14/08/24 # Signals And Systems 11:00Am

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|---|---|---|
| Evaluation: | Attendance | 10% |
| (relative grading) | Quiz | 16% |
| yooooooooooo :fire emoji: | Assignments | 15% |
| | mid sem | 24% |
| | end sem | 35% |

Book: Signals & System 2nd edition by Alan V. oppenheimer

## Lecture : 1 14/08/24 : 11:00AM

## * Linear Algebra Revision

### ⇒ GROUP : consists of ....

- A set G
- A rule / binary operation "$*$"
  - a. associative
    $x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$
  - b. There exists an element "e" called the identity of group G such that
    $e*x = x*e = x \quad \forall x \in G$
  - c. $\forall x \in G, \exists\ x^{-1}$ such that
    $x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$
  - d. if $x*y = y*x \quad \forall x,y \in G$,
    this group is called

**Commutative / Abelian Group**

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abellion Group

eg: set of continuous time periodic signals with time period T with "$*$" = "$+$"

### ⇒ FIELD : consists of the following

- A set F
- Two binary operations "+" and "."
  such that ....
  - $(F,+)$ is an abellian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - left distributive
      $x\cdot(y+z) = xy + xz \quad \forall x,y,z \in F$
    - Right distributive
      $(x+y)\cdot z = xz + yz \quad \forall x,y,z \in F$

eg: F = Real Numbers $\mathbb{R}$

## * Vector Space : A set V with a map...

- '+' : $V \times V \rightarrow V$
  $(v_1, v_2) \rightarrow (v_1 + v_2)$ called vector addition
- '.' : $F \times V \rightarrow V$
  $(a, v_1) \rightarrow (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a\cdot(v_1+v_2) = av_1 + av_2$
- $(a_1+a_2)\cdot v = a_1 v + a_2 v$
- $(\alpha\beta)v = \alpha(\beta v) = \beta(\alpha v)$
- $1 \cdot v = v$
- $a \cdot 0 = 0$
- $0 \cdot v = 0$
- if $v \neq 0$, then $a \cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

## * METRIC SPACE : metric is a map

$$d : X \times X \rightarrow R$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by $(X,d)$

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

## * NORM: let V be a F-vector space.
A map $\|\cdot\| : V \rightarrow R$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a| \, \|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

## Lecture : 2

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### * Inner Product :

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v,v \rangle \geq 0$ and $\langle v,v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ " $-$ " : complex conjugate operation
- it is linear in the first co-ordinate
  $$\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$$
  $\forall\, v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $$\langle v, a_1 w_1 + a_2 w_2 \rangle = \bar{a}_1 \langle v, w_1 \rangle + \bar{a}_2 \langle v, w_2 \rangle$$
  $\forall\, w_1, w_2, v \in V$ and $a_1, a_2 \in F$

measures cosine similarity
" norm(x) · norm(y) · cosθ

eg: dot product

$$\langle x,y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner Product

$$\langle x,y \rangle = \sum_{i=1}^{n} x_i \cdot \bar{y}_i$$

complex conjugate

### * Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \quad \text{implies } a_i = 0 \ \forall i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

weighted linear combination of vector V

if $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ is a basis for $V_n(F)$, then $\forall\, v_i \in V \ \& \ a_i \in F$

$$V = \sum_{i=1}^{n} a_i e_i$$

$V = a_1 e_1 + a_2 e_2 + \dots a_n e_n$

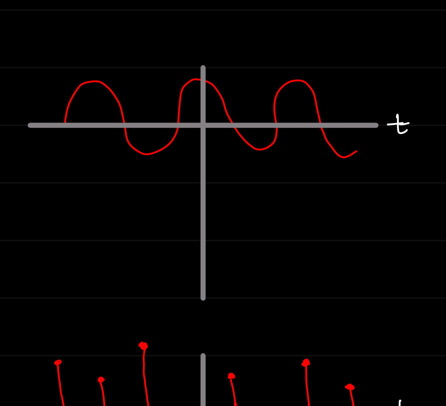
### * ORTHOGONAL & ORTHONORMAL Basis

A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

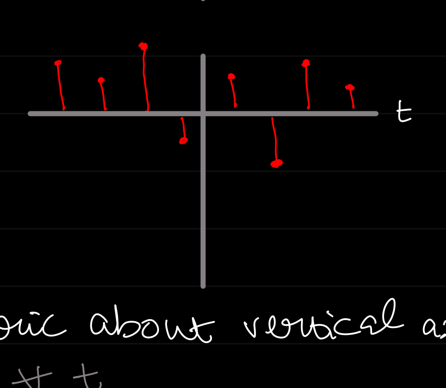
$$v_i \neq 0 \quad \forall i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall\, i \ \& \ j$$

## SIGNALS (CTS & DTS)

- continuous time signal




- Discrete time signal




- Even Signal : symmetric about vertical axis
  $x(t) = x(-t) \ \forall t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t) \ \forall t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

### * Even/odd component of a signal:

$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

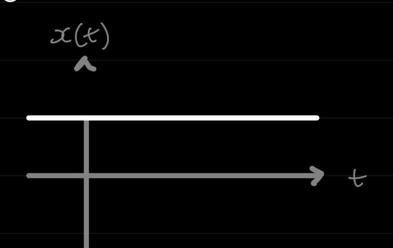
$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

2r

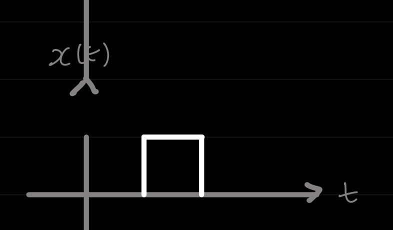
### * PERIODIC : if $x(t) = x(t+T) \ \forall t$ then T is the time period

The smallest time period T defined for this property is called the fundamental time period

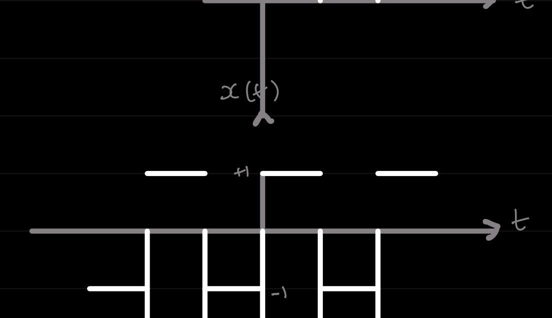
eg:



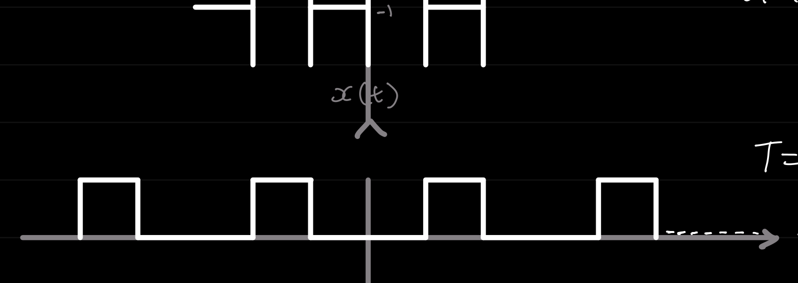

Not periodic



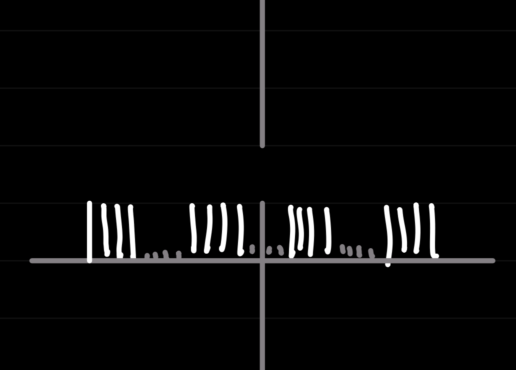

$T \to \infty$
Aperiodic




Periodic with T=2




T=3



T=8

classification { Probabilistic
Dataministic

### * ENERGY OF SIGNAL

Continuous Time signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Discrete Time signal

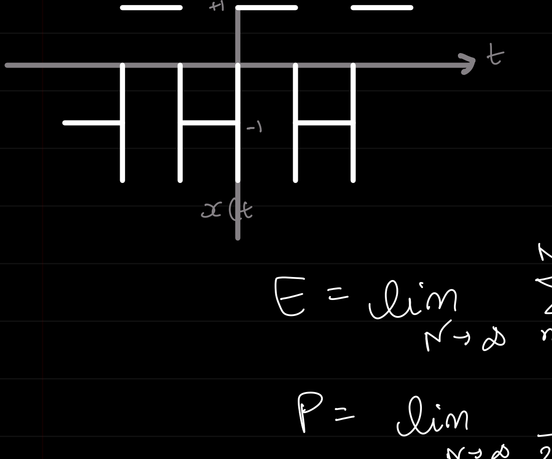
$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Note: Periodic signals are power signals ✓
Aperiodic signals are not power signals ✗

T=∞
so, power = not defined
hence, Aperiodic signals are not power signals

avg. energy in
" a time duration

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$




compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{x=-N}^{N} |x[n]|^2$$

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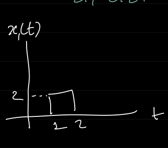
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$



$$= \int_{2}^{2} |2|^2 dt = 4\int_0^2 dt$$

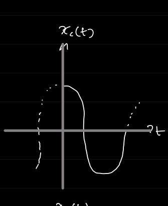
$$= 4 < \infty$$ ↑ energy signal

$x_1(t)$ ≡ energy signal, continuous signal, aperiodic, deterministic

$x_1(t) \neq x_1(t+T) \Rightarrow$ aperiodic signal

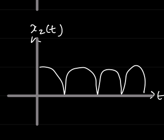
$x_2(t)$ = continuous signal, periodic signal, ← deterministic, power signal

we know its value at every point t ⇓ not probabilistic



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

↓ after mod ⇒



$$E = \underbrace{\lim_{T_0 \to \infty} \int_{-T_0}^{T_0} |x(t)|^2 dt}_{\infty}$$

Since $E \to \infty$ and not $E < \infty$, this signal is not an energy signal

$$P = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} |x(t)|^2 dt$$

intuition { ≈ $\frac{\text{infinite block}}{\text{infinit block}}$ = finite

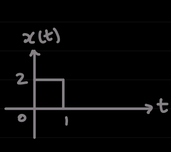
$\boxed{0 < P < \infty}$ ⇒ Power Signal ✓

## * TRANSFORMATION OF SIGNAL

### ① Translation

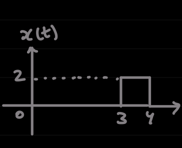
energy of signal doesn't change

(a) DELAY



$y(t) = x(t - t_0)$

let $t_0 = 3$



$y(t) = x(t-3)$

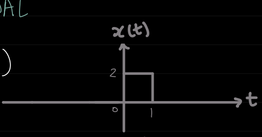
$y(0) = x(-3) = 0$
$y(3) = x(0) = 2$
$y(4) = x(1) = 2$
$y(5) = x(2) = 0$

} verification of transformation

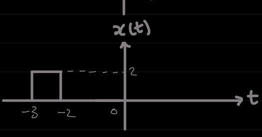
(b) ADVANCE A SIGNAL

$y(t) = x(t + t_0)$

let $t_0 = 3$



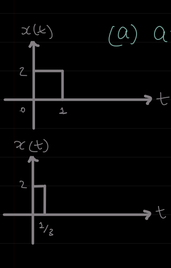
$y(-3) = x(0) = 2$
$y(-2) = x(1) = 2$
$y(-1) = x(2) = 0$
verified



energy of scale changes in accordance with area under curve

### ② Scaling

$y(t) = x(at)$

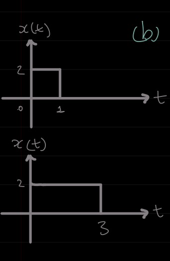


(a) $a > 1$ : compression of the signal on the t-axis

let $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$
$y(\frac{2}{3}) = x(2) = 0$



(b) $0 < a < 1$ : Dilation on t-axis

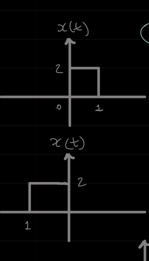
let $a = \frac{1}{3}$

$y(t) = x(at) = x(t/3)$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

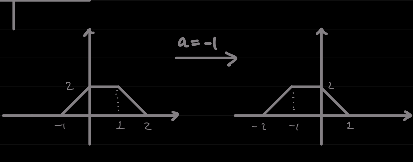
$y(6) = x(2) = 0$



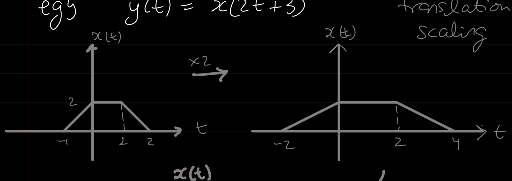
ⓒ SPECIAL CASE: $a = -1$

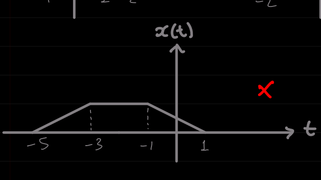
$y(t) = x(-t)$

flipped on the t-axis ≡ time reversal



eg) $y(t) = x(2t+3)$ translation + scaling

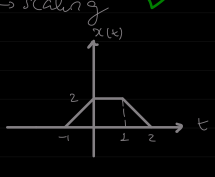




scaling then translation because otherwise we get 2t+6 ✗

here,
scaling → translation ✗
translation → scaling ✓

$x(t)$
↓ translate
$x(t+3)$
↓ scale
$x(2t+3)$



## * FUNDAMENTAL SIGNALS

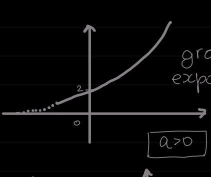
- **Continuous Time Signals**

$$x(t) = ce^{at}$$

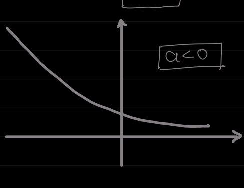
- CASE: 1 c ≡ real constant let c = 2
  a ≡ real constant

$x(t) = 2e^{at}$

① let $a = 1$



② let $a = -1$



③ let $a = 0$

- CASE: 2 C ≡ real constant
  a ≡ purely imaginary = $j\omega$

$x(t) = e^{j\omega t}$ ∀ t

periodic if $x(t) = x(t+T)$

$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t} \cdot e^{j\omega T} = e^{j\omega t} = x(t)$
↳ if $e^{j\omega T} = 1$

$e^{j\omega T} = 1$ if
$\omega T = 2\pi k$

$e^{j2\pi k}$

since we need to find the fundamental time period (smallest interval), let K = 1

so, $\omega T = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$ fundamental time period

$$E = \int_{-\infty}^{\infty} |e^{j\omega t}|^2 dt = \int_{-\infty}^{\infty} 1\,dt = \infty$$

- CASE: 3 C ≡ complex number
  a ≡ real number

$x(t) = |c|e^{j\theta} \cdot e^{at} = |c|e^{at + j\theta}$

$e^{at}$ will define if it is growing / decaying exponential

we can also say $|c|e^{j\theta} = c\cos(\theta) + c \cdot j\sin(\theta)$

$Re\{x(t)\} = |c| \cdot \cos\theta \cdot e^{at}$

$Im\{x(t)\} = |c| \cdot \sin\theta \cdot e^{at}$

- CASE: 4 C ≡ complex number = $|c|e^{j\theta}$
  a ≡ complex number = $r + j\omega$

$x(t) = ce^{at} = |c|e^{j\theta} \cdot e^{(r+j\omega)t}$

$= \underbrace{|c|e^{rt}}_{\text{envelope with either increasing or decaying exponential}} \cdot \underbrace{e^{j(\omega t+\theta)}}_{}$

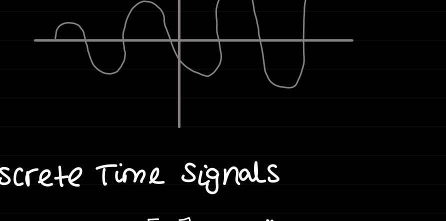
$\cos(\omega t+\theta) + j\sin(\omega t+\theta)$
cosinosoidal signal    sinosoidal signal

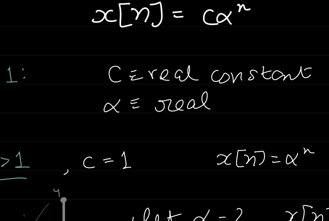
$Re\{x(t)\} = |c|e^{rt} \cdot \cos(\omega t+\theta)$

$Im\{x(t)\} = |c|e^{rt} \cdot \sin(\omega t+\theta)$

let $r = 2$, $|c| = 1.5$

$Re\{x(t)\} = \underbrace{1.5e^{2t}}\underbrace{\cos(\omega t+\theta)}$





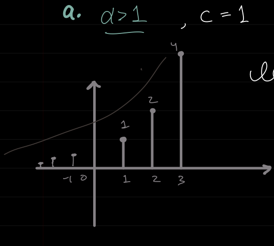
- **Discrete Time Signals**

$$x[n] = c\alpha^n$$

- Case 1: C ≡ real constant
  $\alpha$ ≡ real

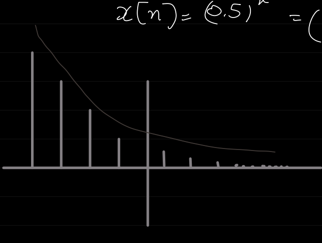
a. $\alpha > 1$, c = 1 $x[n] = \alpha^n$

let $\alpha = 2$, $x[n] = 2^n$



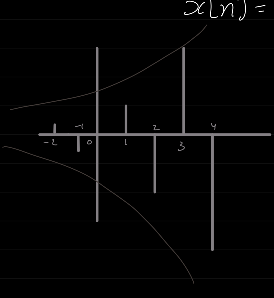
b. $\alpha < 1$ let $\alpha = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$



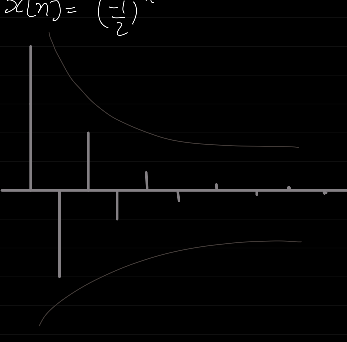
c. $\alpha < -1$ let $\alpha = -2$

$x[n] = (-2)^n$



d. $-1 < \alpha < 0$ let $\alpha = \frac{-1}{2}$

$x[n] = (\frac{-1}{2})^n$



- Case: 2 C ≡ real
  $\alpha = e^{j\omega}$

$x[n] = c\alpha^n = \underbrace{e^{j\omega n}}_{\text{let } c=1}$

if $x[n] = x[n+N]$ ∀ n

then $x[n]$ is periodic

$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N} = e^{j\omega n}$
if $e^{j\omega N} = 1$

$e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi K$

here we cannot say K = 1 for smallest interval necessarily because we want N to be an integer

$N = \frac{2\pi K}{\omega}$ ≡ has to be integer

$\frac{N}{K} = \frac{2\pi}{\omega}$ ≡ has to be rational number

Note: ω has to be a multiple of π