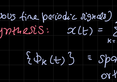
overlap



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(0) = \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

case : 1. $\quad 0 \leq t \leq \frac{1}{2}$



$$y(t) = \int_{t-1}^{-1/2} (1+\tau+t)\,d\tau + \int_{-1/2}^{t} (1+\tau+t)\,d\tau$$

case : 2 ⇒

* Lecture: 11

→ Continuous time periodic signals FOURIER SERIES

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

synthesis equation

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

eg: $x(t) = 3 + 2e^{j\omega t} + e^{j2\omega t}$ synthesis equation

$$= 3 + 2(\cos\omega t + j\sin\omega t) + \cos 2\omega t + j\sin 2\omega t$$

will keep indefinitely
will be real positive beats
and $\sin\omega t$ not disappear → periodic

$$= \sum_{k=0}^{\infty} a_k e^{jk\omega t} \qquad \left.\begin{array}{l} a_0 = 3 \\ a_1 = 2 \\ a_2 = 1 \end{array}\right\} \text{all real coeff}$$

$$\text{coefficient equation} \quad a_k = \langle x(t), \phi_k(t) \rangle \qquad \text{add other } a_k \text{ of } k = 0, 1, 2 \text{ are zero}$$

$$= \frac{1}{T}\int_T x(t)\, \phi_k^*(t)\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-jk\omega t}\, dt$$

$$\{\phi_k(t)\}_{k=\pm}^{\infty} = \text{orthogonal basis} \Rightarrow \int_T \phi_k(t)\phi_k^*(t)\, dt = T\delta[k-k]$$

$$a_k = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\, dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T \phi_m(t)\phi_k^*(t)\, dt\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T e^{j(m-k)\omega t}\, dt\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} a_m \cdot T \cdot \delta[m-k] = a_k = \text{LHS}$$

$\phi_1(t), \phi_{-1}(t) \Rightarrow$ 1st harmonic

$\phi_2(t), \phi_{-2}(t) \Rightarrow$ 2nd harmonic

and so on...

* Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$

* Analysis: $a_k = \frac{1}{T}\int_T x(t) e^{jk\omega t}\, dt$

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let's assume $x(t) = e^{st}$ where $s = j\omega$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \quad \text{let } H(s) \text{ complex}$$

$$\boxed{y(t) = e^{st} \cdot H(s)}$$

$e^{st} \rightarrow$ act as eigenfunctions of an LTI system
$H(s) =$ corresponding eigenvalue

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow e^{st} H(s)$$

↳ here the LTI system is acting as a linear operator

if I feed an LTI system with a continuous signal with some frequencies, then, it outputs a signal with same frequencies but with different amplitudes of the frequencies

output = line spectrum of $y(t)$

existing frequencies can disappear in output but a different frequency will not get added up

eg: set $H(-j\omega) = 0$
So, $a_k = H(-jk\omega) = 0$
this freq. ω will disappear in output

eg) given signal with 4 frequencies

freq domain representation

eg) S: $y(t) = x(t-2)$ and $x(t) = e^{jt}$

Ans) only delays system
observed from eq.

will it alter the signal frequencies? Intuitively, No

$$x(t) = e^{jt}$$

$$\text{Now, } \omega = 2\pi f$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = \pi \qquad \boxed{T = \pi}$$

We have only one freq. component in the line spectrum of $x(t)$

$x(t) = 1 \cdot e^{jt}$
$\frac{T}{\text{exponent}}$

$$y(t) = e^{j(t-2)} = e^{jt} e^{-j2} = e^{jt} \cdot H(j2)$$

$$\boxed{H(j2) = e^{-j2}}$$

where
ω = 2, $s = j\omega$
So, $H(s) = e^{-s}$

line spectrum of $y(t)$

no more freq. will be added

Same freq. comes out with altered amplitude

---

eg) What is $h(t)$? $\qquad h(t) = \delta(t-2)$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \delta(t-2)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-2)\, e^{-s2}\, dt \quad \Rightarrow H(j2) = e^{-s}$$

NOTE: $x(t) * \delta(t) = x(t)$

$x(t) * \delta(t-t_0) = x(t-t_0)$
shifts the signal by t_0 units

---

eg) $x(t) = \sin(\omega t)$

find and plot the line spectrum of $x(t)$

$$\begin{cases} e^{j\omega t} = \cos\omega t + j\sin\omega t \\ e^{-j\omega t} = \cos\omega t - j\sin\omega t \end{cases}$$

$$e^{j\omega t} - e^{-j\omega t} = j2\sin\omega t$$

So, $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{j2} = \frac{-1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$

2 frequencies with amplitude $\frac{-1}{2}$ and $\frac{1}{2}$

has have form $a_k e^{jk\omega t}$

So we have 2 frequencies at $k = 1$ and -1

eg) $x(t) = 1 + \sin(\omega t) + \cos(2\omega t + \frac{\pi}{4})$

Ans) $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$

$$\cos\left(2\omega t + \frac{\pi}{4}\right) = \frac{e^{j(2\omega t + \frac{\pi}{4})} + e^{-j(2\omega t + \frac{\pi}{4})}}{2}$$

$$x(t) = \overset{DC}{1} + j\left(\frac{1}{2j}\right)e^{j\omega t} + \left(\frac{-1}{2j}\right)e^{-j\omega t} + \left(\frac{1}{2}\right)e^{j(2\omega t + \frac{\pi}{4})} + \left(\frac{1}{2}\right)e^{-j(2\omega t + \frac{\pi}{4})}$$

line spectrum $x(t)$
line spectrum of $x(t)$

5 frequencies

if the question asks for absolute value line spectrum then just write the mod of the complex amplitudes

---

eg) $x(t) = \begin{cases} 1 & : |t| < T_1 \\ 0 & : T_1 < |t| < \frac{T}{2} \end{cases}$

① if $T \to \infty$, signal will become aperiodic

② There should be a DC term, so $a_0 \neq 0$

Ans

$$a_0 = \frac{1}{T}\int_T x(t)\, e^{0}\, dt = \frac{1}{T}\int_T x(t)\, dt$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} \quad \text{as } \omega = \frac{2\pi}{T} = \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{jk\omega t}\, dt = \frac{1}{T}\left[\frac{e^{jk\omega t}}{jk\omega}\right]_{-T_1}^{T_1}$$

euler

$$a_k = \frac{1}{k\pi}\sin(k\omega T_1) \qquad : k \neq 0$$

$$a_k = \frac{1}{k\pi}\sin\left(k\frac{2\pi}{4T_1}T_1\right)$$

$$a_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

line spectrum has become more frequency peaks created

LECTURE: 12 — 25/09/21

• LINEARITY

• Prove the linearity property of signals

if $x(t) \xrightarrow{\text{Fourier series coefficient}} a_k$ $\quad$ $x(t)$ and $y(t)$ are periodic with fundamental time period $=T$

and $y(t) \rightleftharpoons b_k$

prove $z(t) = \alpha x(t) + \beta y(t) \rightleftharpoons \alpha a_k + \beta b_k$

and $z(t)$ is same fundamental time period T

sol) let $z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t}$ where $\omega = \frac{2\pi}{T}$

$$C_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_T (\alpha x(t) + \beta y(t))\, e^{-jk\omega t}\, dt$$

$$= \underbrace{\alpha \frac{1}{T}\int_T x(t) e^{-jk\omega t}\, dt}_{a_k} + \underbrace{\beta \frac{1}{T}\int_T y(t) e^{-jk\omega t}\, dt}_{b_k}$$

$$= \alpha a_k + \beta b_k \quad \text{hence proved}$$

• TIME SHIFTING

$$x(t) \xrightarrow{FS} a_k$$

find fourier series coefficients for $z(t-t_0)$ (shifted)

$$z(t) = x(t-t_0) \xrightarrow{F.S} C_k$$

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t-t_0)\, e^{-jk\omega t}\, dt$$

$$C_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega t}\, dt$$

let $\tau = t - t_0 \Rightarrow \tau$
$d\tau = dt$

let limits: $-\frac{T}{2}$ to $\frac{T}{2}$

now: $-\frac{T}{2} - t_0$ to $\frac{T}{2} - t_0$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}-t_0}^{\frac{T}{2}-t_0} x(\tau)\, e^{-jk\omega(\tau+t_0)}\, d\tau$$

$$C_k = e^{-jk\omega t_0} \underbrace{\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega\tau}\, d\tau}_{a_k}$$

$$C_k = a_k \cdot e^{-jk\omega t_0}$$

there is a shift in frequency but the magnitude remains the same

• TIME REVERSAL

$$x(t) \xrightarrow{FS} a_k$$

$$z(t) = x(-t) \xrightarrow{FS} C_k = ?$$

same time period

$$C_k = \frac{1}{T}\int_T x(-t)\, e^{-jk\omega t}\, dt$$

let $\tau = -t$ and $-\frac{T}{2}$ to $\frac{T}{2}$
$d\tau = -dt$ $\quad$ to $\frac{T}{2}$ to $-\frac{T}{2}$

$$C_k = \frac{1}{T}\int_T x(\tau)\, e^{jk\omega\tau}\, d\tau$$

reverse the limits without of $-d\tau$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)\, e^{jk\omega\tau}\, d\tau$$

$$C_{-k} = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega\tau}\, d\tau$$

$$C_k = a_{-k}$$

$$\boxed{C_k = a_{-k}}$$

no change in frequencies
only change in magnitude

• COMPLEX CONJUGATE

$$x(t) \xrightarrow{FS} a_k \qquad \text{time period } T$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

$$x(t)^* = x^*(t) \xrightarrow{FS} ?$$

$$z(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{jk\omega t}$$

$\Rightarrow$ so,

$$x(t)^* \xrightarrow{FS} a_{-k}^*$$

period changes but time period & ω = same

• CASE: 1 $\quad$ let $x(t)$ is real

$$x(t) = x^*(t)$$
$$\downarrow$$
$$a_k = a_{-k}^*$$

So, $\quad$ magnitude will be symmetric about y-axis

$\boxed{a_k = a_{-k}}$

eg $a_0^* = a_0$
$a_1^* = a_{-1}$ provided $x(t)$ is real
$a_{-1}^* = a_{+1}$

• CASE: 2 $\quad$ let $x(t)$ is real and even

$x(t) = x(-t)$
$\downarrow$
$a_k = a_{-k}$

$a_k^* = a_{-k}$

$\boxed{a_k^* = a_k}$

$\Rightarrow$ a_k are even and real

$\Rightarrow$ a_k are also real and even

• CASE: 3 $\quad$ let $x(t)$ is real and odd

$x(t) = -x(-t)$

$a_k^* = -a_{-k}$

$\boxed{a_k^* = -a_k}$

let $a_k = \alpha + j\beta \Rightarrow a_k^* = \alpha - j\beta$
but $a_k^* = -a_k$ only when real part doesn't exist in a_k

$\Rightarrow$ a_k are purely complex

• TIME SCALING

$$x(t) \xrightarrow{FS} a_k$$

$$z(t) = x(\alpha t) \xrightarrow{FS} C_k ? \qquad ; \alpha \neq -1$$

(real number (not density))

$T \to \frac{T}{\alpha}$
$\omega \to \alpha\omega$

$$z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega\alpha t}$$

$$C_k = \frac{1}{T_\alpha}\int_{T_\alpha} x(\alpha t)\, e^{-jk\omega\alpha t}\, dt$$

let $\tau = \alpha t$
$d\tau = \alpha\, dt$
and $-\frac{T}{2}\alpha$ to $\frac{T}{2}\alpha$ $\quad$ to $-\frac{T}{2\alpha}$ to $\frac{T}{2\alpha}$

$$C_k = \frac{1}{\alpha}\frac{1}{T_\alpha}\int_{T_\alpha} x(\tau)\, e^{-jk\omega\tau}\, d\tau = a_k$$

Coefficients are same
fundamental time period
and fundamental freq. changes

• multiplication

$$x(t) \xrightarrow{FS} a_k$$

$$y(t) \xrightarrow{FS} b_k$$

$$z(t) = x(t) \cdot y(t) \xrightarrow{FS} C_k = ?$$

$$C_k = \frac{1}{T}\int_T z(t) y(t)\, e^{-jk\omega t}\, dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}\, dt$$

$$y(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega t}\, dt$$

$$C_k = \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty} a_l b_m e^{j(l+m)\omega t} \cdot e^{-jk\omega t}\, dt$$

$$C_k = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_l b_m \underbrace{\left(\frac{1}{T}\int_T e^{j(k-l-m)\omega t}\, dt\right)}_{=1}$$

$$C_k = \sum_{l=-\infty}^{\infty} a_l b_{m} \qquad \begin{cases} \text{when } k-l-m=0 \\ k = l+m \\ \text{or} \\ l = k-m \end{cases}$$

$$= \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

■ multiplication
in the time domain leads to convolution in the frequency domain and VICE VERSA

• PARSEVALS RELATION

$$x(t) \xrightarrow{FS} a_k$$

$$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Parseval's relation states that the total power of a signal is partitioned in the frequency domain and vice versa

* DISCRETE TIME FOURIER SERIES

given a discrete time periodic signal $x[n]$

SYNTHESIS

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega n} \qquad \omega = \frac{2\pi}{N}$$

fundamental time period $= N$

Some arbitrary time period

$$\text{Domain} = 2\pi \to 2\pi - \omega \to 2\pi - 2\omega \to \dots \to N\omega$$

$$\to 0 \to \frac{2\pi}{N} \to \frac{4\pi}{N} \to \frac{(N-1)2\pi}{N} \to 2\pi$$

frequencies are also periodic

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n]$$

where $\phi_k[n] = e^{jk\omega n}$

$$\phi_{k+N}[n] = \phi_k[n] = e^{jk\omega n}$$

$$\langle x[n], \phi_l[n]\rangle_N = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jl\omega n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\left(\sum_{k=\langle N\rangle} a_k e^{jk\omega n}\right) e^{-jl\omega n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-l)\omega n}$$

$$= \sum_{k=\langle N\rangle} a_k \underbrace{\left(\frac{1}{N}\sum_{n=\langle N\rangle} e^{j(k-l)\omega n}\right)}_{=1}$$

$$\text{when } k-l=0 \Rightarrow k=l$$

$$\boxed{a_l = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, \phi_l^*[n]}$$

Q) $x(t) = \cos\omega_0 t \qquad \omega_0 = \frac{2\pi}{3}$

$a_k = ?$

$= \frac{1}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$

$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

$\Rightarrow a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{1}{2j}$

a_k is non zero only for the complex exponentials present in $x(t)$

Note: NT = 3, we need to see $a_1, a_{-1}, a_2, a_{-2}, a_3, a_{-3}$

$\Rightarrow a_0 = a_3 = (1)\frac{1}{2j} - \frac{1}{2j} = 0$

Conclusion on coeff. : frequency

periodicity in both domains

(both required for true periodicity)

eg3) $x(t) = 1 + \sin\frac{2\pi}{3}t + 1\cos\frac{2\pi}{3}t = 1 + \frac{1}{2}\left(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}\right)$

DC component = 1

So, $a_0 = 1$ and so over $\frac{2\pi}{3}$

also, every AC component has its periodic part related to unique slots which it periodic with but we can adjust slots

so we periodic with N

$x(t) = 1 + \frac{e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}}{2j} + \frac{1}{2}\left(\frac{e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}}{2}\right)$

CASE 3: N = 3, 6

$a_0 = 1$

$a_1 = \frac{1}{2} + \frac{1}{4}$

$a_2 = \frac{1}{2j} + \frac{1}{4}$

$a_3 = \frac{1}{2j}$

$a_4 = \frac{1}{4}$

for N=3 : no sum overlapping occurs in every N

for N=6 : two sum overlapping occurs in every N

for N=12 : two, three, four, five, six, seven, eight, nine, ten, eleven, twelve

$\Rightarrow$ for N = 3, $\frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}$

same as original function

$\frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}$

is $E(x(\frac{2\pi}{3}t))$ periodic?

$z(t) = x(t) + x(-t)$

periodic N=3, $\omega_0 = \frac{2\pi}{3}$

but $E(x(\frac{2\pi}{3}t))$ is not periodic

$3\cos(\frac{2\pi}{3}t) + 3\cos(\frac{2\pi}{3}t)$ is also periodic with N=3, $\omega_0 = \frac{2\pi}{3}$

$\cos\left(\frac{4\pi}{3}t + \frac{\pi}{3}\right) = 2\cos\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)$

$\frac{3}{2}\cos\left(\frac{4\pi}{3}t\right), N=3$

all are periodic

hence conjugate was not

coeff.

$z(t) = 1 + \frac{1}{2j}\left(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}\right) + \frac{1}{2}\left(\frac{e^{j\frac{4\pi}{3}t} + e^{-j\frac{4\pi}{3}t}}{2}\right) - \left(\frac{e^{j\frac{4\pi}{3}t} - e^{-j\frac{4\pi}{3}t}}{2j}\right)$

$N_{new} = 6$

note: $e^{j\frac{4\pi}{3}t}$ is periodic with N=3 but the period before N_{new} has to be 6 as periodic with N as well.

$z(t) = 1 + \left(\frac{1}{2j} + \frac{1}{2} + \frac{1}{2}\right)e^{j\frac{2\pi}{3}t} + \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2}\right)e^{j\frac{4\pi}{3}t}$

new zeros

we have a_0, a_1, a_2

eg4) $x(t)$ = periodic with period N

Some real F.S. coeffs of an periodic signal.

$g(t) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(t-kT)$

and periodic with period N

$g(t) = x(0)\delta(t) + x(1)\delta(t-T) + \dots$

$0, 1, \dots, N-1 \longrightarrow N, N+1, \dots, 2N-1$

So, there know will keep on repeating

so, time periodic with NT

$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\frac{2\pi}{NT}t}$

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt$

like $N=5$

$g(t)$ is not $x(0) + x(1) + x(2) + x(3) + x(4)$ period

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt = \frac{1}{NT}\int_{0}^{NT} x(0)\delta(t) + x(1)\delta(t-T) + \dots + x(N-1)\delta(t-(N-1))\,dt$

So, there know will keep on repeating

so, time periodic with NT

$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\frac{2\pi}{NT}t}$

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt$

like $N=5$

$g(t)$ is not $x(0) + x(1) + x(2) + x(3) + x(4)$ period

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt = \frac{1}{NT}\int_{0}^{NT} x(0)\delta(t) + x(1)\delta(t-T) + \dots + x(N-1)\delta(t-(N-1))\,dt$

$x(n) \xrightarrow{DTFT} b_n$

$a_n = \frac{b_n}{NT}$ Summation is divided by time period

$a_n = \frac{1}{NT}\int_{NT} g(t)\,e^{-jn\frac{2\pi}{NT}t}\,dt$

$a_n = \frac{1}{NT}\int_{NT} \sum_{k=0}^{N-1} x(k)\,\delta(t-kT)\,e^{-jn\frac{2\pi}{NT}t}\,dt$

$= \frac{1}{NT}\int_{NT} \sum_{k=0}^{N-1} x(k)\,e^{-jn\frac{2\pi}{NT}k}\,\delta(t-k)\,dt$

$a_n = \frac{1}{NT}\sum_{k=0}^{N-1} x(k)\,e^{-jn\frac{2\pi}{NT}k}$

$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{x(k)}{T}\right)e^{-jn\frac{2\pi}{N}k}$

eg5) a) $x(t)$ is a real signal

b) $x(t)$ is periodic with T=4

c) $a_{k+3} = 0$ for $|k| \geq 1$

d) A signal with F.S coeff $b_k = e^{jk}\,a_k$

e) $\frac{1}{T}\int_{T} |x(t)|^2\,dt = \frac{1}{2}$

(find?) from (a) we get $a_k = a_{-k}$ (cc)

so, $b_k = e^{jk}\,a_k$ exists as conjugate symmetry

(b) $\rightarrow$ same power in one time period

(using parseval) eq^n

$\frac{1}{T}\int_{T} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

$\Rightarrow |a_{-1}|^2 + |a_0|^2 + |a_1|^2 = \frac{1}{2}$

(c) $\rightarrow$ real signal :

$a_k^* = a_{-k}$

so, $a_k^* = a_k$

and $a_k = a_{-k}$

(d) $\rightarrow$ if $x(t) = x(-t)$

then coeff of $x(t)$ are real

so, if $x(t) = x(-t)$ and $x(t)$

$x(t) \xrightarrow{FS} a_k$

$x(-t) \xrightarrow{FS} a_{-k}$

$x(-t) = x(t) \xrightarrow{FS} ?$

$x(-t) = \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt$

$= \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt$

like $t \to -t$ and $dt \to -dt$

$= \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt$

$= a_k$

so stage 2 = real + odd

$\Rightarrow$ coefficients are purely imaginary

answer : $\operatorname{Im} = 0$

$\operatorname{Im}(a_k) = \frac{1}{2j}$

and b_1, b_2 are purely imaginary

like $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$

so general equation :

$b_1 = \frac{1}{2}j, \quad b_2 = \frac{1}{2}j$

So, $|b_k|^2 = |j|^2 = \frac{1}{4}$

so we have a signal which (2 poles) and $b_k = e^{-jk}\,a_k$

also signal is odd

So we found $z(t+1)$ to be a reversing function

except reversing lower is not showing so positioned opp half for $z(t)$

after $t = 0$ we

$a_{k+3} = 0$ because $|k| \geq 1$ (only 1 is allowed)

So we have only z coeff. Plugging $\frac{1}{2}j$

$\frac{1}{2}j$

(e) we have a signal which (2 poles) and $b_k = e^{-jk}\,a_k$

also signal is odd

So we found $z(t+1)$ to be a reversing function

except reversing lower is not showing so positioned opp half for $z(t)$

after $t = 0$ we

$a_{k+3} = 0$ because $|k| \geq 1$ (only 1 is allowed)

So we have only z coeff. Plugging $\frac{1}{2}j$

$\frac{1}{2}j$

eg9) D $x(t)$ is periodic with $N=6$

1) $\sum_{n=0}^{5} x(n) = 2$

2) $\sum_{n=0}^{5} (-1)^n\,x(n) = 1$

3) $x(n)$ has the max power per period among all $x(t)$ of the $x(t)$ satisfying the frequency & time shift

$a_{k+3} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,e^{-jk\frac{2\pi}{N}n}$

$= \frac{1}{6} + \frac{1}{6}e^{-jk\pi}$

$a_k = \frac{1}{6}$

$a_1 = \frac{1}{6} + \cos(-\pi n) + j\sin(-\pi n)$

$a_1 = \frac{1}{6} + \cos(\pi n) - j\sin(\pi n)$

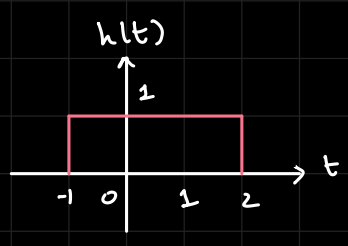
$a_2 = \frac{1}{6} + \cos(2\pi n)$

$a_3 = ?$

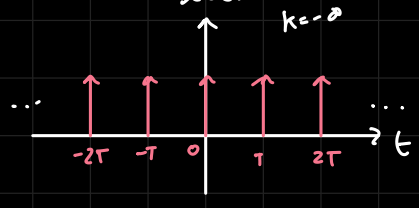
2) $\sum_{n=0}^{5} (-1)^n\,x(n) = 1$

$\Rightarrow \

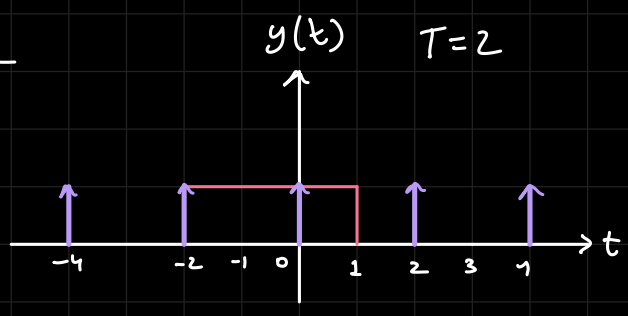
* SNS : midsem 2023

① 

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$



(a) $T = 2$



for one period:

$t = 0$

$y(0) = \begin{cases} 1 & : -2 \\ 1 & : 0 \end{cases}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

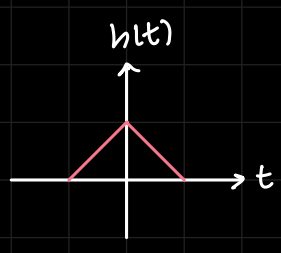
$$y(t) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(\tau-2k)\, h(t-\tau)\, d\tau$$

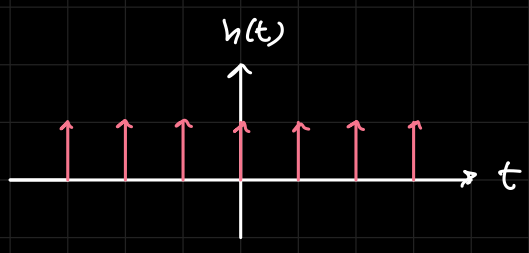
$$= \int_{-2-t}^{1-t} \sum_{k=-\infty}^{\infty} \delta(\tau-2k)\, d\tau$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \int$$

similar question from book:

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$





$$y(t) = x(t) * h(t)$$

$$x(t) * \delta(t) = \int_{\tau} x(\tau)\, \delta(t-\tau)\, d\tau$$

since $\delta(t-\tau) = 0$ for $t \neq \tau$

so, $x(t) * \delta(t) = x(t)$

similarly,

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

given : $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

* Lecture : 14

Filter : $e^{st} \rightarrow e^{st} \cdot H(s)$

$s = j\omega$

attenuates or passes amplitude of certain frequencies

High pass filter
↓
image's edges
freq. attenuated
hence feels blurred

for CTS
$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

DTS:

$$x[n] = z^n \longrightarrow \boxed{LTI} \longrightarrow y[n] = ?$$
$h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}_{H(z)}$$

$$y[n] = z^n H(z)$$

let $z = e^{j\omega_0}$

$$x[n] = e^{j\omega_0 n} \longrightarrow e^{j\omega_0 n} \cdot H(e^{j\omega_0})$$
↓
Z Transform (last chp)

DT FS

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$h[n]$ | LTI system
↓

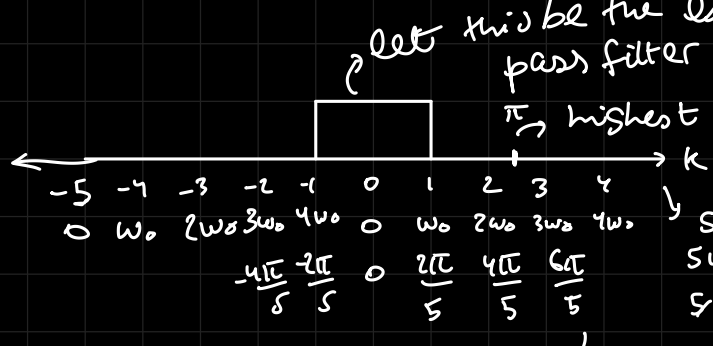
$$y[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \cdot H(e^{jk\omega_0})$$

---

given : $x[n] = \sum_{k=\langle 5 \rangle} a_k e^{jk\frac{2\pi}{5}n}$

$a_0 = 1$
$a_1 = a_{-1} = \frac{1}{2}$
$a_2 = a_{-2} = \frac{3}{4}$

$$x[n] = 1 + \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n} + \frac{3}{4} e^{j2\omega_0 n} + \frac{3}{4} e^{-j2\omega_0 n}$$



$2\pi - \frac{4\pi}{5}$
↓
0

$= -\frac{4\pi}{5} = \frac{6\pi}{5}$

freq. response of filter

let $x = 1$ for this case

$H(0) = x$
$H(e^{j\omega_0}) = x$
$H(e^{-j\omega_0}) = x$
$H(e^{j2\omega_0}) = 0$
$H(e^{-j2\omega_0}) = 0$

Midterm Review

Name:

Group:

Q1 $y = \frac{1}{x^2+1}$ $-\infty < x < \infty$

Q2 $y = \frac{1}{x^2-1}$

① $y = \frac{1}{x^2+1}$

$y' = -\frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

② $y = \frac{1}{x^2-1}$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

③ $y = \frac{1}{x^2+1}$ $-\infty < x < \infty$

$y' = -\frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

④ $y = \frac{1}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q3 $y = \frac{x^2}{x^2+1}$ $-\infty < x < \infty$

$y' = \frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q4 $y = \frac{x^2}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q5 $y = \frac{1}{x^2+1}$ $-\infty < x < \infty$

$y' = -\frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q6 $y = \frac{1}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q7 $y = \frac{x^2}{x^2+1}$ $-\infty < x < \infty$

$y' = \frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q8 $y = \frac{x^2}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q9 $y = \frac{1}{x^2+1}$ $-\infty < x < \infty$

$y' = -\frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q10 $y = \frac{1}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q11 $y = \frac{x^2}{x^2+1}$ $-\infty < x < \infty$

$y' = \frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q12 $y = \frac{x^2}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q13 $y = \frac{1}{x^2+1}$ $-\infty < x < \infty$

$y' = -\frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q14 $y = \frac{1}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Q15 $y = \frac{x^2}{x^2+1}$ $-\infty < x < \infty$

$y' = \frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q16 $y = \frac{x^2}{x^2-1}$ $x \neq \pm 1$

$y' = -\frac{2x}{(x^2-1)^2}$

$y'' = \frac{6x^2+2}{(x^2-1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = -\frac{3}{4}$

Chapter 3.9

Q1 $y = \frac{1}{x^2+1}$ $-\infty < x < \infty$

$y' = -\frac{2x}{(x^2+1)^2}$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \frac{3}{4}$

Q2

- Formulas

odd : $x(-t) = -x(t)$
even : $x(-t) = x(t)$

$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

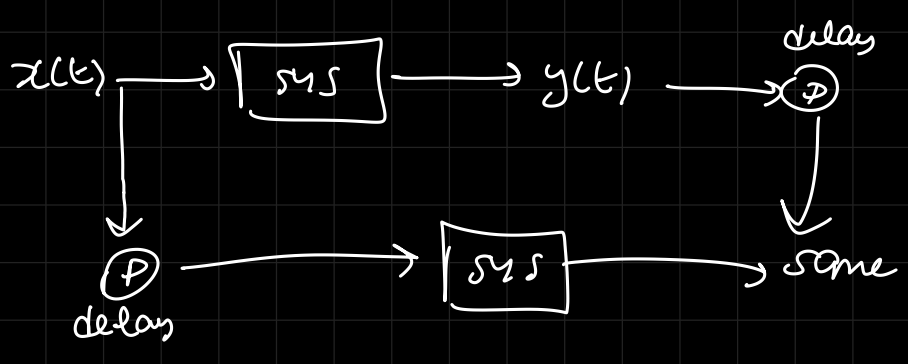
$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

Energy of signal = $\int_{-\infty}^{\infty} |x(t)|^2 dt$ CTS

$\sum_{n=-\infty}^{\infty} |x[n]|^2$ DTS

Power of signal : $\lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x(t)|^2 dt$ CTS

$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} |x[n]|^2$ DTS



time invariance

linear : no time scaling
coeff of i/p & o/p should be constant

time invariance : no added/subtracted term besides input and output

- convolution

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

for memoryless systems: $h[n] = k\delta[n]$
i.e. $k = h[0]$
and $y(t) = kx(t)$

for invertible systems :

if $h_2(t)$ is the imp. response of system which is inverse of sys with $h_1(t)$ as the impulse response then,

$$h_1(t) * h_2(t) = \delta(t)$$

BIBO stability: $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$ ✓

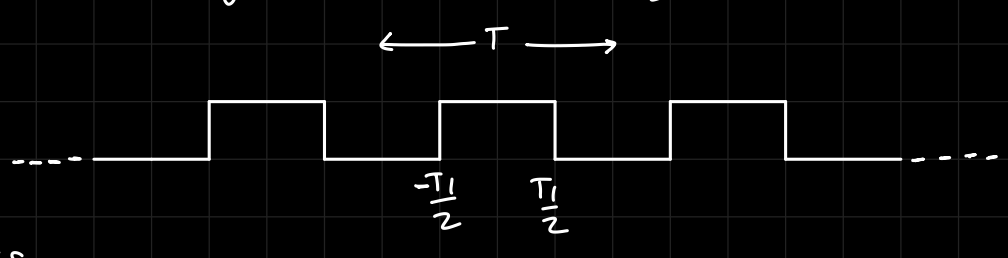
unit step response : $s(t)$

$$s(t) = \int h(\tau)\,d\tau$$

$$h(t) = \frac{d}{dt} s(t)$$

* LECTURE 15 (missed) 16/10/24

lets say we have this square wave



T_1 is fixed btw

let $T_1 = 2$ seconds

when $T = 4T_1 \Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8}$

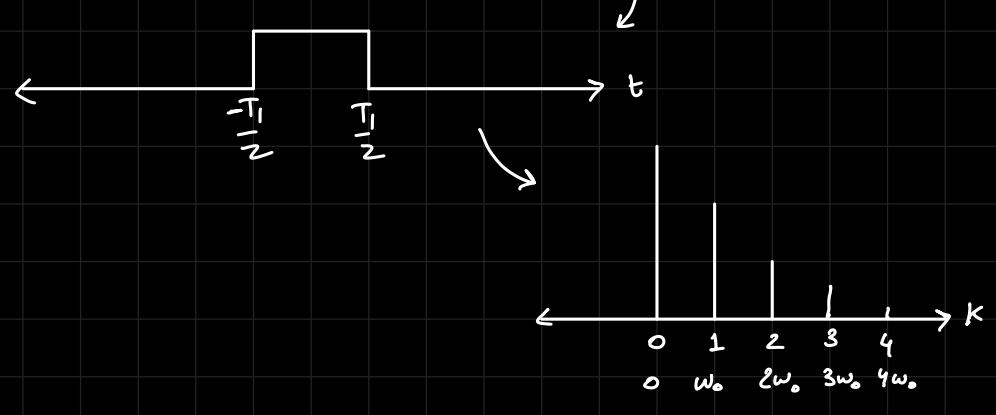
when $T = 8T_1 \Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{16}$

as $T \to \infty \Rightarrow \omega_0 \to 0$

↳ frequency

because $T \propto \frac{1}{\omega_0}$ ofc

when $T \to \infty$



• FOURIER TRANSFORM

→ **SYNTHESIS** eq^n

on inverse fourier transform

represents the amplitude at a certain ω $\equiv X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$K\omega_0 \to \omega$

remember, for FS → $x(t) = \sum_{k=-\infty}^{\infty} a_K e^{jK\omega_0 t}$

→ **ANALYSIS** eq^n

on fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

again, remember analysis in FS

$$a_K = \frac{1}{T}\int_T x(t)e^{-jK\omega_0 t}\,dt$$

→ In terms of f (frequency in hz) $\omega = 2\pi f$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}\,df$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}\,dt$$

---

eg: $x(t) = e^{-at}u(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$X(j\omega) = \left[\frac{e^{-t(a+j\omega)}}{-a-j\omega}\right]_0^{\infty} = 0 - \frac{1}{-a-j\omega}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

note: for $x(t) = u(t)$

$$X(j\omega) = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^{\infty}$$

$$X(j\omega) = \frac{e^{-j\omega(\infty)}}{-j\omega} - \frac{1}{-j\omega}$$

not defined ✗

but it worked out for us in the previous question because we had

$$\underbrace{\frac{e^{-a(\infty)}\,e^{-j\omega(\infty)}}{-a-j\omega}}_{0}$$

0 undefined

magnitude line spectrum

for the previous question where $x(t) = e^{-at}u(t)$

$$|X(j\omega)| = \left|\frac{1}{a+j\omega}\right| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$\times\, a-j\omega$
$\div\, a-j\omega$

$$\left|\frac{a-j\omega}{a^2-j^2\omega^2}\right| \to \frac{\sqrt{a^2+\omega^2}}{a^2+\omega^2}$$

$$\angle X(j\omega) = -\tan^{-1}\frac{\omega}{a}$$

- **FOURIER TRANSFORM**
  - → works for non-periodic signals as well
  - → exists for Energy signals, power signals and even some neither energy nor power signals

Valid only for absolutely integrable signals

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Fourier transform}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{inverse Fourier transform}$$

- **Impulse function**

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

this integral has some non-zero value only when $t = 0$ where $\int \delta(t) = 1$

So $X(j\omega) = e^{-j\omega(0)} = 1$

$|X(j\omega)| = 1, \quad \angle X(j\omega) = 0$

$$\boxed{\delta(t) \overset{F.T.}{\rightleftharpoons} 1}$$

now, let $x(j\omega) = 2\pi\delta(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega$$

$$= \frac{2\pi}{2\pi} \cdot 1 \cdot e^{0} = 1$$

$$\boxed{2\pi\delta(\omega) \rightleftharpoons 1}$$

- **Single sided Real Exponential function**

$$x(t) = e^{-at} u(t) \quad : \quad a > 0$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \left[-\frac{e^{-t(a+j\omega)}}{a+j\omega} \right]_0^{\infty}$$

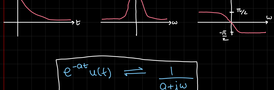
$$0 - \left(\frac{-1}{a+j\omega} \right) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2}$$

magnitude

$$|X(j\omega)| = \frac{1}{|a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}}$$

phase

$$\angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



$$\boxed{e^{-at} u(t) \rightleftharpoons \frac{1}{a+j\omega}}$$

- **DOUBLE SIDED REAL EXPONENTIAL FUNCTION**

$$x(t) = e^{-a|t|} = \begin{cases} e^{at} & : t < 0 \\ e^{-at} & : t > 0 \end{cases}$$

$$= e^{-at} u(t) + e^{at} u(-t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{-t(a+j\omega)} dt + \int_{0}^{\infty} e^{t(a-j\omega)} dt$$

$$= \left[-\frac{e^{-t(a+j\omega)}}{a+j\omega} \right]_0^{\infty} + \left[\frac{e^{t(a-j\omega)}}{a-j\omega} \right]_{-\infty}^{0}$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}$$

$$|X(j\omega)| = \frac{2a}{a^2+\omega^2} \quad \forall \omega$$

$$\angle X(j\omega) = \tan^{-1}\left(\frac{0}{\dots}\right) = 0 \quad \forall \omega$$

$$\boxed{e^{-a|t|} \rightleftharpoons \frac{2a}{a^2+\omega^2}}$$

similar case : $x(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t = 0 \\ -e^{at} & t < 0 \end{cases}$

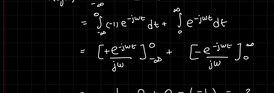
$$X(j\omega) = \int_{0}^{\infty} e^{-t(a+j\omega)} dt - \int_{-\infty}^{0} e^{t(a-j\omega)} dt$$

$$= \frac{a-j\omega - a - j\omega}{a^2+\omega^2} = \frac{-2j\omega}{a^2+\omega^2}$$

purely imaginary

$$|X(j\omega)| = \frac{-2\omega}{a^2+\omega^2}$$

$$\angle X(j\omega) = \begin{cases} \frac{\pi}{2} & : \omega < 0 \\ -\frac{\pi}{2} & : \omega > 0 \end{cases}$$



- **Signum Function**

$$x(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

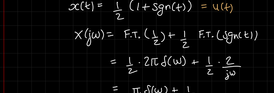
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} (-1) e^{-j\omega t} dt + \int_{0}^{\infty} e^{-j\omega t} dt$$

$$= \left[+\frac{e^{-j\omega t}}{j\omega} \right]_{-\infty}^{0} + \left[-\frac{e^{-j\omega t}}{j\omega} \right]_{0}^{\infty}$$

$$= \frac{1}{j\omega} - 0 + 0 - \left(\frac{-1}{j\omega} \right) = \frac{2}{j\omega}$$

$$|X(j\omega)| = \frac{-2}{\omega}, \quad \angle X(j\omega) = \begin{cases} -\pi/2 & : \omega > 0 \\ \pi/2 & : \omega < 0 \end{cases}$$



another similar case :

$$x(t) = \frac{1}{2}(1 + sgn(t)) = u(t)$$

$$X(j\omega) = \text{F.T.}\left(\frac{1}{2}\right) + \frac{1}{2} \text{F.T.}(sgn(t))$$

$$= \frac{1}{2} \cdot 2\pi\delta(\omega) + \frac{1}{2} \cdot \frac{2}{j\omega}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

$$= \pi\delta(\omega) - j \cdot \frac{1}{\omega}$$

$$|X(j\omega)| = \sqrt{\pi^2\delta(\omega) + \frac{1}{\omega^2}}$$

$$= \begin{cases} \text{N.D.} & : \omega = 0 \\ \frac{1}{\omega} & : \text{else} \end{cases}$$

//doubt $\angle X(j\omega) = -\tan^{-1}\left(\frac{1}{\omega\pi\delta(\omega)}\right)$ = undefined

$$\boxed{u(t) \rightleftharpoons \frac{1}{j\omega} + \pi\delta(\omega)}$$

- Lecture: 16 (missed)

$$\text{rect}\left(\frac{t}{2T}\right) \rightleftharpoons \frac{2\sin(\omega T)}{\omega}$$

$$\frac{1}{2\pi}\, 2\cdot\frac{\sin(Wt)}{t} \rightleftharpoons \text{rect}\left(\frac{\omega}{2W}\right)$$

$$e^{-\alpha t}\cos(\omega_0 t)\,u(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} \cos(\omega_0 t)\, e^{-t(\alpha+j\omega)}\, dt$$

$$\cos\omega_0 t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-t(-j\omega_0+\alpha+j\omega)} + e^{-t(\alpha+j\omega+j\omega_0)}\, dt$$

$$= \frac{1}{2}\left[-e^{-\alpha t}\right]_0^{\infty} + \frac{1}{2}\left[-\frac{e^{-t(\alpha+2j\omega)}}{\alpha+2j\omega}\right]_0^{\infty}$$

$$= \frac{1}{2}\left\{0+1\right\} + \frac{1}{2}\left\{0+\frac{1}{\alpha+2j\omega}\right\}$$

$$e^{-3|t|}\sin 2t \rightarrow \begin{cases} e^{3t}\sin 2t : t<0 \\ e^{-3t}\sin 2t : t>0 \end{cases}$$

$$\sin 2t = e^{j2t} - e^{-j2t}$$

$$A\,\text{rect}\left(\frac{t}{2}\right) \rightleftharpoons \frac{AT\sin(\omega t/2)}{\omega T/2}$$

Lecture 17 — 23/10/24

eg: S: $y[n] = \frac{1}{2}\{x[n] - x[n-1]\}$

find the frequency response of the system

$e^{j\omega n}$ acts as eigen functions for discrete time systems

let $x[n] = e^{j\omega n}$

$$y[n] = \frac{1}{2}\left[e^{j\omega n} - e^{j\omega(n-1)}\right]$$

$$= e^{j\omega n}\left(\frac{1-e^{-j\omega}}{2}\right)$$

$$= e^{j\omega n} \cdot H(e^{j\omega})$$

$$z^n \rightarrow H(z)z^n$$

let $z^n = e^{j\omega n}$: $e^{j\omega n} \rightarrow H(e^{j\omega})e^{j\omega n}$

$$H(e^{j\omega}) = \frac{1-e^{-j\omega}}{2}$$

magnitude response?

$$j\,e^{-j\omega/2}\left\{\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}\right\}$$

$$j\,e^{-j\omega/2} \cdot \sin\left(\frac{\omega}{2}\right)$$

$j = 1$

$$|H(e^{j\omega})| = \left|\sin\left(\frac{\omega}{2}\right)\right|$$

eg: $x(t) = \cos\omega t$

Time | freq.
1 | $2\pi\delta(\omega)$
$\delta(t)$ | 1

find FT of $x(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t} + e^{j(-\omega_0-\omega)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{j(\omega_0-\omega)t}}{(\omega_0-\omega)t}\right]$$

Shifting in the freq. domain

$$1 \rightleftharpoons 2\pi\delta(\omega)$$
$$e^{j\omega_0 t} \rightleftharpoons 2\pi\delta(\omega-\omega_0)$$

let $Y(j\omega)$

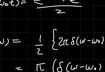
$$Y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

$\omega = \omega_0 \rightarrow \frac{1}{2\pi}$ get $e^{j\omega_0 t} = e^{j\omega_0 t}$

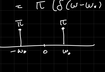
So, $\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$ $x(t)$

$$X(j\omega) = \frac{1}{2}\left\{2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right\}$$

$$= \pi\left\{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right\}$$



eg) $y(t) = x(t)\cos\omega_0 t$



(mod signal (mod signal signal) (from any a system oscillator)) ($\omega = 2\pi f$)

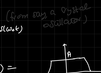
let $X(j\omega) =$

$R(j\omega) = ?$

we know, multiplication in time domain means convolution in freq. domain

So, $R(j\omega) = \frac{1}{2\pi}\{X(j\omega) * F(\cos\omega_0 t)\}$

$$= \frac{\pi}{2\pi}\left\{X(j\omega) * (\delta(j\omega-\omega_0) + \delta(j\omega+\omega_0))\right\}$$



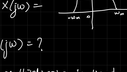
we get $F(r(t))$, now what if we do $y(t) \cdot \cos(\omega_0 t)$ again?

we get two more impulses at $2\omega_0$ & $-2\omega_0$.

frequency selective filter

eg3: $y(t) = v(t)\cos\omega_0$

from prev. given



if we apply a low pass filter, we get the original signal back

• INTEGRATION PROPERTY

$$x(t) \rightleftharpoons X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \rightleftharpoons ?$$

$$x(t) * u(t) \rightleftharpoons X(j\omega) \cdot U(j\omega)$$

$$X(j\omega) \cdot \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{X(j\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \rightleftharpoons \frac{X(j\omega)}{j\omega} + \pi X(0)$$

eg3 $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

periodic signal ✓

we used FT for aperiodic but works for periodic as well

find fundamental period $= T \Rightarrow \omega_0 = \frac{2\pi}{T}$

FS: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

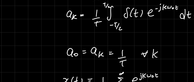
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T}$$

$$a_0 = a_k = \frac{1}{T} \quad \forall k$$

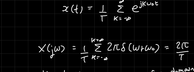
$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

$$X(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0)$$

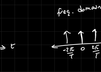
time domain | freq. domain



eg) $x(t) = e^{-a|t|}$



$X(j\omega) = \frac{2a}{a^2+\omega^2}$



if $X(j\omega) \rightleftharpoons x(t)$

$$X(t) \rightleftharpoons 2\pi x(-\omega)$$

if $X(j\omega) \rightleftharpoons x(t)$

$$x(t) \rightleftharpoons \frac{1}{2\pi}X(-\omega)$$

$$\int_{-\infty}^{\infty} X(j\omega)\,d\omega = ?\ \pi_{\omega}$$

$$= 2\pi\int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}\,d\omega$$

$a \rightarrow 0$: $x(t) = 1$

$X(j\omega) \rightarrow 2\pi\delta(\omega)$

$$= \int_{-\pi/2}^{\pi/2}\frac{2 \cdot a \cdot \sec^2\theta\,d\theta}{a^2 \cdot \sec^2\theta}$$

$$= 2\pi$$

* Lecture : 18

→ Discrete time fourier transform

Revision: CTFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$ synthesis IFT

$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ analysis FT

notation: $X(e^{j\omega})$

similar as $X(j\omega)$ but symbolizes that it is periodic since $e^0 = e^{j2\pi} = e^{j4\pi} = e^{j6\pi}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \text{analysis eqn}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{synthesis eqn}$$

2π → one complete period

eg: ① $x[n] = a^n u[n]$: $|a| < 1$ decaying signal

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

since $|a e^{-j\omega}| < 1$, apply GP

$$= \frac{1}{1 - a e^{-j\omega}}$$

eg: ② $|a| > 1 \rightarrow x[n] = a^{|n|}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

+1−1

$$= \sum_{n=-\infty}^{0} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} - 1$$

$$= \sum_{n=0}^{\infty} (a e^{j\omega})^n + \sum_{n=0}^{\infty} (a e^{-j\omega})^n - 1$$

$$= \frac{1}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}} - 1$$

eg: ③ $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n]\, e^{j\omega n}$$

nonzero only at $n=0$

So, $X(e^{j\omega}) = 1$

$x[n] \rightleftharpoons 1$

eg: ④ $X(e^{j\omega}) = \sum_{k=-2}^{2} e^{-j\omega n}$

$$= e^{-j\omega(-2)} + e^{-j\omega(-1)} + 1 + e^{-j\omega(1)} + e^{-j\omega(2)}$$

general: $X(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n}$

$$= e^{j\omega N}\left(1 + \cdots + e^{-j\omega(2N)}\right)$$

$$= e^{j\omega N} \sum_{n=0}^{2N} e^{-j\omega n}$$

$$= e^{j\omega N}\left(\frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}}\right)$$

$$X(e^{j\omega}) = \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\frac{\omega}{2})}$$

eg: ⑤ $x[n] = \delta[n] \rightleftharpoons X(e^{j\omega}) = 1$

now, let $X(e^{j\omega}) = \begin{cases} 1 & : -W < \omega < W \\ 0 & : \text{else} \end{cases}$

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\, d\omega = \frac{1}{j\pi n}\left.\frac{e^{j\omega n}}{2j}\right|_{-W}^{W}$$

$$= \frac{\sin Wn}{\pi n}$$

note: $-W \le \omega \le W$

L'Hospital: for $n=0$

$\frac{W\cos Wn}{\pi} = \frac{W}{\pi} = \frac{W}{\pi} = 1$

Same as ② but in discrete domain

let $W = \frac{\pi}{2}$

$n=0$: $x[0] = \frac{\sin \pi/2}{\pi} = \frac{\pi/2}{\pi} = \frac{1}{2}$

$n=1$: $x[1] = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi}$

$n=2$: $x[2] = 0$

when $W = \pi$, $x[0] = 1$

• DTFT of a periodic signal

Recap: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

(continuous time)

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \omega_k)$$

$\omega_0 = \frac{2\pi}{T}$ / $a_k = \frac{1}{T}$

$\omega_0 = \frac{2\pi}{N}$

$$x[n] = e^{j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-j(\omega - \omega_0)n}$$

$$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0)$$

let $x(n) = \frac{1}{2\pi}\int 2\pi\delta(\omega)\, e^{j\omega n}\, d\omega = \frac{2\pi}{2\pi}$

$= 1$

duality property of fourier transformation

Signals & Systems

let $x[n] = e^{j\omega n}$

$$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0) \quad : -\pi < \omega < \pi$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$$

note: $\delta(\omega)$ and $\delta(\omega)$ are different

$\delta(\omega) = 0 \quad : \omega \neq 0$

and $\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1$

eg: let $x(n) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 n}$

let $N = 3$

$$x(n) = a_0 + a_1 e^{j\frac{2\pi}{3}n} + a_2 e^{j\frac{4\pi}{3}n}$$

$$X(e^{j\omega}) = 2\pi a_0 \delta(\omega) + 2\pi a_1 \delta(\omega - \omega_0) + 2\pi a_2 \delta(\omega - 2\omega_0)$$

$$= 2\pi a_0 \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$2\pi a_1 \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$$

$$2\pi a_2 \sum_{k=-\infty}^{\infty} \delta(\omega - 2\omega_0 - 2\pi k)$$

- Lecture 19

- Quality of Fourier Transform

$x(t) \xrightarrow{CTFT} X(\Omega)$ frequency dom. spectrum

periodicity ↓ ↑ sampling

$x_p(t) \xrightarrow{CTFT} X_p(\Omega)$ Continuous FT

sampling ↓ ↑ periodicity

$x(n) \xrightarrow{DTFT} X(e^{j\Omega})$ Discrete FT

periodicity ↓ ↑ sampling

$x_p(n) \xrightarrow{DTFT} X_p(e^{j\Omega})$ Discrete frequency periodic spectrum

DFT: $x(n) \xrightarrow{DFT} X(k)$ Discrete frequency periodic spectrum

(1) $x(t) = \sum_{k=-\infty}^{\infty} X(k\Omega_0)e^{jk\Omega_0 t}$

DTFT: $x(n) \xrightarrow{DTFT} X(e^{j\Omega})$

$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$

$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$

We know $x(n)$ = periodic, discrete signal with fundamental period N

$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}$

$x(n) = a_0 + a_1 e^{j\Omega_0 n} + a_2 e^{j2\Omega_0 n} \ldots a_{N-1}e^{j(N-1)\Omega_0 n}$

$X(e^{j\Omega}) = 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega-\omega_0) + 2\pi a_2\delta(\omega-2\omega_0) \ldots$

$a_k = \frac{1}{N}$ for $x(n) = \sum_{k=0}^{N-1} \delta(n-kN)$

$X(e^{j\Omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$

note: at $\omega = 0$, ω_0

Result: $x(n) = \sum_{k=-\infty}^{\infty}\delta(n-kN)$

$X(e^{j\Omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$

(when one period, one pulse, only one frequency component)

but at all periods at one point, that has spectrum

and since FT defines frequency, transition in time domain, transformation in freq domain

- Generic Discrete time Periodic Signal

$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}$

$X(e^{j\Omega}) = 2\pi\sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1} a_k\delta(\omega-k\omega_0-l\Omega)$

$0 < \omega < \omega_s$

$x(n)$

or $a_1 = 1$

So, $x(n) = \frac{1}{2}(e^{j\Omega n}+1)$

$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} a^n e^{-j\Omega n} + \sum_{n=-\infty}^{\infty}(e^{-j\Omega})^n$

$= \sum_{n=0}^{\infty}(ae^{-j\Omega})^n - \sum_{n=1}^{\infty}(ae^{-j\Omega})^n + 1$

$= \frac{1}{1-ae^{-j\Omega}} - \frac{1}{1-ae^{-j\Omega}} + 1$

$= \frac{1}{1-ae^{-j\Omega}} - \frac{ae^{-j\Omega}}{1-ae^{-j\Omega}}$

$= \frac{1+a^2-2ae^{j\Omega}}{1-2a\cos\omega+a^2}$

if $a = 0$ $\frac{2-2e^{j\Omega}}{1-2\cos\omega}$

$x(n) \longrightarrow \frac{2}{1-e^{j\Omega}}$

$x(n) = \frac{1}{2}[x(n)+1] \longrightarrow \frac{1}{1-e^{j\Omega}}+\delta(\omega)$

$= U(e^{j\Omega})$ Overall

- Properties

① Periodicity : $X(e^{j\Omega}) = X(e^{j(\Omega+2\pi)})$

② Linearity : if $x(n) \rightleftharpoons X(e^{j\Omega})$ & $y(n) \rightleftharpoons Y(e^{j\Omega})$

then $ax(n)+by(n) \rightleftharpoons aX(e^{j\Omega})+bY(e^{j\Omega})$

③ Time Shifting

if $x(n) \rightleftharpoons X(e^{j\Omega})$

$x(n-n_0) \rightleftharpoons X(e^{j\Omega})e^{-j\Omega n_0}$

$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\omega$

$x(n-n_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}e^{-j\Omega n_0}d\omega$

④ Frequency Shifting

if $x(n) \rightleftharpoons X(e^{j\Omega})$

$e^{j\Omega n_0}x(n) \rightleftharpoons X(e^{j(\Omega-\omega_0)})$

⑤ Differencing in time domain

$x(n) = x(e^{j\Omega})$

$x(n) - x(n-1) \rightleftharpoons X(e^{j\Omega})(1-e^{-j\Omega})$

⑥ Accumulation in Time domain

let $x(n) \rightleftharpoons X(e^{j\Omega})$

and $u(n) \rightleftharpoons U(e^{j\Omega})$

then, $x(n)*u(n) \rightleftharpoons X(e^{j\Omega})U(e^{j\Omega})$

⑦ Accumulation in time domain

let $x(n) \rightleftharpoons X(e^{j\Omega})$

$y(n) = \sum_{m=-\infty}^{n} x(m) = x(n)*u(n) \rightleftharpoons \frac{X(e^{j\Omega})}{1-e^{-j\Omega}} + \pi X(e^{j0})\delta(\omega)$

⑧ Differentiation in Freq domain

let $x(n) \rightleftharpoons X(e^{j\Omega})$

$nx(n) \rightleftharpoons j\frac{d}{d\omega}X(e^{j\Omega})$

$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$

$\frac{d}{d\omega}X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty}(-jn)x(n)e^{-j\Omega n}$

$j\frac{d}{d\omega}X(e^{j\Omega}) = n\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$

⑨ Let $x(n) \rightleftharpoons X(e^{j\Omega})$

$y(n) = x(2n)$

$y_s(n) = \begin{cases} x(n) & n=2k \\ 0 & \text{else} \end{cases}$

let $x(n)$:

Down sampling $y_s(n)$:

$y(n)$ a common step sampling in $2T$ of $x(n)$ → y_s sampling at T → Less resolution

Sampling $y_s(n)$:

General: let $y_s(n) = x[2n]$

Down Sampling

pick 1st sample
skip (2-1) samples
pick 3rd sample
⋮

- Derivation

$y_s(n)$ = inserted 1 zero between every consecutive samples of $x(n)$

General: inserted $(L-1)$ zeros into every consecutive samples of $x(n)$

- Down sampling in the discrete domain

let $x(n) \rightleftharpoons X(e^{j\Omega})$

and $y(n) = x(Ln)$ $\rightleftharpoons Y(e^{j\Omega})$

$L = 2$

$y_s(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-kL)$

$Y(e^{j\Omega}) = \frac{1}{L}\sum_{k=0}^{L-1} X(e^{j(\Omega-\frac{2\pi k}{L})})$

(Correlation in the frequency domain)

$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\Omega-\frac{2\pi k}{N})})$

$x_s(n) = \begin{cases} x[n] & \text{if } n=kL \\ 0 & \text{else} \end{cases}$

$= x[n]\,\delta[n-kL]$

$x_s(n) = x[n]\times\delta[n]$

$y_s[n] = x_s[Ln]$

If we have upsampled version of y_s instead, change in the freq domain

Let $y_s(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\Omega})e^{j\Omega n}d\Omega$

(Correlation in the frequency domain)

$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\Omega-\frac{2\pi k}{N})})$

$X_s(n) = \begin{cases} x[n/L] & \text{if } n=kL \\ 0 & \text{else} \end{cases}$

$X_s(n) = X[n]\times\delta[n]$

$y_s[n] = x_s[Ln]$

If we have upsampled version of y_s instead, change in the freq domain

Let $y_s(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\Omega})e^{j\Omega n}d\Omega$

(Correlation in the frequency domain)

$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\Omega-\frac{2\pi k}{N})})$

$X(e^{j\Omega})$

see 5.96

$P(e^{j\Omega})$

$X_p(e^{j\Omega})$ from x to x_p by sampling in T → $\frac{2\pi}{T}$

$X_p(e^{j\Omega}) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(e^{j(\Omega-\frac{2\pi k}{T})})$

3 copies inserted in one period

$L = 3$

$X(e^{j\Omega})$

$P(e^{j\Omega})$

$X_s(e^{j\Omega})$

$\omega_s = \frac{2\pi}{3}$

2 copies inserted in one period

Oversampling : ω_s

Undersampling : $\omega_s < 2\omega_m$

$x_p(n) \xrightarrow{\text{down sample}} y_s(n)$

* Revision : Fourier Transform

CONTINUOUS :

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{analysis}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \text{synthesis}$$

DISCRETE :

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{analysis}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad \text{synthesis}$$

- **Common Transforms**

① $\delta(t) \rightleftharpoons 1$

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\delta(t)}_{\text{non-zero only at } t=0} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{0}\, dt = 1$$

② $1 \rightleftharpoons 2\pi\delta(\omega)$

$$X(j\omega) = 2\pi\delta(t)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2\pi}\delta(\omega)\, e^{j\omega t}\, d\omega$$

non-zero at $\omega = 0$

$$= \int_{-\infty}^{\infty} \delta(\omega)\, e^{0}\, d\omega = 1$$

③ $e^{-at} u(t) \rightleftharpoons \dfrac{1}{a+j\omega}$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{-e^{-(a+j\omega)t}}{j(a+\omega)} \right]_{0}^{\infty}$$

$$= 0 - \left(\frac{-1}{a+j\omega} \right) = \frac{1}{a+j\omega}$$

④ $e^{-a|t|} \rightleftharpoons \dfrac{2a}{a^2+\omega^2}$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$e^{-a|t|} = \begin{cases} e^{-at} & : t > 0 \\ e^{at} & : t < 0 \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^{0} + \left[\frac{-e^{-(a+j\omega)t}}{a+j\omega} \right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} - 0 + 0 + \frac{1}{a+j\omega}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{a+j\omega+a-j\omega}{a^2+\omega^2} = \frac{2a}{a^2+\omega^2}$$

⑤ $e^{j\omega_0 t} \rightleftharpoons$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t}\, dt \quad \ldots$$

note : $\displaystyle\int_{-\infty}^{\infty} |e^{j\omega_0 t}|\, dt = \int_{-\infty}^{\infty} |\cos\omega_0 t + j\sin\omega_0 t|\, dt$

$$= \int_{-\infty}^{\infty} \sqrt{\cos^2\omega_0 t + \sin^2\omega_0 t}\, dt = \int_{-\infty}^{\infty} 1\, dt \Rightarrow \infty$$

i.e. $e^{j\omega_0 t}$ is not absolutely integrable because it does not satisfy $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$

So, we use the duality property of fourier transform :

if : $F\{x(t)\} = X(j\omega)$

then : $F^{-1}\{X(t)\} = 2\pi x(-\omega)$

eg : $F\{\delta(t)\} = 1$

$F^{-1}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$

modulation property :

let $x(t) \rightleftharpoons X(j\omega)$

$$x(t) \cdot e^{j\omega_0 t} \rightleftharpoons ?$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$x(t) \cdot e^{j\omega_0 t} \rightleftharpoons X($$

* Lecture : 20 08/11/24

Continous fourier transform is a map
$x : \mathbb{R} \to \mathbb{C}$
real domain → complex domain

amplitude == continuous ? analog : digital
↙
it can still be over continous time domain

processing of signals can only take place if the signal is digitalized which can be done by sampling → measuring signal's amplitude in a time interval
↙
T should be such that I get a similar signal after sampling & interpolation

- upsampling

$$y_2[n] = \begin{cases} x[n/L] & : n \% L == 0 \\ 0 & : \text{else} \end{cases}$$

$$Y_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y_2[n] e^{-j\omega n}$$

$$= \sum_{\substack{n:\text{multiple} \\ \text{of } L}} x[n/L] e^{-j\omega n}$$

let $\frac{n}{L} = r$

$n = rL$

$$= \sum x[r] e^{-j(\omega L) r}$$

$$= X(e^{j\omega L})$$

$$\boxed{Y_2(e^{j\omega}) = X(e^{j\omega L})}$$

Plot $Y_1(e^{j\omega})$ for $L = 2, 3$