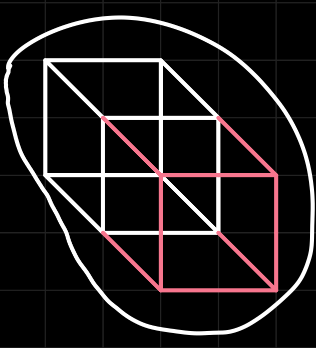
- Lecture 3

★ Gauss' Divergence Theorem

divide the volume into tiny boxes and calculate the outward flux and add them up

$$\int_{V_{big}} (\bar{\nabla}\bar{v})\,d\bar{v} = \sum_{\substack{\text{all tiny}\\ \text{parallelpiped}}} \lim_{dv\to 0} (\bar{\nabla}\bar{v})\,dv$$

$$= \sum \oint_{\substack{\text{small}\\ \text{volume}}} V\cdot ds$$



outward flux from all the shared surfaces cancel out

$$\text{so, final flux} = \sum_{\substack{\text{unshared surfaces}\\ \text{coinciding with the}\\ \text{boundary of the big volume}}} \bar{V}\cdot d\bar{S}$$

closed surface integral

$$= \oint_{\substack{\text{entire}\\ \text{surface}}} \bar{V}\cdot d\bar{S}$$

- LAPLACIAN OPERATOR ($\bar{\nabla}^2$) can operate on both, scalar and vector

$$\bar{\nabla}(\bar{\nabla}U) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}\right)$$

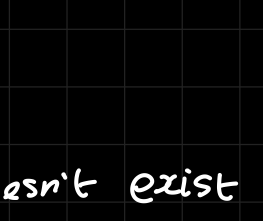
$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$(\bar{\nabla}^2 U) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)U =$$

★ EARNSHAW'S THEOREM

- A scalar field $\phi(x,y)$ that has $\underbrace{\nabla^2}_{\text{laplacian}}\phi = 0$, cannot have a local min/max in that region.

eg: $\phi = x^2+y^2$
$\nabla\phi = 2x+2y$
$\nabla^2\phi = 4$
non-zero
has a local min ✓



local min at (0,0,0)

also works with gravitational field

if $\bar{\nabla}\cdot\bar{E} = 0$
→ $\bar{\nabla}(\bar{\nabla}\phi) = 0$
→ $\bar{\nabla}^2\phi = 0$
then local min/max doesn't exist

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

if $D<0$: saddle point

$\nabla f = f_x + f_y$
$\nabla^2 f = f_{xx} + f_{yy}$
if $\nabla^2 f = 0 \rightarrow f_{xx} + f_{yy} = 0$
and so, $D = \underbrace{-\underbrace{f_{xy}^2}_{+ve}}_{-ve\text{ always}}$

so, $D<0$
↳ saddle point
↳ neither local min nor local max

Simplest Saddle Point

$z = x^2 - y^2$
OR
(Hyperbolic Paraboloid) $z - \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$

not a stable equilibrium

how can you stabilize a football on the saddle point?
↳ constantly rotate it
i.e. changing the electric field
≡ Paul Trap / Ion trapping in quantum computers

Earnshaw's theorem prevents stability on the saddle point

★ CURL operator

$$\underbrace{\bar{\nabla}\times\bar{A}}_{} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

cannot say $\bar{\nabla}$ is $\parallel$ $\bar{A}$
because operator does not has a direction

defn of curl: line integral along an infinitesimal loop
↳ loop can be in any direction

curl of a vector field ≡ closed line integral per unit area

★ STOKE'S THEOREM (finite loop)

$$\int(\bar{\nabla}\times\bar{A})\cdot d\bar{S} = \oint \bar{A}\cdot dl$$

divide the big loop into infinitesimally small loops

- MOBIUS STRIP

cannot apply stoke's Theorem

- Irrotational Field
  ↳ curl : $\bar{\nabla}\times F = 0$
  ↳ conservative field
  ↳ can be written as a gradient of scalar
  $\bar{F} = \bar{\nabla}u$

eg: electrostatic field ✓
not electrodynamic field ×

- SOLENOIDAL FIELD
  ↳ divergence : $\bar{\nabla}\cdot F = 0$
  ↳ solenoidal field over volume V doesn't have any source/sink in that volume
  ↳ $\bar{\nabla}\cdot(\bar{\nabla}\times A) = 0$
  ↳ A solenoidal field can be written as $\bar{\nabla}\times\bar{A}$, $\bar{A}$ ≡ vector potential

eg: magnetic field is solenoidal field
electrostatic field is solenoidal only when there is no charge