14/08/24 # Signals And Systems 11:00Am

Evaluation: (relative grading)		
	Attendance	10%
	Quiz	16%
	Assignments	15%
	mid sem	24%
	end sem	35%

yooooooooooo :fire emoji:

Book: Signals & System 2nd edition by Alan V. oppenheimer

Lecture : 1 — 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ GROUP : consists of ....

- A set G
- A rule / binary operation "$*$"
  - a. associative
    $$x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$$
  - b. There exists an element "e" called the identity of group G such that
    $$e*x = x*e = x \quad \forall x \in G$$
  - c. $\forall x \in G, \exists\, x^{-1}$ such that
    $$x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$$
  - d. if $x*y = y*x \quad \forall x, y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abelian Group

eg: set of continuous time periodic signals with time period T with "$*$" = "+"

⇒ FIELD : consists of the following

- A set F
- Two binary operations "+" and "·" such that ....
  - $(F,+)$ is an abelian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - left distributive
      $$x\cdot(y+z) = xy + xz \quad \forall x,y,z \in F$$
    - Right distributive
      $$(x+y)\cdot z = xz + yz \quad \forall x,y,z \in F$$

eg: F = Real Numbers $\mathbb{R}$

* Vector space : A set V with a map...

- '+' : $V \times V \to V$
  $(v_1, v_2) \to (v_1 + v_2)$ called vector addition
- '·' : $F \times V \to V$
  $(a, v_1) \to (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a\cdot(v_1+v_2) = av_1 + av_2$
- $(a_1+a_2)\cdot v = a_1 v + a_2 v$
- $(\alpha\beta)v = \alpha(\beta v) = \beta(\alpha v)$
- $1\cdot v = v$
- $a \cdot 0 = 0$
- $0 \cdot v = 0$
- if $v \neq 0$, then $a \cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \to R$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X, d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM: let V be a F-vector space.

A map $\|\cdot\| : V \to R$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a| \, \|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

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* Inner Product :

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ " $-$ " : complex conjugate operation
- it is linear in the first co-ordinate
  $$\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$$
  $\forall\, v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $$\langle v, a_1 w_1 + a_2 w_2 \rangle = \bar{a}_1 \langle v, w_1 \rangle + \bar{a}_2 \langle v, w_2 \rangle$$
  $\forall\, w_1, w_2, v \in V$ and $a_1, a_2 \in F$

eg: dot product → measures cosine similarity " norm(x) · norm(y) · $\cos\theta$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner Product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot \bar{y}_i$$

→ complex conjugate

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \qquad \text{implies } a_i = 0 \ \forall i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

if $\begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{matrix}$ is a basis for $V_n(F)$, then $\forall\, v_i \in V$ & $a_i \in F$ → weighted linear combination of vector V

$$V = \sum_{i=1}^{n} a_i e_i$$

$V = a_1 e_1 + a_2 e_2 + \dots a_n e_n$

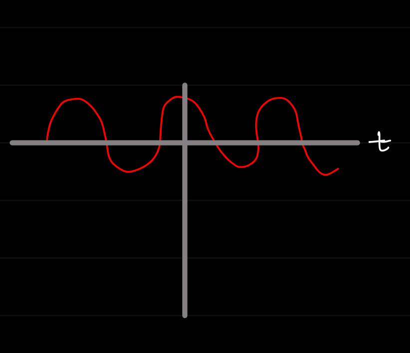
* ORTHOGONAL & ORTHONORMAL Basis

A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

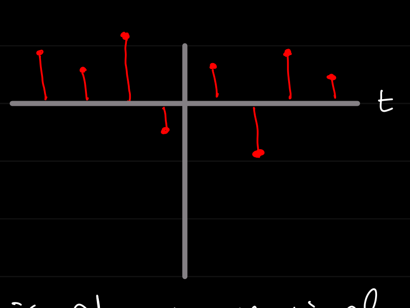
$$v_i \neq 0 \quad \forall i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall i \,\&\, j$$

SIGNALS (CTS & DTS)

- continuous time signal



- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t) \ \forall t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t) \ \forall t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal:

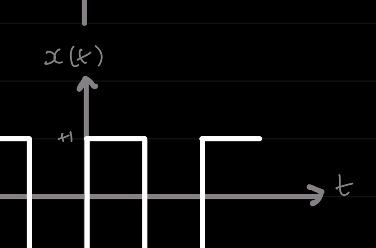
$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

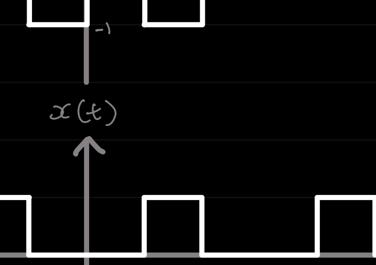
* PERIODIC : if $x(t) = x(t+T) \ \forall t$
then T is the time period

The smallest time period T defined for this property is called the fundamental time period

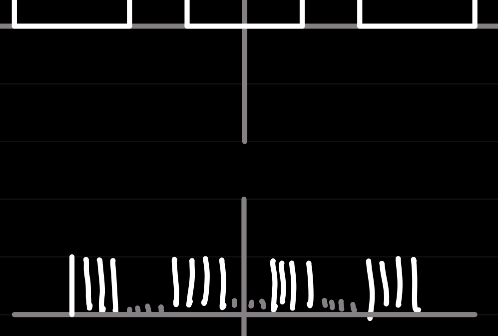
eg:



Not periodic

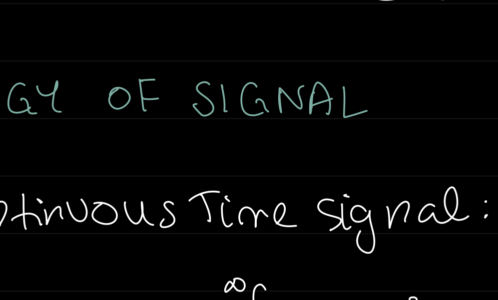


$T \to \infty$
Aperiodic



Periodic with $T = 2$

$T = 3$



$T = 8$

classification $\begin{cases} \text{Probabilistic} \\ \text{Dataministic} \end{cases}$

* ENERGY OF SIGNAL

Continuous Time signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time signal

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

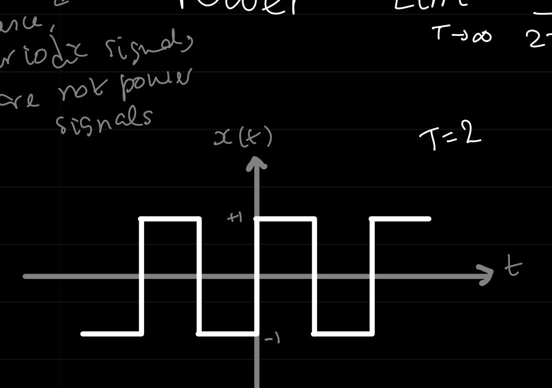
Note: Periodic signals are power signals ✓
Aperiodic signals are not power signals ✗

$T = \infty$
so, power = not defined
hence, Aperiodic signals are not power signals

avg. energy in " a time duration

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$T = 2$



compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{x=-N}^{N} |x[n]|^2$$

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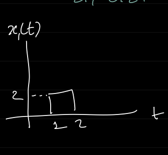
$$E=\int_{-\infty}^{\infty}|x(t)|^2dt$$



$$=\int_{1}^{2}|2|^2dt = 4\int_0^2 dt$$

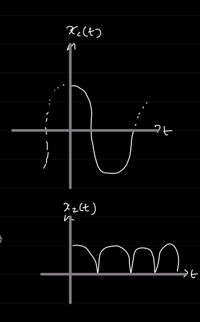
$$= 4 < \infty$$ ↑ energy signal

$x_1(t)$ ≡ energy signal, continuous signal, aperiodic, deterministic

$x_1(t) \neq x_1(t+T) \Rightarrow$ aperiodic signal

$x_2(t)$ = continuous signal, periodic signal, deterministic, power signal

we know its value at every point t → not probabilistic



$$E=\int_{-\infty}^{\infty}|x(t)|^2dt$$

after mod ⇒

$$E=\underbrace{\lim_{T_0\to\infty}\int_{-T_0}^{T_0}|x(t)|^2dt}_{\infty}$$

Since $E\to\infty$ and not $E<\infty$, this signal is not an energy signal

$$P=\lim_{T_0\to\infty}\frac{1}{2T_0}\int_{-T_0}^{T_0}|x(t)|^2dt$$

intuition $\left\{ \approx \frac{\text{infinite block}}{\text{infinit block}} \right.$ = finite

$\boxed{0<P<\infty} \Rightarrow$ Power Signal ✓

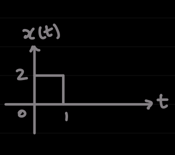
* TRANSFORMATION OF SIGNAL

① Translation

energy of signal doesn't change

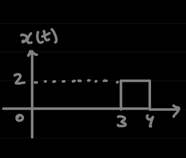
(a) DELAY

$y(t) = x(t-t_0)$



let $t_0 = 3$

$y(t) = x(t-3)$



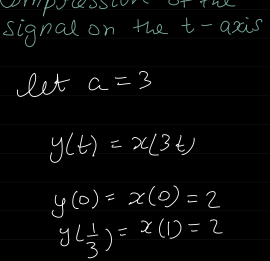
$y(0) = x(-3) = 0$
$y(3) = x(0) = 2$
$y(4) = x(1) = 2$
$y(5) = x(2) = 0$
} verification of transformation

(b) ADVANCE A SIGNAL

$y(t) = x(t+t_0)$

let $t_0 = 3$

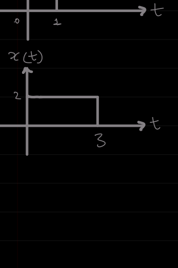
$y(-3) = x(0) = 2$
$y(-2) = x(1) = 2$
$y(-1) = x(2) = 0$
verified



energy of scale changes in accordance with area under curve

② Scaling

$y(t) = x(at)$

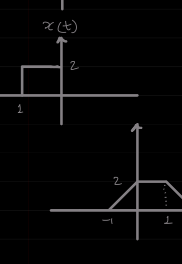


(a) $a>1$: compression of the signal on the t-axis

let $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$
$y(\frac{2}{3}) = x(2) = 0$



(b) $0<a<1$: Dilation on t-axis

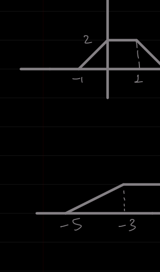
let $a=\frac{1}{3}$

$y(t) = x(at) = x(t/3)$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

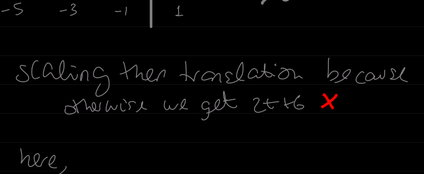
$y(6) = x(2) = 0$



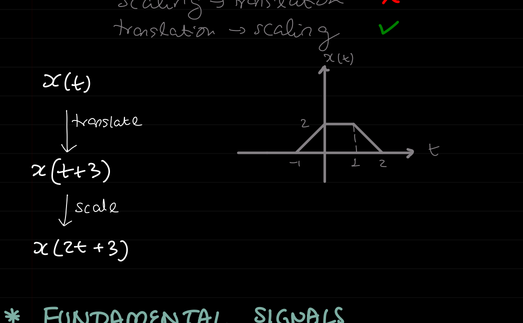
ⓒ SPECIAL CASE: $a=-1$

$y(t) = x(-t)$

flipped on the t-axis ≡ time reversal



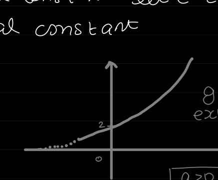
eg) $y(t) = x(2t+3)$ translation + scaling



scaling then translation because otherwise we get $2t+6$ ✗

here,
scaling → translation ✗
translation → scaling ✓

$x(t)$
↓ translate
$x(t+3)$
↓ scale
$x(2t+3)$



* FUNDAMENTAL SIGNALS

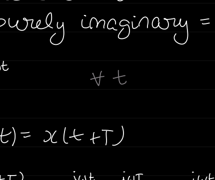
- **Continuous Time Signals**

$$x(t) = Ce^{at}$$

- CASE: 1 C ≡ real constant let $c=2$
  a ≡ real constant

$x(t) = 2e^{at}$

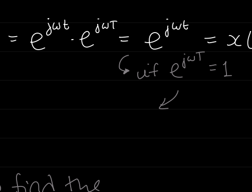
① let $a = 1$



growing exponential

$\boxed{a>0}$

② let $a=-1$



$\boxed{a<0}$

③ let $a = 0$

- CASE: 2 C ≡ real constant
  a ≡ purely imaginary $= j\omega$

$x(t) = e^{j\omega t}$ $\forall t$

periodic if $x(t) = x(t+T)$

$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t}\cdot e^{j\omega T} = e^{j\omega t} = x(t)$ if $e^{j\omega T}=1$

$e^{j\omega T}=1$ if $\omega T = 2\pi k$

$e^{j2\pi k}$

since we need to find the fundamental time period (smallest interval), let $K=1$

So, $\omega T = 2\pi \Rightarrow \boxed{T=\frac{2\pi}{\omega}}$ fundamental time period

$$E=\int_{-\infty}^{\infty}|e^{j\omega t}|^2dt = \int_{-\infty}^{\infty}1\,dt = \infty$$

- CASE: 3 C ≡ complex number
  a ≡ real number

$x(t) = |C|e^{j\theta}\cdot e^{at} = |C|e^{at+j\theta}$

e^{at} will define if it is growing / decaying exponential

we can also say, $|C|e^{j\theta} = c\cdot\cos(\theta)+c\cdot j\sin(\theta)$

$Re\{x(t)\} = |C|\cdot\cos\theta\cdot e^{at}$

$Im\{x(t)\} = |C|\cdot\sin\theta\cdot e^{at}$

- CASE: 4 C ≡ complex number $= |C|e^{j\theta}$
  a ≡ complex number $= r+j\omega$

$x(t) = Ce^{at} = |C|e^{j\theta}\cdot e^{(r+j\omega)t}$

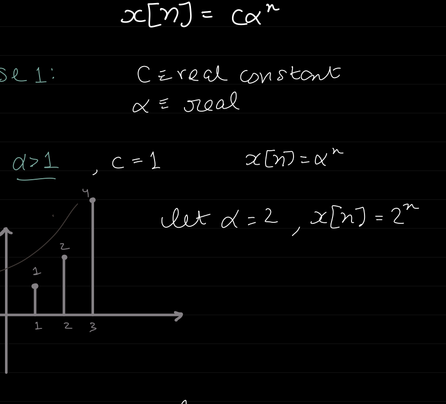
$$= \underbrace{|C|e^{rt}}_{\substack{\text{envelope with either}\\ \text{increasing or decaying}\\ \text{exponential}}}\cdot \underbrace{e^{j(\omega t+\theta)}}_{\substack{\cos(\omega t+\theta)+j\sin(\omega t+\theta)\\ \text{cosinosoidal signal} \quad \text{sinosoidal signal}}}$$

$Re\{x(t)\} = |C|e^{rt}\cdot\cos(\omega t+\theta)$

$Im\{x(t)\} = |C|e^{rt}\cdot\sin(\omega t+\theta)$

let $r=2$, $|C|=1.5$

$Re\{x(t)\} = \underbrace{1.5e^{2t}}\,\underbrace{\cos(\omega t+\theta)}$



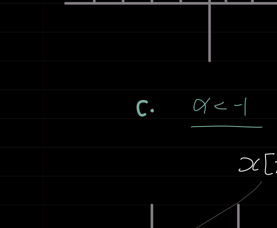
- **Discrete Time Signals**

$$x[n] = C\alpha^n$$

- Case 1: C ≡ real constant
  α ≡ real

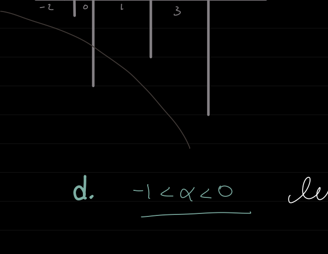
a. $\alpha>1$, $c=1$ $x[n]=\alpha^n$

let $\alpha=2$, $x[n]=2^n$



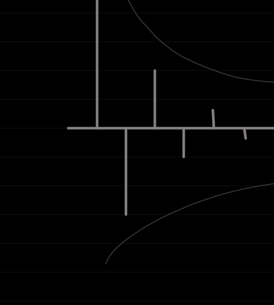
b. $\alpha<1$ let $\alpha=0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$



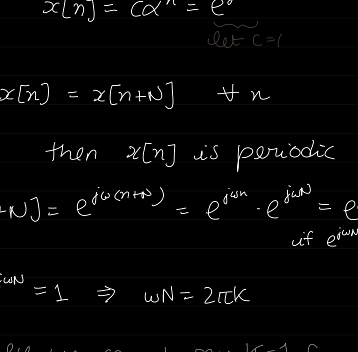
c. $\alpha<-1$ let $\alpha=-2$

$x[n] = (-2)^n$



d. $-1<\alpha<0$ let $\alpha=\frac{-1}{2}$

$x[n] = (\frac{-1}{2})^n$



- Case: 2 C ≡ real
  $\alpha = e^{j\omega}$

$x[n] = C\alpha^n = \underbrace{e^{j\omega n}}_{\text{let } C=1}$

if $x[n] = x[n+N]$ $\forall n$

then $x[n]$ is periodic

$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n}\cdot e^{j\omega N} = e^{j\omega n}$ if $e^{j\omega N}=1$

$e^{j\omega N}=1 \Rightarrow \omega N = 2\pi K$

here we cannot say $K=1$ for smallest interval necessarily because we want N to be an integer

$N=\frac{2\pi K}{\omega}$ ≡ has to be integer

$\frac{N}{K}=\frac{2\pi}{\omega}$ ≡ has to be rational number

Note: ω has to be a multiple of π

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- for continuous time signals
  → frequency is unique $(0 \to \infty)$

- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

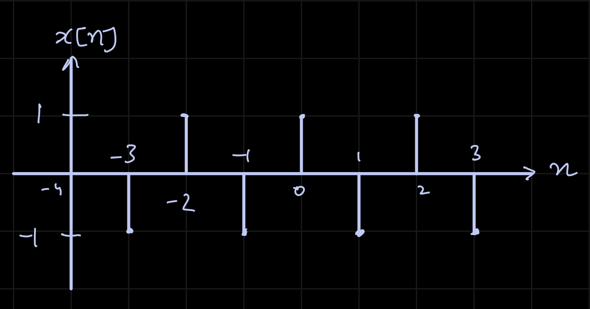
DTS

$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j \cdot 2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

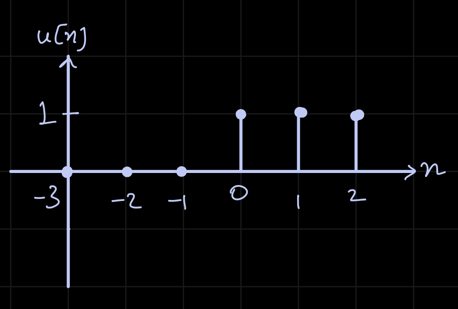
let $\omega_0 = \pi \Rightarrow e^{j \cdot \pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

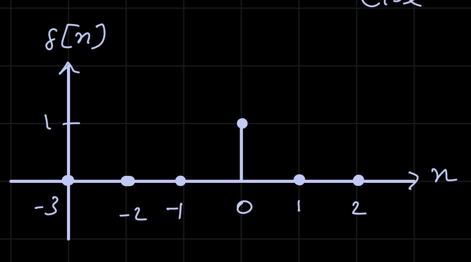
unit step signal $u[n]$

$$u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : \text{else} \end{cases}$$



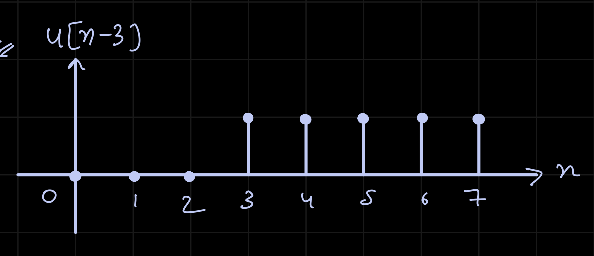
unit impulse function $\delta[n]$

$$\delta[n] = \begin{cases} 1 : n = 0 \\ 0 : \text{else} \end{cases}$$



note:

$1 : n-3 \geq 0$
$n \geq 3$



conversion btw u[ ] and δ[ ]

$$\boxed{\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^{n} \delta[k] \end{aligned}}$$

note: $u[n] = 0 \quad \forall\, n < 0$

so, $u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$

$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$

$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$

$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1$ → 2?

- $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when $n-k = 0$ i.e. $k = -2$
but k is never negative since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n-k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n-k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underset{0}{\delta[1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[-1]} + \ldots$$
$$= \underline{\underline{1}} ✓$$

- **SIFTING / SAMPLING**

$x[n] \cdot \delta[n] = ?$

$$x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

we can also say $x[n]\,\delta[n] = x[0]\,\delta[n]$

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

eg) $x[n] \cdot \delta[n - n_0] = ?$

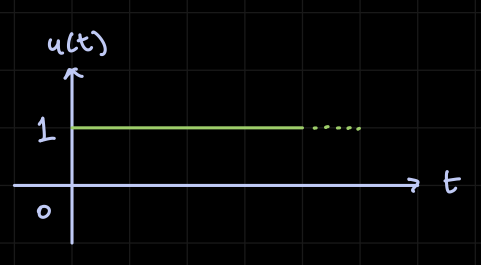
$$x[n] \cdot \delta[n-n_0] = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- unit step function

$$u(t) = \begin{cases} 1 : t \geq 0 \\ 0 : \text{else} \end{cases}$$



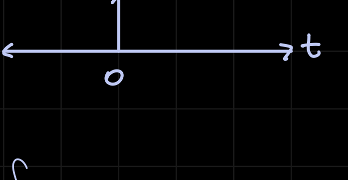
OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function ≡ singularity function

$$\delta(t) = 0 \quad : t \neq 0$$

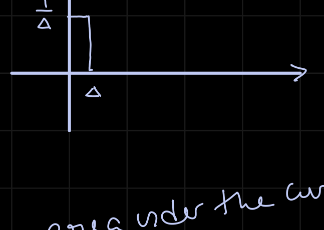
and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

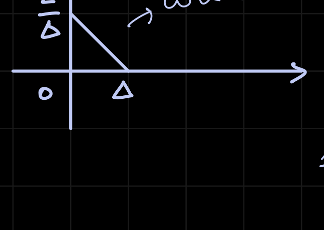


$\delta_\Delta(t)$:

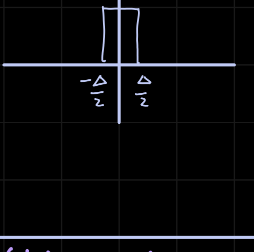
infinitesimally small

$$\delta_\Delta(t) = \begin{cases} 0 : t \geq \Delta \;\&\; t < 0 \\ \frac{1}{\Delta} : \text{otherwise} \end{cases}$$





→ area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
≡ unit impulse func ✓



≡ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t-\tau)\,d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$