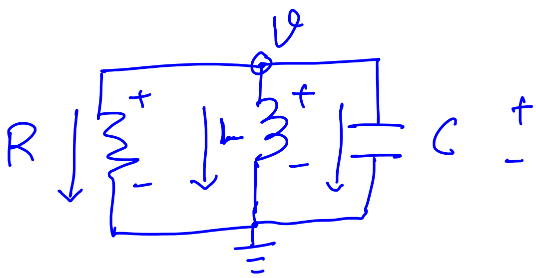
Quick Recap

- natural response of a parallel RLC circuit



$V_R = V_C = V_L = v, \quad \forall t \geq 0$

$\underline{\underline{I_R}} + \underline{\underline{I_L}} + \underline{\underline{I_C}} = 0 \quad \forall t \geq 0$

$V_C(0) = V_0, \quad I_L(0) = I_0$

$$C \frac{d^2 v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L} v = 0$$

$v(t) = A e^{st}$ (Assumption)

$$C s^2 + \frac{1}{R} s + \frac{1}{L} = 0 \quad \text{(characteristic eq.)}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha \rightarrow$ decay rate, $\omega_0 \rightarrow$ resonant frequency

$\alpha = \frac{1}{2RC}, \quad \omega_0^2 = \frac{1}{LC}$

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ (general structure)

$A_1, A_2, S_1, S_2 \longrightarrow$ (complex numbers)

$(\alpha > \omega_0) \rightarrow$ Overdamped system

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\Rightarrow$ real roots $\qquad S_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} < 0$

$$\alpha > \sqrt{\alpha^2 - \omega_0^2} \Rightarrow S_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} < 0$$

negative real roots

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \checkmark$$

$\Rightarrow$ A_1 and A_2 are real numbers.

$v(0) = V_C(0) = V_0$

$t = 0$

$$V_0 = A_1 + A_2 \text{ ——— } (1)$$

$$\frac{dv}{dt} = A_1 S_1 e^{s_1 t} + A_2 S_2 e^{s_2 t}$$

$I_R(t) + I_L(t) + I_C(t) = 0 \qquad \forall t \geq 0$

$I_R(0) + I_L(0) + I_C(0) = 0$

$$I_R(0) = \frac{V_R(0)}{R} = \frac{v(0)}{R} = \frac{V_0}{R}$$

$$I_C(0) = -I_R(0) - I_L(0)$$

$$= -\frac{V_0}{R} - I_0$$

$$\frac{dv(0)}{dt} = \frac{dV_C(0)}{dt} = \frac{1}{C} I_C(0) = -\frac{1}{C}\underbrace{\left(I_0 + \frac{V_0}{R}\right)}$$

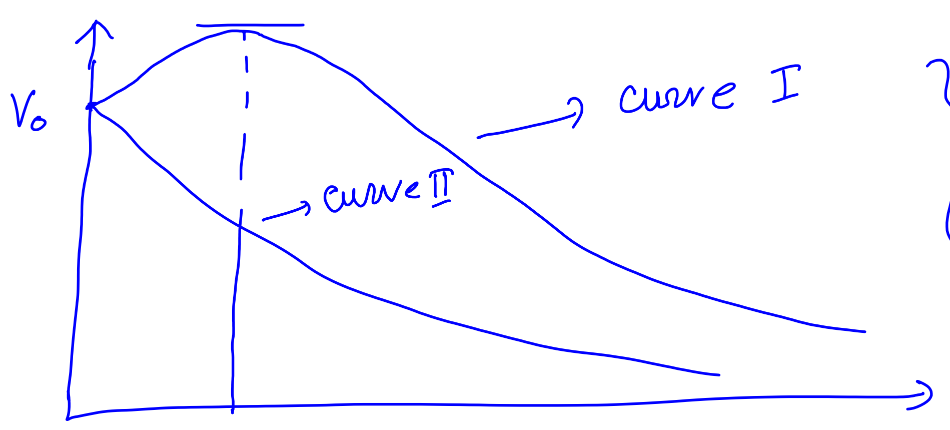
$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$t = 0$

$$-\frac{1}{C}\left(I_0 + \frac{V_0}{R}\right) = A_1 s_1 + A_2 s_2 \quad \text{——— (II)}$$

Solve for A_1 and A_2 from (I) and (II).

$v(\infty) = 0$ because $s_1, s_2 < 0$



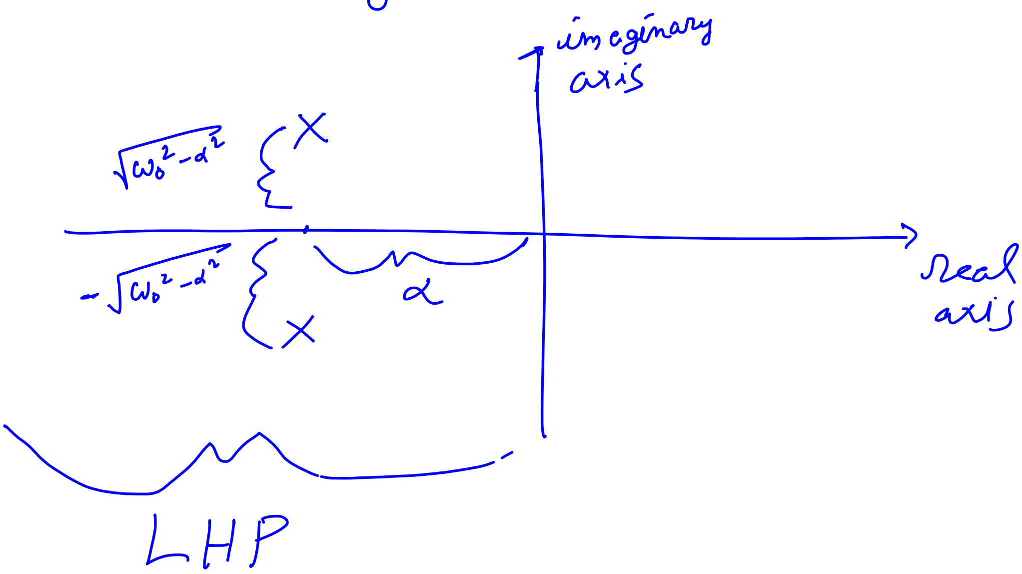
possibilities

$(\alpha < \omega_0) \longrightarrow$ Underdamped System

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha \pm \sqrt{(-1)(\omega_0^2 - \alpha^2)}$$

$$= -\alpha \pm j\sqrt{(\omega_0^2 - \alpha^2)} \quad , \quad (j = \sqrt{-1})$$



$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\boxed{e^{j\omega t} = \cos\omega t + j\sin\omega t}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$\therefore \quad s_{1,2} = -\alpha \pm j\omega_d$$

$$\Rightarrow \quad v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$A_1 = \gamma_1 + j\delta_1$$

$$A_2 = \gamma_2 + j\delta_2$$

$$v(t) = e^{-\alpha t}\left((\gamma_1 + j\delta_1)e^{j\omega_d t} + (\gamma_2 + j\delta_2)e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\Big((\gamma_1 + j\delta_1)(\cos\omega_d t + j\sin\omega_d t)$$

$$+ (\gamma_2 + j\delta_2)(\cos\omega_d t - j\sin\omega_d t)\Big)$$

$$v(0) = (\gamma_1 + j\delta_1) + (\gamma_2 + j\delta_2)$$

$$= (\gamma_1 + \gamma_2) + j(\delta_1 + \delta_2)$$

$$\Rightarrow \quad \delta_1 + \delta_2 = 0 \quad \checkmark$$

$$\delta_1 = \delta \quad \therefore \quad \delta_2 = -\delta$$

$$v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$\frac{dv(t)}{dt} = -\alpha e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$+ e^{-\alpha t}\left(A_1 j\omega_d e^{j\omega_d t} - A_2 j\omega_d e^{-j\omega_d t}\right)$$

$$\frac{dv(0)}{dt} = -\alpha(A_1 + A_2) + A_1 j\omega_d - A_2 j\omega_d$$

$$= -\alpha\left(\gamma_1 + \cancel{j\delta} + \gamma_2 - \cancel{j\delta}\right) + (\gamma_1 + j\delta).j\omega_d$$
$$- (\gamma_2 - j\delta)\, j\,\omega_d$$

$$= -\alpha(\gamma_1 + \gamma_2) + j\,\omega_d\,(\gamma_1 - \gamma_2)$$
$$- \delta\,\omega_d - \delta\,\omega_d$$

$$= -\alpha(\gamma_1 + \gamma_2) - 2\delta\omega_d + j\,\omega_d\,(\gamma_1 - \gamma_2)$$

$$\Rightarrow \quad \gamma_1 - \gamma_2 = 0$$

$$\Rightarrow \quad \gamma_1 = \gamma_2 = \gamma$$

$$\left.\begin{aligned} A_1 &= \gamma + j\delta \\ A_2 &= \gamma - j\delta \end{aligned}\right\}$$ complex conjugates

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$= e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$
$$= e^{-\alpha t}\left(A_1 \cos\omega_d t + j A_1 \sin\omega_d t\right.$$
$$\left. + A_2 \cos\omega_d t - j A_2 \sin\omega_d t\right)$$

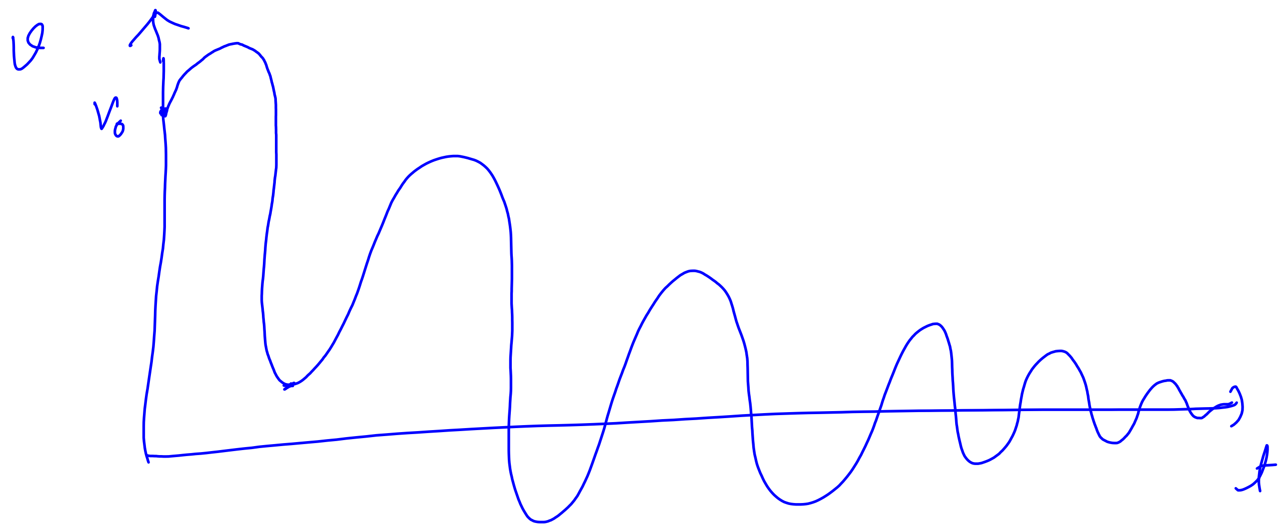
$$= e^{-\alpha t}\left(\underbrace{(A_1 + A_2)}_{2\gamma}\cos\omega_d t + j\underbrace{(A_1 - A_2)}_{-2\delta}\sin\omega_d t\right)$$

$$v(t) = e^{-\alpha t}\left(\underbrace{A_d \cos \omega_d t + B_d \sin \omega_d t}\right)$$

A_d, B_d are real numbers

$(A_d = 24,\ B_d = -28)$

$v(\infty) = 0$



$(\underline{\alpha = \omega_0}) \rightarrow$ (Critically damped)

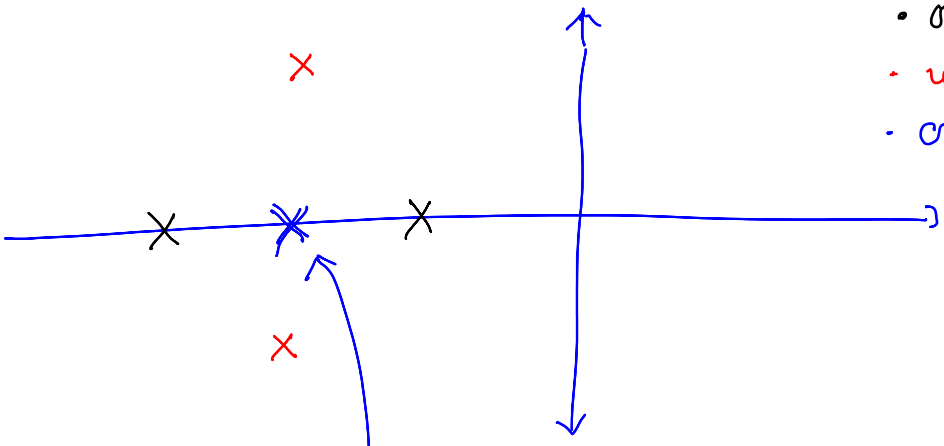
$$C \frac{d^2 v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L} v = 0$$

$$\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC} v = 0$$

$$\alpha = \frac{1}{2RC}, \quad \omega_0^2 = \frac{1}{LC}$$

$$\underline{\frac{d^2 v}{dt^2} + 2\alpha \frac{dv}{dt} + \alpha^2 v = 0}$$

- overdamped
- underdamped
- critically damped



two roots are same (-ve)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha \qquad (\text{since } \alpha = \omega_0)$$

Let's assume,

$$v(t) = A_2 e^{st}$$

It's a valid solution

but, only unknown A_2, however, we have two initial conditions.

This is also a valid solution

$$v(t) = A_1 t e^{st} \qquad (\text{Assumption})$$

$$\frac{dv(t)}{dt} = A_1 t s e^{st} + A_1 e^{st}$$

$$\frac{d^2 v}{dt^2} = A_1 t s^2 e^{st} + A_1 s e^{st} + A_1 s e^{st}$$

$$\frac{d^2 v}{dt^2} + 2\alpha \frac{dv}{dt} + \alpha^2 v$$

$$= A_1 t s^2 e^{st} + A_1 s e^{st} + A_1 s e^{st}$$
$$+ 2\alpha\left(A_1 t s e^{st} + A_1 e^{st}\right) + \alpha^2 A_1 t e^{st}$$

$$= A_1 t \underline{\alpha^2} e^{-\alpha t} - A_1 \alpha \cancel{e^{-\alpha t}} - A_1 \alpha \cancel{e^{-\alpha t}}$$
$$+ 2\alpha\left(\underline{-A_1 t \alpha} e^{-\alpha t} + A_1 \cancel{e^{-\alpha t}}\right) + \alpha^2 A_1 \underline{t} e^{-\alpha t}$$

$$= 0$$

<u>general solution</u>

$$v(t) = A_1 t e^{st} + A_2 e^{st}$$
$$= e^{st}\left(A_1 t + A_2\right)$$
$$= e^{-\alpha t}\left(A_1 t + A_2\right)$$