14/08/24

Signals And Systems

11:00AM

Evaluation: (relative grading) yooooooooooo :fire emoji:

Attendance	10%
Quiz	16%
Assignments	15%
mid sem	24%
end sem	35%

Book: Signals & System 2nd edition by Alan V. Oppenheimer

Lecture : 1 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ GROUP : consists of ....

- A set G
- A rule / binary operation "$*$"
  - a. associative
    $x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$
  - b. There exists an element "e" called the identity of group G such that
    $e*x = x*e = x \quad \forall x \in G$
  - c. $\forall x \in G, \exists\ x^{-1}$ such that
    $x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$
  - d. if $x*y = y*x \quad \forall x,y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abelian Group

eg: set of continuous time periodic signals with time period T with "$*$" = "$+$"

⇒ FIELD : consists of the following

- A set F
- Two binary operations "+" and "·" such that ....
  - $(F,+)$ is an abelian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - left distributive
      $x \cdot (y+z) = xy + xz \quad \forall x,y,z \in F$
    - Right distributive
      $(x+y) \cdot z = xz + yz \quad \forall x,y,z \in F$

eg: F = Real Numbers $\mathbb{R}$

* Vector Space : A set V with a map...

- '+' : $V \times V \to V$
  $(v_1, v_2) \to (v_1 + v_2)$ called vector addition
- '·' : $F \times V \to V$
  $(a, v_1) \to (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a \cdot (v_1 + v_2) = av_1 + av_2$
- $(a_1 + a_2) \cdot v = a_1 v + a_2 v$
- $(\alpha\beta) v = \alpha(\beta v) = \beta(\alpha v)$
- $1 \cdot v = v$
- $a \cdot 0 = 0$
- $0 \cdot v = 0$
- if $v \neq 0$, then $a \cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \to R$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X,d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM : let V be a F-vector space.

A map $\|\cdot\| : V \to R$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a| \, \|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

Lecture : 2

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* Inner Product :

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v,v \rangle \geq 0$ and $\langle v,v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ " $-$ " : complex conjugate operation
- it is linear in the first co-ordinate
  $$\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$$
  $\forall\, v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $$\langle v, a_1 w_1 + a_2 w_2 \rangle = \bar{a}_1 \langle v, w_1 \rangle + \bar{a}_2 \langle v, w_2 \rangle$$
  $\forall\, w_1, w_2, v \in V$ and $a_1, a_2 \in F$

measures cosine similarity
" norm(x) . norm(y) . cos θ

eg: dot product
$$\langle x,y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner Product

complex conjugate
$$\langle x,y \rangle = \sum_{i=1}^{n} x_i \cdot \bar{y}_i$$

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \qquad \text{implies } a_i = 0 \ \forall i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

weighted linear combination of vector V

if $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ is a basis for $V_n(F)$, then $\forall\, v_i \in V$ & $a_i \in F$

$$V = \sum_{i=1}^{n} a_i e_i$$

$V = a_1 e_1 + a_2 e_2 + \dots a_n e_n$

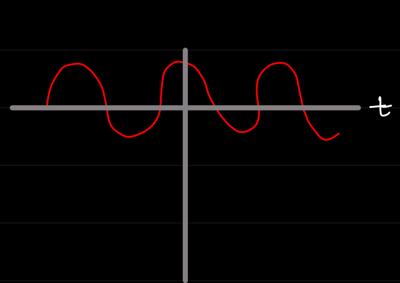
* ORTHOGONAL & ORTHONORMAL Basis

A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

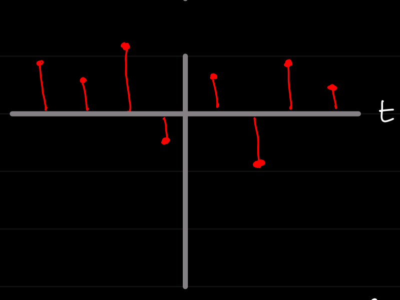
$$v_i \neq 0 \quad \forall i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall i \,\&\, j$$

SIGNALS (CTS & DTS)

- continuous time signal



- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t) \ \forall t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t) \ \forall t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal :

$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

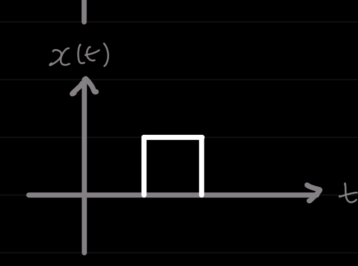
$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

* PERIODIC : if $x(t) = x(t+T) \ \forall t$
then T is the time period

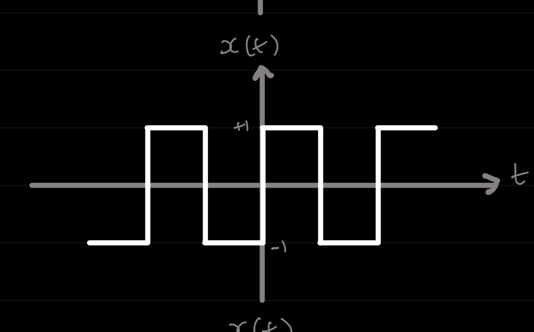
The smallest time period T defined for this property is called the fundamental time period

eg:

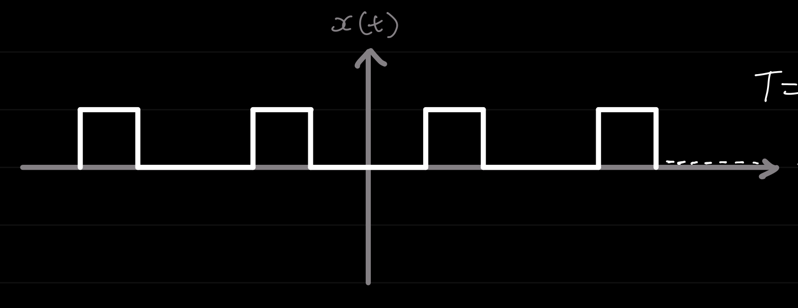
Not periodic



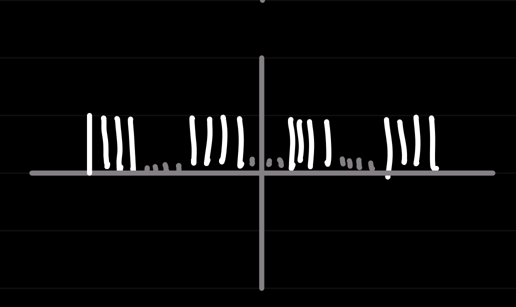
$T \to \infty$
Aperiodic



Periodic with T=2



T=3



T = 8

classification { Probabilistic
Dataministic

* ENERGY OF SIGNAL

Continuous Time signal :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time signal

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

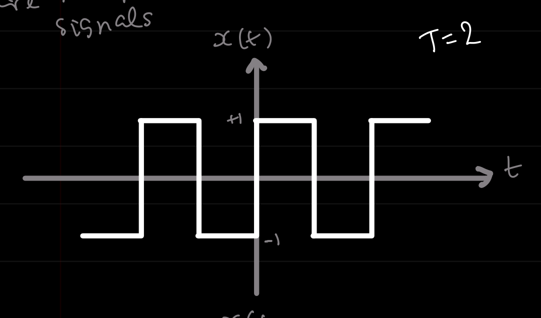
Note: Periodic signals are power signals ✓
Aperiodic signals are not power signals ✗

T=∞
so, power = not defined
hence, Aperiodic signals are not power signals

avg. energy in " a time duration

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

T=2



compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{x=-N}^{N} |x[n]|^2$$

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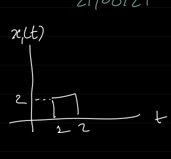
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$



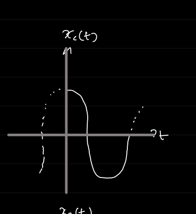
$$= \int_{2}^{2} |2|^2 dt = 4\int_0^2 dt$$

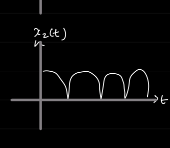
$$= 4 < \infty$$ ↑ energy signal

$x_1(t) \equiv$ energy signal,
continuous signal,
aperiodic,
deterministic

$x_1(t) \neq x_1(t+T) \Rightarrow$ aperiodic signal

$x_2(t) =$ continuous signal,
periodic signal,
deterministic ← we know its value at every point t ⇓ not probabilistic
power signal





$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

↓ after mod ⇒

$$E = \underbrace{\lim_{T_0 \to \infty} \int_{-T_0}^{T_0} |x(t)|^2 dt}_{\infty}$$

Since $E \to \infty$ and not $E < \infty$, this signal is not an energy signal

$$P = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} |x(t)|^2 dt$$

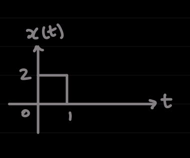
intuition $\left\{ \approx \frac{\text{infinite block}}{\text{infinit block}} \right. =$ finite

$\boxed{0 < P < \infty} \Rightarrow$ Power Signal ✓

* TRANSFORMATION OF SIGNAL

① Translation

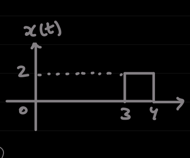
energy of signal doesn't change



(a) DELAY

$y(t) = x(t - t_0)$

let $t_0 = 3$



$y(t) = x(t-3)$

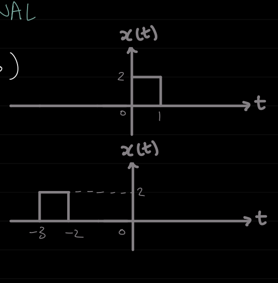
$y(0) = x(-3) = 0$
$y(3) = x(0) = 2$
$y(4) = x(1) = 2$
$y(5) = x(2) = 0$

} verification of transformation

(b) ADVANCE A SIGNAL

$y(t) = x(t + t_0)$

let $t_0 = 3$

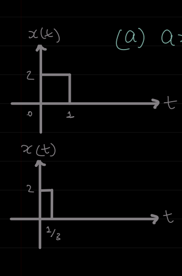


$y(-3) = x(0) = 2$
$y(-2) = x(1) = 2$
$y(-1) = x(2) = 0$
verified

energy of scale changes in accordance with area under curve

② Scaling

$y(t) = x(at)$



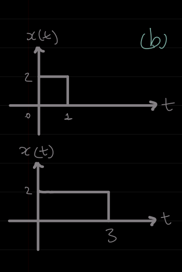
(a) $a > 1$: compression of the signal on the t-axis

let $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$

$y(\frac{2}{3}) = x(2) = 0$



(b) $0 < a < 1$: Dilation on t-axis

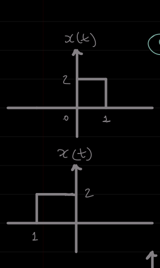
let $a = \frac{1}{3}$

$y(t) = x(at) = x(t/3)$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

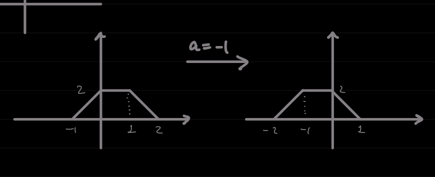
$y(6) = x(2) = 0$



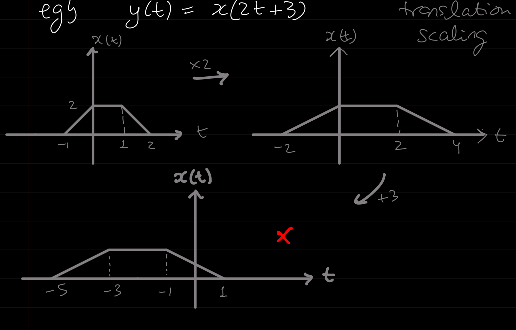
Ⓒ SPECIAL CASE: $a = -1$

$y(t) = x(-t)$

flipped on the t-axis
≡ time reversal



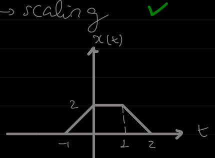
eg} $y(t) = x(2t+3)$ translation + scaling



scaling then translation because otherwise we get $2t+6$ ✗

here,
scaling → translation ✗
translation → scaling ✓

$x(t)$
↓ translate
$x(t+3)$
↓ scale
$x(2t+3)$



* FUNDAMENTAL SIGNALS

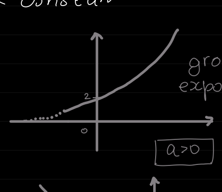
- **Continuous Time Signals**

$$x(t) = Ce^{at}$$

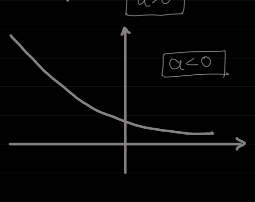
- CASE: 1 $C \equiv$ real constant let $c = 2$
  $a \equiv$ real constant

$x(t) = 2e^{at}$

① let $a = 1$



② let $a = -1$



③ let $a = 0$

- CASE: 2 $C \equiv$ real constant
  $a \equiv$ purely imaginary $= j\omega$

$x(t) = e^{j\omega t}$ $\forall t$

periodic if $x(t) = x(t+T)$

$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t} \cdot e^{j\omega T} = e^{j\omega t} = x(t)$
↳ if $e^{j\omega T} = 1$

$e^{j2\pi k}$ $e^{j\omega T} = 1$ if
$\omega T = 2\pi k$

since we need to find the fundamental time period (smallest interval), let $K = 1$

So, $\omega T = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$ fundamental time period

$$E = \int_{-\infty}^{\infty} |e^{j\omega t}|^2 dt = \int_{-\infty}^{\infty} 1\, dt = \infty$$

- CASE: 3 $C \equiv$ complex number
  $a \equiv$ real number

$x(t) = |C| e^{j\theta} \cdot e^{at} = |C| e^{at + j\theta}$

e^{at} will define if it is growing / decaying exponential

we can also say $|C|e^{j\theta} = C \cdot \cos(\theta) + C \cdot j\sin(\theta)$

$Re\{x(t)\} = |C| \cdot \cos\theta \cdot e^{at}$

$Im\{x(t)\} = |C| \cdot \sin\theta \cdot e^{at}$

- CASE: 4 $C \equiv$ complex number $= |C|e^{j\theta}$
  $a \equiv$ complex number $= r + j\omega$

$x(t) = Ce^{at} = |C|e^{j\theta} \cdot e^{(r+j\omega)t}$

$$= \underbrace{|C|e^{rt}}_{\text{envelope with either increasing or decaying exponential}} \cdot \underbrace{e^{j(\omega t+\theta)}}_{\cos(\omega t+\theta) + j\sin(\omega t+\theta)}$$

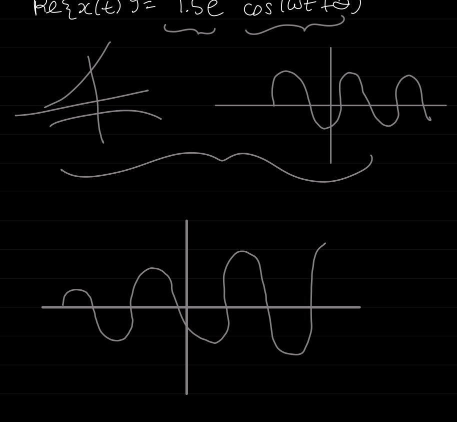
cosinosoidal signal, sinosoidal signal

$Re\{x(t)\} = |C|e^{rt} \cdot \cos(\omega t + \theta)$

$Im\{x(t)\} = |C|e^{rt} \cdot \sin(\omega t + \theta)$

let $r = 2$, $|C| = 1.5$

$Re\{x(t)\} = \underbrace{1.5e^{2t}}\ \underbrace{\cos(\omega t + \theta)}$



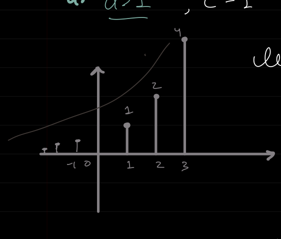
- **Discrete Time Signals**

$$x[n] = C\alpha^n$$

- Case 1: $C \equiv$ real constant
  $\alpha \equiv$ real

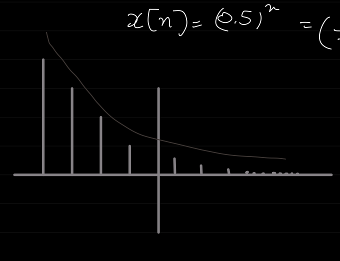
a. $\alpha > 1$, $c = 1$ $x[n] = \alpha^n$

let $\alpha = 2$, $x[n] = 2^n$



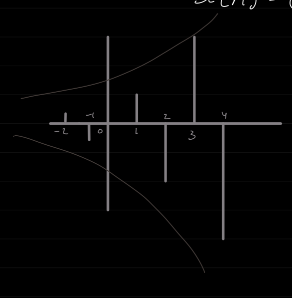
b. $\alpha < 1$ let $\alpha = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$



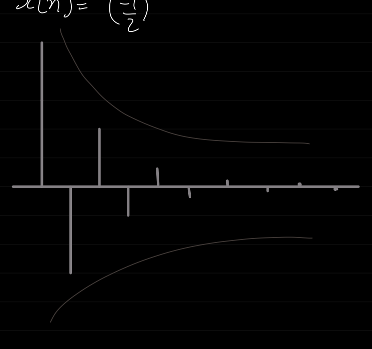
c. $\alpha < -1$ let $\alpha = -2$

$x[n] = (-2)^n$



d. $-1 < \alpha < 0$ let $\alpha = \frac{-1}{2}$

$x[n] = (\frac{-1}{2})^n$



- Case : 2 $C \equiv$ real
  $\alpha = e^{j\omega}$

$x[n] = \underbrace{C\alpha^n}_{\text{let } C=1} = e^{j\omega n}$

if $x[n] = x[n+N]$ $\forall n$

then $x[n]$ is periodic

$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N} = e^{j\omega n}$
if $e^{j\omega N} = 1$

$e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi K$

here we cannot say $K = 1$ for smallest interval necessarily because we want N to be an integer

$N = \frac{2\pi K}{\omega}$ ≡ has to be integer

$\frac{N}{K} = \frac{2\pi}{\omega}$ ≡ has to be rational number

Note: ω has to be a multiple of π

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- for continuous time signals
  → frequency is unique $(0 \to \infty)$

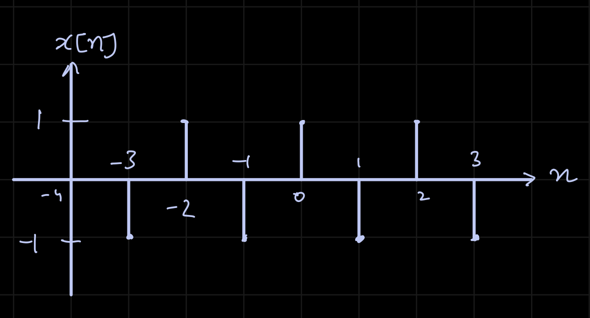
- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

DTS
$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j \cdot 2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

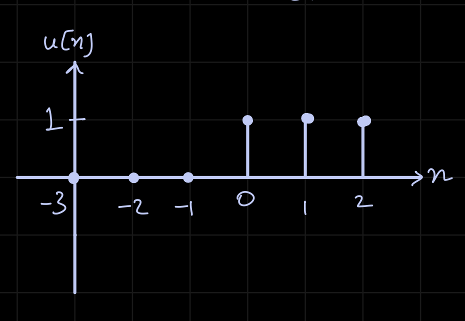
let $\omega_0 = \pi \Rightarrow e^{j \cdot \pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

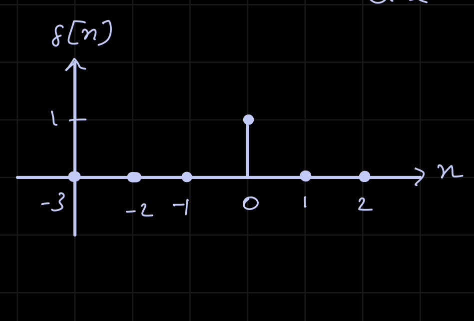
Unit step signal $u[n]$

$$u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : \text{else} \end{cases}$$



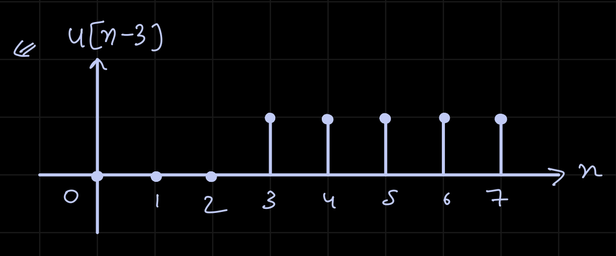
unit impulse function $\delta[n]$

$$\delta[n] = \begin{cases} 1 : n = 0 \\ 0 : \text{else} \end{cases}$$



$1 : n-3 \geq 0$
$n \geq 3$

note:



conversion btw $u[\,]$ and $\delta[\,]$

$$\boxed{\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^{n} \delta[k] \end{aligned}}$$

note: $u[n] = 0 \quad \forall\, n < 0$

so,
$$u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$$
$$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$$
$$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$$
$$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1 \quad ??$$

- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when $n - k = 0$ i.e. $k = -2$
but k is never negative
since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n - k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n - k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underbrace{\delta[1]}_{0} + \underbrace{\delta[0]}_{1} + \underbrace{\delta[-1]}_{0} + \dots$$
$$= \underline{1} \checkmark$$

• SIFTING / SAMPLING

$x[n] \cdot \delta[n] = ?$

$$x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

we can also say $x[n]\,\delta[n] = x[0]\,\delta[n]$

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

eg) $x[n] \cdot \delta[n - n_0] = ?$

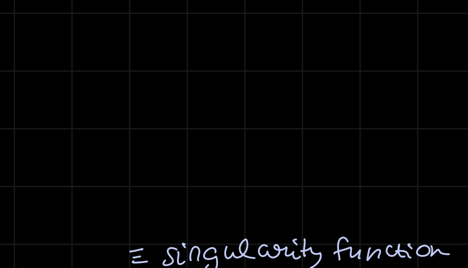
$$x[n] \cdot \delta[n - n_0] = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- Unit step function

$$u(t) = \begin{cases} 1 & : t \geq 0 \\ 0 & : \text{else} \end{cases}$$



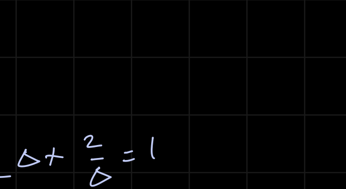
OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function $\equiv$ singularity function

$$\delta(t) = 0 \quad : t \neq 0$$

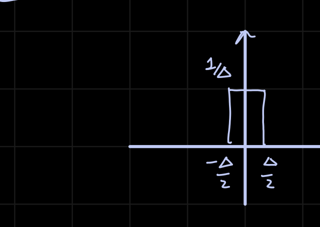
and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

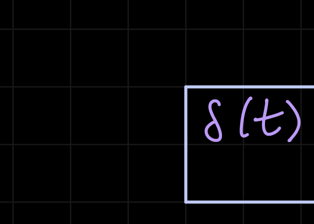


$\delta_\Delta(t)$:

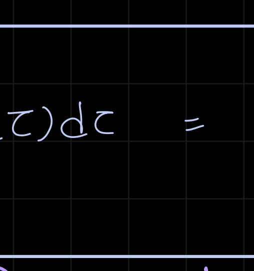
infinitesimally small

$$\delta_\Delta(t) = \begin{cases} 0 : t \geq \Delta \;\&\; t < 0 \\ \frac{1}{\Delta} : \text{otherwise} \end{cases}$$





→ area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
$\equiv$ unit impulse func ✓



$\equiv$ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t - \tau)\,d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

* Tutorial: 1 30/08/24

(7) $\text{even}\{x[t]\} = \dfrac{x(t)+x(-t)}{2}$

$\text{odd}\{x[t]\} = \dfrac{x(t)-x(-t)}{2}$

8) (a) $x_1[n] = \sin(0.02\pi n)$

$\omega = 0.02\pi$

$T = \dfrac{2\pi}{\omega} =$

(3) $p(x,y) = a^2x_1y_1 + abx_1y_2 + b^2x_2y_2 + ab\,x_2y_1$

$A = \begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

$|A| = a^2b^2 - a^2b^2 = 0$

atleast one eigenvalue is zero

So, since not all eigenvalues are +ve

(d) for no values of a, b

eigenvalues are values of λ such that

$|A - \lambda I| = 0$

eigenvalues / eigenvectors

Q17 $B = \{V_1, V_2, \dots V_n\}$

B is span of V

$\bar{x} \in V$

$\bar{x} = K_1V_1 + K_2V_2 + \dots K_nV_n$

$-\ \bar{x} = C_1V_1 + C_2V_2 + \dots C_nV_n$

$0 = (K_1 - C_1)V_1 + (K_2 - C_2)V_2 + \dots (K_n - C_n)V_n$

since $\bar{V_i} \neq 0$, $(K_i - C_i)$ need to be 0

i.e. $K_i = C_i$

$\dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\times 10^{-2}\pi} = 100 \Leftarrow T$

(8) $x_1[n] = \sin(0.02\pi n)$

Periodic ✓

(11) $y[n] = \sum_{n=-5}^{5} \sin(2n)\,\delta[n+7]$

but range is $-5 \to 5$, so ans = 0

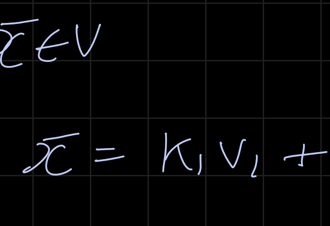
(12) $x(t) = u(t-2) - u(t-4)$

$u(t) \Rightarrow t \geq 0 : 1$

$u(t-2) \Rightarrow t \geq 2 : 1$

$u(t-4) \Rightarrow t \geq 4 : 1$

$\Rightarrow$



but since $\delta(t) = 1$ at $t = 0$

So, ans = 0

(5) (b) $x_2[n] = \cos\left(\dfrac{\pi}{2} + 0.3n\right)$

$f = \dfrac{0.3}{2\pi}$ rational / irrational ?

$\sin = \cos\left(\dfrac{\pi}{2} - \theta\right)$

(c) $x_3[n] = 1 - e^{j2\pi n/5} + e^{j4\pi n/7}$

note: $e^{j\theta} = \cos\theta + j\sin\theta$

$\Rightarrow 1 - \cos\left(\dfrac{2\pi n}{5}\right) - j\sin\left(\dfrac{2\pi n}{5}\right)$

$+ \cos\left(\dfrac{4\pi n}{7}\right) + j\sin\left(\dfrac{4\pi n}{7}\right)$

5 5 7/2 7/2

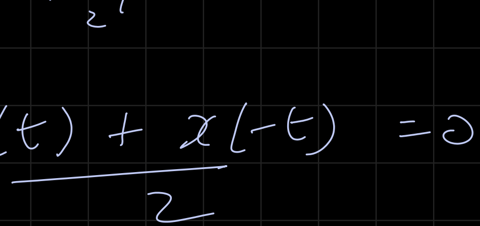
$\Rightarrow \text{LCM}\left(5, \dfrac{7}{2}\right) \Rightarrow$

(a) $\dfrac{x(t) + x(-t)}{2} = 0$

$x(t) = -x(-t)$

$\forall t$

(b) $e^{-5t}u(t+2)$



$e^{-5t}u(t+2) = -\left[e^{5t}u(-t+2)\right]$

$\left(e^{-5t}\right)_{(-2,\infty)} + \left(e^{5t}\right)_{(-\infty,2)}$

$\left(e^{-5t}\right)_{(-2,2)} + \left(e^{5t}\right)_{(2,2)}$

(5) $P = tu(t)$

$\lim_{T\to\infty} \dfrac{1}{2T}\int_{-T}^{T} |t\,u(t)|^2\,dt$

$\lim_{T\to\infty} \dfrac{1}{2T}\int_0^T t^2\,dt$

$\lim_{T\to\infty} \dfrac{1}{2T}\left[\dfrac{t^3}{3}\right]_0^T$

$\lim_{T\to\infty} \dfrac{1}{2T} \times \dfrac{T^3}{3} \Rightarrow \dfrac{T^2}{6}$

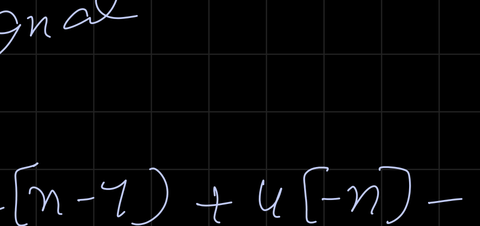
$\to \infty$

neither energy nor power signal

(9) (c) $\dfrac{u[n] - u[n-4] + u[-n] - u[-n-4]}{2} = 0$

$u[n] + u[-n] = u[n-4] + u[-n-4]$

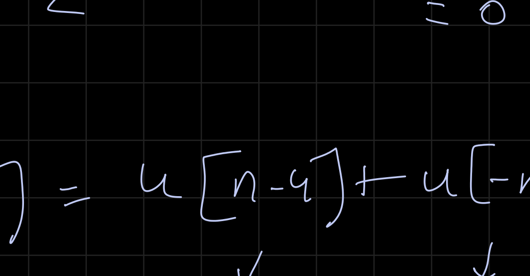
$1 : n \geq 0$ $1 : n \leq 0$ $n \geq 4$ $n \leq 4$



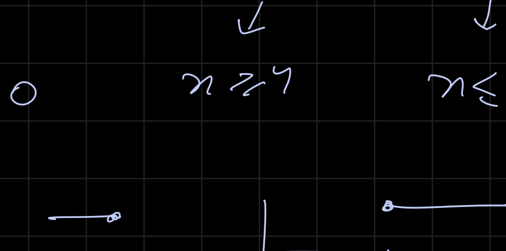
$-3 < n < 3$

$\boxed{|n| < 3}$ ✓

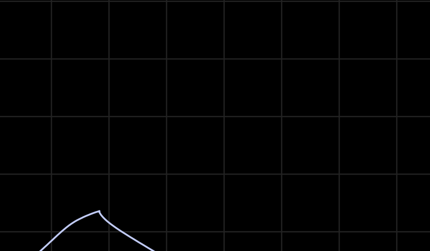
(7) $x(t) =$



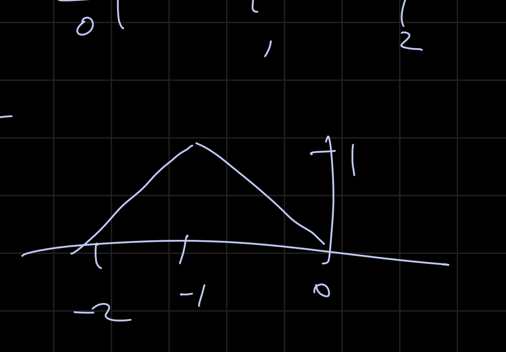
$x(-t) =$



$\text{even} = \dfrac{x(t) + x(-t)}{2} =$



odd =



* Lecture: 5

⇒ Systems and their properties

input $x(t)$ → system → output $y(t)$

① System: with memory or without memory

$x[n] \to y[n]$

$S_1: y(t) = x(t)$ no memory
$S_2: y(t) = x^2(t)$ no memory
$S_3: y(t) = x(t-2)$ memory ✓ requires past value
$S_4: y(t) = \frac{1}{2}\{x(t) + x(t-1)\}$ mem ✓

output depends on just present inputs → no memory required

output needs to store previous outputs → requires memory

$S_5: y(t) = x(t+2)$: requires memory

• Invertible / Non-invertible Systems

$x(t) \to \boxed{S_1} \xrightarrow{w(t)} \boxed{S_2} \to y(t)$

$y(t) = x(t)$ iff $S_2 = \text{inverse}(S_1)$

ⓐ let $S_1: w(t) = 2x(t)$
$S_2: y(t) = \frac{1}{2} w(t) = \frac{1}{2} \times 2x(t) = x(t)$

S_1 is invertible and S_2 is the inverse system

ⓑ let $S_1: w[n] = \sum_{k=-\infty}^{n} x[n]$

then $S_2: y[n] = w[n] - w[n-1]$ invertible ✓

$= \{\cancel{x[-\infty]} + \dots \cancel{x[n-1]} + x[n]\} - \{\cancel{x[-\infty]} + \dots \cancel{x[n-1]}\}$

$= x[n]$

ⓒ let $S_1: w[n] = x[n] - x[n-1]$

$S_2: y[n] = \sum_{k=-\infty}^{n} w[k] = x[n]$ invertible ✓

$\cancel{x[-\infty]} - x[-\infty-1] + \dots \cancel{x[1]} - \cancel{x[2]} + \cancel{x[2]} - x[3] + x[3] - x[4] + \dots - x[n-1] + x[n-1] - x[n]$

ⓔ $S_1: w(t) = x^2(t)$
$S_2: y(t) = \sqrt{w(t)}$ } S_1 is not invertible

* System is invertible if and only if distinct inputs lead to distinct outputs

(very casual)

• Causal Systems

$S_1: y(t) = x(t - t_0)$: t_0 is a +ve real number

$S_2: y[n] = x[n+3]$

System output is depending on past system output

$S_1 \equiv$ causal system
$S_2 \equiv$ non-causal system

Causality: unless there is no cause, there should be no effect.

OR unless there is a non-zero input, there should not be non-zero output

OR a causal system depends on current and past input

eg) Causal system: $y[n] = \sum_{k=-\infty}^{n} x[k]$

image processing: $S_4: y[n,m] = \frac{1}{5}\{x[n,m] + x[n-1,m] + x[n+1,m] + x[n,m-1] + x[n,m+1]\}$

non-causal system

• Stability System

A bounded input $\equiv |x(t)| < M_x < \infty$ (finite value $\forall t$)

not on the duration (y input) → but on the amplitude (x axis) ↓

OR $|x[n]| < M_x < \infty$

This bounded input is fed to a system. If the system generates an output which is also bounded, we call it a stable system.

eg) $x(t) = u(t)$; $x[n] = u[n]$
$[0,1]$

unbounded: $x(t) = tu(t)$ OR $x[n] = tu[n]$

if $|y(t)| < M_y < \infty \;\forall t$, for bounded input,
↳ finite number

then the system is stable

eg) $S_1: y[n] = \frac{1}{3}\{x[n] - x[n-1] - x[n+1]\}$

$|y[n]| = |\frac{1}{3}x[n] - \frac{1}{3}x[n-1] - \frac{1}{3}x[n+1]|$

since $|x[n]| < M_x < \infty \;\forall n$

and the sum & difference of 3 finite terms is finite, so,

$|y[n]| < M_y < \infty \;\forall n$

$S_3: y[n] = nx[n]$

$|x[n]| < M_x < \infty \;\forall n$
bounded input assumed

$\Rightarrow |y[n]| = |nx[n]| \to \infty$ as $n \to \infty$

unstable system

$S_4: y(t) = x(2t)$

no manipulation on the y axis
2t means manipulation on the x-axis

• Time Invariant System

if $x(t) \xrightarrow{\text{system}} y(t)$

then $x(t-t_0) \to y(t-t_0)$

and $x(t+t_0) \to y(t+t_0)$

system output does not change if there is a change in time

if $x_1(t) \to y_1(t)$
and $x_2(t) \to y_2(t)$
and $x_2(t) = x_1(t-t_0)$

then $y_2(t)$ must be equal to $y_1(t-t_0)$

if it is a time invariant system

eg) $S: y(t) = x(2t)$

⇒ let $t_0 = 4$: $y(t-t_0) = x(2t-8)$

⇒ time variant system

$x_1(t) \to y_1(t) = x_1(2t)$
$x_2(t) \to y_2(t) = x_2(2t)$

$y_1(t-t_0) = x_1(2t-2t_0)$
first scaling : $x_1(2t)$
then delay : $x_1(2(t-t_0))$ ($t \to t-t_0$)

$y_2(t-t_0) = x(2t-t_0)$
first delay : $x(t-t_0)$
then scaling : $x(2t-t_0)$

$S_2: y[n] = nx[n]$

$x_1[n] \to y_1[n] = nx_1[n]$
↓ delayed output
$y_1[n-n_0] = (n-n_0)x[n-n_0]$

$x_2[n] \to y_2[n] = nx_2[n]$

let $x_2[n] = x_1[n-n_0]$ be delayed input

$y_2[n] = (n)x[n-n_0]$

since $y_1[n] \neq y_2[n]$, system is time varying

note: 1st system had delayed output and we gave delayed input

• Linearity

A system is linear if the properties of additivity & homogeneity holds true

$x_1(t) \to y_1(t)$
$x_2(t) \to y_2(t)$

$\Rightarrow ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$
then system is linear

eg) $y(t) = 2x(t) + 3$

$x_1(t) \to y_1(t) = 2x_1(t) + 3$

$x_2(t) \to y_2(t) = 2x_2(t) + 3$

$x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t) = 2(x_1 + x_2) + 2(3)$

but $y(x_1 + x_2) = 2(x_1 + x_2) + 3$

Since, $y_1(t) + y_2(t) \neq y(x_1 + x_2)$

not a linear system

• A system which is linear and time invariant $\equiv$ LTI
↓
linear & time invariant system

* Lecture: SNS 6 — 4/9/24

Q) $x[n] = \cos(\frac{\pi}{8}n^2)$

periodic if $x(n) = x[n+N]\ \forall n$

$\cos(\frac{\pi}{8}(n^2+N^2+2nN)) = \cos(\frac{\pi}{8}n^2)$

$\cos(\frac{\pi}{8}n^2+\frac{\pi}{8}N^2+\frac{\pi}{4}nN) = \cos(\frac{\pi}{8}n^2)$

we know $\cos(2\pi\theta+K) = \cos\theta\ \ \forall K \in \mathbb{Z}$

$\frac{\pi}{8}N^2 \Rightarrow$ can it be a multiple of 2π?

if $N=4 \Rightarrow 2\pi = K2\pi$ ✓ ($K=1$)	$\frac{\pi}{4}nN \to K2\pi$?
true for $N=8$ as well	$N=4 \Rightarrow \pi 4n \to$ ✗
but not for $N=2$	$N=8 \Rightarrow$ ✓ $2\pi nN$ = ✓

---

* Discrete Time convolution

$x[n] \to$ LTI system $\to y[n]$

$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K]$$

weighted linear combination of delayed signals

* Impulse Response of an LTI System

$x[n] \to$ LTI system $\to y[n]$

$\delta[n] \to y[n] = h[n]$ ≡ Impulse response of the LTI system

$\delta[n-1] \to h[n-1]$

$x[1]\,\delta[n-1] \to x[1]\,h[n-1]$

$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K] \longrightarrow \sum_{K=-\infty}^{\infty} x[K]\,h[n-K] = y[n]$$

discrete time convolution of

$x[n] \times h[n] = y[n]$

Valid for LTI systems only

convolution $\to$ linear & commutative

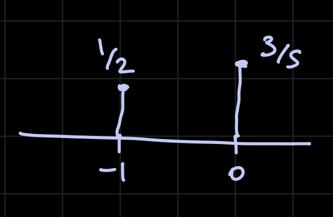
$x[n] \times h[n] = h[n] \times x[n]$

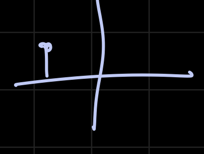
$(x_1[n] + x_2[n])\,h[n] = x_1[n]\,h[n] + x_2[n]\,h[n]$

eg) given an LTI system whose impulse response is

$$h[n] = \begin{cases} 1 & n = 0,1,2 \\ 0 \end{cases}$$

and $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$





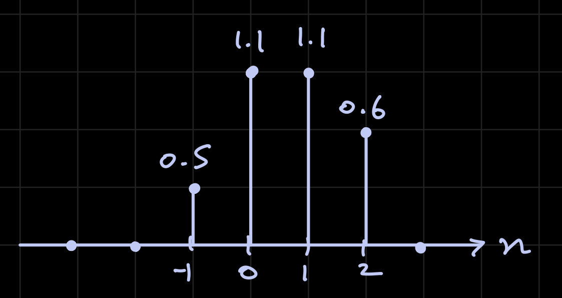
compute $y[n]$

Ans) $y[n] = h[n] * x[n]$

$y[n] = \frac{3}{5}h[n] + \frac{1}{2}h[n+1]$

because for $x[n] \to y[n]$,
$\delta[n] \to h[n]$

$$y[n] = \frac{3}{5}\left[\delta[n]+\delta[n-1]+\delta[n-2]\right] + \frac{1}{2}\left[\delta[n+1]+\delta[n]+\delta[n-1]\right]$$



eg) LTI System (← given)

$$h[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$$

$x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

Ans) $y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$

OR ≡ $\sum_{-\infty}^{\infty} h[K]\,x[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} x[n-K] = \sum_{K=0}^{\infty} \alpha^{(n-K)} u[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} \frac{\alpha^n}{\alpha^K} u[n-K]$ ($1 : n \geq K$)

$y[n] = \sum_{K=0}^{n} \frac{\alpha^n}{\alpha^K}$ ✓ $\qquad \frac{a(1-r^n)}{1-r}$

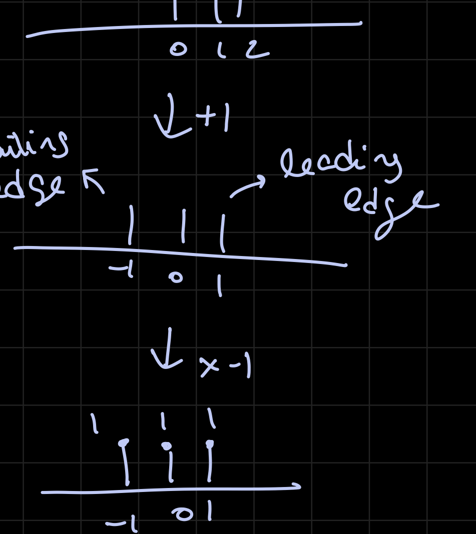
$x\text{-}1$ $\ -K \xrightarrow{+1} -K+1$

$K \to (1-K)$

$\downarrow +1$

$K+1$

$x\text{-}1$ $\ -K+1$



* LECTURE: 7

06/09/24

Note: $y[n] \rightarrow$ function $\equiv$ variable
$y[10] \rightarrow$ scalar $\equiv$ constant

$h[n]$ = characteristic impulse response of the system

$$y[n] = x[n]\,h[n]$$

An LTI system is uniquely charaterized by its $h[n]$ - hence two systems are same if they are LTI and their $h[n]$ are same

bonus question} $\int_{-\infty}^{t} \cos(\tau)\,x(\tau)\,d\tau$

let $x(\tau) = \cos(\tau)$

$$\Rightarrow \int_{-\infty}^{t} \cos^2(\tau)\,d\tau \Rightarrow \int_{-\infty}^{t} \frac{1+\cos(2\tau)}{2}\,d\tau$$

$$\Rightarrow \left[\underbrace{\frac{t}{2}}_{} + \underbrace{\overbrace{\frac{\sin(2t)}{4}}^{\in[-1,1]}}_{\text{bounded}}\right]_{-\infty}^{t} \Rightarrow$$

becomes unbounded as $t \to \infty$ ($\equiv$ unstable)

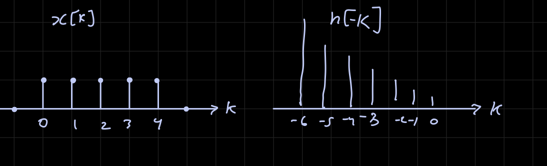
Q1) $\delta(8t-2) = 1$ at $t = \frac{1}{4}$ ✗

$\frac{1}{8}\delta(t - \frac{1}{4}) = 1$ at $t = \frac{1}{4}$ ✓

eg} $x[n] = \begin{cases} 1 : & 0 \le n \le 4 \\ 0 : & \text{else} \end{cases}$

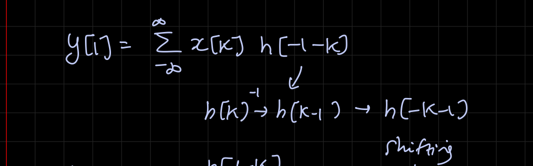
$x[n] = u[n] - u[n-4]$

$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{else} \end{cases}$ & $\alpha > 1$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$y[0] = \sum_{-\infty}^{\infty} x[k]\,h(-k)$$



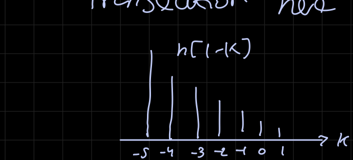
only common point : $k=0$

$$y[n] = x[0]\,h[-0] = h[n] = \alpha^0 = \underline{1}$$

$$y[1] = \sum_{-\infty}^{\infty} x[k]\,h[-1-k]$$

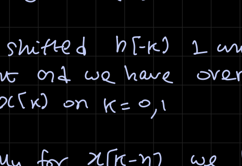
$h[k] \xrightarrow{-1} h(k-1) \rightarrow h(-k-1)$

shifting ↓ scaling ↓ reversing



$h[1-k) = 1$ at $1-k=0 \Rightarrow k=1$

but we do scaling then translation here



Check: $h[1-k) = 1$ at $1-k=0$ because $\alpha^k = \alpha^0 = 1$

$k=1$

$\Rightarrow y[k] = x[k]\,h[1-k)$

now, we shifted $h[-k]$ 1 unit to the right and we have overlap with $x[k]$ on $k=0,1$

Similarly for $x[k-n]$, we have overlap with $k=0,1 \ldots\ldots n-1, n$

404: Brain not found
2nd attempt initiated ...

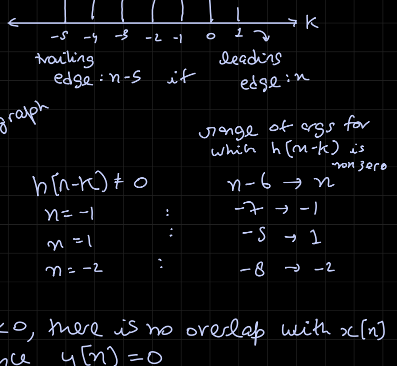
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$x[k]$

we are not shifting $x[k]$ in this case but we could

$h[-k]$ $(n=0)$

$h[1-k]$ $(n=1)$



verification: $h(0)$ at $1-k=0$, $k=1$ $\equiv$ same as $h[-0]$ in previous graph (argument is zero)

trailing edge : $n-5$ if leading edge : n

range of args for which $h[n-k]$ is non zero

$h[n-k] \neq 0$		
		$n-6 \rightarrow n$
$n=-1$	:	$-7 \rightarrow -1$
$n=1$	:	$-5 \rightarrow 1$
$n=-2$	:	$-8 \rightarrow -2$

① when $n<0$, there is no overlap with $x[n]$ hence $y[n]=0$

②
$n=11$	:	5 to 11
$n=10$	:	4 to 10

So, no overlap with $x[n]$ when $n>10$

hence, $y[n] = 0$: $n<0$ and $n>10$

③ $0 \le n < 4$: $y[n] = \sum_{k=0}^{n} x[k]\,h(n-k]$

overlap	n	range
$k=0$	0	$-6 \rightarrow 0$
$k=0,1$	1	$-5 \rightarrow 1$
$k=0,1,2$	2	$-4 \rightarrow 2$
$k=0,1,2,3$	3	$-3 \rightarrow 3$
$k=0,1,2,3,4$	4	$-2 \rightarrow 4$
$k=0,1,2,3,4$	5	$-1 \rightarrow 5$
$k=0,1,2,3,4$	6	$0 \rightarrow 6$
$k=1,2,3,4$	7	$1 \rightarrow 7$
$k=2,3,4$	8	$2 \rightarrow 8$
$k=3,4$	9	$3 \rightarrow 9$
$k=4$	10	$4 \rightarrow 10$

of $h[$ leading edge left $x[n]$

So, $y[n]$ is defined for $0 \le n \le 10$

② $0 \le n \le 4 \Rightarrow y[n] = \sum_{k=0}^{n} x[k]\,h[n-k]$

③ $4 < n \le 6 \Rightarrow y[n] = \sum_{k=0}^{4} x(k)\,h[n-k]$

④ $6 < n \le 10 \Rightarrow y[n] = \sum_{k=n-6}^{4} x[k]\,h[n-k]$

overlap: $1,2,3,4 \rightarrow 2,3,4 \rightarrow 3,4 \rightarrow 4$
n: $7 \rightarrow 8 \rightarrow 9 \rightarrow 10$

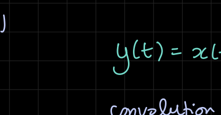
final $\rightarrow$ $y[n] = \sum_{k=0}^{n} \alpha^{n-k}$: $0 \le n \le 4$
$+ \sum_{k=0}^{4} \alpha^{n-k}$: $4 < n \le 6$
$+ \sum_{k=n-6}^{4} \alpha^{n-k}$: $6 < n \le 10$

convolution for CTS = DTS

$$\sum \alpha^{n-k} = \alpha^n \sum_{\ldots}^{\ldots} \frac{1}{\alpha^k} = \alpha^n \{GP\}$$

$\frac{a(1-r^n)}{1-r}$

$\boxed{\text{CTS}}$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

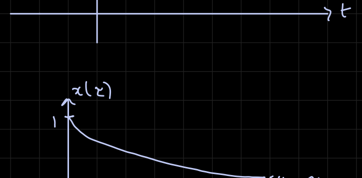
useful when $h(t) = u(t)$

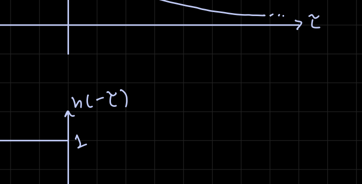
$y(t) = x(t) * h(t)$

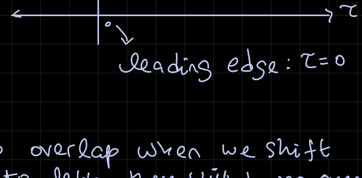
convolution operation

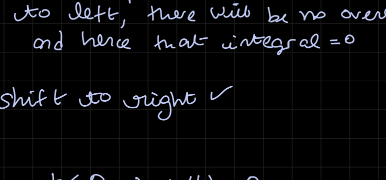
eg) $x(t) = e^{-at}u(t)$: $a>0$

$h(t) = u(t)$









leading edge : $\tau = 0$

no overlap when we shift $h(-\tau)$ to left, there will be no overlap and hence that integral $=0$ ✗

shift to right ✓

① $t<0 \Rightarrow y(t) = 0$

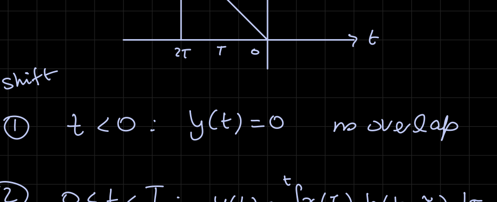
② $t>0 \Rightarrow y(t) = \int_0^t x(\tau)\,h(t-\tau)\,d\tau$ ✓

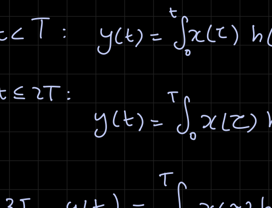
integral $-\frac{e^{-at}}{a}$

starts at 0 then saturates as $t \to \infty$

eg} $x(t) = \begin{cases} 1 & : 0<t<T \\ 0 & : \text{else} \end{cases}$

$h(t) = \begin{cases} t & : 0<t<2T \\ 0 & : \text{else} \end{cases}$





t = shift

① $t<0$: $y(t) = 0$ no overlap

② $0 \le t < T$: $y(t) = \int_0^t x(\tau)\,h(t-\tau)\,d\tau$

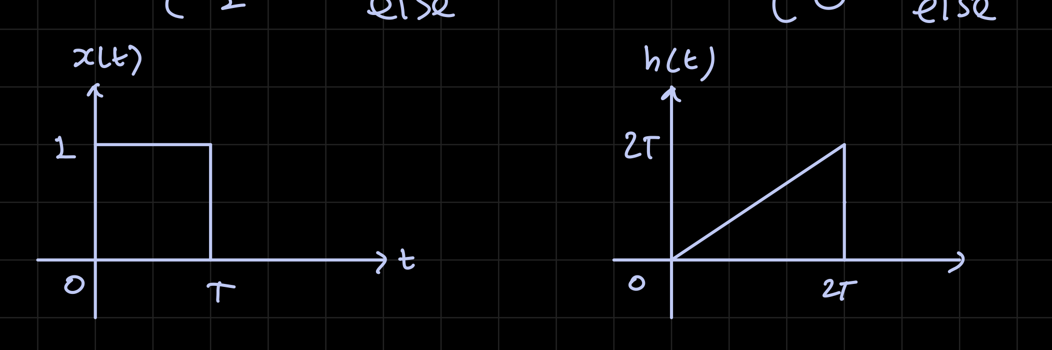
③ $T \le t \le 2T$: $y(t) = \int_0^T x(\tau)\,h(t-\tau)\,d\tau$

④ $2T < t \le 3T$ $y(t) = \int_{t-2T}^{T} x(\tau)\,h(t-\tau)\,d\tau$

* Lecture : 8 11/09/24

⇒ Convolution

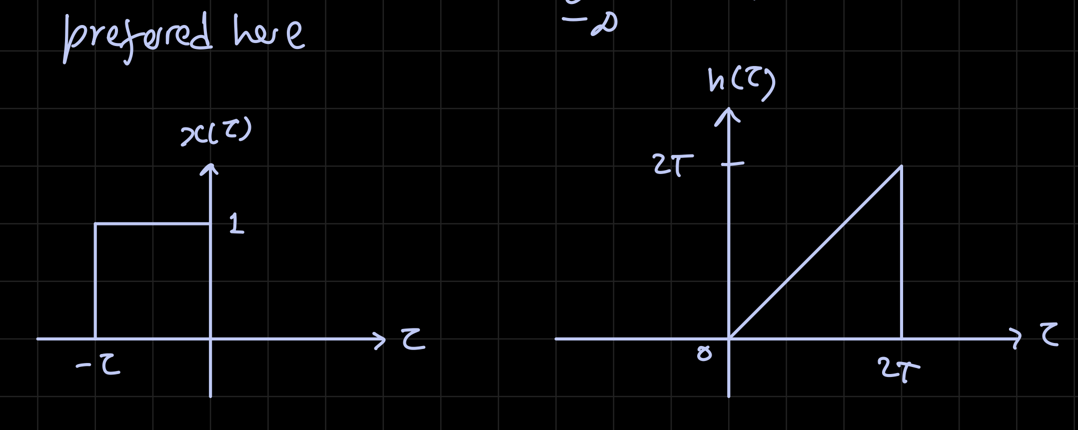
$x(t) = \begin{cases} 0 & 0 \le t \le T \\ 1 & \text{else} \end{cases}$ $h(t) = \begin{cases} t & 0 \le t \le 2t \\ 0 & \text{else} \end{cases}$



easier to reverse $x(t)$

so, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

this one preferred here ⇐ $= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$



at $y(0)$ ⇒ overlap = none

so, $y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

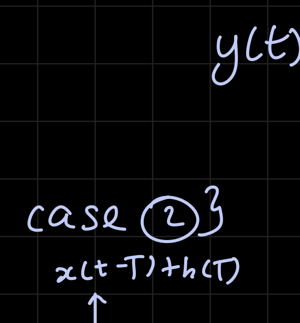
moving forward, $y(1) \dots y(n)$
if we shift $x(-\tau)$ to left
i.e. $t<0$, it will never overlap

So, we move to right, $t>0$
when plotting $x(t-\tau)$

case ① } $t<0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$$

case ② } $t>3t$ again no overlap

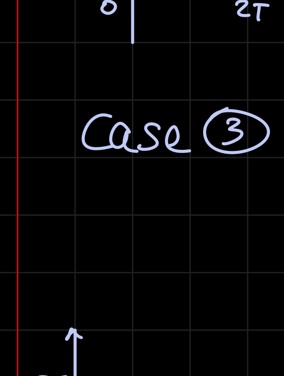


$y(t) = 0$

no non-zero overlap

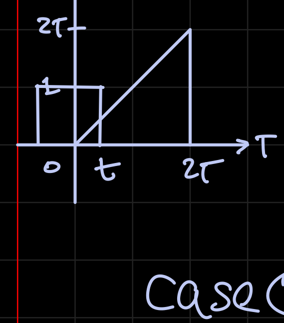
Case ③ } $0<t<T$

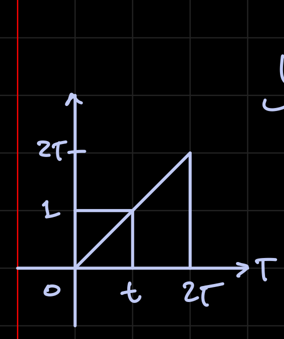
$$y(t) = \int_0^t h(\tau)\, d\tau = \int_0^t \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$



Case ④ : $T \le t \le 2T$

$$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{-T^2}{2} + tT$$





case ⑤ } $2T \le t \le 3T$

$$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$$

$$= \frac{4T^2}{2} - \frac{(t-T)^2}{2}$$

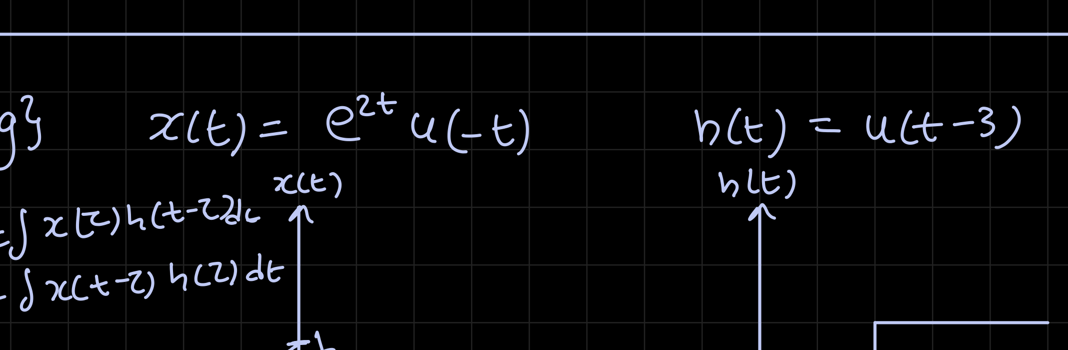
$$= \frac{4T^2 - t^2 - T^2 + 2Tt}{2}$$

$$= \frac{3T^2 + 2tT - t^2}{2}$$

---

eg } $x(t) = e^{2t} u(-t)$ $h(t) = u(t-3)$

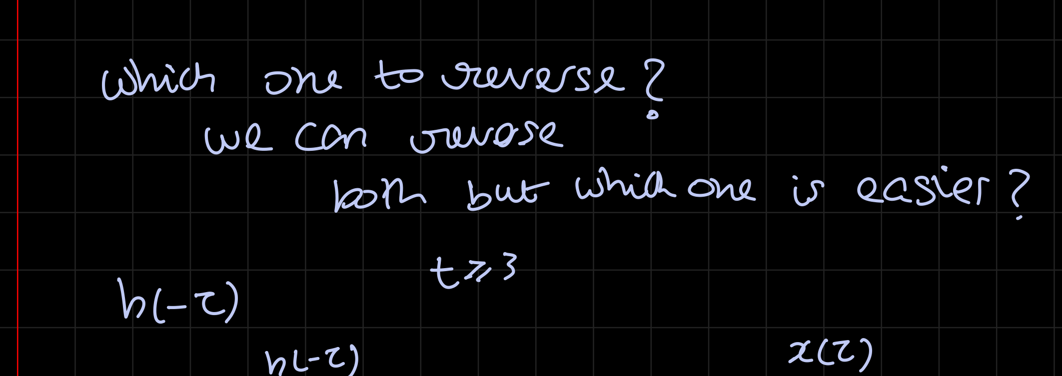
$y(t) = \int x(\tau) h(t-\tau) d\tau$
$= \int x(t-\tau)\, h(\tau)\, dt$



Which one to reverse ?
we can reverse both but which one is easier ?

$t \geq 3$

$h(-\tau)$

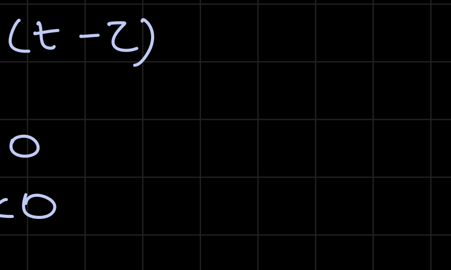


Case 1 : $t \le 3$ in $h(t-\tau)$

moves to right if $t>0$
left if $t<0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t-3} x(\tau)\, d\tau$$



$\int_{-\infty}^{t-3} e^{2\tau} d\tau$

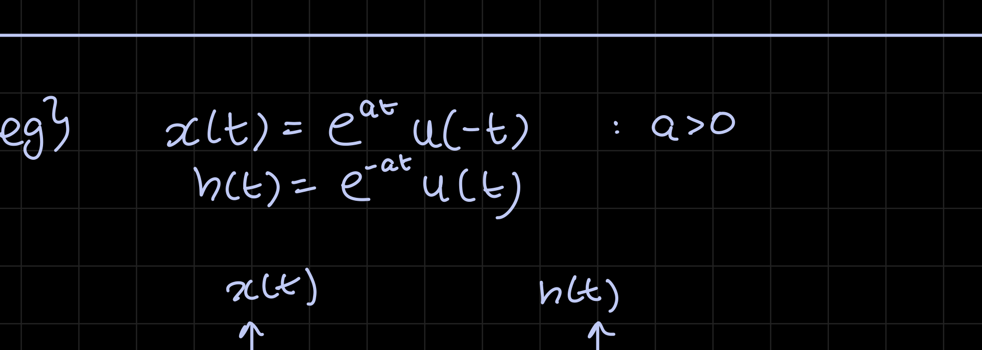
$$\Rightarrow \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t-6}}{2} = \frac{e^{2t}}{2e^6}$$

case ② } $t>3$: constant: $e^{2\tau}$

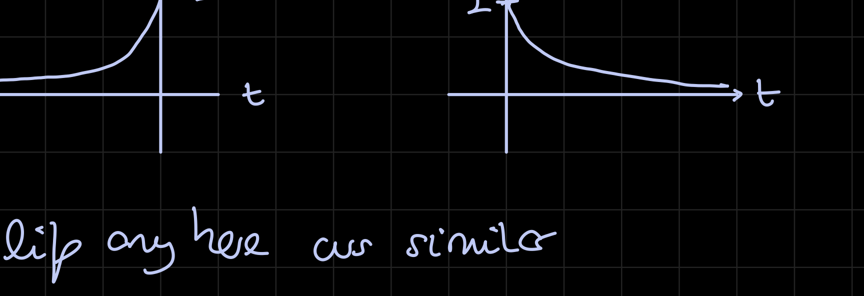
$$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \int_{-\infty}^{0} \frac{e^{2\tau}}{2} = \frac{1}{2}$$

---

eg } $x(t) = e^{at} u(-t)$: $a>0$
$h(t) = e^{-at} u(t)$



flip any here are similar



Shifting by t

Ranges: $t>0$, $t \le 0$
OR $t \geq 0$, $t<0$

case ① : $t=0$

$$y(0) = \int_0^{\infty} x(-\tau)\, h(\tau) = \int_0^{\infty} e^{-2a\tau} d\tau$$

$$\Rightarrow \frac{e^{-2a\tau}}{-2a}\Big|_0^{\infty}$$

$$\Rightarrow 0 + \frac{1}{2a} = \frac{1}{2a}$$

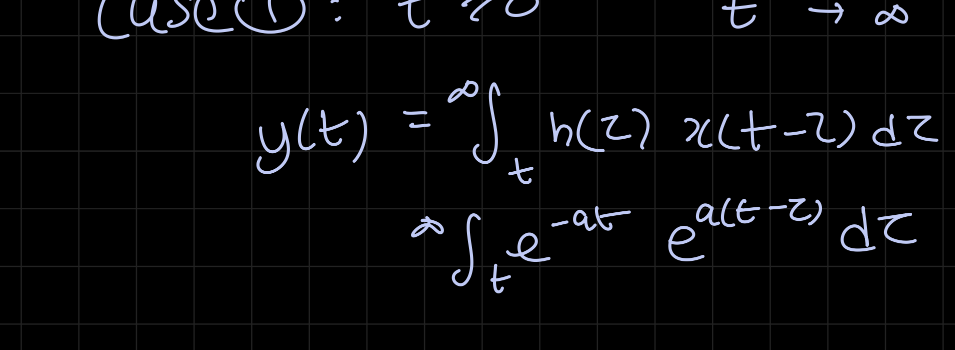
case ③ : $t<0$

$$y(t) = \int_0^{\tau - t} x(t-\tau)\, h(\tau)\, d\tau$$

$t - \tau \to \infty$

$$= \int_0^{\tau - t}$$

left edge of $x(-\tau)$ will always be equal to t



for $t \geq 0$: $t \to \infty$

for $t<0$: $0 \to \infty$

Case ① : $t \geq 0$ $t \to \infty$

$$y(t) = \int_t^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$
$$\int_t^{\infty} e^{-at}\, e^{a(t-\tau)}\, d\tau$$

case ② : $t<0$ $0 \to \infty$

$$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^{\infty} e^{-at} \cdot e^{a(t-\tau)}\, d\tau$$

* SYSTEM PROPERTIES

given an LTI system and

$$h[n] = \begin{cases} 1 & : n = 0, 1 \\ 0 & : \text{else} \end{cases}$$

defaults: input : $x[\,]$
output : $y[\,]$
impulse response : $h[\,]$

S: $y[n] = ?$ ⇒ System output = ?

apply convolution

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K] = \sum_{K=-\infty}^{\infty} x[n-K]\, h[K]$$

from the given def^n, $h[n] = \delta[n] + \delta[n-1]$

$$\Rightarrow h[K] = \delta[K] + \delta[K-1]$$

so, $$y[n] = \sum_{K=-\infty}^{\infty} \left(\delta[K] + \delta[K-1]\right) x[n-K]$$

$$= \sum_{-\infty}^{\infty} \delta[K]\, x[n-K] + \sum_{-\infty}^{\infty} \delta[K-1]\, x[n-K]$$

$$= x[n] + x[n-1]$$

eg } find impulse response of systems

S_2 : $y[n] = (x[n] + x[n-1])^2$

S_3 : $y[n] = \max\{x[n], x[n-1]\}$

we know: $y[n] = \sum_{K=-\infty}^{\infty} x[n-K]\, h[K]$

There is only one LTI system that gives the same impulse response

these systems S_2, S_3 give same response because they are not LTI

* Lecture: 9 13/09/24

- **SYSTEM PROPERTIES**

② (i) Commutativity

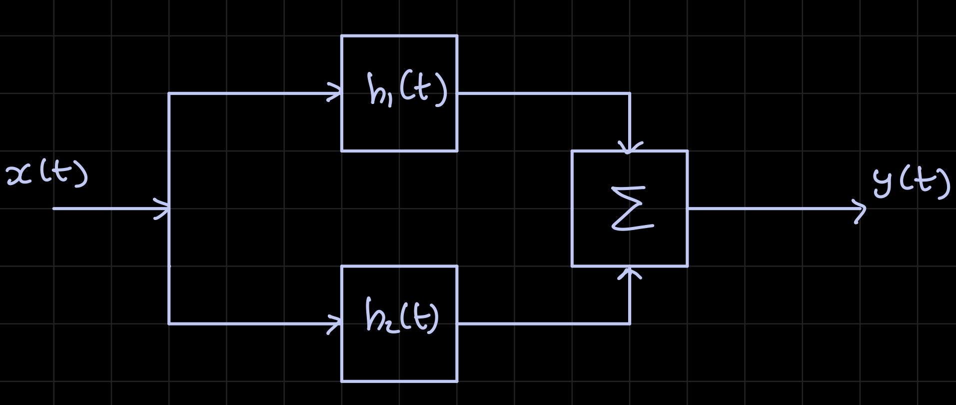
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Prove

(ii) Convolution distributes over addition

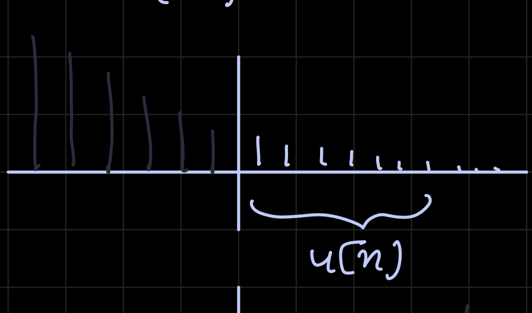
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



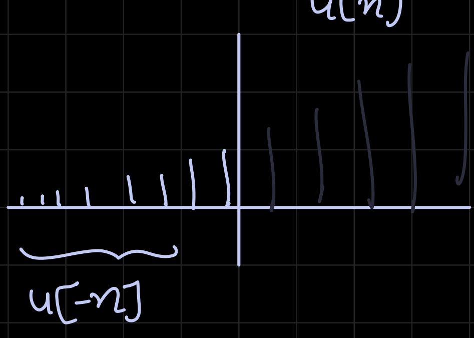
holds true for both CTS and DTS

eg3 $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$

$x_1[n] = 2^n u[n]$



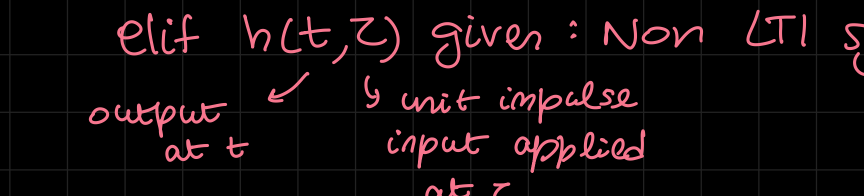
$x_2[n] = 2^n u[-n]$



easier: $y[n] = x_1[n] * h[n] + x_2[n] * h[n]$

(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



* only LTI

for LTI systems, in cascading, order of these systems can be swapped

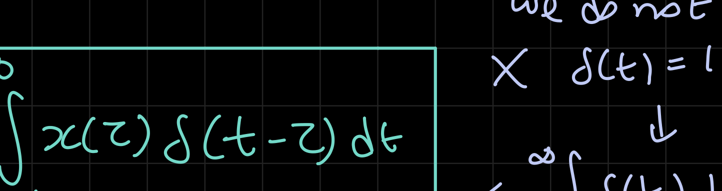
③ Invertibility :

Note: if $h(t)$ given: LTI system ✓

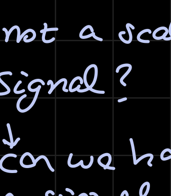
elif $h(t,\tau)$ given: Non LTI system ✓

output at t

unit impulse input applied at τ



eg3 if $h_1(t) * h_2(t) = \delta(t)$ → continuous delta(t)



$$y(t) = x(t) * (h_1(t) * h_2(t))$$

$$= x(t)\,\delta(t) \quad \} \text{ operational definition of } \delta(t)$$

$$= x(t)$$

we do not write

× $\delta(t) = 1$ at $t=0$

✓ $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, dt}$$

not a scalar signal?

can we have a signal with ∞ value at a single point?

Discrete Time

$$x[n] = \sum_{-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$= x[n] * \delta[n]$$

for this we can say $\delta[n] = 1 : n = 0$

④ Causality

claim: if a system is causal,

$h(t) = 0$ for $t<0$ (CTS)
$h[n] = 0$ for $n<0$ (DTS)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

because x cannot have future values so $k>0$

hence, for causal systems,

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

OR

$$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

if $h[n] = 0$ at $n<0$

integrator : S if $h(n) = u(n)$ CTS
accumulator : S if $h[n] = u[n]$ DTS

⑤ memory / memoryless system

$S_1: y(t) = Kx(t) \Rightarrow h(t) = K\delta(t)$
$S_2: y[n] = Kx[n] \Rightarrow h[n] = K\delta[n]$

⑥ stability (BIBO Stability Criteria)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

let input signal is bounded and $|x(t)| < M_x \ \forall t$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)\, d\tau| \le M_x \times \left(\text{some finite number}\right) < \infty \ \forall t$$

if the system is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)\, d\tau| < \infty$$

DTS: $|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

if the system holds absolute summability,

then the system is stable

Note: $h[n] = u[n]$, $h(t) = u(t)$ } LTI, unstable, causal, with memory

but still we use these for integrators & accumulators because we use clipper and clamper since we know the upper and lower bound

Identity : system with $h(t) = \delta(t)$

* Constant Coefficient difference eqn for causal LTI systems

$$\sum_{k=0}^{N} b_k\, y[n-k] = \sum_{k=0}^{M} a_k\, x[n-k]$$

a_k, b_k: not functions of time ..× constant coeff

① N ≡ order of the system

② The system is at rest if :

if, $x[n] = 0 : n < n_0$
then, $y[n] = 0 : n < n_0$
$y[n-1] = 0 : n < n_0$
⋮
$y[n-N] = 0 : n < n_0$

} auxilliary conditions

note: $N > 0$

MA

→ case: 1 ⇒ $N = 0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots a_m x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_1} x[n-1] + \dots \frac{a_m}{b_m} x[n-M]$$

causal system ✓ ≡ moving average system → weighted sum

$n \to n-1 \to n-2 \to n-3 \cdots$

system is called finite impulse response system

≡ FIR system

IIR / AR

→ case 2: $N = 1$ and $M = 0$ (simplicity)

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] - \alpha_1 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\alpha_1 = \frac{b_1}{b_0}$

if $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$h[0] = \alpha_0$
$h[1] = -\alpha_1 \alpha_0$
$h[2] = +\alpha_1^2 \alpha_0$
⋮ ⋮

} FIR? ×

Infinite impulse response system (IIR) ✓

* Auto Regressive System

≡ AR system

output depends on regresses of its previous/past inputs

ARMA

→ case: 3 $N > 0$, $M > 0$

* Lecture : 10 18/09/24

⇒ constant coeff differential eqⁿ representation of causal LTI systems

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k x(t)}{dt^k}$$

N ≡ order of the system

let $N = 2$ & $M = 3$

$$b_2 \frac{d^2 y(t)}{dt^2} + b_1 \frac{dy(t)}{dt} + b_0 y(t) =$$
$$a_0 x(t) + a_1 \frac{d\,x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a^3 \frac{d^3 x(t)}{dt^3}$$

Initial condition ↓

if $x(t) = 0 \quad \forall \; t < t_0$

then $\frac{d^k y(t)}{dt^k} = 0 \quad \forall \; t < t_0$

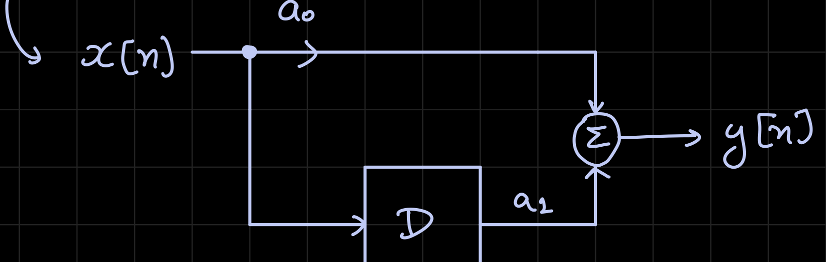
$\forall \; k \in [0, N]$

* Block diagram representation
useful for DTS

let $a_0 x[n]$ ≡ scalar multiplier

then $x[n-1]$ ≡ delay element

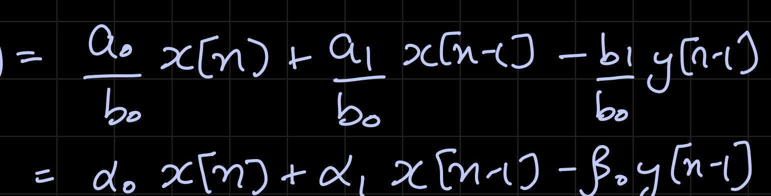
let S_1: $y[n] = a_0 x[n] + a_1 x[n-1]$



S_2: $b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$
$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$
and $\beta_0 = \frac{b_1}{b_0}$



* FOURIER SERIES

we know, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

and $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)$

$x[k]$ are scalars / constants
we can write them as a_k

$$x[n] = \sum_{k=-\infty}^{\infty} a_k\,\delta[n-k]$$

$\boxed{\text{CT}}$ $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

weighted sum of frequencies

Synthesis equation $\left\{ \; x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \right.$ finite basis

weighted linear combination of vectors

$\phi_k(t) = e^{jk\omega_0 t}$ (freq. of oscillation)

$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T}$$

fundamental time period of $x(t)$

$f = \omega_0$ rad/s
freq ≠ −ve
Time Period = T
counter clockwise $\phi_1(t) = e^{j\omega_0 t}$

completes 2 cycles in one T time period
freq = $2\omega_0$ rad/s
$\phi_2(t) = e^{2j\omega_0 t}$

Time Period = T
clockwise $\phi_{-1}(t) = e^{-j\omega_0 t}$
freq ≠ −ve
$f = \omega_0$ rad/s

$\phi_{-2}(t) = e^{-2j\omega_0 t}$

$\phi_0(t) = e^{j0} = e^0 = 1 = \cos 0 + j\sin 0$ ✓

DC component of signal
no fluctuation

can we write $\phi_2(t)$ in terms of $\phi_1(t)$?

No

they are independent of each other

inner product

$$\langle \phi_k(t), \phi_L(t) \rangle_T = \frac{1}{T}\int_0^T \phi_k(t)\,\phi_L^*(t)\,dt$$
$$= \frac{1}{T}\int_0^T e^{jk\omega_0 t} \cdot e^{-jL\omega_0 t}\,dt$$

integers; let = m

$$= \frac{1}{T}\int_0^T e^{j(k-L)\omega_0 t}\,dt$$
$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt$$
$$= \begin{cases} 0 & : k \neq L \\ 1 & : k = L \end{cases}$$

defining the inner product

$$\langle \phi_k(t), \phi_L(t) \rangle_T = \frac{1}{T}\int_T \phi_k(t)\,\phi_L^*(t)\,dt$$

(continuous time periodic signals)

⇒ Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

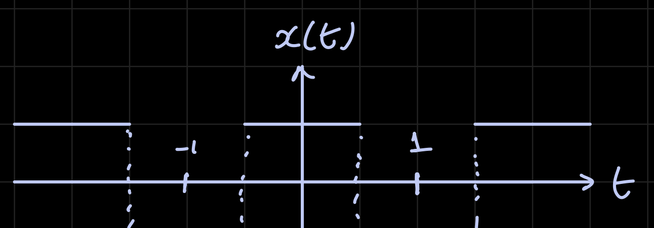
$\{\phi_k(t)\}$ ≡ span a vector space from an orthonormal basis

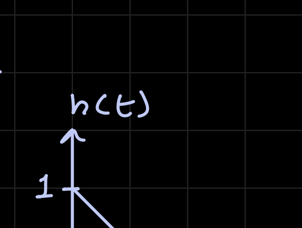
⇒ Analysis: $a_k = \langle x(t), \phi_u(t) \rangle_T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_u(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_\ell(t) \right\rangle_T$$
$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\langle \phi_k(t), \phi_L(t) \rangle_T}_{\delta[k-L]}$$

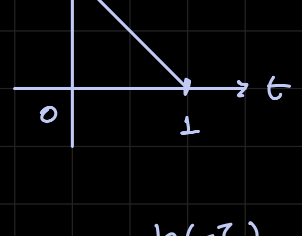
eg}



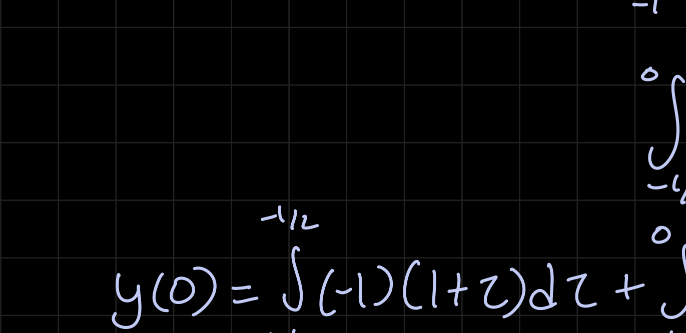


convolute $x(t)$ & $h(t)$

Ans} $h(\tau) = 1 - \tau$
$h(-\tau) = 1 + \tau$ reflect



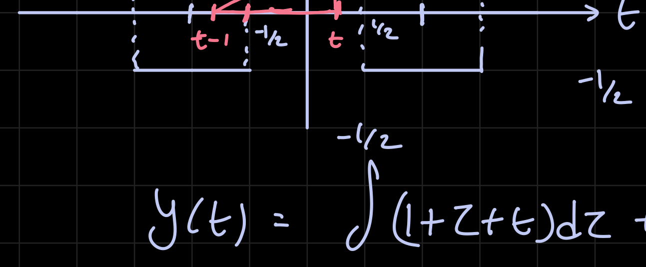
overlap



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(0) = \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

case : 1 : $\quad 0 \le t \le \frac{1}{2}$



$$y(t) = \int_{t-1}^{-1/2} (1+\tau+t)\,d\tau + \int_{-1/2}^{t} (1+\tau+t)\,d\tau$$

case : 2 ⇒

* Lecture: 11

⇒ Continuous time periodic signals FOURIER SERIES

$e^{jk\omega_0 t} \to e^{j2k\omega_0 t}$ time period is decreasing

constant → pure imaginary exponential (no real part) (in power)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

synthesis equation

$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$ rad/s; $f_0 = \frac{1}{T}$ per sec or Hz

(DC) 3, amplitude 2, 1

eg: $x(t) = 3 + 2e^{j\omega_0 t} + e^{j2\omega_0 t}$ synthesis equation

$= 3 + 2(\cos\omega_0 t + j\sin\omega_0 t) + \cos 2\omega_0 t + j\sin 2\omega_0 t$

will keep oscillating till ∞ and pattern holds and since it is not decaying ⇒ periodic

$$= \sum_{k=0}^{2} a_k e^{jk\omega_0 t}$$

$a_0 = 3$, $a_1 = 2$, $a_2 = 1$ } in this case

all other a_k ∀ $k \neq 0, 1, 2$ are zero

analysis equation

$a_k = \langle x(t), \phi_k(t) \rangle_T$ (2nd coordinate → hence we take conjugate)

$$= \frac{1}{T}\int_T x(t)\,\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$$

$\{\phi_k(t)\}_{k\in\mathbb{Z}}$ ≡ orthogonal basis ⇒ $\int_T \phi_m(t)\phi_k^*(t)\,dt = T\delta[m-k]$

$$a_k = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m\left(\int_T \phi_m(t)\phi_k^*(t)\,dt\right)\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T e^{j(m-k)\omega_0 t}\,dt\right]$$

$$= \frac{1}{\cancel{T}}\sum_{m=-\infty}^{\infty} a_m \cdot \cancel{T} \cdot \delta[m-k] = a_k = \text{LHS}$$

$\phi_1(t), \phi_{-1}(t)$ ⇒ 1st harmonic

$\phi_2(t), \phi_{-2}(t)$ ⇒ 2nd harmonic

and so on...

* Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

* Analysis: $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt$

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

let's assume $x(t) = e^{st}$ where $s = j\omega_0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,e^{st}\cdot e^{-s\tau}\,d\tau$$

$$= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau}_{\text{let } H(s)\text{, constant}}$$

$$\boxed{y(t) = e^{st}\cdot H(s)}$$

e^{st} = act as eigenfunctions of an LTI system
$H(s)$ = corresponding eigenvalue

$A\bar{v} = \lambda\bar{v}$

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow e^{st}\cdot H(s)$$

here the LTI system is acting as a linear operator

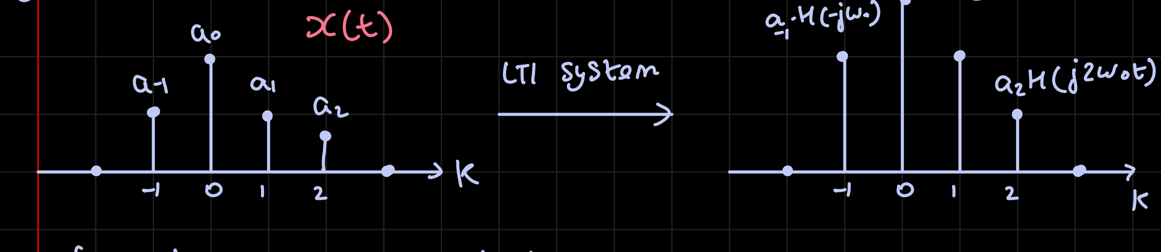
if i feed an LTI system with a continuous signal with some frequencies, then, it outputs a signal with same frequencies but with different amplitudes of the frequencies

output ≡ **Line Spectrum** of $y(t)$

existing frequencies can disappear in output but a different frequency will not get added up

eg: set $H(-j\omega_0) = 0$
so, $a_{-1}\cdot H(-j\omega_0) = 0$
this freq. will disappear in output

eg) given signal with 4 frequencies



freq domain representation

eg) S: $y(t) = x(t-2)$ and $x(t) = e^{j2t}$

Ans) only delaying system observed from eqn

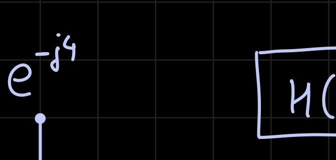
will it alter the signal frequencies? Intuitively, No

$x(t) = e^{j2t}$, $\omega_0 = 2$ rad/s

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$ $\boxed{T = \pi}$

we have only one freq. component in the line spectrum of $x(t)$

$x(t) = 1\cdot e^{j2t}$ (1 = amplitude)

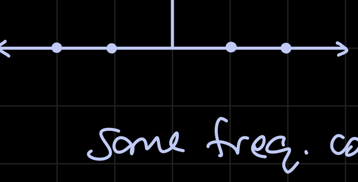


$$y(t) = e^{j2(t-2)} = e^{j2t}e^{-j4} = e^{j2t}\cdot H(j2)$$

$\boxed{H(j2) = e^{-j4}}$

note: in $H(s)$, $s = j\omega$, so, $H(j2) = e^{-j4}$

line spectrum of $y(t)$



no more freq. will be added

same freq. comes out with altered amplitude

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eg) what is $h(t)$? $h(t) = \delta(t-2)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau = \int_{-\infty}^{\infty}\delta(\tau-2)e^{-j2\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty}\delta(\tau-2)e^{-j2\cdot 2}\,d\tau \Rightarrow H(j2) = e^{2}$$

NOTE: $x(t) * \delta(t) = x(t)$

$x(t) * \delta(t-t_0) = x(t-t_0)$

shifts the signal by t_0 units

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eg) $x(t) = \sin(\omega_0 t)$

find and plot the line spectrum of $x(t)$

$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$
$e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$

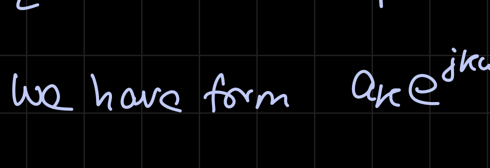
$e^{j\omega_0 t} - e^{-j\omega_0 t} = j2\sin\omega_0 t$

so, $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2} = \frac{-j}{2}\cdot e^{j\omega_0 t} + \frac{j}{2}e^{-j\omega_0 t}$

2 frequencies with amplitude $\frac{-j}{2}$ and $\frac{j}{2}$

we have form $a_k e^{jk\omega_0 t}$

so we have 2 frequencies at $k = 1$ and -1



eg) $x(t) = 1 + \sin(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

Ans) $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$\cos\left(2\omega_0 t + \frac{\pi}{4}\right) = \frac{e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{j(-2\omega_0 t + \frac{\pi}{4})}}{2}$$

$$x(t) = \underbrace{1}_{DC} + \left(\frac{1}{2j}\right)e^{j(1)\omega_0 t} + \left(\frac{-1}{2j}\right)e^{j(-1)\omega_0 t} + \left(\frac{1}{2}\right)e^{j2\omega_0 t}\cdot e^{j\frac{\pi}{4}} + \left(\frac{1}{2}\right)e^{j(-2)\omega_0 t}e^{j\frac{\pi}{4}}$$

line spectrum of $x(t)$