14/08/24 # Signals And Systems 11:00AM

Evaluation : (relative grading) yoooooooooooo :fire emoji:	Attendance	10%
	Quiz	16%
	Assignments	15%
	mid sem	24%
	end sem	35%

Book: Signals & System 2nd edition by Alan V. Oppenheimer

Lecture : 1 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ GROUP : consists of ....

- A set G
- A rule / binary operation "*"
  - a. associative
    $$x * (y * z) = (x * y) * z \quad \forall x, y, z \in G$$
  - b. There exists an element "e" called the identity of group G such that
    $$e * x = x * e = x \quad \forall x \in G$$
  - c. $\forall x \in G, \exists\ x^{-1}$ such that
    $$x * x^{-1} = e = x^{-1} * x \quad \forall x \in G$$
  - d. if $x * y = y * x \quad \forall x, y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abelian Group

eg: set of continuous time periodic signals with time period T with "*" = "+"

⇒ FIELD : consists of the following

- A set F
- Two binary operations "+" and "·" such that ....
  - $(F, +)$ is an abelian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - left distributive
      $x \cdot (y + z) = xy + xz \quad \forall x, y, z \in F$
    - Right distributive
      $(x + y) \cdot z = xz + yz \quad \forall x, y, z \in F$

eg: F = Real Numbers $\mathbb{R}$

* Vector Space : A set V with a map...

- '+' : $V \times V \rightarrow V$
  $(v_1, v_2) \rightarrow (v_1 + v_2)$ called vector addition
- '·' : $F \times V \rightarrow V$
  $(a, v_1) \rightarrow (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied :

- $(V, +)$ is an abelian group
- $a \cdot (v_1 + v_2) = av_1 + av_2$
- $(a_1 + a_2) \cdot v = a_1 v + a_2 v$
- $(\alpha \beta) v = \alpha (\beta v) = \beta (\alpha v)$
- $1 \cdot v = v$
- $a \cdot 0 = 0$
- $0 \cdot v = 0$
- if $v \neq 0$, then $a \cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \rightarrow R$$

satisfies the following : $(\forall x, y, z \in X)$

- $d(x, y) \geq 0$ and $d(x, y) = 0$ if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X, d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM : let V be a F-vector space.

A map $\|\cdot\| : V \rightarrow R$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a| \, \|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

Lecture: 2

16/08/24 : 9:30Am

* Inner Product:

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ "$-$" : complex conjugate operation
- it is linear in the first co-ordinate
  $$\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$$
  $\forall\, v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $$\langle v, a_1 w_1 + a_2 w_2 \rangle = \bar{a}_1 \langle v, w_1 \rangle + \bar{a}_2 \langle v, w_2 \rangle$$
  $\forall\, w_1, w_2, v \in V$ and $a_1, a_2 \in F$

measures cosine similarity
" norm(x) · norm(y) · cos θ

eg: dot product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner Product

complex conjugate

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot \bar{y}_i$$

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \qquad \text{implies } a_i = 0 \ \forall i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition: The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

weighted linear combination of vector V

if $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ is a basis for $V_n(F)$, then $\forall\, v_i \in V$ & $a_i \in F$

$$V = \sum_{i=1}^{n} a_i e_i$$

$V = a_1 e_1 + a_2 e_2 + \dots a_n e_n$

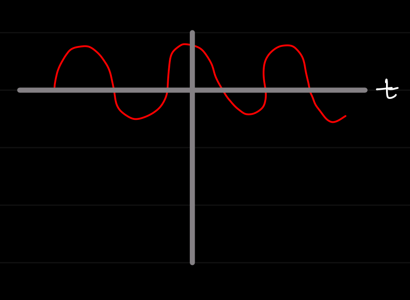
* ORTHOGONAL & ORTHONORMAL Basis

A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if:

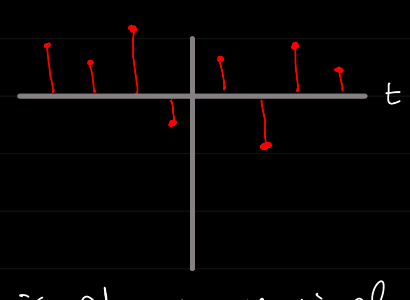
$$v_i \neq 0 \quad \forall i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall i \,\&\, j$$

SIGNALS (CTS & DTS)

- continuous time signal



- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t) \ \forall t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t) \ \forall t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal:

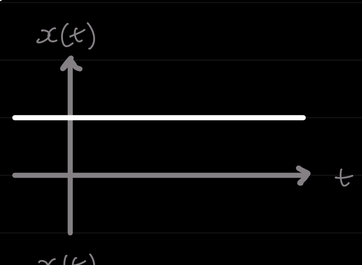
$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

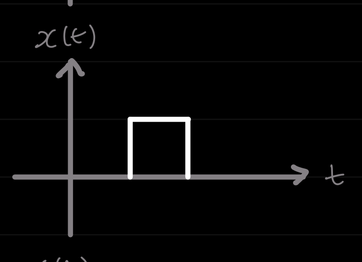
* PERIODIC : if $x(t) = x(t+T) \ \forall t$ then T is the time period

The smallest time period T defined for this property is called the fundamental time period

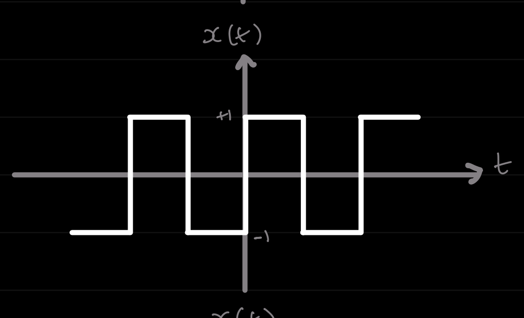
eg:



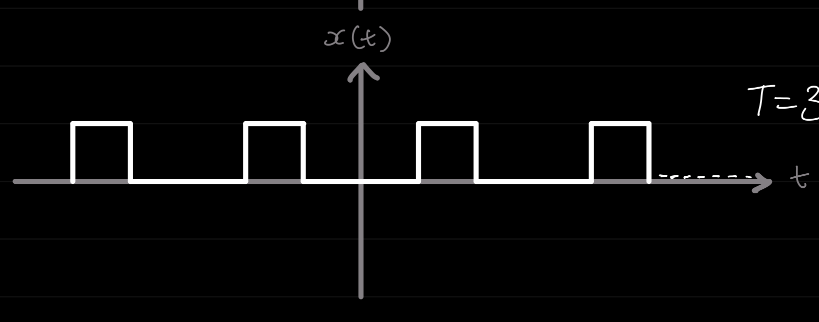
Not periodic



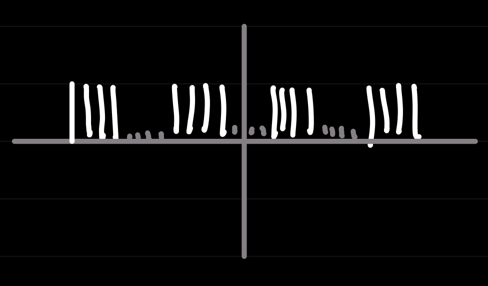
$T \to \infty$
Aperiodic



Periodic with $T=2$



$T=3$



$T=8$

classification $\begin{cases} \text{Probabilistic} \\ \text{Dataministic} \end{cases}$

* ENERGY OF SIGNAL

Continuous Time signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time signal

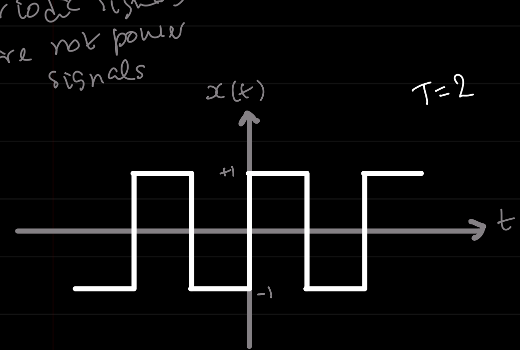
$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Note: Periodic signals are power signals ✓
Aperiodic signals are not power signals ✗

$T = \infty$
so, power = not defined
hence, Aperiodic signals are not power signals

avg. energy in
" a time duration

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$



$T=2$

compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

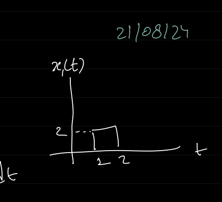
$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{x=-N}^{N} |x[n]|^2$$

Lecture 3

$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

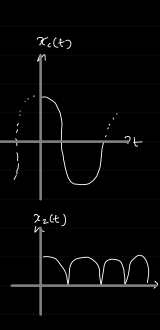
$= 4 < \infty$ → energy signal



$x(t)$ = energy signal, aperiodic, deterministic

$x(t) \neq x(t+T)$ → aperiodic signal

$x(t)$ = continuous signal, periodic signal, deterministic, power signal



$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

∞

Since $E \to \infty$ and not $E < \infty$, that signal is not an energy signal.

$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

= finite

$0 < P < \infty$ → Power Signal

* TRANSFORMATION OF SIGNAL

① Translation

(a) DELAY

$y(t) = x(t - t_0)$

let $t_0 = 3$

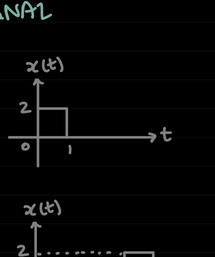
$y(t) = x(t-3)$

$y(0) = x(-3) = 0$

$y(3) = x(0) = 2$

$y(4) = x(1) = 2$

$y(5) = x(2) = 0$



(b) ADVANCE A SIGNAL

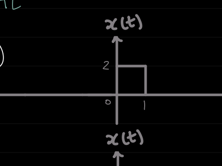
$y(t) = x(t + t_0)$

let $t_0 = 3$

$y(-3) = x(0) = 2$

$y(-2) = x(1) = 2$

$y(-1) = x(2) = 0$



② Scaling

$y(t) = x(at)$

(a) $a > 1$: compression of the signal on the t-axis

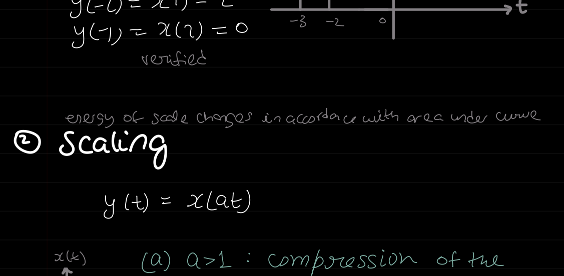
let $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$

$y(\frac{1}{3}) = x(1) = 2$

$y(\frac{2}{3}) = x(2) = 0$



(b) $0 < a < 1$: Dilation on t-axis

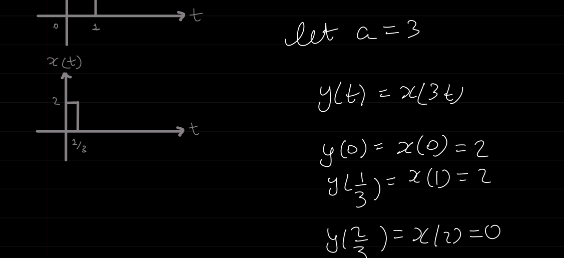
let $a = \frac{1}{3}$

$y(t) = x(\frac{t}{3})$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

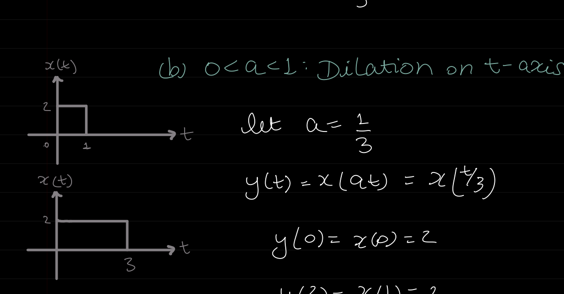
$y(6) = x(2) = 0$



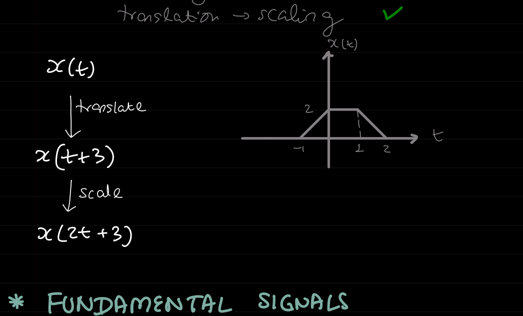
③ SPECIAL CASE: $a = -1$

$y(t) = x(-t)$

flipped on the t-axis / time reversal



eg) $y(t) = x(2t+3)$ translation + scaling



scaling then translation because otherwise we get $2t+6$ ✗

here,

scaling → translation ✗

translation → scaling ✓

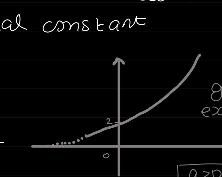
$x(t)$

$\downarrow$ translate

$x(t+3)$

$\downarrow$ scale

$x(2t+3)$



* FUNDAMENTAL SIGNALS

• Continuous Time Signals

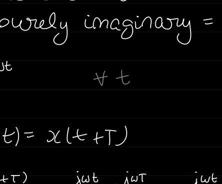
$x(t) = ce^{at}$

• CASE: 1 $c \equiv$ real constant let $c = 2$

$a \equiv$ real constant

$x(t) = 2e^{at}$

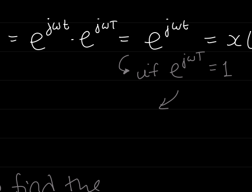
① let $a = 1$



growing exponential

$\boxed{a>0}$

② let $a = -1$



$\boxed{a<0}$

③ let $a = 0$

• CASE: 2 $c \equiv$ real constant

$a \equiv$ purely imaginary $= j\omega$

$x(t) = e^{j\omega t}$ $\forall t$

periodic if $x(t) = x(t+T)$

$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t} \cdot e^{j\omega T} = e^{j\omega t} = x(t)$

if $e^{j\omega T} = 1$

$e^{j\omega T} = 1$ if

$\omega T = 2\pi k$

$e^{j2\pi k}$

since we need to find the fundamental time period (smallest interval), let $K = 1$

So, $\omega T = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$ fundamental time period

$E = \int_{-\infty}^{\infty} |e^{j\omega t}|^2 dt = \int_{-\infty}^{\infty} 1\, dt = \infty$

• CASE: 3 $C \equiv$ complex number

$a \equiv$ real number

$x(t) = |c| e^{j\theta} \cdot e^{at} = |c| e^{at + j\theta}$

e^{at} will define if it is growing / decaying exponential

we can also say, $|c| e^{j\theta} = c\cos(\theta) + c \cdot j\sin(\theta)$

$Re\{x(t)\} = |c| \cdot \cos\theta \cdot e^{at}$

$Im\{x(t)\} = |c| \cdot \sin\theta \cdot e^{at}$

• CASE: 4 $C \equiv$ complex number $= |c| e^{j\theta}$

$a \equiv$ complex number $= r + j\omega$

$x(t) = ce^{at} = |c| e^{j\theta} \cdot e^{(r+j\omega)t}$

$= \underbrace{|c| e^{rt}}_{\text{envelope with either increasing or decaying exponential}} \cdot \underbrace{e^{j(\omega t + \theta)}}_{\text{cosinosoidal signal}}$

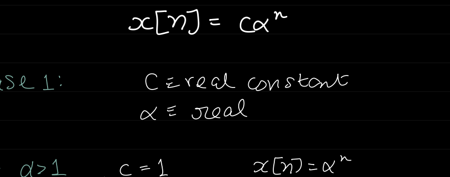
$\cos(\omega t + \theta) + j\sin(\omega t + \theta)$ sinosoidal signal

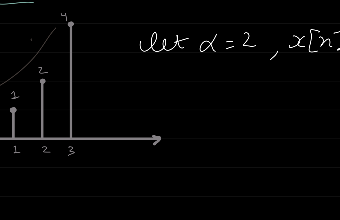
$Re\{x(t)\} = |c| e^{rt} \cdot \cos(\omega t + \theta)$

$Im\{x(t)\} = |c| e^{rt} \cdot \sin(\omega t + \theta)$

let $r = 2$, $|c| = 1.5$

$Re\{x(t)\} = \underbrace{1.5e^{2t}}\, \underbrace{\cos(\omega t + \theta)}$





• Discrete Time Signals

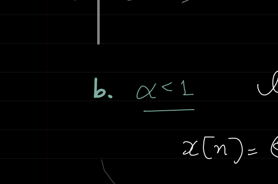
$x[n] = c\alpha^n$

• Case 1: $C \equiv$ real constant

$\alpha \equiv$ real

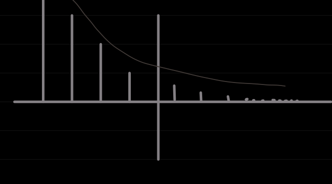
a. $\alpha > 1$, $c = 1$ $x[n] = \alpha^n$

let $\alpha = 2$, $x[n] = 2^n$



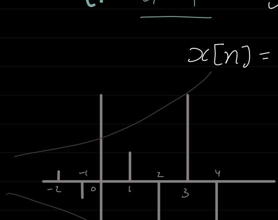
b. $\alpha < 1$ let $\alpha = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$



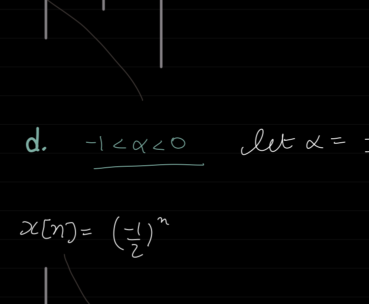
c. $\alpha < -1$ let $\alpha = -2$

$x[n] = (-2)^n$



d. $-1 < \alpha < 0$ let $\alpha = \frac{-1}{2}$

$x[n] = (\frac{-1}{2})^n$



• Case: 2 $C \equiv$ real

$\alpha = e^{j\omega}$

$x[n] = c\alpha^n = \underbrace{e^{j\omega n}}_{\text{let } c=1}$

if $x[n] = x[n+N]$ $\forall n$

then $x[n]$ is periodic

$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N} = e^{j\omega n}$

if $e^{j\omega N} = 1$

$e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi K$

here we cannot say $K = 1$ for smallest interval necessarily because we want N to be an integer

$N = \frac{2\pi K}{\omega}$ ≡ has to be integer

$\frac{N}{K} = \frac{2\pi}{\omega}$ ≡ has to be rational number

Note: ω has to be a multiple of π

* Lecture: 4

28/08/24

- for continuous time signals
  → frequency is unique $(0 \to \infty)$

- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

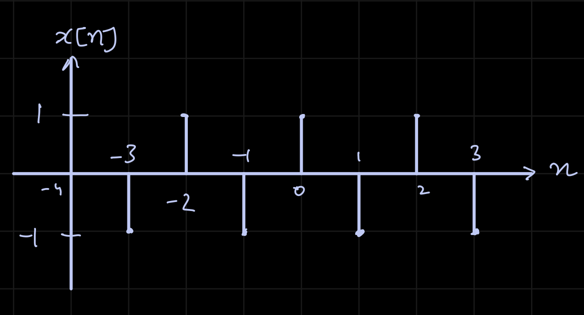
DTS

$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j \cdot 2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

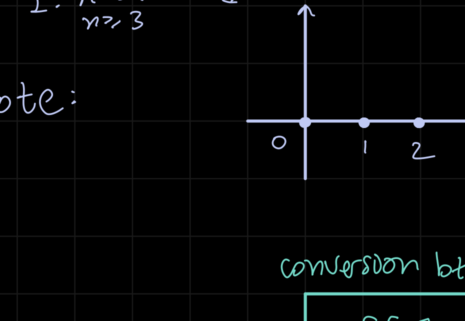
let $\omega_0 = \pi \Rightarrow e^{j \cdot \pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

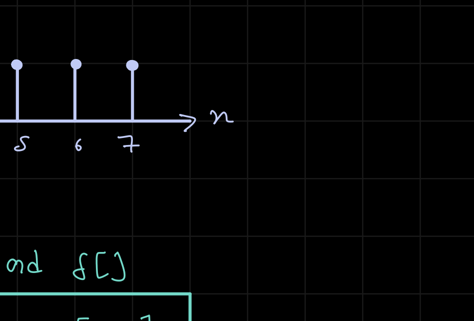
unit step signal $u[n]$

$$u[n] = \begin{cases} 1 & : n \geq 0 \\ 0 & : \text{else} \end{cases}$$



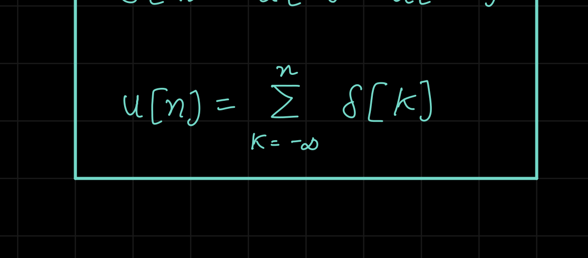
unit impulse function $\delta[n]$

$$\delta[n] = \begin{cases} 1 & : n = 0 \\ 0 & : \text{else} \end{cases}$$



note:

$1 : n-3 \geq 0$
$n \geq 3$



conversion btw u[ ] and δ[ ]

$$\boxed{\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^{n} \delta[k] \end{aligned}}$$

note: $u[n] = 0 \quad \forall\, n < 0$

so,

$$u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$$

$$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$$

$$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$$

$$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1 \quad ??$$

- $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when $n - k = 0$ i.e. $k = -2$
but k is never negative
since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n - k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n - k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underset{0}{\delta[1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[-1]} + \ldots$$
$$= \underline{1} \checkmark$$

- **SIFTING / SAMPLING**

$x[n] \cdot \delta[n] = ?$

$$x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

we can also say $x[n]\,\delta[n] = x[0]\,\delta[n]$

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

eg) $x[n] \cdot \delta[n - n_0] = ?$

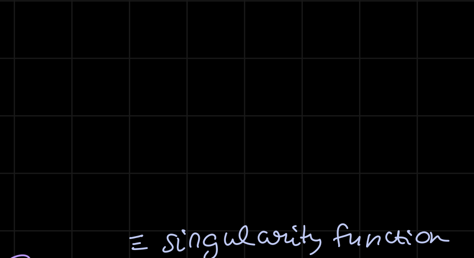
$$x[n] \cdot \delta[n - n_0] = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- unit step function

$$u(t) = \begin{cases} 1 & : t \geq 0 \\ 0 & : \text{else} \end{cases}$$



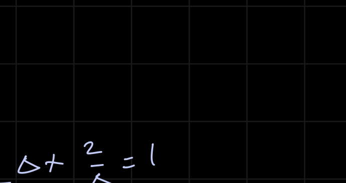
OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function ≡ singularity function

$$\delta(t) = 0 \quad : t \neq 0$$

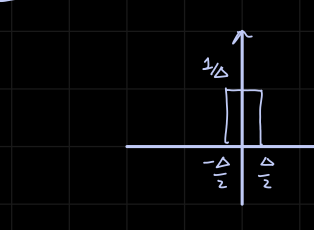
and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

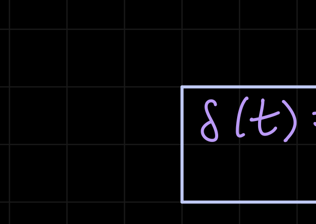


$\delta_\Delta(t)$:

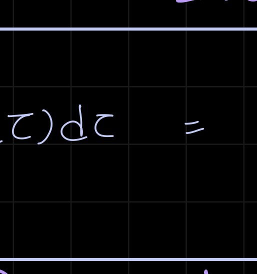
infinitesimally small

$$\delta_\Delta(t) = \begin{cases} 0 & : t \geq \Delta \ \&\ t < 0 \\ \frac{1}{\Delta} & : \text{otherwise} \end{cases}$$





area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
≡ unit imp ulse func ✓



≡ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t - \tau)\,d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

* Tutorial : 1 30/08/24

(7) $\text{even}\{x[t]\} = \dfrac{x(t)+x(-t)}{2}$

$\text{odd}\{x[t]\} = \dfrac{x(t)-x(-t)}{2}$

8) (a) $x_1[n] = \sin(0.02\pi n)$

$\omega = 0.02\pi$

$T = \dfrac{2\pi}{\omega} =$

(3) $p(x,y) = a^2x_1y_1 + abx_1y_2 + b^2x_2y_2 + ab\,x_2y_1$

$A = \begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

$|A| = a^2b^2 - a^2b^2 = 0$

atleast one eigenvalue is zero

So, since not all eigenvalues are +ve

(d) for no values of a, b

eigenvalues are values of λ such that

$|A - \lambda I| = 0$

eigenvalues / eigenvectors

Q17 $B = \{V_1, V_2, \dots V_n\}$

B is span of V

$\bar{x} \in V$

$\bar{x} = K_1V_1 + K_2V_2 + \dots K_nV_n$

$-\ \bar{x} = C_1V_1 + C_2V_2 + \dots C_nV_n$

$0 = (K_1 - C_1)V_1 + (K_2 - C_2)V_2 + \dots (K_n - C_n)V_n$

since $\bar{V_i} \neq 0$, $(K_i - C_i)$ need to be 0

i.e. $K_i = C_i$

$\dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\times10^{-2}\pi} = 100 \in T$

(8) $x_1[n] = \sin(0.02\pi n)$

Periodic ✓

(11) $y[n] = \sum_{n=-5}^{5} \sin(2n)\,\delta[n+7]$

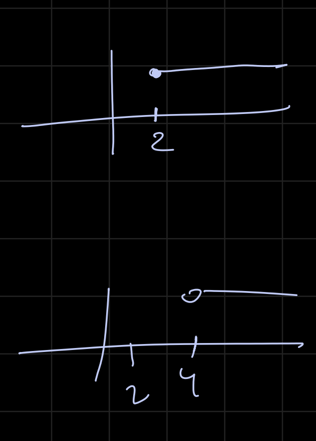
but range is $-5 \to 5$, so ans = 0

(12) $x(t) = u(t-2) - u(t-4)$

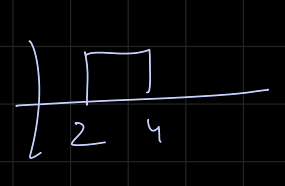
$u(t) \Rightarrow t \geq 0 : 1$

$u(t-2) \Rightarrow t \geq 2 : 1$

$u(t-4) \Rightarrow t \geq 4 : 1$



$\Rightarrow$



but since $\delta(t) = 1$ at $t = 0$

So, ans = 0

rational / irrational ?

$f = \dfrac{0.3}{2\pi}$

(5) (b) $x_2[n] = \cos\left(\dfrac{\pi}{2} + 0.3n\right)$

$\sin = \cos\left(\dfrac{\pi}{2} - \theta\right)$

(c) $x_3[n] = 1 - e^{j2\pi n/5} + e^{j4\pi n/7}$

note: $e^{j\theta} = \cos\theta + j\sin\theta$

$\Rightarrow 1 - \cos\left(\dfrac{2\pi n}{5}\right) - j\sin\left(\dfrac{2\pi n}{5}\right)$

$+ \cos\left(\dfrac{4\pi n}{7}\right) + j\sin\left(\dfrac{4\pi n}{7}\right)$

5 5 7/2 7/2

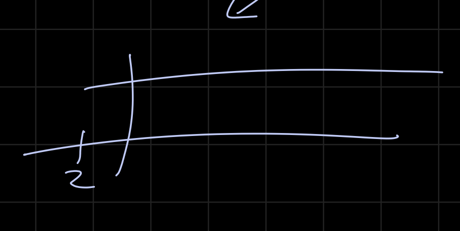
$\Rightarrow \text{LCM}\left(5, \dfrac{7}{2}\right) \Rightarrow$

(a) $\dfrac{x(t) + x(-t)}{2} = 0$

$x(t) = -x(-t)$

$\forall t$

(b) $e^{-5t}u(t+2)$



$e^{-5t}u(t+2) = -[e^{5t}u(-t+2)]$

$(e^{-5t})_{(-2,\infty)} + (e^{5t})_{(-\infty,2)}$

$(e^{-5t})_{(-2,2)} + (e^{5t})_{(-2,2)}$

(5) $P = tu(t)$

$\lim_{T\to\infty} \dfrac{1}{2T}\int_{-T}^{T} |t\,u(t)|^2\,dt$

$\lim_{T\to\infty} \dfrac{1}{2T}\int_{0}^{T} t^2\,dt$

$\lim_{T\to\infty} \dfrac{1}{2T}\left[\dfrac{t^3}{3}\right]_0^T$

$\lim_{T\to\infty} \dfrac{1}{2T} \times \dfrac{T^3}{3} \Rightarrow \dfrac{T^2}{6}$

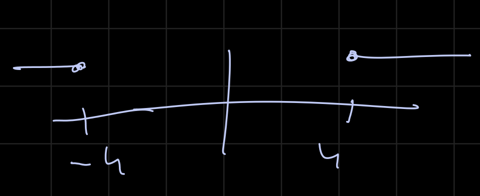
$\to \infty$

neither energy nor power signal

(9) (c) $\dfrac{u[n] - u[n-4] + u[-n] - u[-n-4]}{2} = 0$

$u[n] + u[-n] = u[n-4] + u[-n-4]$

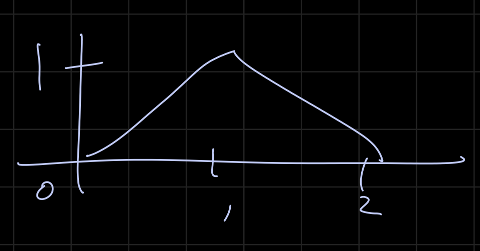
$1 : n \geq 0$ $\quad 1 : n \leq 0$ $\quad n \geq 4$ $\quad n \leq 4$



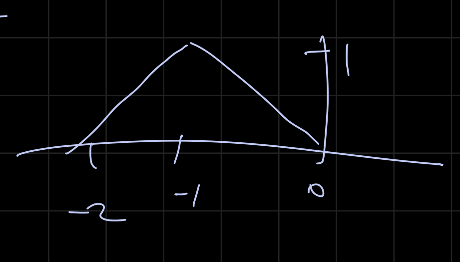
$-3 < n < 3$

$\boxed{|n| < 3}$ ✓

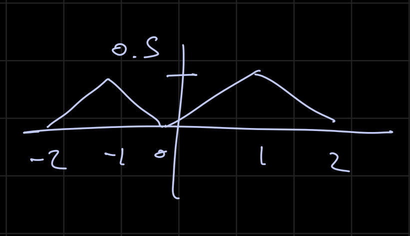
(7) $x(t) =$



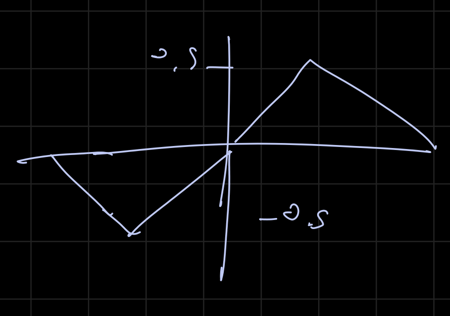
$x(-t) =$



$\text{even} = \dfrac{x(t) + x(-t)}{2} =$



odd =



* Lecture: 5

⇒ Systems and their properties

$$x(t) \xrightarrow{\text{input}} \boxed{\text{system}} \xrightarrow{\text{output}} y(t)$$

$$x[n] \rightarrow y[n]$$

① System: with memory or without memory

$S_1: y(t) = x(t)$ no memory
$S_2: y(t) = x^2(t)$ no memory
$S_3: y(t) = x(t-2)$ memory ✓ requires past value
$S_4: y(t) = \frac{1}{2}\{x(t) + x(t-1)\}$ mem ✓

output depends on just present inputs → no memory required

output needs to store previous outputs → requires memory

$S_5: y(t) = x(t+2)$: requires memory

• Invertible / Non-invertible Systems

$$x(t) \rightarrow \boxed{S_1} \xrightarrow{w(t)} \boxed{S_2} \rightarrow y(t)$$

$y(t) = x(t)$ iff $S_2 = \text{inverse}(S_1)$

ⓐ let $S_1: w(t) = 2x(t)$
$S_2: y(t) = \frac{1}{2} w(t) = \frac{1}{2} \times 2x(t) = x(t)$

S_1 is invertible and S_2 is the inverse system

ⓑ let $S_1: w[n] = \sum_{k=-\infty}^{n} x[n]$

then $S_2: y[n] = w[n] - w[n-1]$ invertible ✓

$= \{\cancel{x[-\infty]} + \dots \cancel{x[n-1]} + x[n]\} - \{\cancel{x[-\infty]} + \dots \cancel{x[n-1]}\}$

$= x[n]$

ⓒ let $S_1: w[n] = x[n] - x[n-1]$

$S_2: y[n] = \sum_{k=-\infty}^{n} w[k] = x[n]$ invertible ✓

$\cancel{x[-\infty]} - x[-\infty-1] + \dots \cancel{x[1]} - \cancel{x[2]} + \cancel{x[2]} - x[3] + x[3] - x[4] + \dots - x[n-1] + x[n-1] - x[n]$

ⓔ $S_1: w(t) = x^2(t)$
$S_2: y(t) = \sqrt{w(t)}$ } S_1 is not invertible

* System is invertible if and only if distinct inputs lead to distinct outputs

(very casual)

• Causal Systems

$S_1: y(t) = x(t - t_0)$: t_0 is a +ve real number

$S_2: y[n] = x[n+3]$

System output is depending on past system output

$S_1 \equiv$ causal system
$S_2 \equiv$ non-causal system

Causality: unless there is no cause, there should be no effect.

OR unless there is a non-zero input, there should not be non-zero output

OR a causal system depends on current and past input

eg) causal system: $y[n] = \sum_{k=-\infty}^{n} x[k]$

image processing: $S_4: y[n,m] = \frac{1}{5}\{x[n,m] + x[n-1,m] + x[n+1,m] + x[n,m-1] + x[n,m+1]\}$

non-causal system

• Stability System

A bounded input $\equiv |x(t)| < M_x < \infty$ (finite value $\forall t$)

not on the duration (y input) → but on the amplitude (x axis) ↓

OR $|x[n]| < M_x < \infty$

This bounded input is fed to a system If the system generates an output which is also bounded, we call it a stable system.

eg) $x(t) = u(t)$; $x[n] = u[n]$
$[0,1]$

unbounded: $x(t) = tu(t)$ OR $x[n] = tu[n]$

if $|y(t)| < M_y < \infty \; \forall t$, for bounded input,
↳ finite number

then the system is stable

eg) $S_1: y[n] = \frac{1}{3}\{x[n] - x[n-1] - x[n+1]\}$

$|y[n]| = |\frac{1}{3}x[n] - \frac{1}{3}x[n-1] - \frac{1}{3}x[n+1]|$

since $|x[n]| < M_x < \infty \; \forall n$

and the sum & difference of 3 finite terms is finite, so,

$|y[n]| < M_y < \infty \; \forall n$

$S_3: y[n] = nx[n]$

$|x[n]| < M_x < \infty \; \forall n$
bounded input assumed

$\equiv |y[n]| = |nx[n]| \rightarrow \infty$ as $n \rightarrow \infty$

unstable system

$S_4: y(t) = x(2t)$
↓
no manipulation on the y axis 2t means manipulation on the x-axis

• Time Invariant System

if $x(t) \xrightarrow{\text{system}} y(t)$

then $x(t-t_0) \rightarrow y(t-t_0)$

and $x(t+t_0) \rightarrow y(t+t_0)$

system output does not change if there is a change in time

if $x_1(t) \rightarrow y_1(t)$
and $x_2(t) \rightarrow y_2(t)$
and $x_2(t) = x_1(t-t_0)$

then $y_2(t)$ must be equal to $y_1(t-t_0)$

if it is a time invariant system

eg) $S: y(t) = x(2t)$

⇒ let $t_0 = 4$: $y(t-t_0) = x(2t-8)$

⇒ time variant system

$x_1(t) \rightarrow y_1(t) = x_1(2t)$
$x_2(t) \rightarrow y_2(t) = x_2(2t)$

$y_1(t-t_0) = x_1(2t-2t_0)$
first scaling : $x_1(2t)$
then delay : $x_1(2(t-t_0))$ ($t \rightarrow t-t_0$)

$y_2(t-t_0) = x(2t-t_0)$
first delay : $x(t-t_0)$
then scaling : $x(2t-t_0)$

$S_2: y[n] = nx[n]$

$x_1[n] \rightarrow y_1[n] = nx_1[n]$
↓ delayed output
$y_1[n-n_0] = (n-n_0)x[n-n_0]$

$x_2[n] \rightarrow y_2[n] = nx_2[n]$

let $x_2[n] = x_1[n-n_0]$ be delayed input
↓
$y_2[n] = (n)x[n-n_0]$

since $y_1[n] \neq y_2[n]$, system is time varying

note: 1st system had delayed output and we gave delayed input

• Linearity

: A system is linear if the properties of additivity & homogenity holds true

$x_1(t) \rightarrow y_1(t)$
$x_2(t) \rightarrow y_2(t)$

⇒ $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
then system is linear

eg) $y(t) = 2x(t) + 3$

$x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$

$x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$

$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) = 2(x_1 + x_2) + 2(3)$

but $y(x_1 + x_2) = 2(x_1 + x_2) + 3$

Since, $y_1(t) + y_2(t) \neq y(x_1 + x_2)$

not a linear system

• A system which is linear and time invariant $\equiv$ LTI
↓
linear & time invariant system

* Lecture: SNS 6 4/9/24

q) $x[n] = \cos(\frac{\pi}{8} n^2)$

periodic if $x(n) = x[n+N] \ \forall n$

$\cos(\frac{\pi}{8}(n^2 + N^2 + 2nN)) = \cos(\frac{\pi}{8} n^2)$

$\cos(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{4} nN) = \cos(\frac{\pi}{8} n^2)$

we know $\cos(2\pi\theta + K) = \cos\theta \quad \forall K \in \mathbb{Z}$

$\frac{\pi}{8} N^2 \Rightarrow$ can it be a multiple of 2π?

if $N=4 \Rightarrow 2\pi = K2\pi$ ✓ ($K=1$)

true for $N=8$ as well

but not for $N=2$

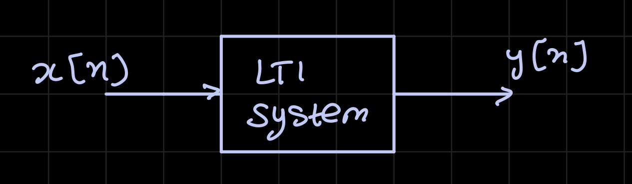
$\frac{\pi}{4} nN \to K2\pi$?

$N=4 \Rightarrow \pi 4n \to$ ✗

$N=8 \Rightarrow$ ✓ $2\pi nN$ = ✓

---

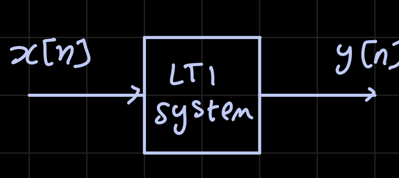
* Discrete Time convolution



$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K]$$

weighted linear combination of delayed signals

* Impulse Response of an LTI System



$\delta[n] \to y[n] = h[n]$

≡ Impulse response of the LTI System

$\delta[n-1] \to h[n-1]$

$x[1]\,\delta[n-1] \to x[1]\,h[n-1]$

$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K] \longrightarrow \sum_{K=-\infty}^{\infty} x[K]\,h[n-K] = y[n]$$

discrete time convolution of

$x[n] \times h[n] = y[n]$

Valid for LTI systems only

convolution → linear & commutative

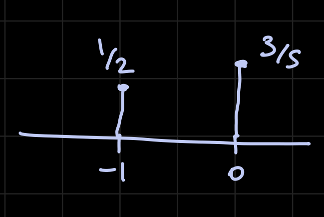
$x[n] \times h[n] = h[n] \times x[n]$

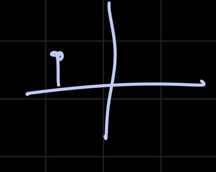
$(x_1[n] + x_2[n])\,h[n] = x_1[n]\,h[n] + x_2[n]\,h[n]$

eg) given an LTI System whose impulse response is

$h[n] = \begin{cases} 1 & n = 0,1,2 \\ 0 & \end{cases}$

and $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$





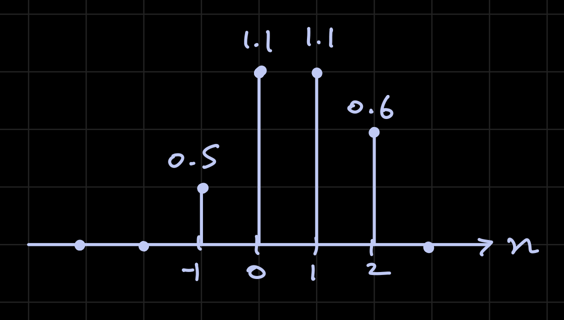
compute $y[n]$

Ans) $y[n] = h[n] * x[n]$

$y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1]$

because for $x[n] \to y[n]$, $\delta[n] \to h[n]$

$$y[n] = \frac{3}{5}\left[\delta(n) + \delta(n-1) + \delta(n-2)\right] + \frac{1}{2}\left[\delta[n+1] + \delta[n] + \delta[n-1]\right]$$



eg) LTI System (← given)

$h[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$

$x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

Ans) $y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$

OR $\equiv \sum_{-\infty}^{\infty} h[K]\,x[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} x[n-K] = \sum_{K=0}^{\infty} \alpha^{(n-K)} u[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} \frac{\alpha^n}{\alpha^K} u[n-K]$ ($1 : n \geq K$)

$y[n] = \sum_{K=0}^{n} \frac{\alpha^n}{\alpha^K}$ ✓

$\frac{a(1-r^n)}{1-r}$

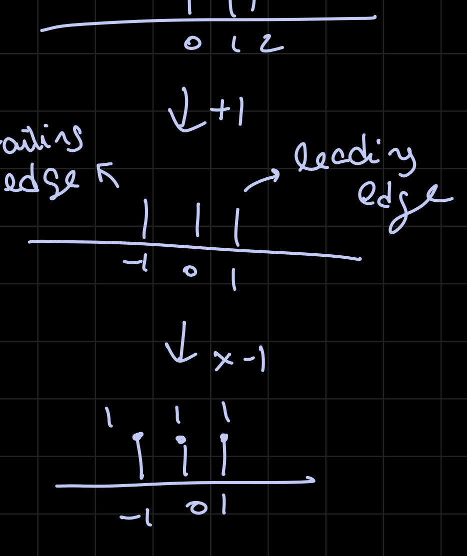
$\times -1 \quad -K \xrightarrow{+1} -K+1$

$K \to (1-K)$

$\downarrow +1$

$K+1$

$\times -1 \quad -K+1$



* LECTURE: 7 06/09/24

Note: $y[n] \to$ function $\equiv$ variable
$y[10] \to$ scalar $\equiv$ constant

$h(n)$ = characteristic impulse response of the system

$$y[n] = x[n]\,h[n]$$

An LTI system is uniquely charaterized by its $h[n]$ - hence two systems are same if they are LTI and their $h[n]$ are same

bonus question} $\int_{-\infty}^{t} \cos(\tau)\,x(\tau)\,d\tau$

let $x(\tau) = \cos(\tau)$

$$\Rightarrow \int_{-\infty}^{t} \cos^2(\tau)\,d\tau \Rightarrow \int_{-\infty}^{t} \frac{1+\cos(2\tau)}{2}\,d\tau$$

$$\Rightarrow \Big[\underbrace{\frac{t}{2}}_{} + \underbrace{\overbrace{\frac{\sin(2t)}{4}}^{\in[-1,1]}}_{\text{bounded}}\Big]_{-\infty}^{t} \Rightarrow$$

becomes unbounded as $t \to \infty$ ($\equiv$ unstable)

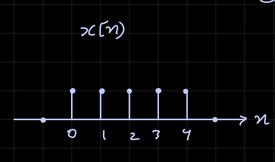
Q1) $\delta(8t-2) = 1$ at $t = \frac{1}{4}$ ✗

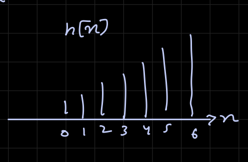
$\frac{1}{8}\,\delta(t-\frac{1}{4}) = 1$ at $t = \frac{1}{4}$ ✓

eg} $x[n] = \begin{cases} 1 : & 0 \le n \le 4 \\ 0 : & \text{else} \end{cases}$

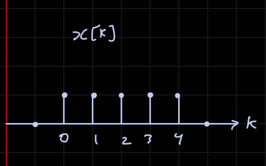
$x[n] = u[n] - u[n-4]$

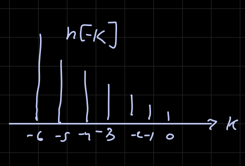
$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{else} \end{cases}$ & $\alpha > 1$





$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$$
$$y[0] = \sum_{-\infty}^{\infty} x[K]\,h[-K]$$





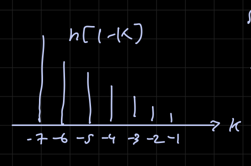
only common point : $k = 0$

$$y[n] = x[0]\,h[-0] = h[n] = \alpha^0 = \underline{1}$$

$$y[1] = \sum_{-\infty}^{\infty} x[K]\,h[-1-K]$$

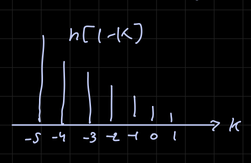
$h[K] \xrightarrow{-1} h[K-1] \to h[-K-1]$

shifting → scaling → reversing



$h[1-K] = 1$ at $1-K = 0 \Rightarrow K = 1$

but we do scaling then translation here



Check: $h[1-K] = 1$ at $1-K=0$ because $\alpha^K = \alpha^0 = 1$

$K = 1$

$\Rightarrow y[K] = x[K]\,h[1-K]$

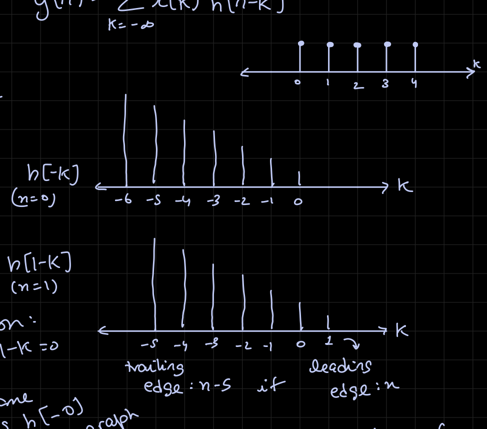
now, we shifted $h[-K]$ 1 unit to the right and we have overlap with $x[K]$ on $K = 0, 1$

Similarly for $x[K-n]$, we have overlap with $K = 0, 1 \ldots\ldots n-1, n$

404: Brain not found
2nd attempt initiated ...

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$$

we are not shifting $x[K]$ in this case but we could



trailing edge : $n-5$ if leading edge : n

verification: $h[0]$ at $1-K=0$, $K=1$ $\equiv$ same as $h[-0]$ in previous graph (argument is zero)

$h[n-K] \ne 0$		range of args for which $h[n-K]$ is non zero
		$n-6 \to n$
$n = -1$	:	$-7 \to -1$
$n = 1$	:	$-5 \to 1$
$n = -2$	:	$-8 \to -2$

① when $n < 0$, there is no overlap with $x[n]$ hence $y[n] = 0$

② $n = 11$: 5 to 11
$n = 10$: 4 to 10

So, no overlap with $x[n]$ when $n > 10$

hence, $y[n] = 0$: $n < 0$ and $n > 10$

③ $0 \le n < 4$: $y[n] = \sum_{K=0}^{n} x[K]\,h[n-K]$

overlap	n	range
$K = 0$	0	$-6 \to 0$
$K = 0,1$	1	$-5 \to 1$
$K = 0,1,2$	2	$-4 \to 2$
$K = 0,1,2,3$	3	$-3 \to 3$
$K = 0,1,2,3,4$	4	$-2 \to 4$
$K = 0,1,2,3,4$	5	$-1 \to 5$
$K = 0,1,2,3,4$	6	$0 \to 6$
$K = 1,2,3,4$	7	$1 \to 7$
$K = 2,3,4$	8	$2 \to 8$
$K = 3,4$	9	$3 \to 9$
$K = 4$	10	$4 \to 10$

leading edge of h left $x(n)$

So, $y[n]$ is defined for $0 \le n \le 10$

② $0 \le n \le 4 \Rightarrow y[n] = \sum_{K=0}^{n} x[K]\,h[n-K]$

③ $4 < n \le 6 \Rightarrow y[n] = \sum_{K=0}^{4} x[K]\,h[n-K]$

④ $6 < n \le 10 \Rightarrow y[n] = \sum_{K=n-6}^{4} x[K]\,h[n-K]$

overlap: $1,2,3,4 \to 2,3,4 \to 3,4 \to 4$
n: $7 \to 8 \to 9 \to 10$

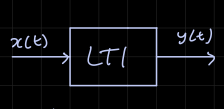
$$\text{final} \to y[n] = \sum_{K=0}^{n} \alpha^{n-K} : 0 \le n \le 4$$
$$+ \sum_{K=0}^{4} \alpha^{n-K} : 4 < n \le 6$$
$$+ \sum_{K=n-6}^{4} \alpha^{n-K} : 6 < n \le 10$$

convolution for CTS = DTS

$$\sum \alpha^{n-K} = \alpha^n \sum \frac{1}{\alpha^K} = \alpha^n \{GP\}$$

$$\frac{a(1-r^n)}{1-r}$$

$\boxed{\text{CTS}}$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

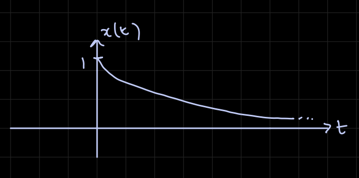
useful when $h(t) = u(t)$

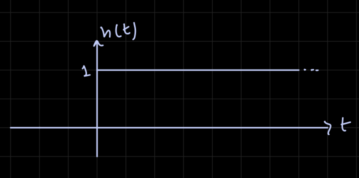
$$y(t) = x(t) * h(t)$$

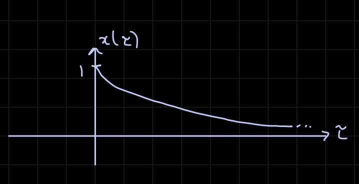
convolution operation

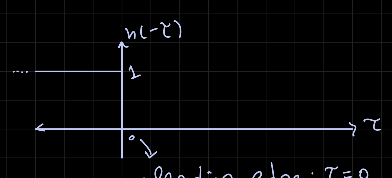
eg? $x(t) = e^{-at}\,u(t)$: $a > 0$

$h(t) = u(t)$









leading edge : $\tau = 0$

no overlap when we shift $h(-\tau)$ to left, there will be no overlap and hence that integral = 0 ✗

shift to right ✓

① $t < 0 \Rightarrow y(t) = 0$

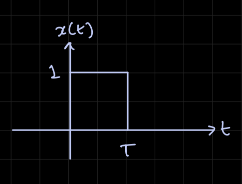
② $t > 0 \Rightarrow y(t) = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$ ✓

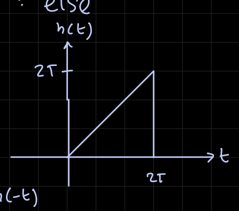
integral $-\frac{e^{-at}}{a}$

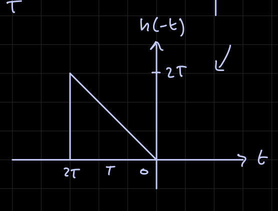
starts at 0 then saturates as $t \to \infty$

eg} $x(t) = \begin{cases} 1 & : 0 < t < T \\ 0 & : \text{else} \end{cases}$

$h(t) = \begin{cases} t & : 0 < t < 2T \\ 0 & : \text{else} \end{cases}$







t = shift

① $t < 0$: $y(t) = 0$ no overlap

② $0 \le t < T$: $y(t) = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$

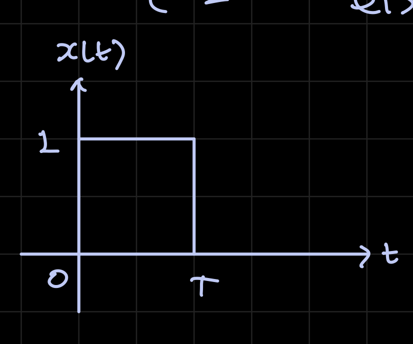
③ $T \le t \le 2T$: $y(t) = \int_{0}^{T} x(\tau)\,h(t-\tau)\,d\tau$

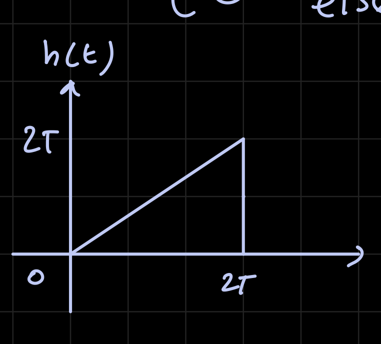
④ $2T < t \le 3T$ $y(t) = \int_{t-2T}^{T} x(\tau)\,h(t-\tau)\,d\tau$

* Lecture: 8 11/09/24

⇒ convolution

$x(t) = \begin{cases} 0 & 0 \le t \le T \\ 1 & \text{else} \end{cases}$ $h(t) = \begin{cases} t & 0 \le t \le 2t \\ 0 & \text{else} \end{cases}$

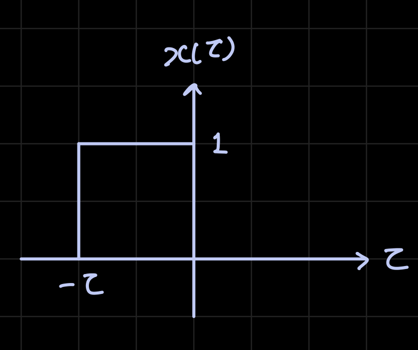


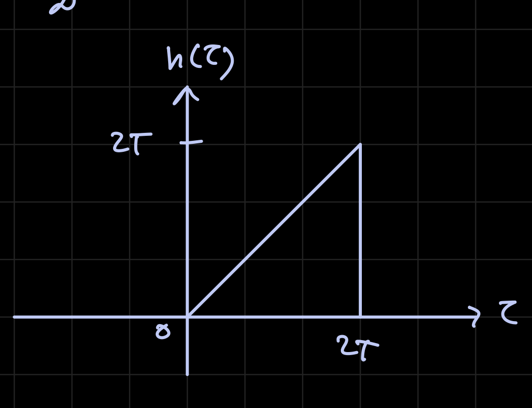


easier to reverse $x(t)$

so, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

this one prefered here ⇐ $= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$





at $y(0)$ ⇒ overlap = none

so, $y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

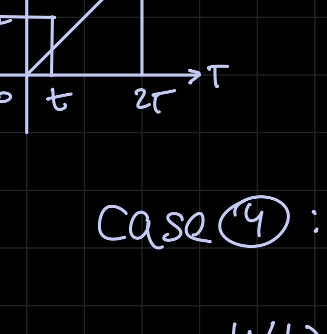
moving forward, $y(1) \dots y(n)$
if we shift $x(-\tau)$ to left
i.e. $t<0$, it will neve overlap

So, we move to right,
$t>0$
when plotting $x(t-\tau)$

case ① $t<0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$$

case ② $t>3t$ again no overlap

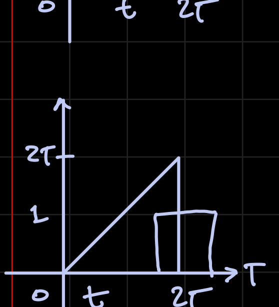


$y(t) = 0$

no non-zero overlap

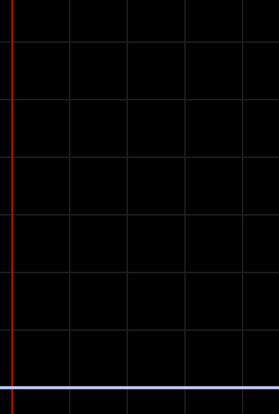
Case ③ $0<t<T$

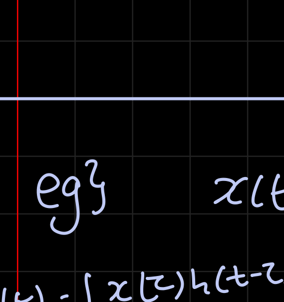
$$y(t) = \int_0^t h(\tau)\, d\tau = \int_0^t \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$



Case ④: $T \le t \le 2T$

$$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{-T^2}{2} + tT$$





case ⑤ $2T \le t \le 3T$

$$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$$

$$= \frac{4T^2}{2} - \frac{(t-T)^2}{2}$$

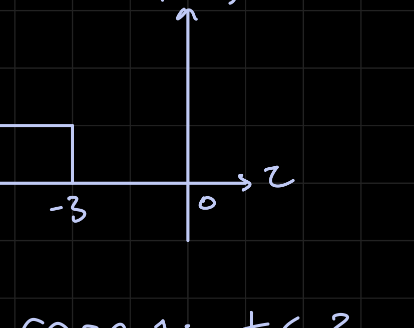
$$= \frac{4T^2 - t^2 - T^2 + 2Tt}{2}$$

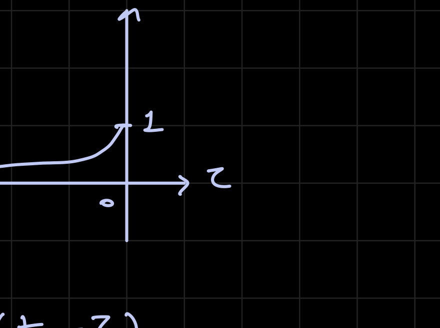
$$= \frac{3T^2 + 2tT - t^2}{2}$$

---

eg} $x(t) = e^{2t} u(-t)$ $h(t) = u(t-3)$

$y(t) = \int x(\tau) h(t-\tau) d\tau$
$= \int x(t-\tau) h(\tau) dt$

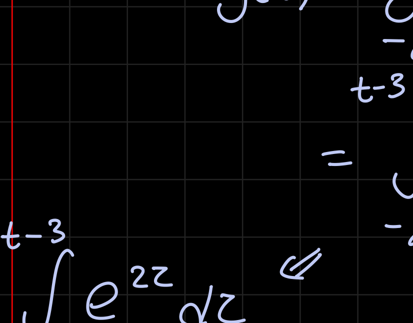


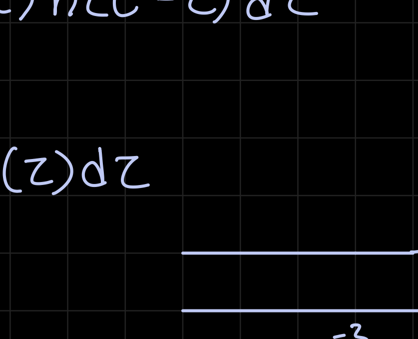


Which one to reverse?
we can reverse
both but which one is easier?

$t \ge 3$

$h(-\tau)$

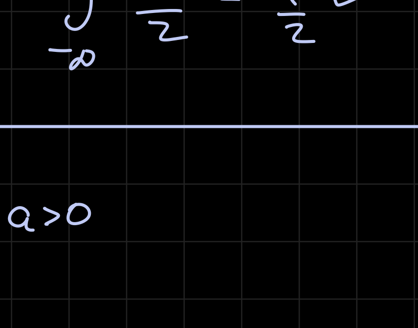




Case 1: $t \le 3$ in $h(t-\tau)$

moves to right if $t>0$
left if $t<0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{t-3} x(\tau)\, d\tau$$



$\int_{-\infty}^{t-3} e^{2\tau} d\tau$

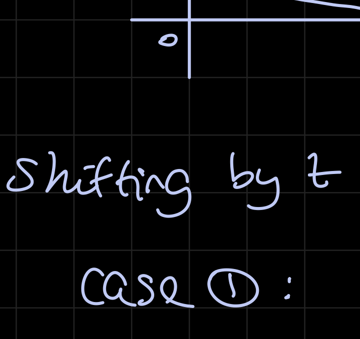
$$\Rightarrow \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t-6}}{2} = \frac{e^{2t}}{2e^6}$$

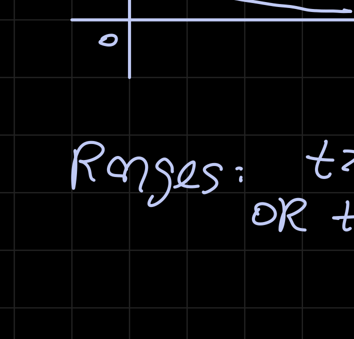
case ② $t>3$: constant: $e^{2\tau}$

$$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \int_{-\infty}^{0} \frac{e^{2\tau}}{2} = \frac{1}{2}$$

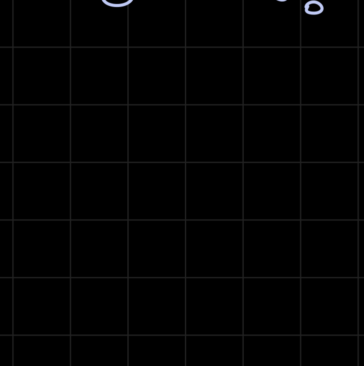
---

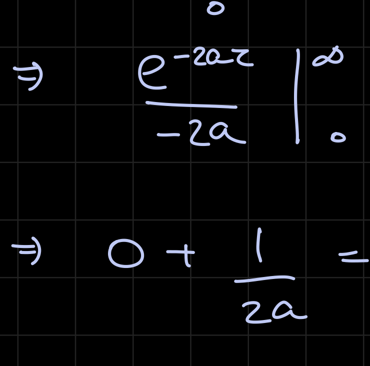
eg} $x(t) = e^{at} u(-t)$: $a>0$
$h(t) = e^{-at} u(t)$





flip any here are similar





Shifting by t Ranges: $t>0$, $t\le 0$
OR $t \ge 0$, $t<0$

case ①: $t=0$

$$y(0) = \int_0^{\infty} x(-\tau)\, h(\tau) = \int_0^{\infty} e^{-2a\tau} d\tau$$

$$\Rightarrow \frac{e^{-2a\tau}}{-2a}\Big|_0^{\infty}$$

$$\Rightarrow 0 + \frac{1}{2a} = \frac{1}{2a}$$

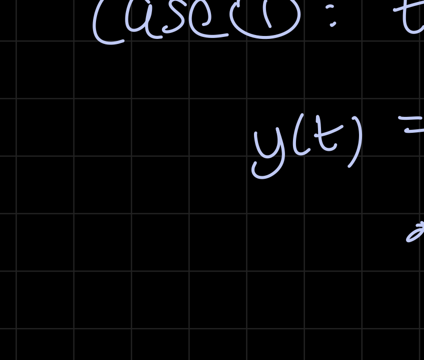
case ③: $t<0$

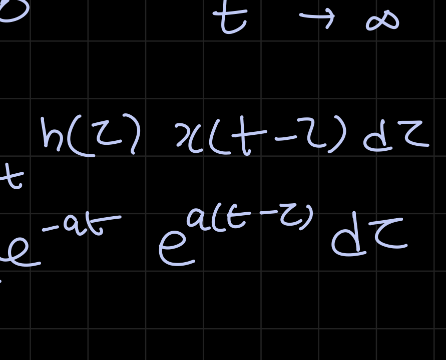
$$y(t) = \int_0^{\tau-t} x(t-\tau)\, h(\tau)\, d\tau$$

$t-\tau \to \infty$

$$= \int_0^{\tau-t}$$

left edge of $x(-\tau)$ will
always be equal to t





for $t \ge 0$: $t \to \infty$

for $t<0$: $0 \to \infty$

case ①: $t \ge 0$ $t \to \infty$

$$y(t) = \int_t^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$
$$\int_t^{\infty} e^{-at}\, e^{a(t-\tau)}\, d\tau$$

case ②: $t<0$ $0 \to \infty$

$$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^{\infty} e^{-at} \cdot e^{a(t-\tau)}\, d\tau$$

* SYSTEM PROPERTIES

given an LTI system and

$$h[n] = \begin{cases} 1 & : n = 0,1 \\ 0 & : \text{else} \end{cases}$$

defaults: input: $x[\,]$
output: $y[\,]$
impulse response: $h[\,]$

S: $y[n] = ?$ ⇒ System output = ?

apply convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

from the given defn, $h[n] = \delta[n] + \delta[n-1]$

$$\Rightarrow h[k] = \delta[k] + \delta[k-1]$$

so, $y[n] = \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1])\, x[n-k]$

$$= \sum_{-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{-\infty}^{\infty} \delta[k-1]\, x[n-k]$$

$$= x[n] + x[n-1]$$

eg} find impulse response of systems

S_2: $y[n] = (x[n] + x[n-1])^2$

S_3: $y[n] = \max\{x[n], x[n-1]\}$

we know: $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

There is only one LTI system
that gives the same impulse response

these systems S_2, S_3 give same response
because they are not LTI

* Lecture: 9 13/09/24

- SYSTEM PROPERTIES

② (i) Commutativity

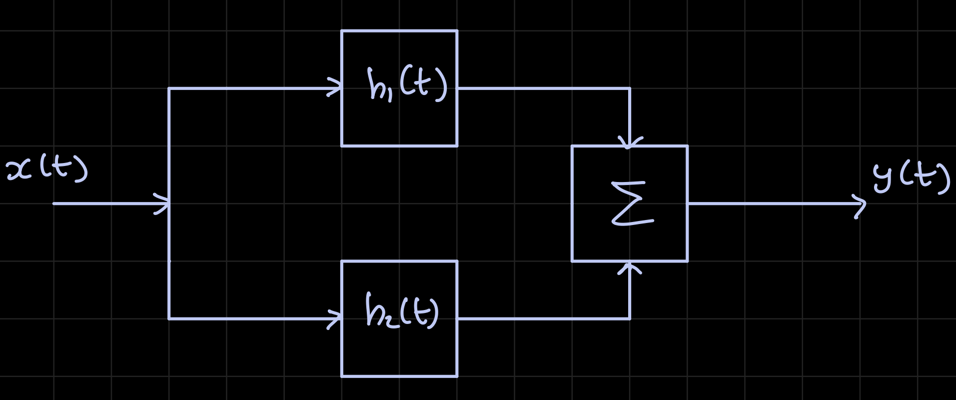
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Prove

(ii) Convolution distributes over addition

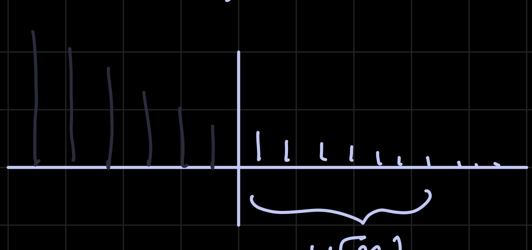
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



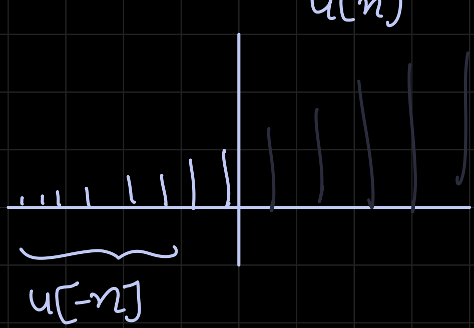
holds true for both CTS and DTS

eg} $x[n] = (\frac{1}{2})^n u[n] + 2^n u[-n]$

$x_1[n] = 2^n u[n]$



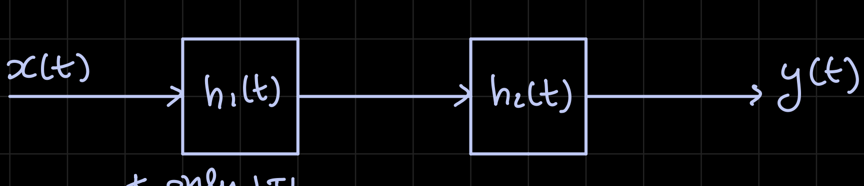
$x_2[n] = 2^n u[-n]$



easier : $y[n] = x_1[n] * h[n] + x_2[n] * h[n]$

(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x_1(t) * h_1(t)] * h_2(t)$$



* only LTI

for LTI systems, in cascading, order of these systems can be swapped

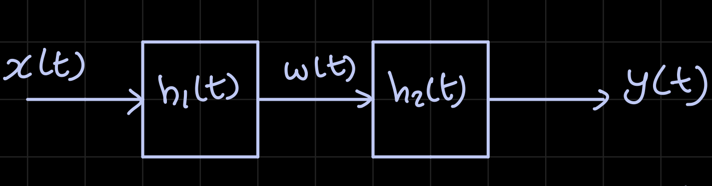
③ Invertibility :

Note: if $h(t)$ given: LTI system ✓

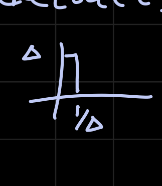
elif $h(t,\tau)$ given : Non LTI system ✓

output at t

unit impulse input applied at τ



eg} if $h_1(t) * h_2(t) = \delta(t)$ → continuous delta(t)



$$y(t) = x(t) * (h_1(t) * h_2(t))$$

$= x(t)\, \delta(t)$ } operational definition of $\delta(t)$

$= x(t)$

↓ we do not write

✗ $\delta(t) = 1$ at $t=0$

↓

✓ $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

↳ not a scalar signal?

↳ can we have a signal with ∞ value at a single point?

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, dt}$$

Discrete Time

$$x[n] = \sum_{-\infty}^{\infty} x[k]\, \delta[n-k]$$

$$= x[n] * \delta[n]$$

↓ for this we can say $\delta[n] = 1 : n = 0$

④ Causality

claim: if a system is causal,

$h(t) = 0$ for $t<0$ (CTS)

$h[n] = 0$ for $n<0$ (DTS)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

because x cannot have future values so $k>0$

hence, for causal systems,

$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$

OR $y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$

} if $h[n] = 0$ at $n<0$

integrator : S if $h(n) = u(n)$ CTS

accumulator : S if $h[n] = u[n]$ DTS

⑤ memory / memoryless system

$S_1 : y(t) = Kx(t) \Rightarrow h(t) = K\delta(t)$

$S_2 : y[n] = Kx[n] \Rightarrow h[n] = K\delta[n]$

⑥ stability (BIBO Stability Criteria)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

let input signal is bounded and $|x(t)| < M_x \;\forall t$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau) d\tau| \le M_x \times (\text{some finite number})$$

if the system is absolutely integrable $< \infty \;\forall t$

$$\int_{-\infty}^{\infty} |h(\tau)\, d\tau| < \infty$$

DTS : $|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

if the system holds absolute summability,

then the system is stable

Note: $h[n] = u[n]$, $h(t) = u(t)$ } LTI, unstable, causal, with memory

but still we use these for integrators & accumulators because we use clipper and clamper since we know the upper and lower bound

Identity : system with $h(t) = \delta(t)$

* Constant Coefficient difference eqⁿ for causal LTI systems

$$\sum_{k=0}^{N} b_k\, y[n-k] = \sum_{k=0}^{M} a_k\, x[n-k]$$

a_k, b_k : not functions of time .. constant coeff ✓

① $N \equiv$ order of the system

② The system is at rest if :

if, $x[n] = 0 : n < n_0$

then, $y[n] = 0 : n < n_0$

$y[n-1] = 0 : n < n_0$

⋮

$y[n-N] = 0 : n < n_0$

} auxiliary conditions

note: $N > 0$

MA

→ case : 1 ⇒ $N = 0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots a_m x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_1} x[n-1] + \dots \frac{a_m}{b_m} x[n-M]$$

causal system ✓ ≡ moving average system → weighted sum

$n \to n-1 \to n-2 \to n-3 \dots$

system is called finite impulse response system

≡ FIR system

IIR / AR

→ case 2: $N = 1$ and $M = 0$ (simplicity)

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] - \alpha_1 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\alpha_1 = \frac{b_1}{b_0}$

if $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$h[0] = \alpha_0$

$h[1] = -\alpha_1 \alpha_0$

$h[2] = +\alpha_1^2 \alpha_0$

⋮

} FIR? ✗

Infinite impulse response system ✓ (IIR)

* Auto Regressive System

≡ AR system

output depends on regresses of its previous/past inputs

ARMA

→ Case : 3 $N > 0$, $M > 0$

* Lecture : 10 18/09/24

⇒ Constant coeff differential eqⁿ representation of causal LTI Systems

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k x(t)}{dt^k}$$

N ≡ order of the system

let $N = 2$ & $M = 3$

$$b_2 \frac{d^2 y(t)}{dt^2} + b_1 \frac{dy(t)}{dt} + b_0 y(t) =$$
$$a_0 x(t) + a_1 \frac{d\,x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a^3 \frac{d^3 x(t)}{dt^3}$$

Initial condition ↲

if $x(t) = 0 \quad \forall \; t < t_0$

then $\frac{d^k y(t)}{dt^k} = 0 \quad \forall \; t < t_0$

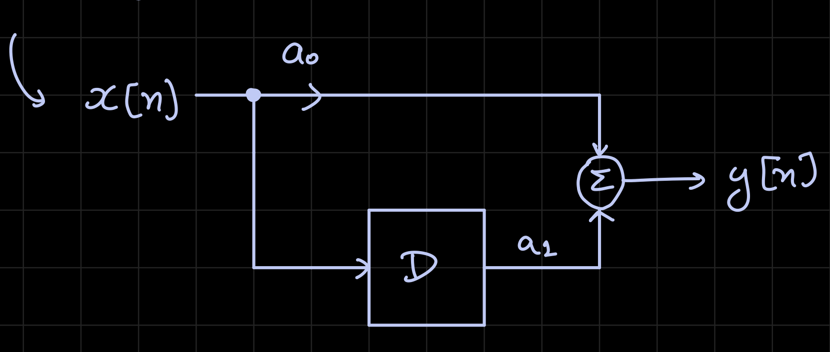
$\forall \; k \in [0, N]$

* Block diagram representation
useful for DTS

let $a_0 x[n]$ ≡ scalar multiplier

then $x[n-1]$ ≡ delay element

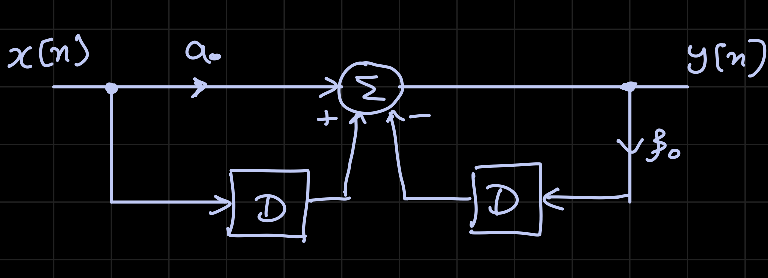
let $S_1 : \; y[n] = a_0 x[n] + a_1 x[n-1]$



$S_2 : \; b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$
$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$
and $\beta_0 = \frac{b_1}{b_0}$



* FOURIER SERIES

we know, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

and $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)$

$x[k]$ are scalars / constants
we can write them as a_k

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k]$$

$\boxed{CT}$ $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

weighted sum of frequencies

Synthesis equation $\left\{ x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \right.$ finite basis

weighted linear combination of vectors

$\phi_k(t) = e^{jk\omega_0 t}$
(freq. of oscillation)

$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T}$$

↳ fundamental time period of $x(t)$

$f = \omega_0$ rad/s
freq ≠ −ve
Time Period = T
counter clockwise $\phi_1(t) = e^{j\omega_0 t}$

completes 2 cycles in one T time period
freq = $2\omega_0$ rad/s
$\phi_2(t) = e^{2j\omega_0 t}$

Time Period = T
clockwise $\phi_{-1}(t) = e^{-j\omega_0 t}$ $\quad \phi_{-2}(t) = e^{-2j\omega_0 t}$
freq ≠ −ve
$f = \omega_0$ rad/s

$\phi_0(t) = e^{j0} = e^0 = 1 = \cos 0 + j \sin 0$ ✓

↳ DC component of signal
no fluctuation

can we write $\phi_2(t)$ in terms of $\phi_1(t)$?

No

they are independent of each other

inner product $\langle \phi_k(t), \phi_l(t) \rangle_T = \frac{1}{T}\int_0^T \phi_k(t)\,\phi_l^*(t)\,dt$

$$= \frac{1}{T}\int_0^T e^{jk\omega_0 t} \cdot e^{-jl\omega_0 t}\,dt$$

integers; let = m

$$= \frac{1}{T}\int_0^T e^{j(k-l)\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt$$

$$= \begin{cases} 0 & : k \neq l \\ 1 & : k = l \end{cases}$$

defining the inner product

$$\langle \phi_k(t), \phi_l(t) \rangle_T = \frac{1}{T}\int_T \phi_k(t)\,\phi_l^*(t)\,dt$$

⇒ Synthesis: $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$\{\phi_k(t)\}$ ≡ span a vector space from an orthonormal basis

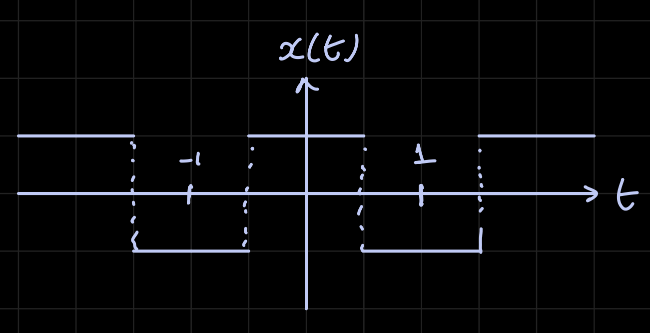
⇒ Analysis: $\quad a_k = \langle x(t), \phi_k(t) \rangle_T$

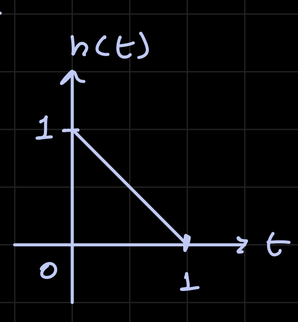
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_l(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_l(t) \right\rangle_T$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\langle \phi_k(t), \phi_l(t) \rangle_T}_{\delta[k-l]}$$

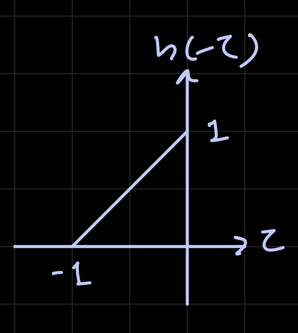
eg3



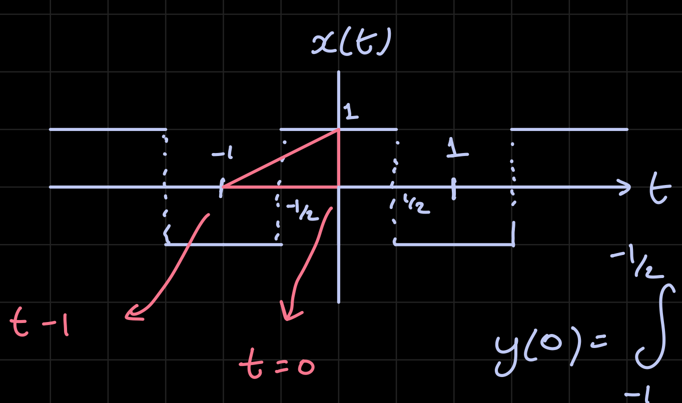


convolute $x(t)$ & $h(t)$

Ans3 $\quad h(\tau) = 1 - \tau$
$\quad h(-\tau) = 1 + \tau$ reflect



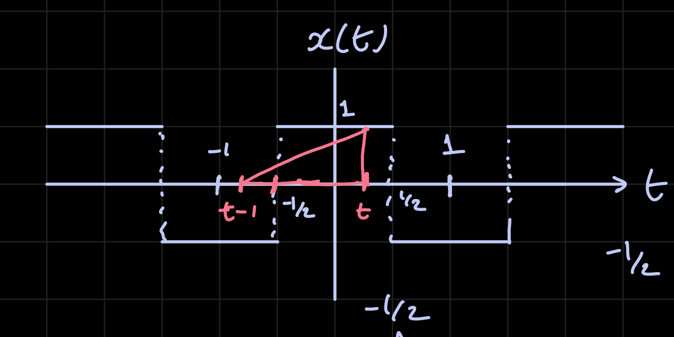
overlap



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(0) = \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

case : 1 : $\quad 0 \leq t \leq \frac{1}{2}$



$-1/2$

$$y(t) = \int_{t-1}^{-1/2} (1+\tau+t)\,d\tau + \int_{-1/2}^{t} (1+\tau+t)\,d\tau$$

case : 2 ⇒