

Sinusoidal steady state Analysis

Given circuit diagram:

$$V_s = V_m \cos \omega t$$

$$V_s = V_m (\cos \omega t + j\theta)$$

$$i(t) = i(t) \cdot R + L \frac{di}{dt}$$

$$i = I_1 \cos \omega t + I_2 \sin \omega t$$

$$I_1 = \frac{R V_m}{R^2 + \omega^2 L^2}, \quad I_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

$$\Rightarrow i(t) = \frac{R V_m}{R^2 + \omega^2 L^2} \cos \omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2} \sin \omega t \quad \text{--- (1)}$$

(Can you write $i(t)$ in compact form? $\rightarrow i(t) = A \cos(\omega t + \phi) = A(\cos \omega t \cos \phi - \sin \omega t \sin \phi)$) --- (2)

$$\left\{ \text{Use: } \cos(C+D) = \cos C \cos D - \sin C \sin D \right\}$$

$$A = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Comparing eqn (1) & (2)

$$A \cos \phi = R V_m / (R^2 + \omega^2 L^2)$$

$$A \sin \phi = -\omega L V_m / (R^2 + \omega^2 L^2)$$

$$A^2 \cos^2 \phi + A^2 \sin^2 \phi = A^2$$

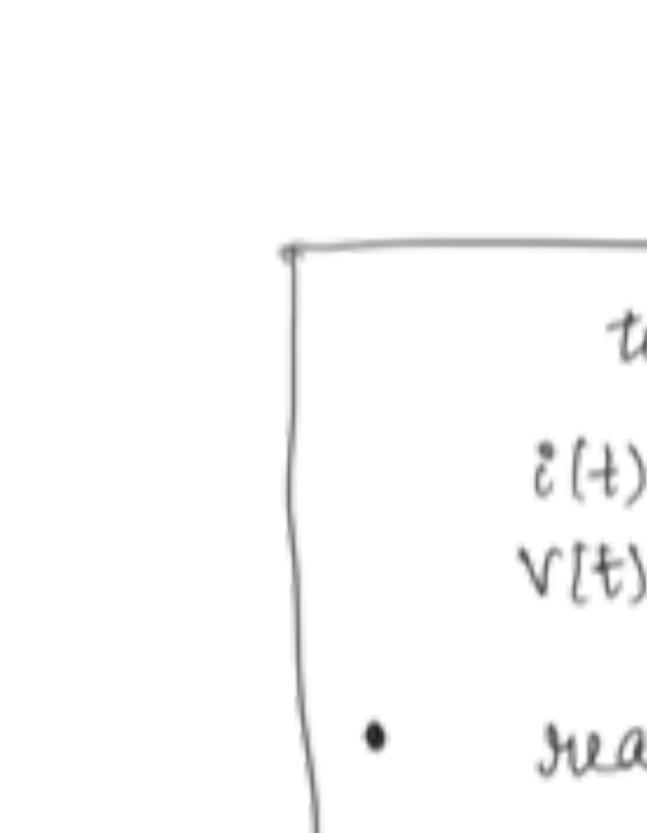
$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right))$$

Consider a complex source

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$V_m \cos \omega t + j V_m \sin \omega t = V_m e^{j\omega t}$$

$$\rightarrow \text{Real V source} = \underline{V_m \cos \omega t} = \underline{\text{Re}\{V_m e^{j\omega t}\}}$$



$$V_m e^{j\omega t} = i R + L \frac{di}{dt}$$

$$\text{Assume } i = \underline{I_m e^{j\omega t + \phi}}$$

$$\Rightarrow V_m e^{j\omega t} = R \cdot I_m e^{j\omega t + \phi} + L(j\omega) I_m e^{j\omega t + \phi}$$

$$\Rightarrow \underline{I_m e^{j\omega t + \phi}} = \frac{V_m e^{j\omega t}}{R + j\omega L}$$

$$\text{real } i(t) = \underline{\text{Re}\{I_m e^{j\omega t + \phi}\}} = \underline{\text{Re}\left\{\frac{V_m}{R + j\omega L} e^{j\omega t}\right\}}$$

$$\rightarrow \frac{V_m}{R + j\omega L} \frac{(R - j\omega L)}{(R - j\omega L)} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j \tan^{-1}\left(\frac{\omega L}{R}\right)}$$

$$\begin{array}{c} x+jy \leftrightarrow r e^{j\theta} \rightarrow r \angle \theta \\ x = r \cos \theta, \quad y = r \sin \theta \\ r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x) \end{array}$$

$$i(t) = \underline{\text{Re}\left\{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j \tan^{-1}\left(\frac{\omega L}{R}\right)} \cdot e^{j\omega t}\right\}}$$

$$= \underline{\text{Re}\left\{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right))}\right\}}$$

$$= \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right))$$

Phasor How we define phasor

$$i(t) = I_m \cos(\omega t + \phi) = \underline{\text{Re}\{I_m e^{j\omega t + j\phi}\}}$$

\downarrow remove $e^{j\omega t}$, $\underline{\text{Re}\{ \}}$

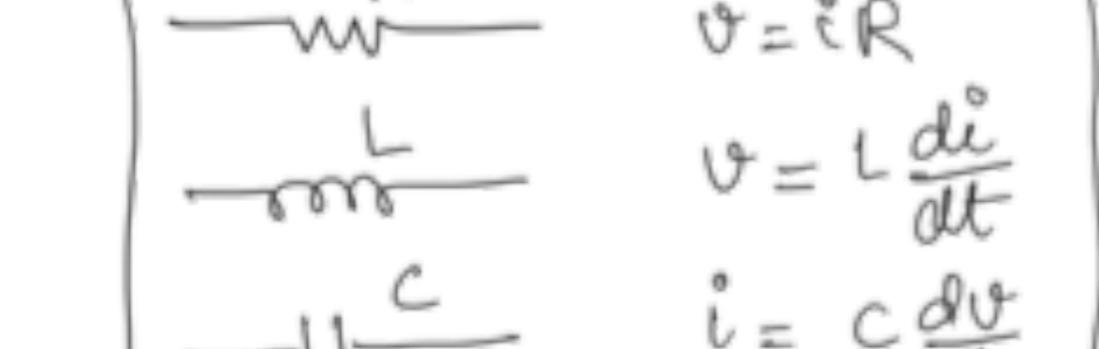
$$\underline{I_m e^{j\phi}} = \underline{I_m \angle \phi}$$

phasor

time domain	Phasor representation
$i(t) = I_m \cos(\omega t + \phi)$	$I_m \angle \phi$
$V(t) = V_m \cos(\omega t + \theta)$	$V_m \angle \theta$

- real, function of t
- notations: small letters

- complex, constant
- Bold capital letter



$$\checkmark i(t) = I_m \cos(\omega t + \phi), \quad V(t) = V_m \cos(\omega t + \theta)$$

$$\checkmark i(t) = \underline{\text{Re}\{I_m e^{j\omega t + j\phi}\}}$$

$$\times i(t) = \underline{I_m \angle \phi}$$

$$\times i(t) = \underline{I_m e^{j\omega t + \phi}}$$

$$\times i(t) = \underline{I_m \angle \omega t + \phi}$$

$$\checkmark \text{Phasor representation of } i(t) = \underline{I_m \angle \phi}$$

$$\checkmark \text{Phasor representation of } V(t) = \underline{V_m \angle \theta}$$

from previous notes

$$V_m e^{j\omega t + j\theta} = R \underline{\text{Im}\{e^{j\omega t + j\phi}\}} + L(j\omega) \underline{\text{Im}\{e^{j\omega t + j\phi}\}}$$

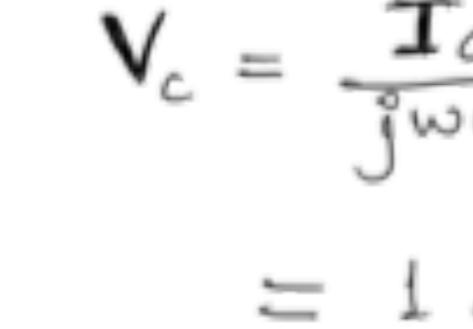
$$\Rightarrow \underline{V_m e^{j\theta}} = \underline{(R + j\omega L) I_m e^{j\phi}}$$

$$\underline{V} = \underline{(R + j\omega L) I}$$

$$\rightarrow \boxed{\underline{V} = \underline{(R + j\omega L) I}}$$

Phasor representation of relationship between i and V for R, L, C

Resistor



$$V = iR$$

Consider complex voltage & current

$$V \rightarrow V_m e^{j\omega t + j\theta}$$

$$i \rightarrow I_m e^{j\omega t + j\phi}$$

$$\Rightarrow V_m e^{j\theta} = R \underline{\text{Im}\{e^{j\omega t + j\phi}\}} + i \omega L \underline{\text{Im}\{e^{j\omega t + j\phi}\}}$$

$$\Rightarrow \underline{V_m e^{j\theta}} = \underline{R I_m e^{j\phi}}$$

$$\boxed{V = R I}$$

$$\rightarrow \boxed{V = R I}$$

Inductor

$$V = L \frac{di}{dt}$$

$$V \rightarrow V_m e^{j\omega t + j\theta}$$

$$i \rightarrow I_m e^{j\omega t + j\phi}$$

$$\Rightarrow V_m e^{j\theta} = L \underline{\text{Im}\{e^{j\omega t + j\phi}\}} + i \omega L \underline{\text{Im}\{e^{j\omega t + j\phi}\}}$$

$$\Rightarrow \underline{V_m e^{j\theta}} = \underline{j\omega L I_m e^{j\phi}}$$

$$\boxed{V = j\omega L I}$$

Capacitor

$$i = C \frac{dv}{dt}$$

$$\underline{V} = \frac{1}{j\omega C} \underline{I}$$

$$V_s = V_m e^{j\theta}$$

$$V = V_R + V_L$$

$$= I R + (j\omega L) I$$

$$= I (R + j\omega L)$$

$$I = \frac{V_s}{R + j\omega L} = \frac{V_m \angle \theta}{R + j\omega L}$$

Time domain Phasor (frequency domain)

R	$V = iR$	$V = IR$
L	$V = L \frac{di}{dt}$	$V = (j\omega L) I$
C	$i = C \frac{dv}{dt}$	$V = \frac{1}{j\omega C} I$

$$V_s = V_m e^{j\theta}$$

$$V = V_R + V_L$$

$$V = (j\omega L) I$$

$$V = \frac{1}{j\omega C} I$$

$$= V_R + V_L$$

$$= (R + j\omega L) I$$

$$= V_m e^{j\theta}$$

$$= V_m \angle \theta$$

$$= V_m \cos \theta + j V_m \sin \theta$$

$$= V_m \cos \theta + j V_m \sin \theta$$

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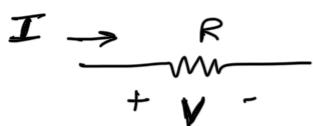
$$= V_m \cos \theta + j V_m \sin \theta$$

Phasor

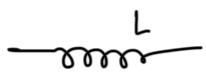
$$i(t) = I_m \cos(\omega t + \phi) = \operatorname{Re} \{ I_m e^{j\omega t + j\phi} \}$$

$$\underbrace{I_m e^{j\phi}}_{\text{phasor}} = \underbrace{I_m \angle \phi}_{\text{phasor}}$$

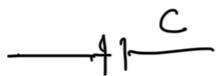
↓ Remove $e^{j\omega t}$, $\operatorname{Re} \{ \}$ ↑



$$V = IR$$



$$V = j\omega L I$$



$$V = \frac{1}{j\omega C} I$$

$$Z_R = R$$

voltage &
current in
phase

$$Z_L = j\omega L$$

voltage
leads
current
by 90°

$$Z_C = \frac{1}{j\omega C}$$

voltage lags current
by 90°

Impedance

$$Z = \frac{V}{I} \quad \Omega$$

$$Z = \underbrace{R}_{\text{resistance}} + j\underbrace{X}_{\text{reactance}}$$

Admittance

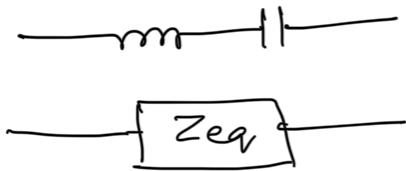
$$Y = \frac{1}{Z} ; \quad \operatorname{Re} \{ Y \} = \text{conductance} \\ \operatorname{Im} \{ Y \} = \text{susceptance}$$

Impedance Combinations

$$\text{Series : } Z_{eq} = Z_1 + Z_2 + \dots + Z_N$$

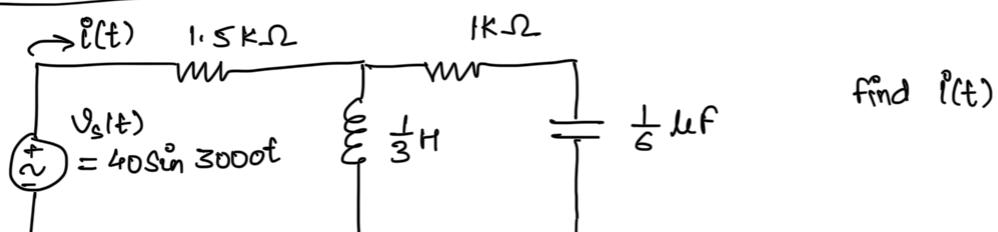
$$\text{Parallel : } \frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \dots + \frac{1}{Z_N}$$

Example: find impedance of a series combination of an inductor L ($L = 5\text{mH}$) and capacitor C ($C = 100\text{nF}$) $\omega = 10^4 \text{ rad/s}$

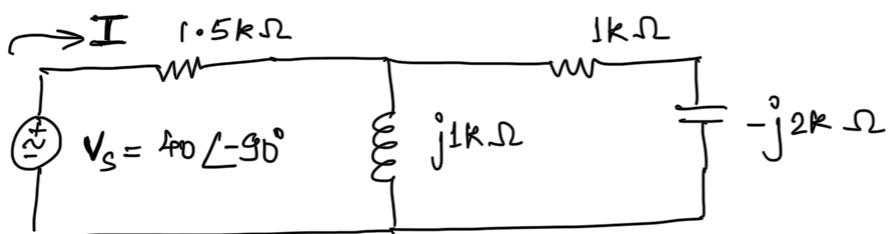


$$\begin{aligned} Z_{\text{eq}} &= Z_L + Z_C \\ &= j50 - j \\ &= j49 = 0 + j49 \\ &= 49 e^{j\pi/2} \end{aligned}$$

Example :-



Step 1 Convert to freq. domain (write in terms of phasor & Z)



$$\begin{aligned} \text{Sin } \omega t \\ &= \cos(90^\circ - \omega t) \\ &= \cos(\omega t - 90^\circ) \end{aligned}$$

$$\begin{aligned} Z_C &= \frac{1}{j\omega C} \\ &= \frac{1}{j3000 \times 10^{-6}} \\ &= \frac{2 \times 10^6}{j \times 10^3} \\ &= -j2 \text{k}\Omega \end{aligned}$$

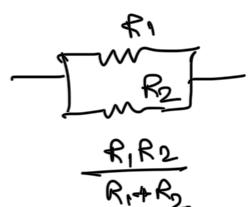
Step 2



$$I = \frac{V_s}{Z_{\text{eq}}}$$

Find Z_{eq} .

$$\begin{aligned} Z_{\text{eq}} &= 1.5 + \frac{(j1)(1-j2)}{1-j1} \text{ k}\Omega \\ &= 2.5 \angle 36.87^\circ \text{ k}\Omega \end{aligned}$$

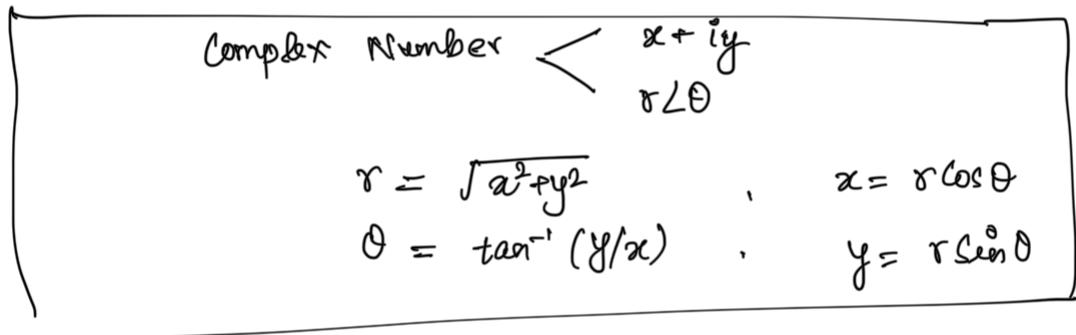


$$I = \frac{V_s}{Z_{\text{eq}}} = \frac{40 \angle -90^\circ}{2.5 \times 10^3 \angle 36.87^\circ} = 16 \angle -126.87^\circ \text{ mA}$$

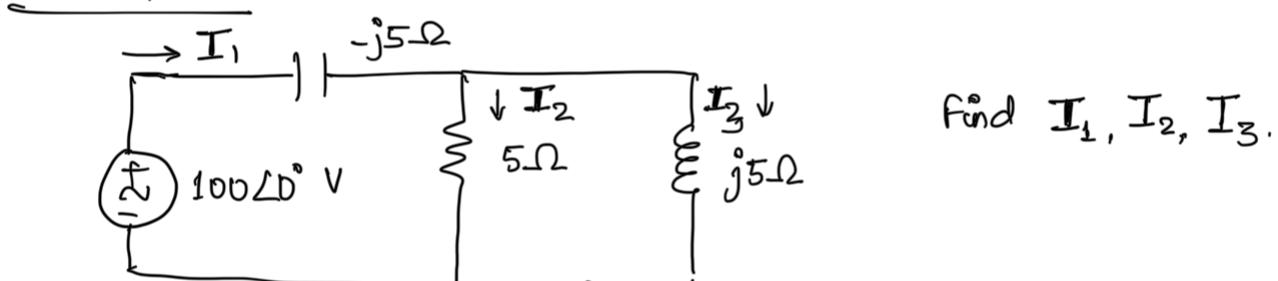
$$i(t) = \operatorname{Re} \left\{ 16 e^{-j126.87^\circ} e^{j\omega t} \right\}$$

$$= 16 \cos(\omega t - 126.87^\circ) \text{ mA}$$

↳ 3000



Example



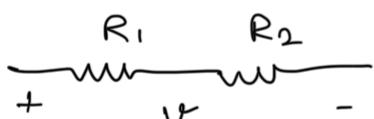
Review

Voltage Division

$$V_K = V \frac{R_K}{R_1 + \dots + R_N}$$

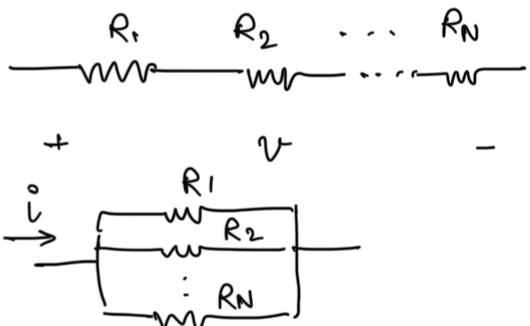
Current Division

$$I_K = i \frac{1/R_K}{1/R_1 + 1/R_2 + \dots + 1/R_N}$$



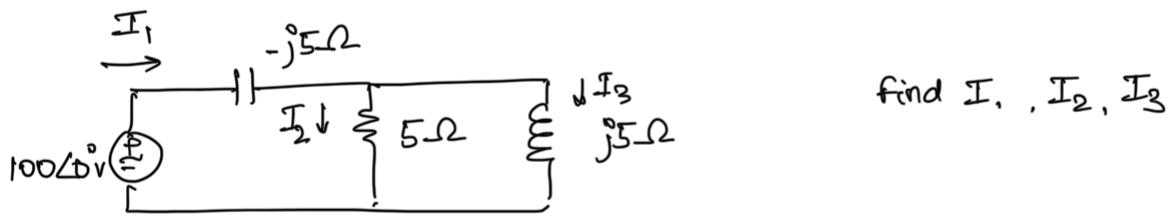
$$V_1 = V \frac{R_1}{R_1 + R_2}$$

$$V_2 = V \frac{R_2}{R_1 + R_2}$$



$$i_1 = i \frac{R_2}{R_1 + R_2}$$

$$i_2 = i \frac{R_1}{R_1 + R_2}$$

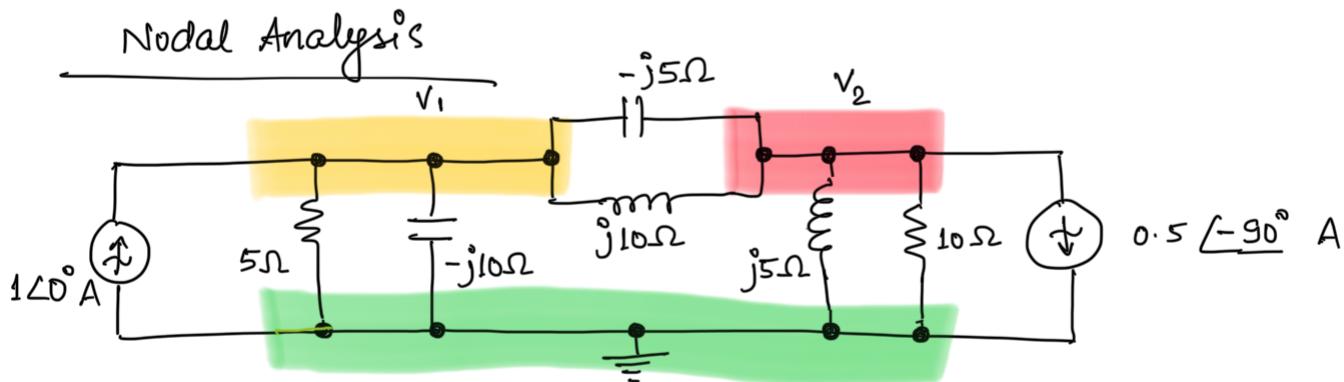


find I_1, I_2, I_3

$$I_1 = \frac{100 \angle 0^\circ}{Z_{eq.}} , \quad I_2 = I_1 \left(\frac{j5}{5+j5} \right), \quad I_3 = I_1 \left(\frac{5}{5+j5} \right)$$

$$Z_{eq.} = 3.53 \angle -45^\circ \Omega = 2.5 - j2.5 \Omega$$

$$I_1 = 28.3 \angle 45^\circ A$$



Find $V_1(t)$ and $V_2(t)$

at node 1 ;

$$1 \angle 0^\circ = \frac{V_1}{5} + \frac{V_1}{-j10} + \frac{V_1 - V_2}{-j5} + \frac{V_1 - V_2}{j10}$$

$$\Rightarrow (0.2 + j0.2)V_1 - j0.1V_2 = 1 \quad \text{--- } ①$$

at node 2 ;

$$-0.5 \angle -90^\circ = \frac{V_2}{10} + \frac{V_2}{j5} + \frac{V_2 - V_1}{-j5} + \frac{V_2 - V_1}{j10}$$

$$\Rightarrow -j0.1V_1 + (0.1 - j0.1)V_2 = j0.5 \quad \text{--- } ②$$

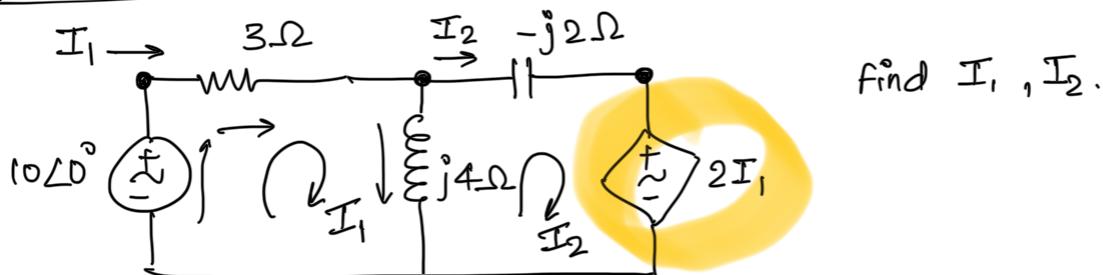
$$V_1 = 1 - j2 = 2.24 \angle -63.4^\circ V$$

$$V_2 = -2 + j4 = 4.47 \angle 116.6^\circ V$$

$$V_1(t) = 2.24 \cos(\omega t - 63.4^\circ) \text{ V}$$

$$V_2(t) = 4.47 \cos(\omega t + 116.6^\circ) \text{ V}$$

Mesh Analysis



$$-10\angle 0^\circ + 3I_1 + (I_1 - I_2)j4 = 0$$

$$(I_2 - I_1)j4 + (-j2)I_2 + 2I_1 = 0$$

$$I_1 = 1.24 \angle 29.7^\circ \text{ A}$$

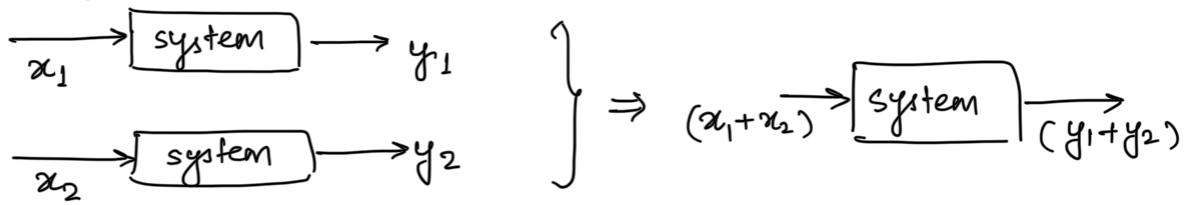
$$I_2 = 2.77 \angle 56.3^\circ \text{ A}$$

$$i_1(t) = 1.24 \cos(\omega t + 29.7^\circ) \text{ A}$$

$$i_2(t) = 2.77 \cos(\omega t + 56.3^\circ) \text{ A}$$

Superposition Principle:-

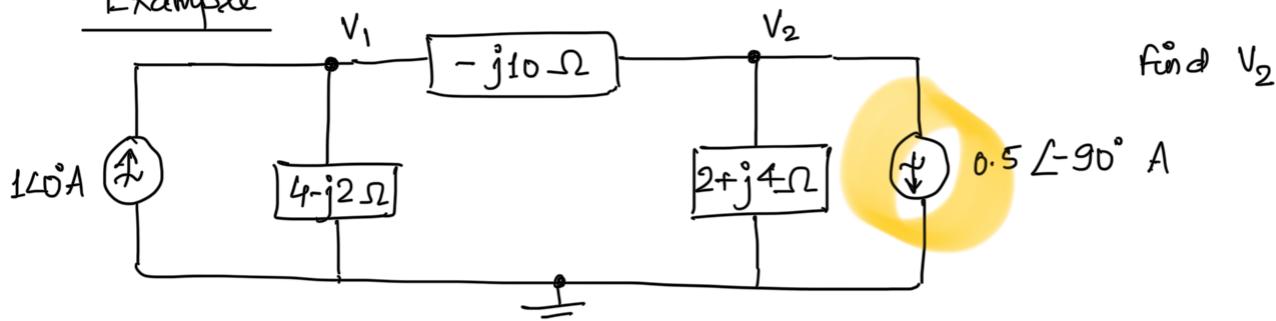
linearity \leftrightarrow superposition



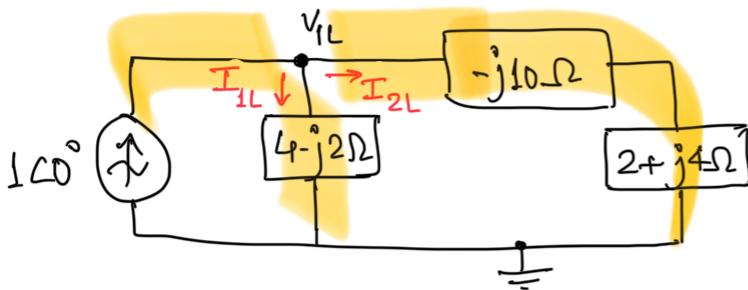
* Multiple independent sources

- consider separate independent sources "acting alone"
- add all responses.

Example



Consider left side source:

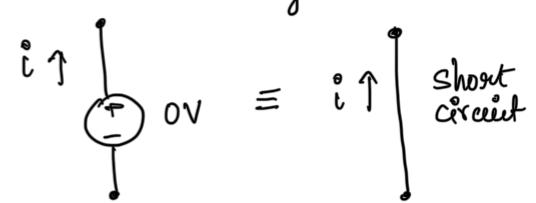
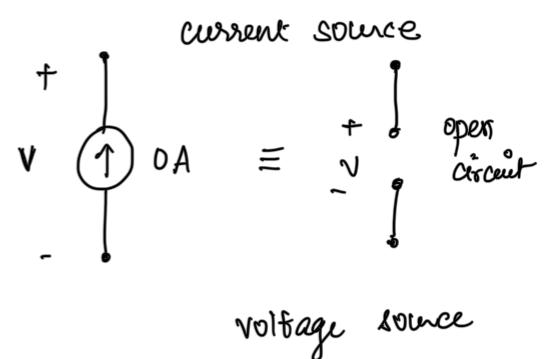


$$V_{1L} = I_{1L} (4 - j2) \quad \leftarrow \text{Ohm's law}$$

$$= 1\angle 0^\circ \frac{(-j10 + 2 + j4)}{(4 - j2) + (-j10 + 2 + j4)} (4 - j2)$$

current division

$$= \frac{-4 - j28}{6 - j8} = 2 - j2 \text{ V}$$

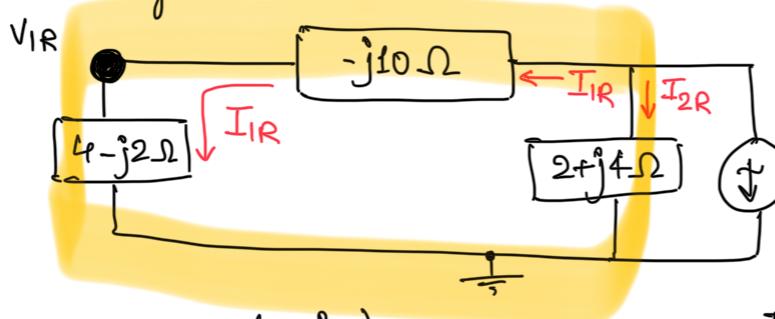


$1\angle 0^\circ$

$$= 1 \cos 0^\circ + j \sin 0^\circ$$

$$= 1$$

Consider right side source

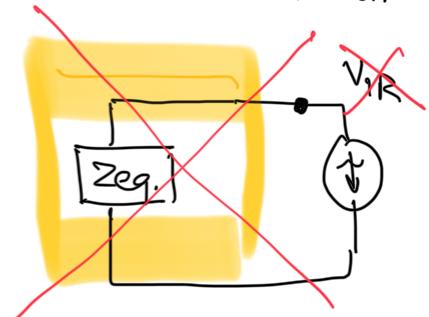


$$V_1 = V_{IL} + V_{IR}$$

$$0.5 \angle -90^\circ A$$

$$\begin{aligned} V_{IR} &= I_{IR} (4-j2) \\ &= \frac{(j0.5)(2+j4)(4-j2)}{(6-j0)} \\ &= -1 V \end{aligned}$$

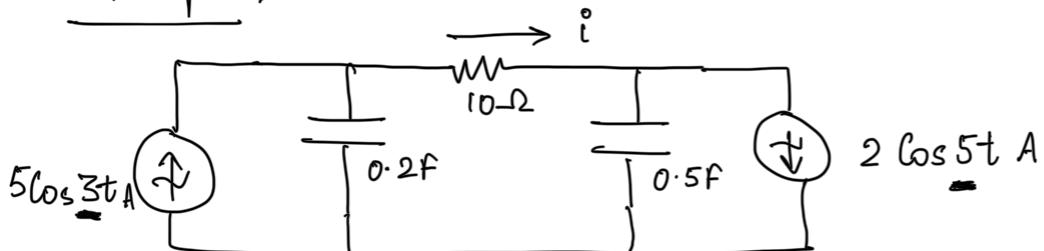
$$I_{IR} = \underbrace{\frac{(-0.5 \angle -90^\circ)(2+j4)}{(-j10 + 4-j2 + 2+j4)}}_{\text{current division}}$$



Applying superposition: $V_1 = V_{IL} + V_{IR}$

$$\begin{aligned} &= 2-j2 - 1 \\ &= 1-j^2 \quad V \end{aligned}$$

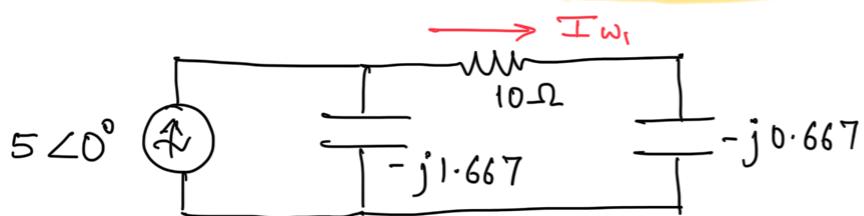
Example :-



find $i(t)$

$$\frac{1}{j\omega C}$$

Consider left source: $\omega_1 = 3 \text{ rad/s}$

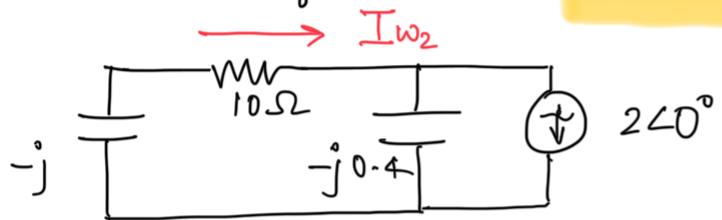


$$\frac{1}{j(3)(0.2)} = \frac{-j}{0.6}$$

$$\frac{1}{j(3)(0.5)} = \frac{-j}{1.5}$$

$$I_{\omega_1} = 5 \angle 0^\circ \left(\frac{-j1.667}{-j1.667 + 10 - j0.667} \right) = 811.7 \angle -76.86^\circ \text{ mA}$$

Consider right source: $\omega_2 = 5 \text{ rad/s}$



$$\frac{1}{j(5)(0.2)} = -j$$

$$\frac{1}{j(5)(0.5)} = -j0.4$$

$$I_{\omega_2} = +2\angle 0^\circ \left(\frac{-j0.4}{-j+10 -j0.4} \right) = 79.2 \angle -82.03^\circ \text{ mA}$$

Applying Superposition:

$$\sum I = \cancel{\overline{I_{\omega_1}} + I_{\omega_2}} \quad \cancel{\overline{3 \text{ rad/s}} + \overline{5 \text{ rad/s}}}$$

$$\overset{\circ}{i}(t) = \overset{\circ}{i}_{\omega_1}(t) + \overset{\circ}{i}_{\omega_2}(t)$$

Phasor

$$\begin{aligned} i(t) &= \text{Im } \cos(\omega t + \phi) \\ &= \text{Re} \{ \text{Im } e^{j\omega t + j\phi} \} \\ &\quad \downarrow \text{Remove } e^{j\omega t} \quad \text{Re} \{ \cdot \} \\ \text{Im } e^{j\phi} &= \text{Im } \angle \phi \end{aligned}$$

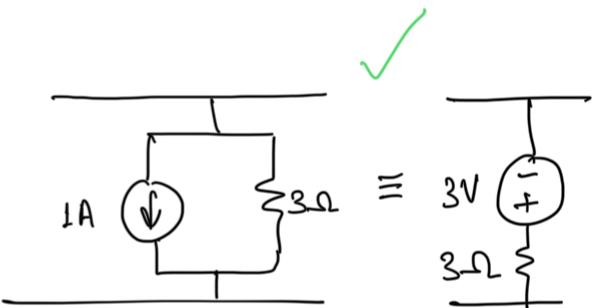
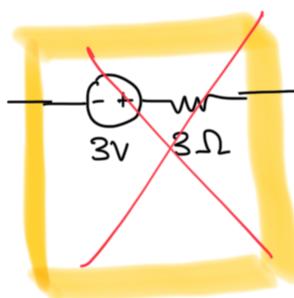
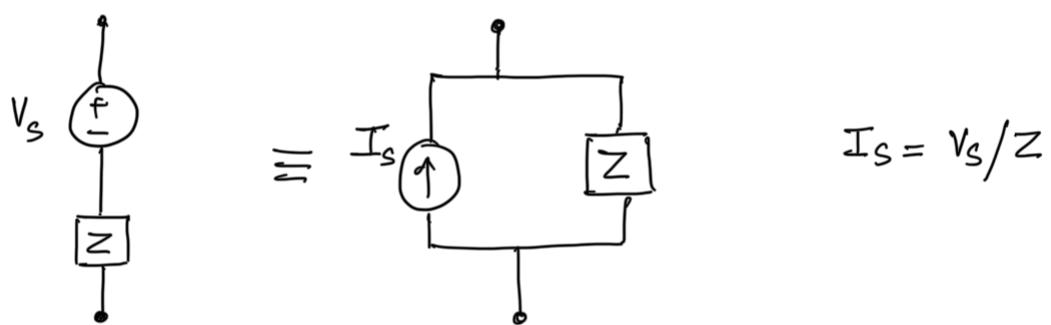
$$\overset{\circ}{i}_{\omega_1}(t) = 811.7 \cos(3t - 76.86^\circ) \text{ mA}$$

$$\overset{\circ}{i}_{\omega_2}(t) = 79.23 \cos(5t - 82.03^\circ) \text{ mA}$$

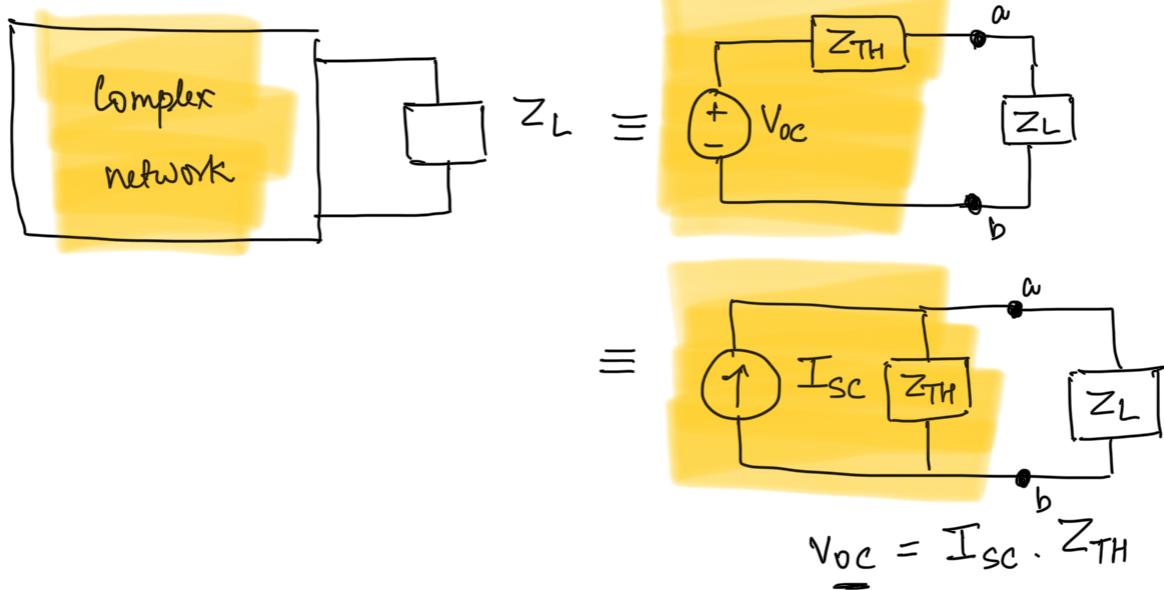
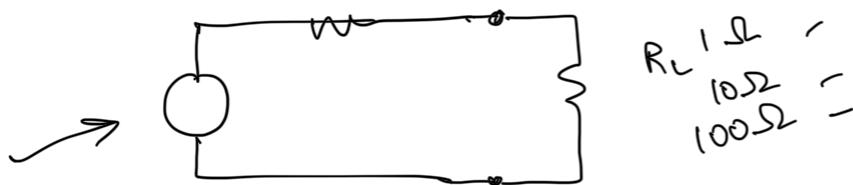
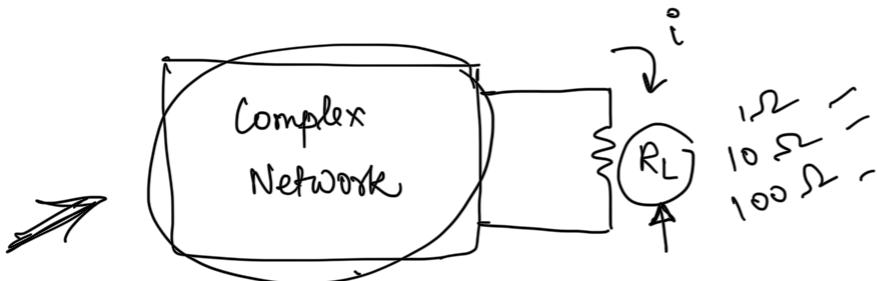
$$\overset{\circ}{i}(t) = [811.7 \cos(3t - 76.86^\circ) + 79.23 \cos(5t - 82.03^\circ)] \text{ mA}$$

Phasors can only be added when they correspond to same frequency

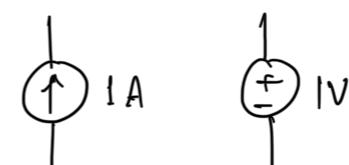
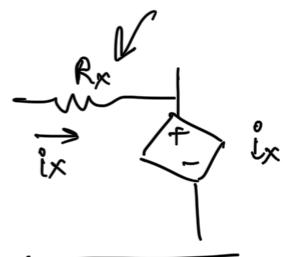
Source Transformation :-



Thevenin & Norton Equivalent Circuits :



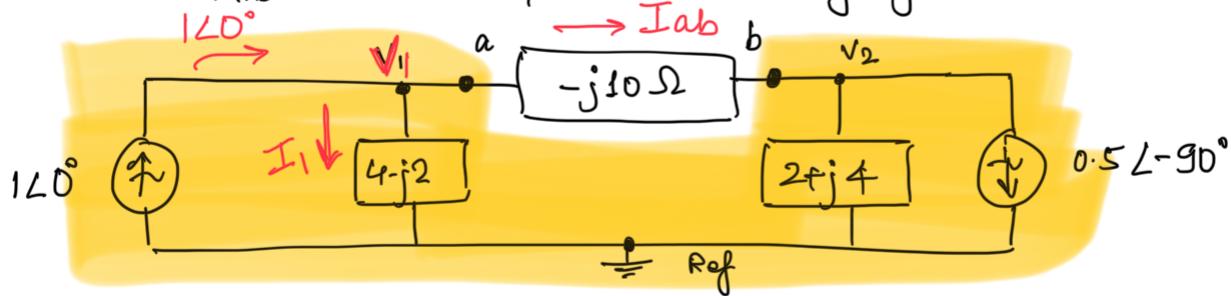
- | | V_{oc} | I_{sc} | Z_{th} |
|---|----------|----------|----------|
| ① Independent sources only | ✓ | ✓ | ✓ |
| ② Independent + dependent sources | ✓ | ✓ | ✓ |
| ③ Dependent sources only \Rightarrow Test sources | | | |



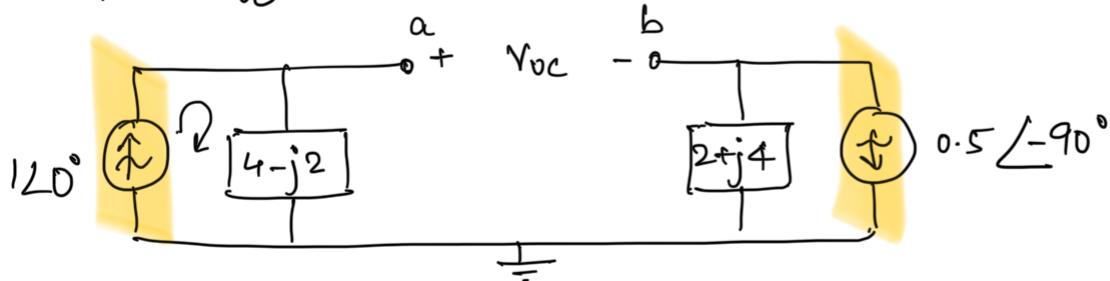
$$V_{oc}, I_{sc}, Z_{th} = V_{oc} / I_{sc}$$

Example:

Find Thvenin Equivalent seen by $-j10\Omega$. Find V_1 .



find V_{oc}



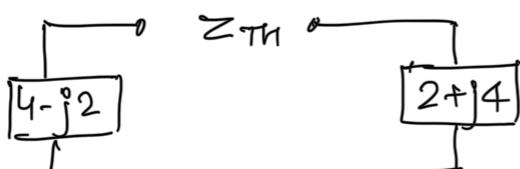
$$V_{oc} = V_a - V_b = 1\angle 0^\circ (4-j2) - (-0.5 \angle -90^\circ) (2+j4)$$

$$= 4-j2 + 2-j1 = 6-j3 \text{ V}$$

find Z_{th}

$v_s \rightarrow$ short circuit

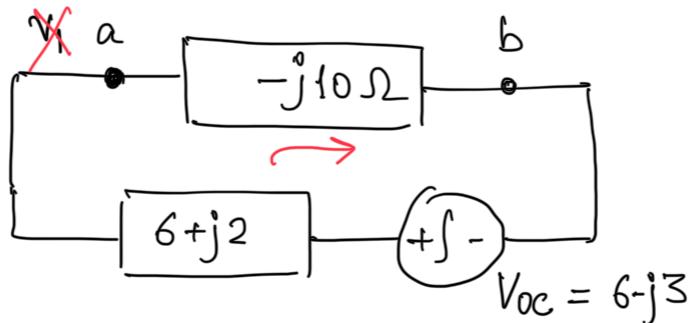
$I_s \rightarrow$ open circuit



$$Z_{th} = (4-j2) + (2+j4)$$

$$= 6+j2 \Omega$$

Equivalent Circuit



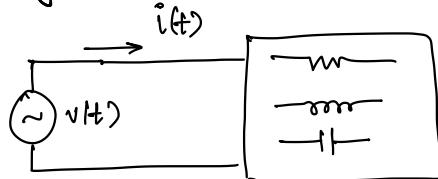
$$I_{ab} = \frac{V_{oc}}{(6+j2) + (-j10)}$$

$$I_1 = 1 \angle 0^\circ - I_{ab}$$

$$\text{Then } V_1 = I_1 (4-j2) = 1-j2 \text{ V}$$

Module 3

AC Power Analysis



$$\cos A \cdot \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

Instantaneous Power : $p(t) = v(t) i(t)$

$$v(t) = V_m \cos(\omega t + \theta)$$

$$i(t) = I_m \cos(\omega t + \phi)$$

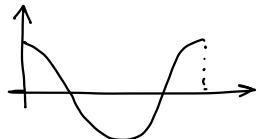
$$p(t) = V_m \cos(\omega t + \theta) \cdot I_m \cos(\omega t + \phi)$$

$$= \underbrace{\frac{V_m I_m}{2} \cos(\theta - \phi)}_{\text{constant}} + \underbrace{\frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi)}_{\text{twice frequency (harmonic)}}$$

Average Power :

$$\begin{aligned} P_{avg} &= \frac{1}{T} \int_0^T p(t) dt \\ &= \underbrace{\frac{1}{T} \int_0^T \frac{V_m I_m}{2} \cos(\theta - \phi) dt}_{\text{constant}} + \underbrace{\frac{1}{T} \int_0^T \frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi) dt}_{0} \\ &= \frac{V_m I_m}{2} \cos(\theta - \phi) + 0 \end{aligned}$$

Average of \cos



$$P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi)$$

$$P_{avg} = \frac{1}{2} \operatorname{Re} \{ \mathbf{V} \mathbf{I}^* \}$$

$$\mathbf{I} = I_m e^{j\phi}$$

$$\mathbf{I}^* = I_m e^{-j\phi}$$

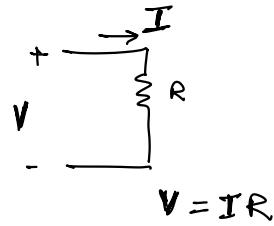
~~$P_{avg} = \mathbf{V} \mathbf{I}^*$~~

No phasor for power!

Average power absorbed by a
purely resistive element

$$P_{avg} = \frac{V_m I_m}{2} \cos(\Theta - \phi)$$

\uparrow \uparrow



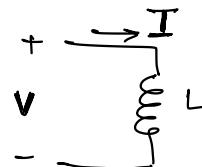
$$P_{avg} = \frac{V_m I_m}{2} = \frac{I_m^2 R}{2} = \frac{V_m^2}{2R}$$

Purely inductive element

$$P_{avg} = \frac{V_m I_m}{2} \cos(\Theta - \phi)$$

$$= \frac{V_m I_m}{2} \cos(90^\circ)$$

$P_{avg} = 0$



$$V = j\omega L I$$

$$(\Theta - \phi) =$$

$$\Rightarrow V_m e^{j\Theta} = j\omega L I_m e^{j\phi}$$

$$\Rightarrow \underbrace{\frac{V_m}{I_m}}_{e^{j\Theta-\phi}} = j\omega L \quad \Theta - \phi = 90^\circ$$

Purely capacitive element

$$P_{avg} = \frac{V_m I_m}{2} \cos(\Theta - \phi)$$

$$\Theta - \phi = -90^\circ$$

$P_{avg} = 0$

$p(t)$

$$P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi) \quad \checkmark$$

$$P_{avg} = \operatorname{Re} \left\{ \frac{V I^*}{2} \right\}$$

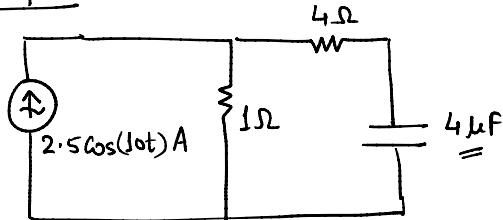
$$v(t) = V_m \cos(\omega t + \theta)$$

$$v(t) = \operatorname{Re} \left\{ V_m e^{j\omega t + j\theta} \right\}$$

$$e^{j\omega t}$$

$$V = V_m \angle \theta$$

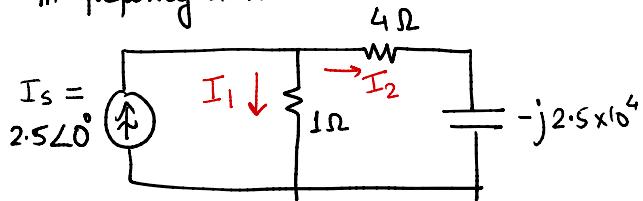
Example



① Find power delivered to each element at $t = 0, 10, 20$ ms

② Find average power delivered to each element.

In frequency domain



Power delivered to 1Ω

$$P_{1\Omega}(t) = V_{1\Omega}(t) i_{1\Omega}(t)$$

$$I_{1\Omega} = I_1 = I_S \frac{\frac{4-j2.5 \times 10^4}{1+4-j2.5 \times 10^4}}{\approx I_S \Rightarrow i_{1\Omega}(t) = 2.5 \cos(10t) A}$$

$$V_{1\Omega} = (I_1)(1\Omega) \Rightarrow V_{1\Omega}(t) = 2.5 \cos(10t) V$$

$$P_{1\Omega}(t) = 2.5 \cos(10t) \cdot 2.5 \cos(10t)$$

$$P_{1\Omega}(t) = 3.125 + 3.125 \cos(20t) \text{ Watts}$$

$$P_{1\Omega}(t=0) = 6.25 \text{ W}$$

$$P_{1\Omega}(t=10\text{ms}) = 6.1877 \text{ W}$$

$$P_{1\Omega}(t=20\text{ms}) = 6.0033 \text{ W}$$

Average power delivered to 1Ω

$$P_{avg, 1\Omega} = 3.125 \text{ W}$$

Power delivered to 4Ω :-

$$P_{4\Omega}(t) = V_{4\Omega}(t) \cdot i_{4\Omega}(t)$$

$$I_{4\Omega} = I_2 = I_s \frac{1}{S - j2.5 \times 10^4} = 10^4 \angle 90^\circ A \rightarrow i_{4\Omega}(t)$$

$$V_{4\Omega} = 4(I_{4\Omega}) = 4 \times 10^4 \angle 90^\circ V \rightarrow v_{4\Omega}(t)$$

$$P_{4\Omega}(t) = 2 \times 10^{-8} + 2 \times 10^{-8} \cos(20t) W$$

$$P_{4\Omega}(t=0) = 4 \times 10^{-8} W$$

$$P_{4\Omega}(t=10ms) = 3.96 \times 10^{-8} W$$

$$P_{4\Omega}(t=20ms) = 3.84 \times 10^{-8} W$$

$$P_{avg, 4\Omega} = 2 \times 10^{-8} W$$

Power delivered to capacitor :-

$$P_C(t) =$$

$$P_{avg, C} =$$

$$I_c = I_2 = 10^4 \angle 90^\circ A \leftarrow \rightarrow i_c(t) \Rightarrow p(t) = i_c(t)v_c(t) \checkmark$$

$$V_c = (j2.5 \times 10^4) I_2 = 2.5 V \rightarrow v_c(t)$$

$$\underbrace{i_c \times v_c}_{\text{Phasor of power}} \rightarrow p(t)$$

$$i_c(t) = 10^4 \cos(10t + 90^\circ)$$

$$v_c(t) = 2.5 \cos(10t)$$

$$P_C(t) = 1.25 \times 10^{-4} \cos(20t + 90^\circ) W$$

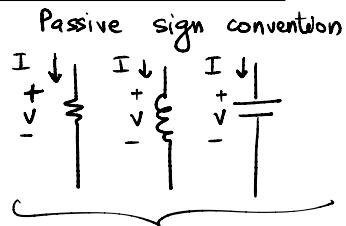
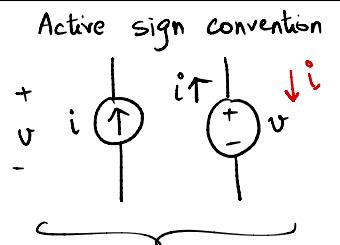
$$P_C(t=0) = 0 W$$

$$P_C(t=10ms) = 2.48 \times 10^{-5} W$$

$$P_C(t=20ms) = 4.86 \times 10^{-5} W$$

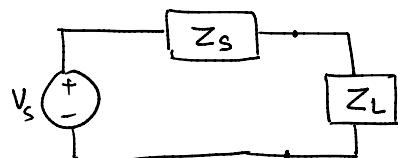
$$P_{avg, C} = 0$$

- ① $P_{1\Omega}(t) + P_{4\Omega}(t) + P_C(t) = \text{constant 1}$ X
- ② $P_{\text{avg}, 1\Omega} + P_{\text{avg}, 4\Omega} + P_{\text{avg}, C} = \text{constant 2}$ \checkmark $= P_{\text{avg, source}}$



$$v \cdot i = -20 \text{ W}$$

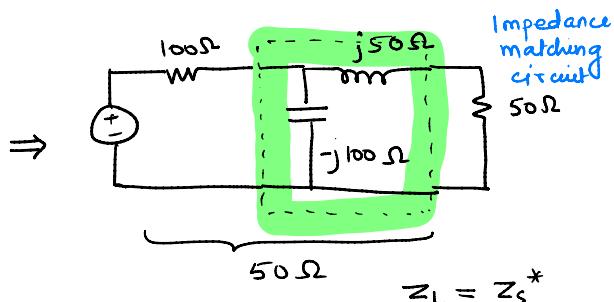
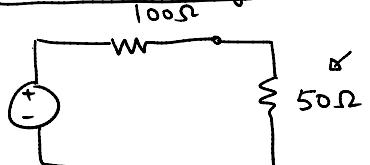
Maximum Power Transfer :



$$R_s = R_L$$

Max power delivered to load,
when $Z_L = Z_s^*$

Impedance Matching :-



Effective value (RMS value) of voltage and current:-

$$v(t) \xrightarrow{i(t)} R \quad \xrightarrow{V_{\text{eff}}} \quad I_{\text{eff}} \xrightarrow{R}$$

$$v(t) \leftarrow P_{\text{avg}} = P_{\text{avg}}$$

$$\Rightarrow \frac{1}{T} \int_0^T i^2(t) R dt = I_{\text{eff}}^2 R = \frac{V_{\text{eff}}^2}{R}$$

$$\Rightarrow I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T i^2(t) dt} = \text{Root mean square}$$

Sinusoidally varying sources

$$i(t) = I_m \cos(\omega t + \phi)$$

$$I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T I_m^2 \cos^2(\omega t + \phi) dt}$$

$$I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$$

$2 \cos^2 x = \cos 2x + 1$

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Suppose, $I = 10 \angle 90^\circ A$

$$\Rightarrow I = \underbrace{\frac{10}{\sqrt{2}}}_{\text{rms}} \angle 90^\circ A \text{ rms}$$

$$V_{\text{eff}} = \frac{V_m}{\sqrt{2}} \text{ rms value}$$

$$V_1 = 100 \angle -60^\circ V \text{ rms}$$

$$V_1 = 100\sqrt{2} \angle -60^\circ V$$

Apparent Power

$$P_{\text{apparent}} = I_{\text{eff}} V_{\text{eff}}$$

Power factor

$$\text{PF} = \frac{\text{average power}}{\text{apparent power}} = \frac{(V_m I_m / 2) \cos(\theta - \phi)}{V_{\text{eff}} I_{\text{eff}}} = \cos(\theta - \phi)$$

$$\text{PF} = \underbrace{\cos(\theta - \phi)}_{\text{Power factor angle}}$$

Review

Instantaneous Power : $p(t) = v(t) i(t)$

Average Power : $P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi) = \frac{1}{2} \operatorname{Re}\{VI^*\}$

$$P_{avg, sources} = \sum P_{avg, elements}$$

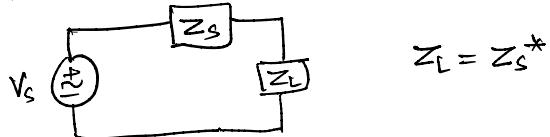
P_{avg} for resistive load : $P_{avg} \neq 0$

P_{avg} for reactive load : $P_{avg} = 0$

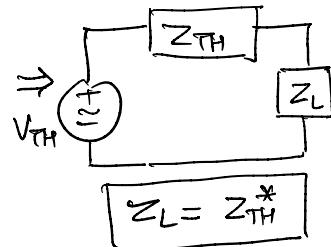
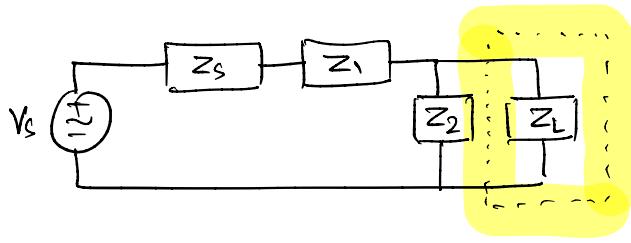
$$Z = R + jX$$

↑ resistive
↓ reactive

Max. power transfer :



$$Z_L = Z_S^*$$



$$Z_L = Z_{TH}^*$$

Effective V and I values (RMS values)

$$I_{eff} = \sqrt{\frac{1}{T} \int_0^T i^2(t) dt}$$

$$i(t) = I_m \cos(\omega t + \phi)$$

$$I_{eff} = \frac{I_m}{\sqrt{2}}, \quad V_{eff} = \frac{V_m}{\sqrt{2}}$$

$$I = I_m \angle \phi = \sqrt{2} I_{eff} \angle \phi$$

Apparent Power

$$P_{apparent} = I_{eff} \cdot V_{eff}$$

$$\begin{array}{ll} I_{eff} & A_{rms} \\ V_{eff} & V_{rms} \\ \hline VA & (\text{volt-Ampere}) \end{array}$$

Power Factor

$$PF = \cos(\theta - \phi) = \frac{\text{Power}}{\text{Apparent}}$$

for purely resistive load: $PF = 1$ (max)

for purely reactive load: $PF = 0$ (min)

$$\begin{array}{ll} \theta - \phi & 90^\circ \\ \theta - \phi & -90^\circ \end{array}$$

Assume PF of a load = 0.5

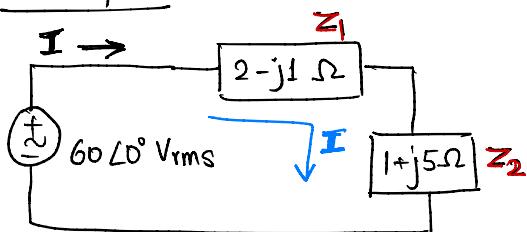
$$\cos(\theta - \phi) = 0.5$$

$$\Rightarrow (\theta - \phi) = \pm 60^\circ$$

$PF = 0.5$ leading \rightarrow capacitive $(\theta - \phi) < 0$
 0.5 lagging \rightarrow inductive $(\theta - \phi) > 0$

I w.r.t. V

Example

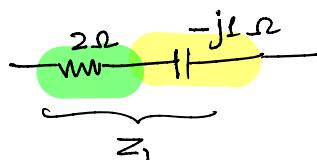


Find

- ① Average power delivered to each loads
- ② Average power supplied by source
- ③ Apparent power supplied by source
- ④ PF of combined load.

$$\text{① } P_{\text{avg}, z_1} = \frac{1}{2} V_m I_m \cos(\theta - \phi) = \frac{1}{2} \text{Re}\{VI^*\}$$

$$= I_{\text{eff}}^2 R$$



$$I =$$

Writing KVL:

$$- 60\angle 0^\circ V_{\text{rms}} + I (2 - j1 + 1 + j5) = 0$$

$$\Rightarrow I = \frac{60\angle 0^\circ V_{\text{rms}}}{3 + j4 \Omega} = \underbrace{12}_{I_{\text{eff}}} \angle -53.13^\circ \text{ A rms}$$

$$P_{\text{avg}, z_1} = (12)^2 2 = 288 \text{ W}$$

$$I = 12 \angle -53^\circ \text{ A rms}$$

$$i(t) = \frac{12}{I_{\text{eff}}} \cos(\omega t - 53^\circ) \text{ A rms}$$

I_{eff} (A rms) versus I_m (A)

$$I_{\text{eff}} = \frac{I_m}{\sqrt{2}} \Rightarrow I_m = I_{\text{eff}} \sqrt{2}$$

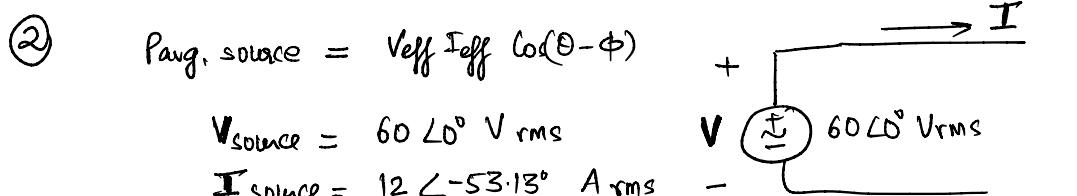
$$I_{\text{eff}} = 12$$

$$I_m = I_{\text{eff}} \sqrt{2} = \frac{12\sqrt{2}}{1}$$

$$i(t) = \frac{12\sqrt{2}}{I_m} \cos(\omega t - 53^\circ) \text{ A}$$

$$P_{\text{avg}} = \frac{1}{2} V_m I_m \cos(\theta - \phi) = V_{\text{eff}} I_{\text{eff}} \cos(\theta - \phi)$$

$$P_{\text{avg}, z_2} = I_{\text{eff}}^2 R = (12)^2 1 = 144 \text{ W}$$



$$P_{\text{avg, source}} = (60)(12) \cos(0 - (-53.13^\circ)) = 432 \text{ W}$$

$$P_{\text{avg}, z_1} = 288 \text{ W}, \quad P_{\text{avg}, z_2} = 144 \text{ W}$$

$$P_{\text{avg}, z_1} + P_{\text{avg}, z_2} = 432 \text{ W}$$

③ $P_{\text{apparent, source}} = I_{\text{eff}} V_{\text{eff}} = (60)(12) = 720 \text{ VA}$

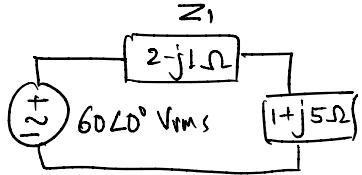
④ PF of combined loads = PF of source

$$\text{PF} = \cos(\theta - \phi) = \cos(\underbrace{0^\circ + 53.13^\circ}_{(\theta - \phi)} = 0.6 \text{ lagging}$$

$(\theta - \phi) > 0 \rightarrow \text{lagging}$

$(\theta - \phi) < 0 \rightarrow \text{leading}$

$$P_{avg,z_1} = I_{eff}^2 R$$



$$P_{avg,z_1} = V_{eff} I_{eff} \cos(\theta - \phi)$$

$$V_{Z_1} = 60 \angle 0^\circ \text{ Vrms} \times \frac{2 - j1}{3 + j4}$$

$$= 26.83 \angle -79.7^\circ \text{ Vrms}$$

$$I_{z_1} = I = 12 \angle -53.13^\circ \text{ Arms}$$

$$P_{\text{avg}, Z_1} = 26.83 \times 12 \times \cos(-79.7^\circ - (-53.13^\circ)) = 288 \text{ W}$$

$$\text{Parage} = \frac{1}{2} \operatorname{Re} \{ \mathbf{V} \mathbf{I}^* \} ; \quad \mathbf{V} = V_m \angle \theta \\ \mathbf{I} = I_m \angle \phi$$

$$V_{eff} = V_{eff} \angle \theta = \frac{V_m}{\sqrt{2}} \angle \theta$$

$$I_{eff} = I_{eff} \angle \phi = \frac{Im}{\sqrt{2}} \phi$$

$$P_{\text{average}} = \operatorname{Re} \left\{ \mathbf{V}_{\text{eff}}^* \mathbf{I}_{\text{eff}} \right\}$$

Complex Power

$$S = V_{\text{eff}} I_{\text{eff}}^*$$

 { Not a phasor of power.
This is a complex number.

$$S = (V_{eff} \angle \theta^\circ) (I_{eff} \angle -\phi) = V_{eff} \cdot I_{eff} e^{j(\theta-\phi)}$$

$$S = P + jQ$$

$$|S| = V_{\text{eff}} I_{\text{eff}} = \text{Papparent (VA)}$$

$$Z = R + jX$$

Purely resistive load $Z = R, X = 0$

$$S = P + jQ \rightarrow V_{\text{eff}} I_{\text{eff}} \frac{\sin(\theta - \phi)}{0}$$

\downarrow
 $V_{\text{eff}} I_{\text{eff}} \cos(\theta - \phi)$

$$P \neq 0$$

$$Q = 0$$

Purely reactive load $Z = jX, R = 0$

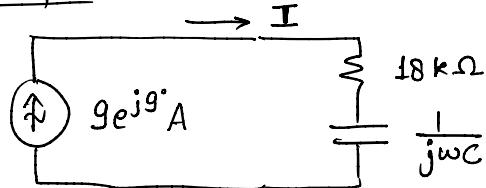
$$S = P + jQ$$

$$P = 0$$

$$Q \neq 0$$

Q signifies energy flow rate into or out of reactive component of load.

Example



Assume $C = 1 \mu\text{F}, \omega = 45 \text{ rad/s}$

Find:

- (1) Complex power provided by source
- (2) Time-average power absorbed by combined load.

Find

- (3) Reactive power absorbed by combined load
- (4) Apparent power absorbed by " "
- (5) Power factor of " "

$$S = 1.158 \times 10^6 \angle -50.99^\circ \text{ VA}$$

$$P_{\text{avg}} = 7.29 \times 10^5 \text{ W}$$

$$Q = -9 \times 10^5 \text{ VAR}$$

$$|S| = P_{\text{apparent}} = 1.158 \times 10^6 \text{ VA}$$

$$\text{PF} = 0.629 \text{ leading}$$

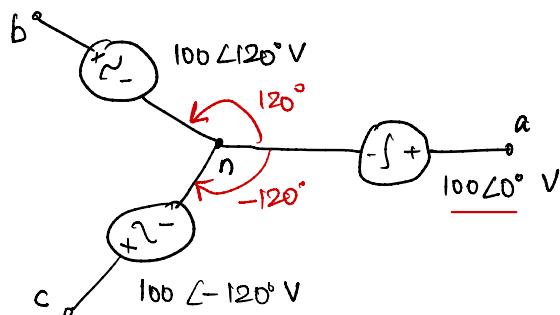
$$p(t) = 7.29 \times 10^5 + 1.158 \times 10^6 \cos(90t - 33^\circ) \text{ W}$$

Module 4

Polyphase Circuits

(Chapter 12 of textbook)

Three-phase source



$$V_{an} = 100\angle 0^\circ \text{ V}$$

$$V_{bn} = 100\angle 120^\circ \text{ V}$$

$$V_{cn} = 100\angle -120^\circ \text{ V}$$

$$\left. \begin{aligned} |V_{an}| &= |V_{bn}| = |V_{cn}| \\ \rightarrow V_{an} + V_{bn} + V_{cn} &= 0 \end{aligned} \right\} \text{Balanced Source}$$

Freq. Domain

$$100\angle 0^\circ$$

$$100\angle 120^\circ$$

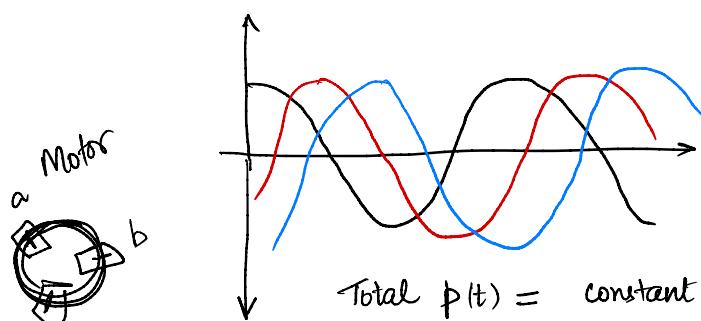
$$100\angle -120^\circ$$

Time Domain

$$100 \cos(\omega t)$$

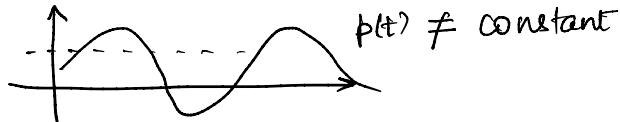
$$100 \cos(\omega t + 120^\circ)$$

$$100 \cos(\omega t - 120^\circ)$$



Total $p(t) = \text{constant}$ for three-phase sources

$$\text{for single phase source: } p(t) = \frac{1}{2} V_m I_m \cos(\theta - \phi) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta - \phi)$$

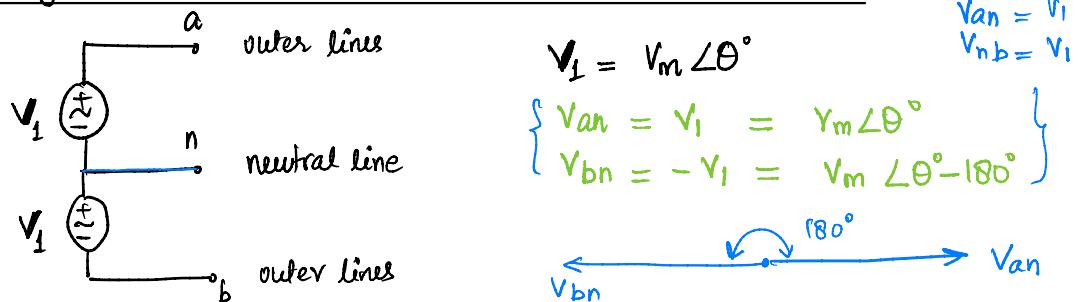


Advantages of three-phase sources are

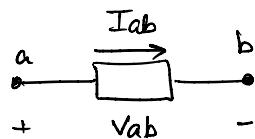
- constant power and constant torque to large motors.
- more economical since motors are more efficient.

$\theta^\circ, \theta - 180^\circ$

Single-Phase Three-Wire Source (Two-phase source)



Double Script Notations:



V_{an} = voltage at point a w.r.t. voltage at point n

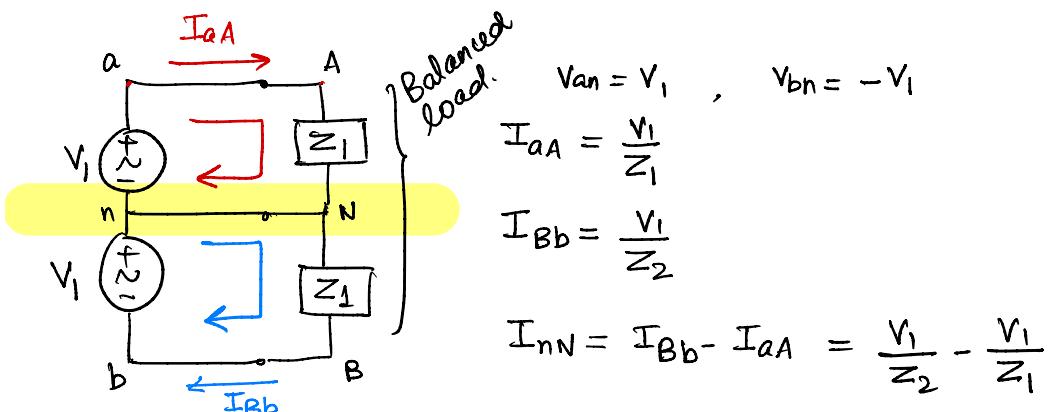
$$V_{ab} = V_{an} - V_{bn} = V_{an} + V_{nb}$$

$$V_{bn} = -V_{nb}$$

I_{ab} = current flowing from point a to point b.

Balanced Single-Phase Three-Wire Source:

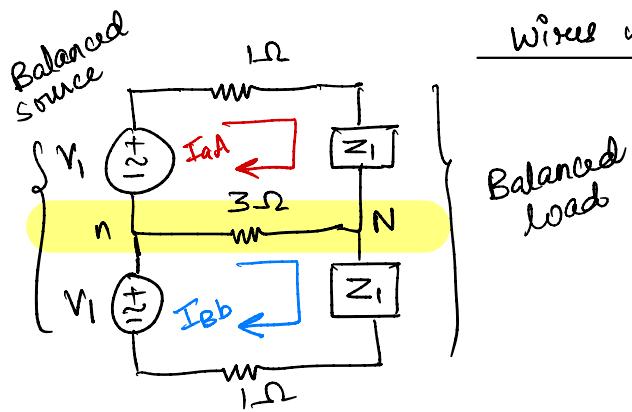
$$\left. \begin{array}{l} |V_{an}| = |V_{bn}| \\ V_{an} + V_{bn} = 0 \end{array} \right\} \text{Balanced source}$$



Assume $Z_1 = Z_2$ (Balanced load)

$$I_{AN} = \frac{V_1}{Z_1} - \frac{V_1}{Z_1} = 0$$

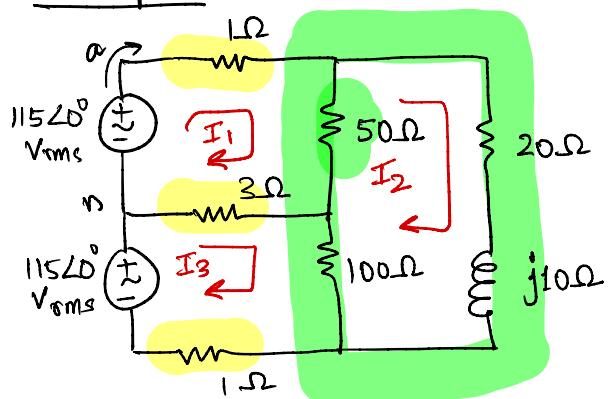
For balanced source } current in neutral line $I_{AN} = 0$
and balanced load }



Wires with non-zero resistance

$$I_{BN} = 0 = I_{Bb} - I_{aA}$$

Example



Line resistances

loads

Find

- ① Average power delivered to each of three loads
- ② Avg power lost in outerlines and neutral line.
- ③ Avg power provided by source.

Using mesh analysis

$$-115\angle 0^\circ + 1(I_1) + 50(I_1 - I_2) + 3(I_1 - I_3) = 0$$

$$50(I_2 - I_1) + (20 + j10)I_2 + 100(I_2 - I_3) = 0$$

$$-115\angle 0^\circ + (I_3 - I_1)3 + (I_3 - I_2)100 + I_3(1) = 0$$

$$\rightarrow I_1 = 11.24 \angle -19.83^\circ \text{ Arms}$$

$$I_2 = 9.389 \angle -24.47^\circ \text{ Arms}$$

$$I_3 = 10.37 \angle -21.80^\circ \text{ Arms}$$

$$I_1 - I_2 = 11.24 - 9.389$$

Avg power delivered to loads:

$$P_{avg, 50\Omega} = |I_2 - I_1|^2 (50) \\ = 206 \text{ W}$$

$$P_{avg, 100\Omega} = 117 \text{ W}$$

$$P_{avg, 20+j10\Omega} = I_2^2 (20) = 1763 \text{ W}$$

$$\text{Total power delivered to loads} = 2086 \text{ W}$$

Avg power lost in lines

$$P_{avg, aa} = |I_1|^2 1 = 126 \text{ W}$$

$$P_{avg, bb} = |I_3|^2 1 = 108 \text{ W}$$

$$P_{avg, nn} = |I_3 - I_1|^2 3 = 3 \text{ W}$$

$$\text{Total power lost in lines} = 237 \text{ W}$$

Power supplied by sources

$$P_{avg, an} = V_{an} I_{na} \cos(\theta - \phi) \\ = 115 \times 11.24 \cos(0 - (-19.83^\circ)) \\ = 1216 \text{ W}$$

$$V_{an} = 115 \angle 0^\circ \text{ V}_{rms}$$

$$I_{na} = I_1$$

$$P_{avg, nb} = 115 \times 10.37 \cos(21.80^\circ) \\ = 1107 \text{ W}$$

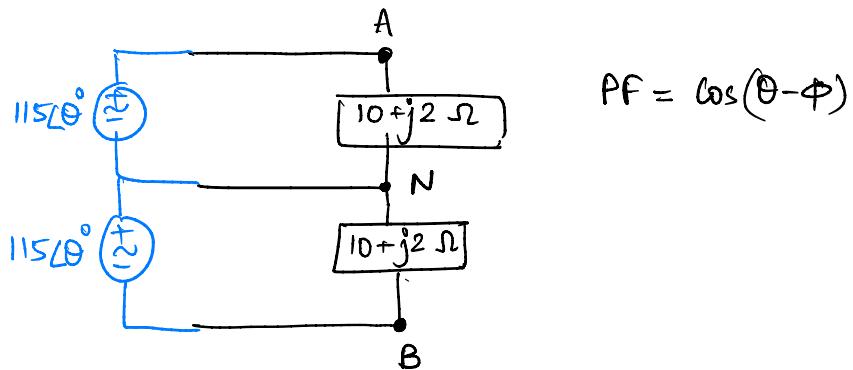
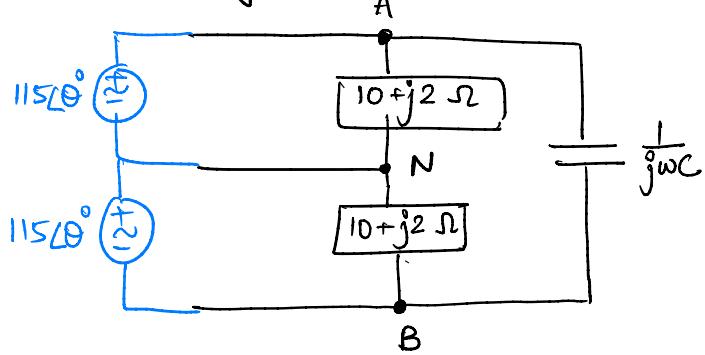
$$\text{Total power supplied} = 2323 \text{ W}$$

$$P_{supplied} = P_{delivered} + P_{loss}$$

① Example : Consider $V_{12} = 9 \angle 30^\circ$, $V_{32} = 3 \angle 130^\circ$
 find $V_{21} = 9 \angle -150^\circ$, $V_{13} = V_{12} + V_{23}$

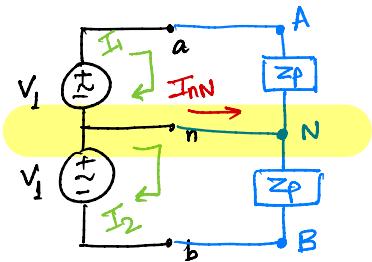
② Consider a single-phase three-wire balanced source connected to a balanced load. $f = 50 \text{ Hz}$, $V_{an} = 115 \angle 0^\circ \text{ V}$

- Find power factor of the load if capacitor is omitted.
- find value of C that will lead to a unity power factor of total load.



Review

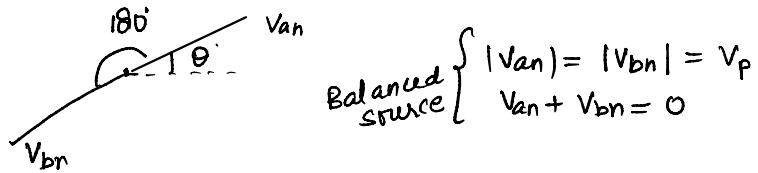
Single-phase Three-wire sources (Two-phase)



$$V_1 = V_p \angle \theta^\circ$$

$$V_{an} = V_a - V_n = V_1 = V_p \angle \theta^\circ$$

$$V_{bn} = -V_1 = -V_p \angle \theta^\circ = V_p \angle \theta^\circ - 180^\circ$$



$$I_{nN} = 0 \quad \leftarrow \text{when source and load are balance.}$$

Time-domain	freq. domain
$V_{an}(t) = V_p \cos(\omega t + \theta^\circ)$	$V_{an} = V_1 = V_p \angle \theta^\circ$
$V_{bn}(t) = V_p \cos(\omega t + \theta^\circ - 180^\circ)$	$V_{bn} = -V_1 = V_p \angle \theta^\circ - 180^\circ$

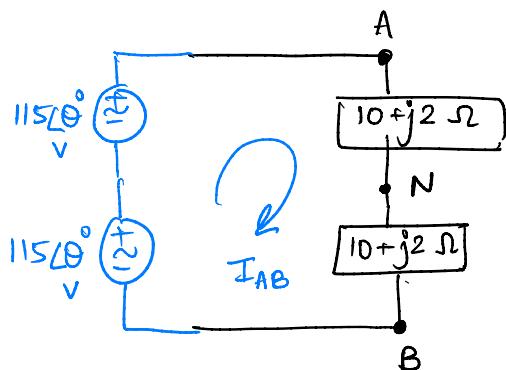
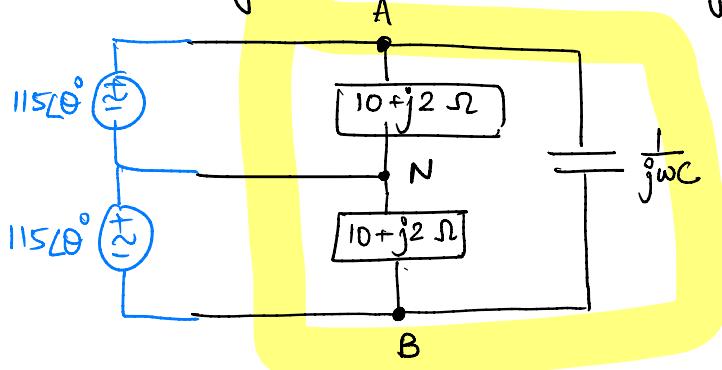
$$p(t) = V(t) i(t)$$

$$P_{avg} = V \cdot I \quad X$$

$$P_{avg} = \frac{1}{2} \operatorname{Re} \{ \underbrace{V \cdot I^*} \} \quad \checkmark$$

① Example : Consider $V_{12} = 9 \angle 30^\circ$, $V_{32} = 3 \angle 130^\circ$
 find $V_{21} = 9 \angle -150^\circ$, $V_{13} = V_{12} + V_{23}$

② Consider a single-phase three-wire balanced source connected to a balanced load. $f = 50 \text{ Hz}$, $V_{an} = 115 \angle 0^\circ \text{ V}$
 → • Find power factor of the load if capacitor is omitted.
 • find value of C that will lead to a unity power factor of total load.



$$PF = \cos(\theta - \phi)$$

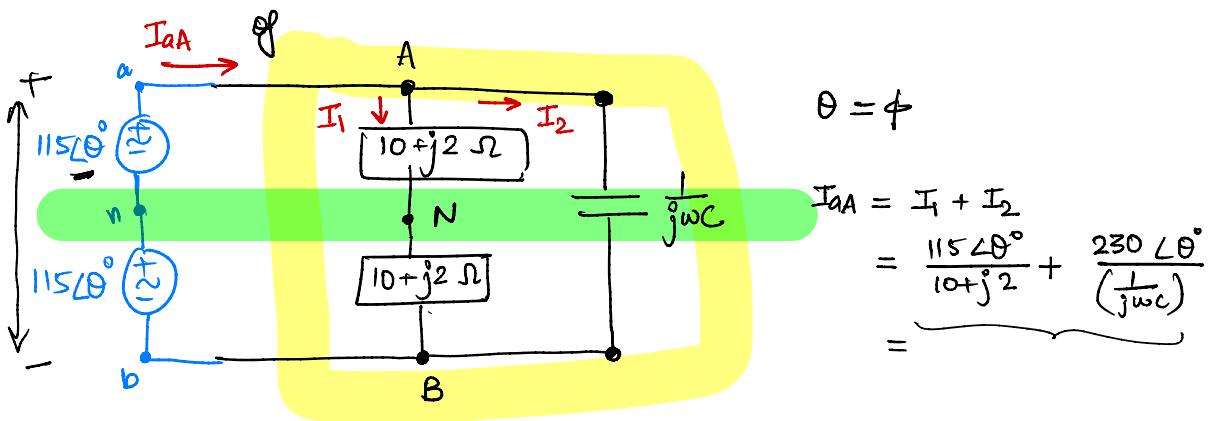
$$I_{AB} = \frac{115 \angle \theta^\circ + 115 \angle \theta^\circ}{2(10 + j2)} = \frac{230 \angle \theta^\circ}{2(10 + j2)}$$

$$= 11.27 \angle \theta^\circ - 11.3^\circ \text{ A}$$

$$\phi = \theta - 11.3^\circ$$

$$PF = \underbrace{\cos(\theta - \theta + 11.3^\circ)}_{\text{lagging}} = 0.98 \text{ lagging}$$

$$PF = 1 \Rightarrow \cos(\theta - \phi) = 1 \Rightarrow \theta - \phi = 0 \Rightarrow \theta = \phi$$



$$\theta = \phi$$

$$\begin{aligned} I_{AA} &= I_1 + I_2 \\ &= \frac{115 \angle 0^\circ}{10 + j2} + \frac{230 \angle 0^\circ}{(\frac{1}{j\omega C})} \\ &= \end{aligned}$$

$$S = \text{Complex power of total load} = \frac{1}{2} VI^*$$

$$\text{PF} = \frac{\text{Re}\{S\}}{|S|} \Rightarrow \text{PF} = 1 \Rightarrow |S| = \text{Re}\{S\} \Rightarrow \text{Im}\{S\} = 0$$

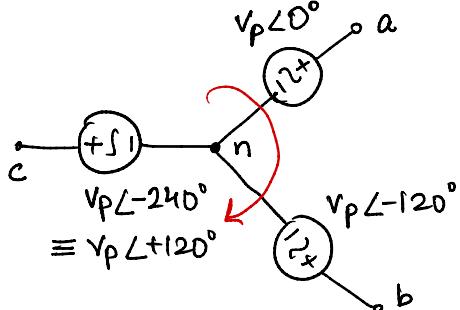
$$\begin{aligned} S &= \frac{1}{2} V_{AN} I_{AN}^* + \frac{1}{2} V_{NB} I_{NB}^* + \frac{1}{2} V_{AB} I_{AB}^* \\ &= \frac{1}{2} V_{AN} \left(\frac{V_{AN}}{Z_p} \right)^* + \frac{1}{2} V_{NB} \left(\frac{V_{NB}}{Z_p} \right)^* + \frac{1}{2} V_{AB} \left(\frac{V_{AB}}{Z_c} \right)^* \\ &= \underbrace{\frac{1}{2} \frac{|V_{AN}|^2}{|Z_p|^2} Z_p}_{} + \underbrace{\frac{1}{2} \frac{|V_{NB}|^2}{|Z_p|^2} Z_p}_{} + \underbrace{\frac{1}{2} \frac{|V_{AB}|^2}{|Z_c|^2} Z_c}_{} \\ &= \frac{|V_{AN}|^2}{|Z_p|^2} Z_p + \frac{1}{2} \frac{|V_{AB}|^2}{|Z_c|^2} Z_c \end{aligned}$$

$Z_p = 10 + j2$
$Z_c = \frac{1}{j\omega C}$

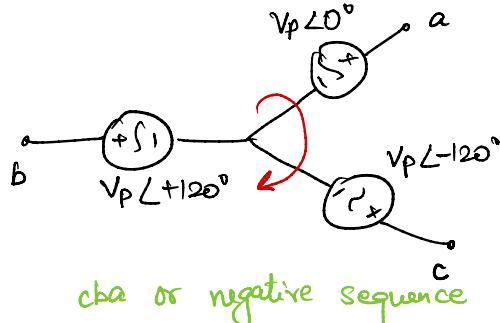
$$S = \frac{115^2}{104} (10 + j2) + \frac{1}{2} \frac{(230)^2}{(\frac{1}{j\omega C})^2} (-j)$$

$$\text{Im}(S) = 0 \Rightarrow C = \frac{1}{\omega \times 104} = \frac{1}{100\pi \times 104} = 30.6 \mu F$$

Three-phase Sources



→ abc or positive sequence



cba or negative sequence

$$V_{an} = V_p \angle 0^\circ$$

$$V_{bn} = V_p \angle -120^\circ$$

$$V_{cn} = V_p \angle +120^\circ$$

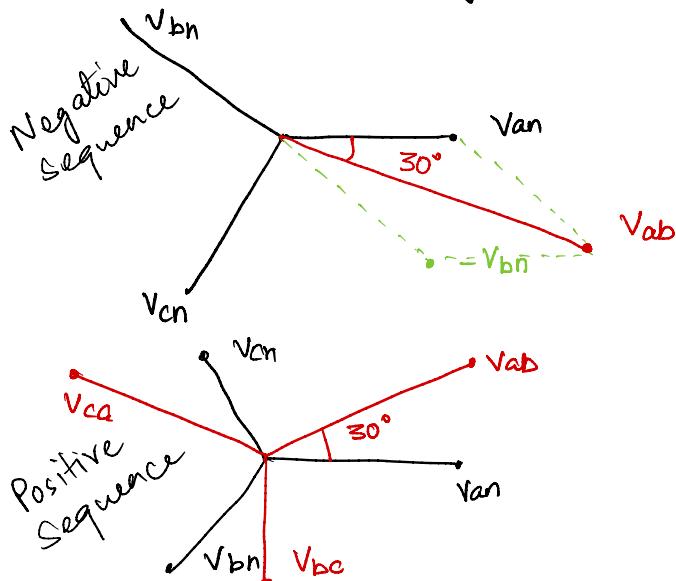
$$\left\{ \begin{array}{l} |V_{an}| = |V_{bn}| = |V_{cn}| = V_p \\ V_{an} + V_{bn} + V_{cn} = 0 \end{array} \right.$$

Balanced source.

Balanced source

Phase voltages: V_{an} , V_{bn} , V_{cn}

Line-to-line or line voltages: V_{ab} , V_{bc} , V_{ca}



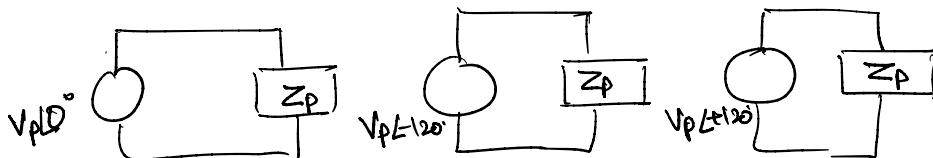
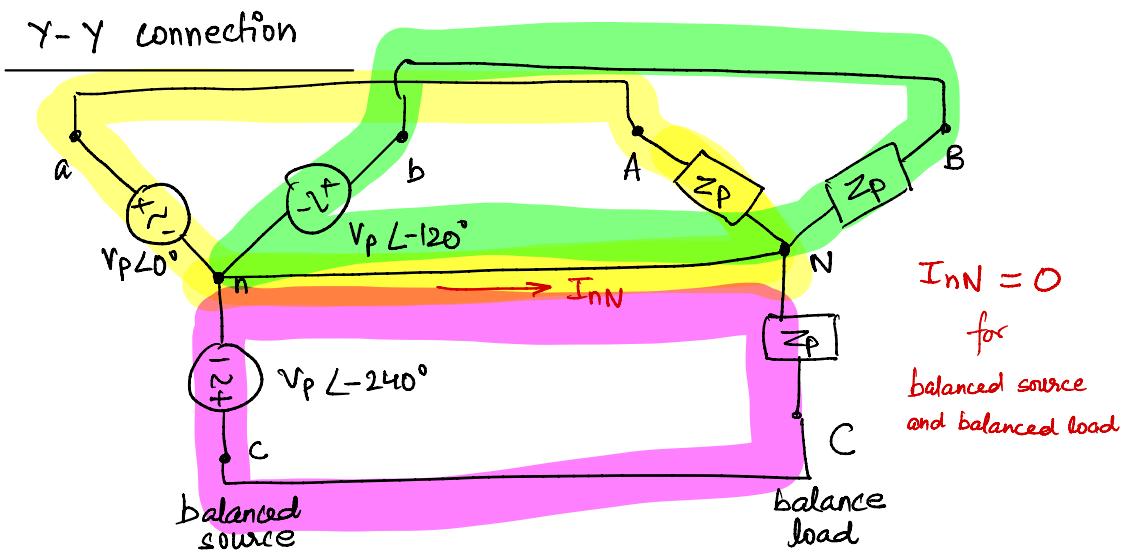
$$V_{ab} = V_{an} - V_{bn} = V_{an} + (-V_{bn})$$

$$V_L = \sqrt{3} V_p$$

$$\begin{aligned} V_{ab} &= V_{an} + V_{nb} \\ &= V_{an} - V_{bn} \\ &= \sqrt{3} V_p \angle 30^\circ \end{aligned}$$

$$V_{bc} = \sqrt{3} V_p \angle -90^\circ$$

$$V_{ca} = \underbrace{\sqrt{3} V_p \angle -210^\circ}_{V_L}$$



"per-phase" analyses

$$I_{AA} = \frac{V_{an}}{Z_p}$$

$$I_{BB} = \frac{V_{bn}}{Z_p} = \frac{V_{an} \angle -120^\circ}{Z_p} = I_{AA} \angle -120^\circ$$

$$I_{CC} = \frac{V_{cn}}{Z_p} = \frac{V_{an} \angle +120^\circ}{Z_p} = I_{AA} \angle +120^\circ$$

$$I_{nn} = I_{AA} + I_{BB} + I_{CC} = 0$$

$$V_{an} = V_p \angle 0^\circ$$

$$\begin{aligned} V_{bn} &= V_p \angle -120^\circ \\ &= V_{an} \angle -120^\circ \end{aligned}$$

$$= V_{an} (1 \angle -120^\circ)$$

Example : Consider three-phase balanced $\gamma - \gamma$ connected system.

$$V_{an} = 200 \angle 0^\circ \text{ V rms}$$

$$Z_p = 100 \angle 60^\circ \Omega$$

find phase and line voltages .

find phase and line currents.

find total average power delivered to load.

Phase Voltages :

$$V_{an} =$$

$$V_{bn} = 200 \angle -120^\circ \text{ Vrms}$$

$$V_{cn} = 200 \angle +120^\circ \text{ Vrms}$$

Line voltages: $V_{ab} = \sqrt{3} V_p \angle 30^\circ = 200\sqrt{3} \angle 30^\circ \text{ Vrms}$

$$V_{bc} = 200\sqrt{3} \angle -90^\circ \text{ Vrms}$$

$$V_{ca} = 200\sqrt{3} \angle -210^\circ \text{ Vrms}$$

Line currents:

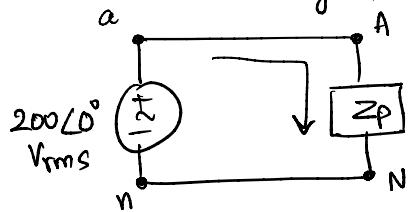
$$I_{aA} = \frac{V_{an}}{Z_p} = \frac{200 \angle 0^\circ}{100 \angle 60^\circ} = 2 \angle -60^\circ \text{ Arms}$$

$$I_{bB} = 2 \angle -120^\circ \text{ Arms}$$

$$I_{cC} = 2 \angle -300^\circ \text{ Arms}$$

Total Average power :

Average power in phase A



$$P_{avg, A} = \frac{1}{2} \operatorname{Re}\{VI^*\} = \operatorname{Re}\{V_{eff} I_{eff}^*\}$$

$$= \operatorname{Re}\{V_{an} I_{AN}\}$$

$$= \operatorname{Re}\{200 \angle 0^\circ \times 2 \angle -60^\circ\} \text{ W}$$

$$= \operatorname{Re}\{400 \angle -60^\circ\}$$

$$= 400 \cos(-60^\circ)$$

$$= 200 \text{ W}$$

$$P_{avg, B} = 200 \text{ W}$$

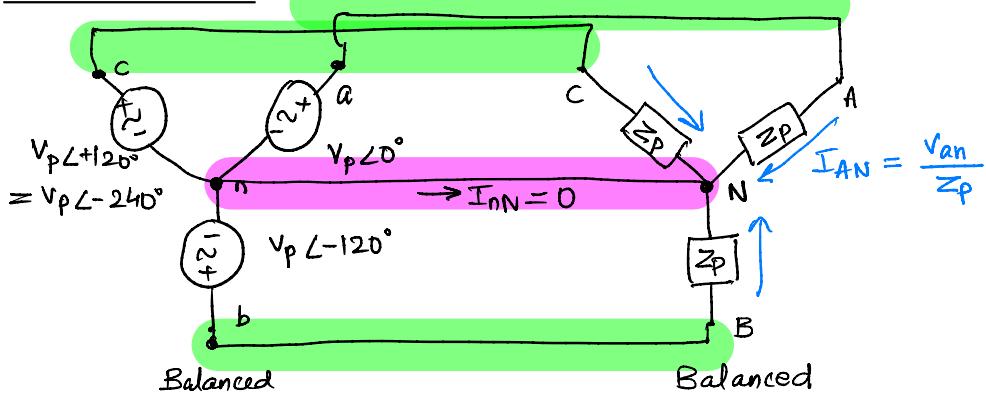
$$P_{avg, C} = 200 \text{ W}$$

$$P_{avg, total} = 600 \text{ W.}$$

Three phase systems

Y-Y connection

"star-star" connection



Phase Voltages : $V_{AN} = V_{an}$, $V_{BN} = V_{bn}$, $V_{CN} = V_{cn}$

Line Voltages : V_{ab} , V_{bc} , V_{ca}

Line Currents : I_{aA} , I_{bB} , I_{cC}

Phase currents : $I_{AN} = I_{aA}$, $I_{BN} = I_{bB}$, $I_{CN} = I_{cC}$

$$V_{an} = V_p \angle 0^\circ$$

$$V_{bn} = V_p \angle -120^\circ$$

$$V_{cn} = V_p \angle -240^\circ$$

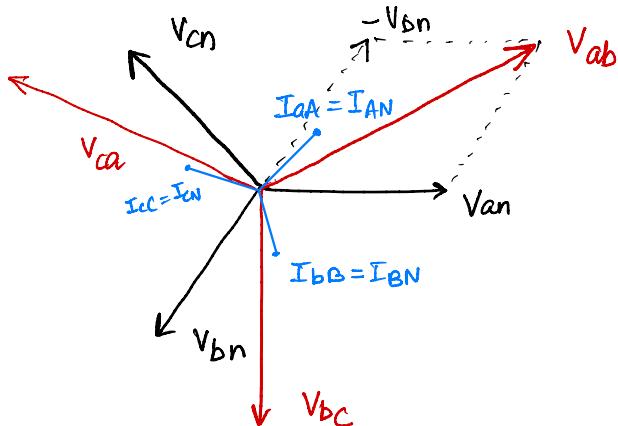
phase voltages

$$V_{ab} = V_{an} - V_{bn} = \sqrt{3} V_p \angle 30^\circ$$

$$V_{bc} = V_{bn} - V_{cn} = \sqrt{3} V_p \angle -90^\circ$$

$$V_{ca} = V_{cn} - V_{an} = \sqrt{3} V_p \angle -210^\circ$$

line voltages



V_p = magnitude of phase voltages = $|V_{AN}| = |V_{BN}| = |V_{CN}| = V_p$
 V_L = magnitude of line voltages = $|V_{ab}| = |V_{bc}| = |V_{ca}| = \sqrt{3}V_p$

$$V_L = \sqrt{3}V_p$$

Line currents = phase currents.

$$I_{aA} = I_{AN} = \frac{V_{AN}}{Z_p} = \frac{V_p}{Z_p}$$

$$I_{bB} = I_{BN} = \frac{V_{BN}}{Z_p}$$

$$I_{cC} = I_{CN} = \frac{V_{CN}}{Z_p}$$

$$\begin{aligned} Z_p &= \frac{V_{AN}}{I_{AN}} \\ &= \frac{V_p \angle \theta^\circ}{I_p \angle \phi^\circ} \\ &= \frac{V_p}{I_p} \angle \theta - \phi^\circ \end{aligned}$$

I_p = magnitude of phase currents = $|I_{AN}| = |I_{BN}| = |I_{CN}|$

I_L = magnitude of line currents = $|I_{aA}| = |I_{bB}| = |I_{cC}|$

$$I_p = I_L$$

$$\begin{aligned} P_{avg} \text{ for any phase} &= \frac{1}{2} \operatorname{Re} \{ V I^* \} & V_{AN} &= V_p \angle \theta^\circ \\ P_{avg, A} &= \frac{1}{2} \operatorname{Re} \{ V_{AN} I_{AN}^* \} & I_{AN} &= I_p \angle \phi^\circ \\ &= \frac{1}{2} \operatorname{Re} \{ V_p \angle \theta^\circ (I_p \angle \phi^\circ)^* \} \\ &= \frac{1}{2} V_p I_p \cos(\theta - \phi) \\ &= \frac{1}{2} \frac{1}{\sqrt{3}} V_L I_L \cos(\theta - \phi) \end{aligned}$$

P_{avg} per phase = P_{avg} Aphase = P_{avg} Bphase = P_{avg} Cphase = $\frac{1}{2} V_p I_p \cos(\theta - \phi)$

$$P_{avg} \text{ per phase} = \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi), \text{ if } V_L, I_L \text{ are rms values.}$$

$$P_{avg} \text{ total} = \frac{3}{2} V_p I_p \cos(\theta - \phi)$$

$$= 3 V_p I_p \cos(\theta - \phi), \text{ if } V_p \& I_p \text{ are rms values.}$$

$$P_{avg} \text{ total} = \sqrt{3} V_L I_L \cos(\theta - \phi), \text{ if } V_L \& I_L \text{ are rms values.}$$

Total instantaneous power

$$P_{\text{total}}(t) = P_A(t) + P_B(t) + P_C(t)$$

$$P_A(t) = V_{AN}(t) i_{AN}(t)$$

$$V_{AN}(t) = V_p \cos(\omega t + \theta)$$

$$i_{AN}(t) = I_p \cos(\omega t + \phi)$$

$$P_A(t) = V_p I_p \cos(\omega t + \phi) \cos(\omega t + \theta)$$

$$= \frac{V_p I_p}{2} \left[\underbrace{\cos(\theta - \phi)}_{\beta} + \underbrace{\cos(2\omega t + \theta + \phi)}_{\alpha} \right]$$

$$= \frac{V_p I_p}{2} [\cos \alpha + \cos \beta]$$

$$P_B(t) = V_{BN}(t) i_{BN}(t) = \frac{V_p I_p}{2} [\cos \beta + \cos(\alpha - 240^\circ)]$$

$$P_C(t) = \frac{V_p I_p}{2} [\cos \beta + \cos(\alpha + 240^\circ)]$$

$$P_{\text{total}}(t) = \frac{V_p I_p}{2} \left[3 \cos \beta + \cos \alpha + \underbrace{\cos(\alpha - 240^\circ)}_{\alpha} + \underbrace{\cos(\alpha + 240^\circ)}_{\alpha} \right]$$

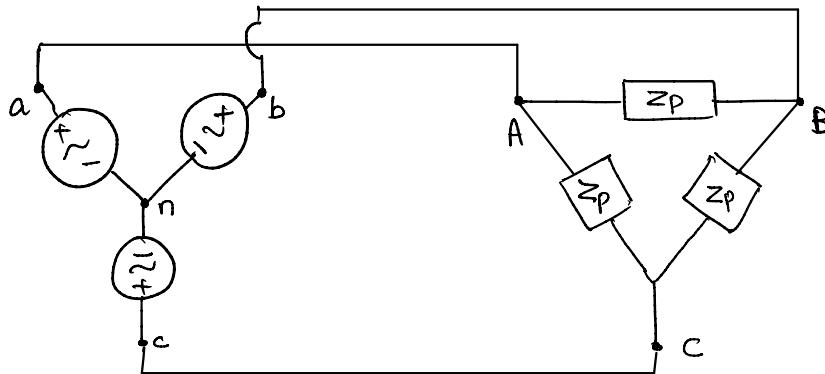
$$P_{\text{total}}(t) = 3 \frac{V_p I_p}{2} \cos \beta = \frac{3 V_p I_p}{2} \cos(\theta - \phi) - \cos \alpha$$

$$P_{\text{avg}} = \frac{1}{T} \int_0^T P(t) dt$$

Y-Δ Connections

Y-connected source

Δ-connected load (no neutral node)



Phase voltages: $V_{AB} = V_{ab}$, $V_{BC} = V_{bc}$, $V_{CA} = V_{ca}$

Line voltages: V_{ab} , V_{bc} , V_{ca} } ← same as phase voltages

$$\left. \begin{array}{l} V_{AB} = V_{ab} = V_{an} - V_{bn} = \sqrt{3} V_{an} \angle 30^\circ \\ V_{BC} = V_{bc} = \sqrt{3} V_{an} \angle -90^\circ \\ V_{CA} = V_{ca} = \sqrt{3} V_{an} \angle -210^\circ \end{array} \right\} \begin{array}{l} \text{Line voltages} \\ = \text{phase voltages} \end{array}$$

V_p = magnitude of phase voltage = $|V_{AB}| = |V_{BC}| = |V_{CA}| = \sqrt{3} |V_{an}|$

V_L = magnitude of line voltage = $(V_{ab}) = |V_{bc}| = |V_{ca}| = \sqrt{3} |V_{an}|$

$$V_p = V_L$$

Line currents: $I_{aA} = I_{AB} - I_{CA} = \sqrt{3} I_{AB} \angle -30^\circ$

$$I_{bB} = \sqrt{3} I_{BC} \angle -30^\circ$$

$$I_{cC} = \sqrt{3} I_{CA} \angle -30^\circ$$

Phase currents: $I_{AB} = V_{AB} / Z_P$

$$I_{BC} = V_{BC} / Z_P$$

$$I_{CA} = V_{CA} / Z_P$$

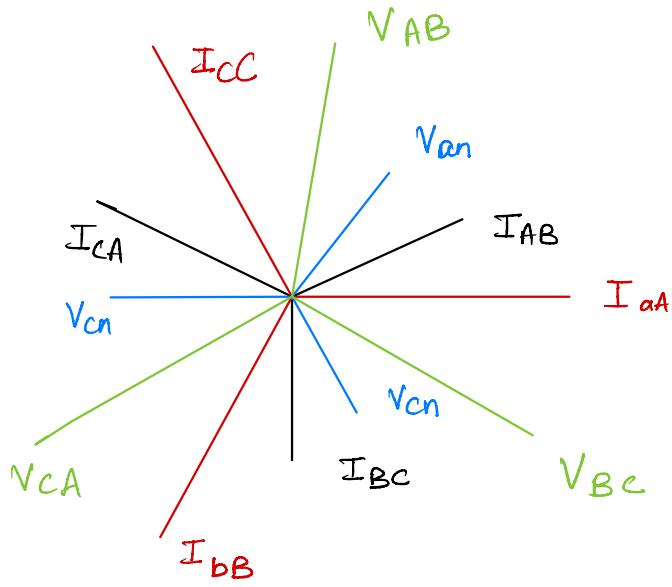
$$\begin{aligned} Z_P &= \frac{V_{AB}}{I_{AB}} \\ &= \frac{V_p \angle \theta^\circ}{I_p \angle \phi^\circ} \\ &= \frac{V_p}{I_p} \angle \theta - \phi^\circ \end{aligned}$$

I_p = magnitude of phase currents = $|I_{AB}| = |I_{BC}| = |I_{CA}|$

I_L = magnitude of line currents = $|I_{aA}| = |I_{bB}| = |I_{cC}|$

$$\Rightarrow I_L = \sqrt{3} \underbrace{|I_{AB}|}_{I_p}$$

$$\Rightarrow I_L = \sqrt{3} I_p$$



Similar to Δ - Δ connection:

$$P_{avg \text{ per phase}} = \frac{1}{2} R_p \left\{ V_{AB} I_{AB}^* \right\} = \frac{1}{2} V_p I_p \cos(\theta - \phi)$$

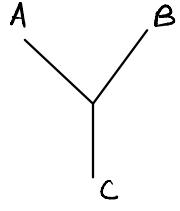
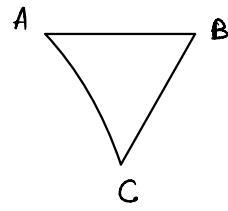
$$P_{avg \text{ per phase}} = \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi) , \text{ if } V_L, I_L \text{ are rms values.}$$

$$P_{avg \text{ total}} = \frac{3}{2} V_p I_p \cos(\theta - \phi)$$

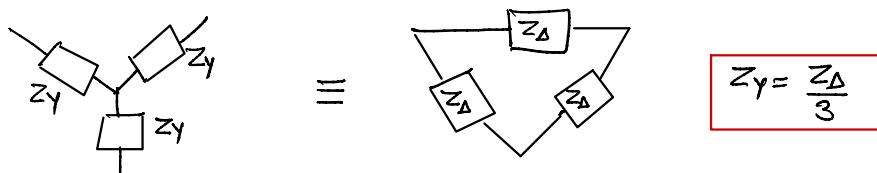
$$= 3 V_p I_p \cos(\theta - \phi) , \text{ if } V_p \text{ & } I_p \text{ are rms values.}$$

$$P_{avg \text{ total}} = \sqrt{3} V_L I_L \cos(\theta - \phi) , \text{ if } V_L \text{ & } I_L \text{ are rms values.}$$

Y-Y connection versus Y-Δ connection

		
Phase voltages Line voltages	V_{AN}, V_{BN}, V_{CN} V_{ab}, V_{bc}, V_{ca}	$V_{AB}, V_{BC}, V_{CA} \rightarrow$ same $V_{ab}, V_{bc}, V_{ca} \leftarrow$ same
Phase currents Line currents	$I_{AN}, I_{BN}, I_{CN} \leftarrow$ same $I_{aA}, I_{bB}, I_{cC} \leftarrow$ same	I_{AB}, I_{BC}, I_{CA} I_{aA}, I_{bB}, I_{cC}
	$V_L = \sqrt{3} V_p$ $I_L = I_p$	$V_L = V_p$ $I_L = \sqrt{3} I_p$
Z_p	$Z_p = \frac{V_p}{I_p} \angle \theta - \phi$	$Z_p = \frac{V_p}{I_p} \angle \theta - \phi$
Power per phase	$\frac{1}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$	$\frac{1}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$
Power total	$\frac{3}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$	$\frac{3}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$

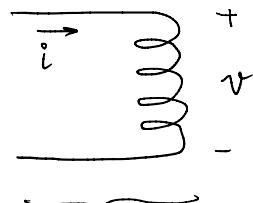
Star to delta load transformation :-



Module 5

Magnetically coupled circuits

(Chapter 13 of textbook)



"Pure linear inductors"
→ no resistance

$$v = L \frac{di}{dt}$$

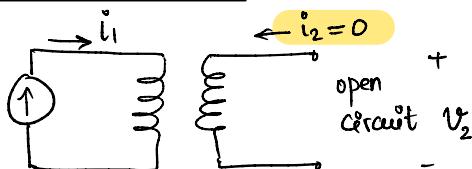
i leads to magnetic flux $i \propto \phi$

Magnetic flux leads to induced v $v \propto d\phi/dt \Rightarrow v \propto di/dt$

$$v = L \frac{di}{dt}$$

DC \rightarrow const $i \rightarrow$ const mag. flux $\rightarrow v = 0$

Mutual Inductance:



$$v_2 \propto \frac{di_1}{dt}$$

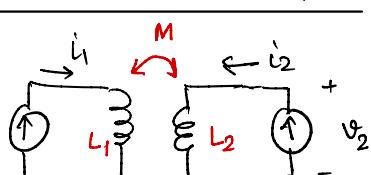
$$\Rightarrow v_2 = M \frac{di_1}{dt}$$

mutual inductance

L : self inductance

Henry

Now consider: $i_2 \neq 0$

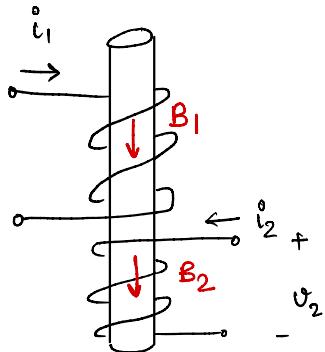


$$v_2 \propto \frac{di_1}{dt} , v_2 \propto \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Self inductance

Mutual inductance



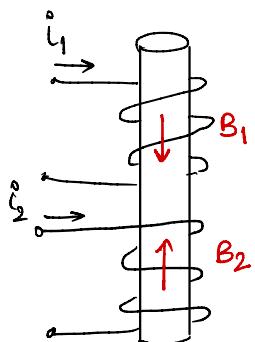
$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

B_2 B_1

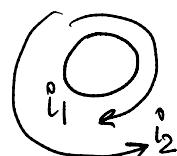
Top view



Additive magnetic flux

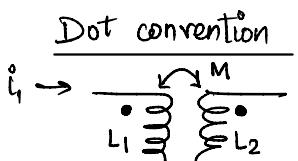


Top view



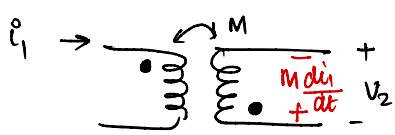
Subtractive magnetic flux

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$



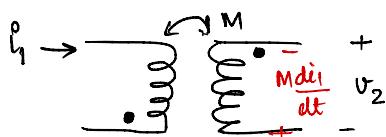
Assume $i_2 = 0$

$$v_2 = M \frac{di_1}{dt}$$

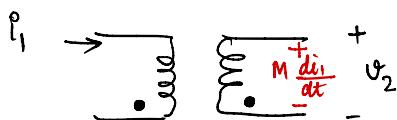


$$v_2 = -M \frac{di_1}{dt}$$

} Current entering at dotted terminal leads to +ve voltage reference at dotted terminal.



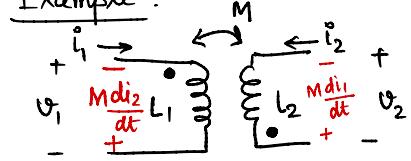
$$v_2 = -M \frac{di_1}{dt}$$



$$v_2 = M \frac{di_1}{dt}$$

} Current entering at undotted terminal leads to +ve voltage reference at undotted terminal.

Example :



(A) find v_2 if $i_1 = -8e^{-t} A$, $i_2 = 0$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

0

$$v_2 = -M \frac{di_1}{dt} = -2 \frac{d}{dt} (-8e^{-t}) = -2 \times (-8) \times (-1) e^{-t}$$

$$= -16 e^{-t} V$$

(B) find v_1 if $i_1 = 0$ and $i_2 = 5 \sin 45t A$.

$$v_1 = \underbrace{L_1 \frac{di_1}{dt}}_0 - M \frac{di_2}{dt}$$

$$v_1 = -M \frac{d}{dt} (5 \sin 45t) = -2 \times 5 \times 45 \cos 45t$$

$$= -450 \cos 45t V$$

Sign Convention for self inductance:

$$v = L \frac{di}{dt}$$

(current entering from
+ve terminal)

$$v = -L \frac{di}{dt}$$

(current leaving from
+ve terminal)

Phasor Representation :

$$V_1 = L_1 \frac{di^*}{dt} + M \frac{di^*}{dt}$$

$$V_2 = L_2 \frac{di^*}{dt} + M \frac{di^*}{dt}$$

$$v = L \frac{di}{dt}$$

time domain

$$V = j\omega L I_1$$

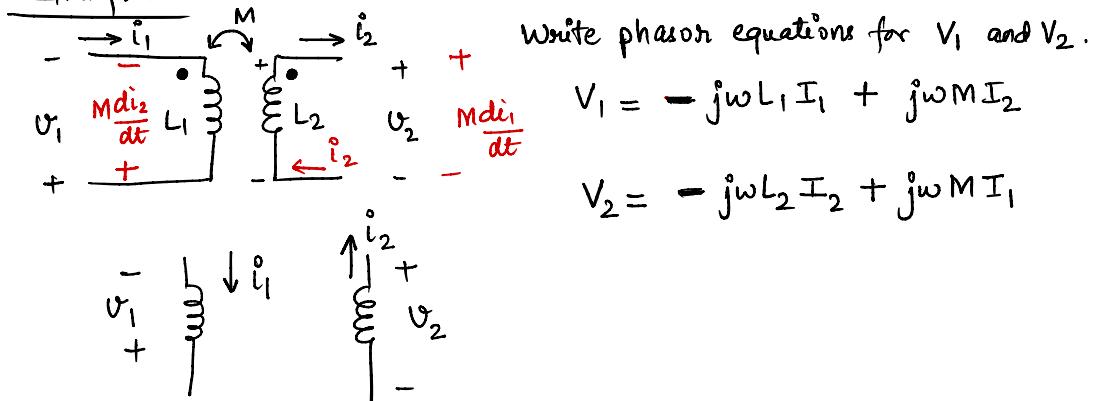
freq.
domain

for sinusoidally varying sources

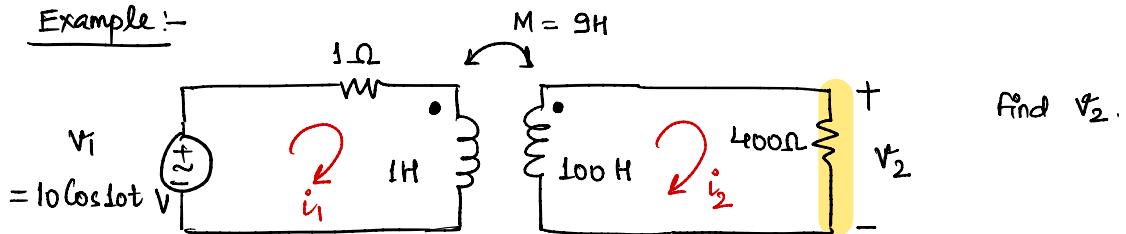
$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$

$$V_2 = j\omega L_2 I_2 + j\omega M I_1$$

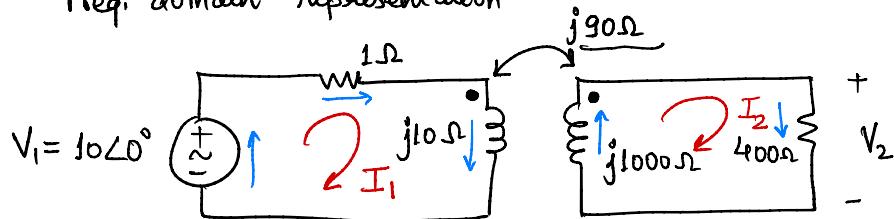
Example



Example :-

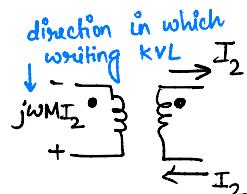


freq. domain representation



$$\text{Mesh 1: } -10 \angle 0^\circ + I_1 (1) + (j10) I_1 - (j90) I_2 = 0$$

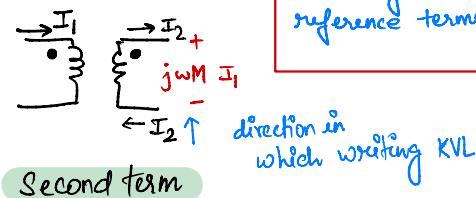
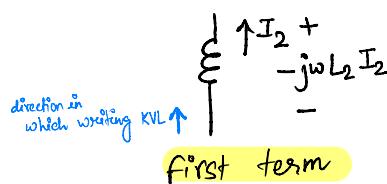
$$\Rightarrow 10 \angle 0^\circ = (1 + j10) I_1 - j90 I_2$$



Last term

$$\text{Mesh 2: } -(-j1000)I_2 - (j90)I_1 + (400)I_2 = 0$$

$$\Rightarrow -j90I_1 + (400 + j1000)I_2 = 0$$



Always remember to put a minus sign if while writing KVL terms you encounter a -ve reference terminal first.

$$(1 + j10)I_1 - j90I_2 = 10$$

$$(-j90)I_1 + (400 + j1000)I_2 = 0$$

$$\text{Cramer's rule: } I_1 = \frac{D_1}{D_0}, \quad I_2 = \frac{D_2}{D_0}$$

$$D_0 = \begin{vmatrix} 1+j10 & -j90 \\ -j90 & 400 + j1000 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} 10 & -j90 \\ 0 & 400 + j1000 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 1+j10 & 10 \\ -j90 & 0 \end{vmatrix}$$

$$V_2 = 400 I_2$$