*Lecture : 2 17/08/24 4:30PM

Partial derivatives

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Note: y is assumed as a constant here

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Note: here x is constant

eg: find f_x, f_y if $f(x,y) = x^2+3xy+y-1$ at $(4,-5)$

Ans} $f_x = [2x+3y]_{(4,-5)} = 8-15 = \boxed{-7}$

$f_y = 3x+1 = [3x+1]_{(4,-5)} = \boxed{13}$

eg2: find Z_x if 'Z' is defined by the eqn :

$\frac{\partial z}{\partial x}$ $yz = x+y+\ln z$ implicit eqn

Ans} $yz - \log z = x+y$ z from x & y

$\log z = \cancel{x+y}\ yz - x - y$

$\cancel{\frac{e^{yz}\cdot e^{y}}{e^{x}}} = \cancel{z}$ ⇒ Note: cannot separate and then differentiate

so, carry out implicit derivation by applying product rule

$yz - \ln z = x+y$

$yz_x - \frac{1}{z}\cdot z_x = 1+0$

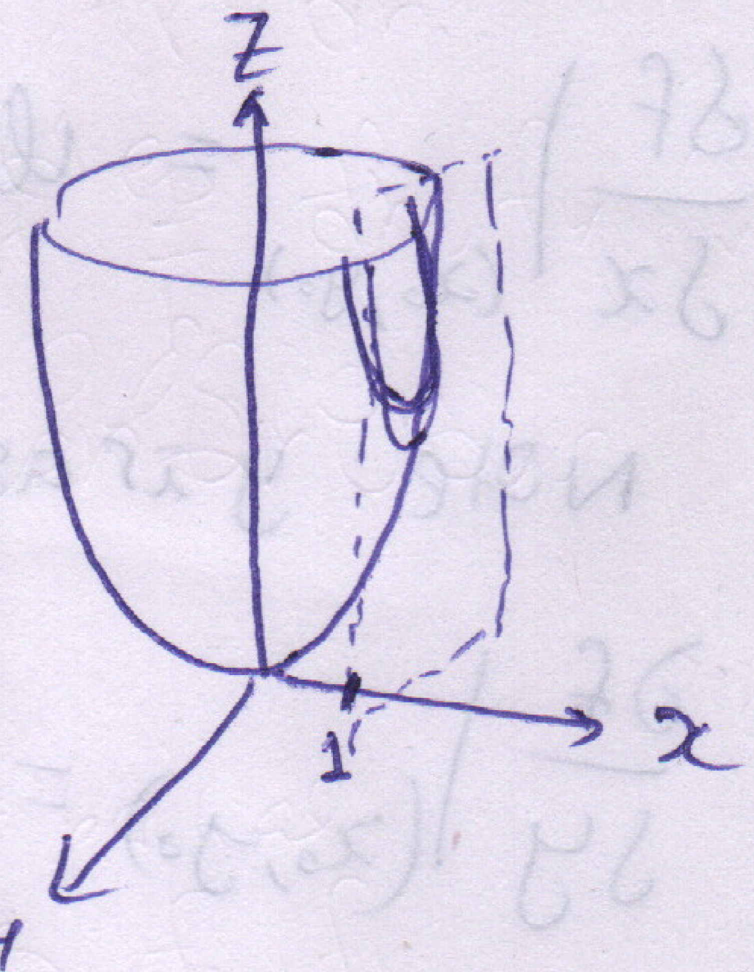
$z_x\left(y-\frac{1}{z}\right) = 1$

$z_x = \frac{\partial z}{\partial x} = \frac{z}{zy-1}$

eg3: The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola. Find the slope of the parabola at $(1,2,5)$.

Ans) We need to find

$\frac{\partial z}{\partial y}$ since x is constant $\{x=1\}$ and since slope is a derivative.



$$z_y = 0 + 2y\Big|_{y=2} = 4.$$

eg: find f_z if $f = x\sin(y+3z)$ and (x,y,z) are independent variables

Ans) $f_z = 3x\cos(y+3z)$

* Higher derivatives: f_x, f_{xx}, f_{yy}

* mixed derivatives: f_{xy}, f_{yx}

eg. find f_{xy} and f_{yx} if $f = x\cos y + ye^x$

Ans) $f_x = \cos y + \cancel{(\ldots)} + ye^x$

$f_{xy} = \cancel{(\ldots)}$

$-\sin y \cancel{(\ldots)} + \cancel{y}e^x$

$\Rightarrow -\sin y + e^x$

$f_y = -x\sin y + e^x$

$f_{yx} = -(\sin y) + e^x$

* THEOREM : Mixed derivative Theorem

if $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing (x_0, y_0) are all continuous

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

eg? $w = xy + \frac{e^y}{y^2+1} \Rightarrow w_{xy} = \cancel{x}\, y + 0$

* THEOREM : Increment theorem for 2 var. functions.

if f, f_x, f_y are defined in open region R at (x_0, y_0) then the total change in f is given by

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

we can also say,

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) + f(x_0, y_0 + \Delta y) - f(x_0, y_0)$$

$$\Delta f = f_x \Big|_{(x_0,y_0)} \Delta x + f_y \Big|_{(x_0,y_0)} \Delta y$$

~~All~~ Questions $\underline{\underline{14.3}} \Rightarrow 14, 25, 38, 43, 51 (f), 54, 57, 59, 61$

let $V = \pi r^2 h$

then total change $= \Delta V = 2\pi r h \cdot dr + \pi r^2 \cdot dh$

and relative change $= \frac{\Delta V}{V} = \frac{2}{r} \cdot dr + \frac{1}{h} \cdot dh$

CHAIN RULE (14.4)

let $w = f(x,y)$ ← has continuous partial derivative f_x, f_y

if x and y are functions of t
i.e. $x \Rightarrow x(t)$ and $y \Rightarrow y(t)$

then $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} = w_x \cdot x_t + w_y \cdot y_t$

w is a function of t implicitly whereas x and y are a func of t explicitly

eg: if $w = x^2 y - y^2$. find w_t at $t=0$ along the path $x = \sin t, y = e^t$

$w_x = \cancel{2x} \, 2xy$

$w_x \cdot x_t = 2xy \frac{dx}{dt} \Rightarrow 2xy \cdot \cos(t)$

$w_y = x^2 - 2y \Rightarrow w_y \cdot y_t = (x^2 - 2y) e^t$

$\left.\frac{dw}{dt}\right|_{t=0} = 2xy \cdot \frac{dx}{dt} + (x^2 - 2y) \frac{dy}{dt} = 2xy \cos(t) + (x^2 - 2y) e^t$

$\Rightarrow 2xy + x^2 - 2y$

① differentiate wrt x, y using $x=0, y=1$ OR
② differentiate wrt t using $t=0$

SAME THING

* Functions on a surface

suppose $w = f(x,y,z)$ [3d space]

$$x = g(r,s) \; ; \; y = h(r,s) \; ; \; z = k(r,s)$$

$$w_r = \cancel{dw}\ \frac{\partial w}{\partial r} = w_x \cdot x_r + w_y \cdot y_r + w_z \cdot z_r$$

similarly for w_s

eg: find w_r if $w = x + 2y + z^2$

$$: x = \frac{r}{s} \quad , y = r^2 + \log s \; ; \; z = 2r$$

Ans: $w_x = \frac{\partial w}{\partial x} = 1 \qquad w_y = \frac{\partial w}{\partial y} = 2 \qquad w_z = \frac{\partial w}{\partial z} = 2z$

$x_r = \frac{dx}{dr} = \frac{1}{s} \qquad y_r = \frac{dy}{dr} = 2r \qquad z_r = 2$

$$\Rightarrow \frac{1}{s} + 4r + 4z \Rightarrow \frac{1}{s} + 4r + 8r = \boxed{\frac{1}{s} + 12r}$$

* Implicit differentiation

<u>Theorem</u>: Suppose $F(x,y)$ is differentiable and that satisfies $F(x,y) = 0$ [defines y as a function of x]

i.e. y is dependent var on x

and $F_y \neq 0$ then,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

<u>Proof</u>: Apply Implicit differentiation of $F(x,y) = 0$

$$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} dy = -\frac{\partial F}{\partial x} \cdot dx \Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}$$

eg: find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin(yx) = 0$

Note: cannot seperate x & y so,

Ans3 $\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{2x + y\cos(yx)}{2y - x\cos(yx)}$ ✓

$F_x = -2x - y\cos(yx)$

$F_y = 2y \cancel{+} \cancel{co}\, x\cos(yx)$

Functions of many variables : $w = f(x_1, .. x_2, ... x_n)$

$$\frac{\partial w}{\partial p} = \frac{\partial f}{\partial x} x_p + \frac{\partial f}{\partial y} y_p + \cdots\cdots \frac{\partial f}{\partial v} v_p$$

Practice Problems: 4, 6, 7, 10, 11, 28, 32, 38, 39, 43
(14.4)