14/08/24

Signals And Systems

11:00AM

Evaluation : (relative grading)

:fire emoji:

Attendance	10%
Quiz	16%
Assignments	15%
mid sem	24%
end sem	35%

Book: Signals & System 2nd edition by Alan V. oppenheimer

LECTURE : 1 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ **GROUP** : consists of ....

- A set G
- A rule / binary operation "*"
  - a. associative
    $x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$
  - b. There exists an element "e" called the identity of group G such that
    $e*x = x*e = x \quad \forall x \in G$
  - c. $\forall x \in G, \exists x^{-1}$ such that
    $x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$
  - d. if $x*y = y*x \quad \forall x,y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abelian Group

eg: set of continuous time periodic signals with time period T with "*" = "+"

⇒ **FIELD** : consists of the following

- A set F
- Two binary operations "+" and "·" such that ....
  - $(F,+)$ is an abelian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - Left distributive
      $x\cdot(y+z) = xy + xz \quad \forall x,y,z \in F$
    - Right distributive
      $(x+y)\cdot z = xz + yz \quad \forall x,y,z \in F$

eg: F = Real Numbers $\mathbb{R}$

* Vector Space : A set V with a map...

- '+' : $V \times V \to V$
  $(v_1, v_2) \to (v_1 + v_2)$ called vector addition
- '·' : $F \times V \to V$
  $(a, v_1) \to (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a\cdot(v_1+v_2) = av_1 + av_2$
- $(a_1+a_2)\cdot v = a_1 v + a_2 v$
- $(\alpha\beta)v = \alpha(\beta v) = \beta(\alpha v)$
- $1\cdot v = v$
- $a\cdot 0 = 0$
- $0\cdot v = 0$
- if $v \neq 0$, then $a\cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \to \mathbb{R}$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X,d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM : let V be a F-vector space.

A map $\|\cdot\| : V \to \mathbb{R}$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a|\,\|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

Lecture : 2

16/08/24 : 9:30am

* Inner Product :

let V be a F-vector space
A map,

$$\langle , \rangle : V \times V \to F$$

is called an inner product if ...

- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ ... complex conjugate operation
- it is linear in the first co-ordinate
  $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$
  $\forall\, v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $\langle v, a_1 w_1 + a_2 w_2 \rangle = \overline{a_1} \langle v, w_1 \rangle + \overline{a_2} \langle v, w_2 \rangle$
  $\forall\, w_1, w_2, v \in V$ and $a_1, a_2 \in F$

eg: dot product
(numerical case similarly complex) $\min(\{x\}) \geq 0$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner product
(complex conjugate)

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot \overline{y_i}$$

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \quad \text{implies } a_i = 0 \ \forall\, i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

(maximal linearly independent set of vectors in V)

if $\begin{cases} e_1 \\ e_2 \\ \vdots \\ e_n \end{cases}$ is a basis for $V_n(F)$, then $\forall\, v \in V$ & $a \in F$

$$v = a_1 e_1 + a_2 e_2 + \dots a_n e_n \qquad V = \sum_{i=1}^{n} a_i e_i$$

* ORTHOGONAL & ORTHONORMAL BASIS

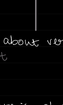
A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

$$v_i \neq 0 \quad \forall\, i$$
$$\langle v_i, v_j \rangle \neq 0 \quad \forall\, i \,\&\, j$$

SIGNALS (133 & 203)

- continuous time signal

- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t) \ \forall\, t$
  eg: cosine function
- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t) \ \forall\, t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal :

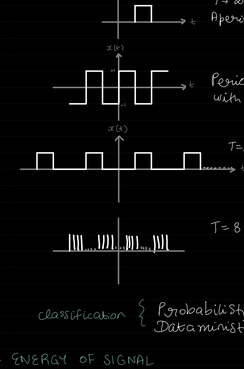
$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

* PERIODIC : if $x(t) = x(t+T) \ \forall\, t$ then T is the time period

The smallest time period T defined for this property is called the fundamental time period

eg:



classification { Probabilistic / Deterministic

* ENERGY OF SIGNAL

Continuous Time Signal :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time Signal

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

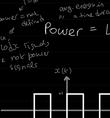
Note: Periodic signals are power signals ✓
Aperiodic signals are not power signals ✗

$T \to \infty$ ← as power is not defined here

avg energy in a time period

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Aperiodic signals are not power signals



compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

★ Lecture 3 [illegible]

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= 4 + 4 \quad \uparrow \text{energy signal}$$

$x_1(t)$ = energy signal, constant signal, periodic signal, aperiodic, power signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$\infty$$

Since $E \to \infty$ and $\text{not } E = \infty$, that signal is not an energy signal.

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$\text{attention} \quad \begin{cases} = \text{finite value} \\ = \text{finite value} \end{cases}$$

$0 < P < \infty$ ⇒ Power Signal ✓

★ TRANSFORMATION OF SIGNAL

① Translation

(a) DELAY

$y(t) = x(t-4)$

like $x_0 = 3$

$y(t) = 2(t-2)$

$y(0) = 2(-3) = 0$
$y(3) = 2(1) = 2$
$y(4) = 2(2) = 4$
$y(5) = 2(3) = 6$

(b) ADVANCE A SIGNAL

$y(t) = 2(t+4)$

like $x_0 = 3$

$y(-1) = 2(3) = 2$
$y(-2) = 2(1) = 2$
$y(-3) = 2(1) = 0$
$y(-4) = 0$

② Scaling

$y(t) = x(2t)$

(a) $a > 1$: Compression of the signal on the t-axis

like $a = 3$

$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$
$y(\frac{2}{3}) = x(2) = 0$

(b) $0 < a < 1$: Dilation on t-axis

like $a = \frac{1}{3}$

$y(t) = x(\frac{1}{3}t) = x(\frac{t}{3})$

$y(0) = x(0) = 2$

$y(3) = x(1) = 2$

$y(6) = x(2) = 0$

(c) DEFLECT $y(t) = x(-t)$

$y(t) = x(-t)$

Reflect on the t-axis a line passed.

$a = -1$

$y(3) = x(-3)$

$y(3) = x(3) = 0$

$y(-3) = x(3) = 0$

$y(t) = x(t) = 0$

$y(t) = x(2t) = 0$ ✗ $y(t) = x(2t)$ ✗ execution ✗ scaling

Scaling then translation, because answer of left $2t \neq t$ ✗

Not, scaling → translation → reflection ✓

$x(t)$
translate
$x(t+3)$
scale
$x(3t+3)$

★ FUNDAMENTAL SIGNALS

• COMPLEXES TIME SIGNALS

$x(t) = Ce^{at}$

• CASE 1 : C real constant and a real constant

$x(t) = Ce^{at}$

① $a = 0$

② let $a > 1$

③ let $a < 1$

④ let $a = 0$

• CASE 2 : C real constant a + purely imaginary $= j\omega$

$x(t) = Ce^{j\omega t}$ $\forall t$

periodic if $x(t) = x(t+T)$

$x(t+T) = Ce^{j\omega(t+T)} = Ce^{j\omega t} e^{j\omega T} = x(t)$

$e^{j\omega T} = 1$ $\omega T = 2\pi k$

Step we need to find the fundamental time period (smallest period) for $k = 1$

So, $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$ fundamental time period

$C = \frac{1}{2}(1e^{j\omega t} + 1e^{-j\omega t}) \quad \frac{1}{2}j\omega = \omega$

• CASE 3 : C + complex number a + real number

$x(t) = 1C1e^{j\theta} e^{at} = 1C1e^{j\theta + at}$

$e^{j\theta}$ will stay it is purely imaginary exponential

we can write $x(t) = 1C1e^{at} = \cos(\omega t) + j\sin(\omega t)$

$Re[x(t)] = 1C1 \cos(\omega t)$

$Im[x(t)] = 1C1 \sin(\omega t)$

• CASE 4 : C + complex number $= 1C1e^{j\theta}$ a complex number $= r + j\omega$

$x(t) = 1C1e^{j\theta} = 1C1e^{(r+j\omega)t}$

$= 1C1e^{rt} \underbrace{e^{j\omega t}}_{\text{purely imaginary exponential}}$

$Re[x(t)] = 1C1e^{rt} \cos(\omega t + \theta)$

$Im[x(t)] = 1C1e^{rt} \sin(\omega t + \theta)$

let $r = 2$, $1C1 = 1.5$

$Re[x(t)] = 1.5e^{2t} \cos(\omega t + \theta)$

• Discrete Time Signals

$x[n] = Ca^n$

• CASE 1 : C + real constant a + real

a. $a > 1$, $C = 1$ $x[n] = a^n$

let $a = 2$, $x[n] = 2^n$

b. $0 < a < 1$ let $a = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$

c. $a < -1$ let $a = -2$

$x[n] = (-2)^n$

d. $-1 < a < 0$ let $a = -\frac{1}{2}$

$x[n] = (-\frac{1}{2})^n$

• CASE 2 : C + real a + $e^{j\theta}$

$x[n] = Ca^n = C e^{j\theta n}$

if $x[n] = x[n+N]$ $\forall n$

then $x[n]$ is periodic

$x[n+N] = C e^{j\theta(n+N)} = C e^{j\theta n} e^{j\theta N} = x[n]$

$e^{j\theta N} = 1$ $\Rightarrow$ $\omega N = 2\pi k$

Note we cannot say $N = 1$ for smallest common exponent because we need N to be an integer

$N = \frac{2\pi k}{\omega}$ ← k is an integer

$\frac{N}{k} = \frac{2\pi}{\omega}$ ← $\frac{N}{k}$ is a rational number

ω must be π times a multiple of 2

* Lecture: 4

28/08/24

- for continuous time signals
  → frequency is unique $(0 \to \infty)$

- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

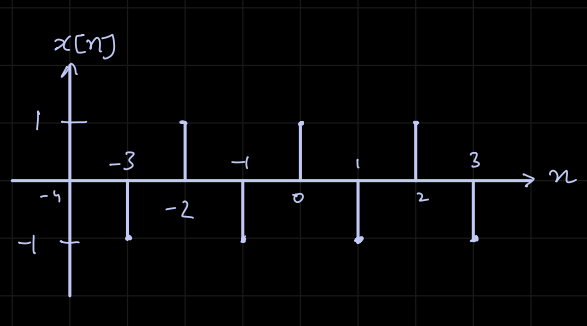
DTS

$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

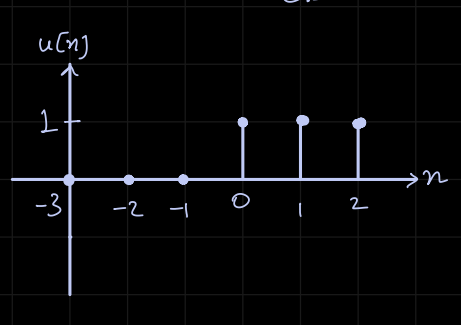
let $\omega_0 = \pi \Rightarrow e^{j\pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

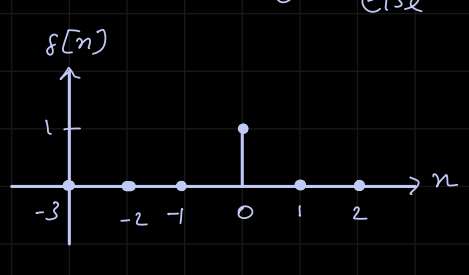
unit step signal
$u[n]$

$$u[n] = \begin{cases} 1 & : n \geq 0 \\ 0 & : \text{else} \end{cases}$$



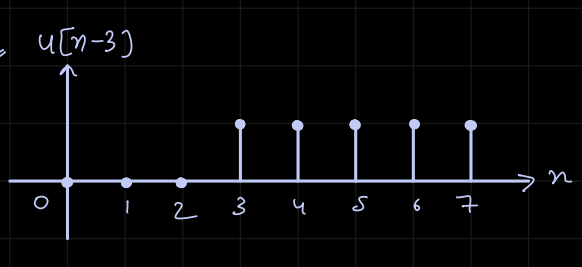
unit impulse function
$\delta[n]$

$$\delta[n] = \begin{cases} 1 & : n = 0 \\ 0 & : \text{else} \end{cases}$$



note:

$1 : n-3 \geq 0$
$n \geq 3$



conversion btw $u[\,]$ and $\delta[\,]$

$$\boxed{\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^{n} \delta[k] \end{aligned}}$$

note: $u[n] = 0 \quad \forall\, n < 0$

So,
$$u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$$
$$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$$
$$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$$
$$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1$$

- $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when
$n - k = 0$ i.e. $k = -2$
but k is never negative
since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n - k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n - k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underset{0}{\delta[1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[-1]} + \dots$$
$$= \underline{\underline{1}} \checkmark$$

- **SIFTING / SAMPLING**

$x[n] \cdot \delta[n] = ?$

$$x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$$

we can also say $x[n]\delta[n] = x[0]\,\delta[n]$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

eg) $x[n] \cdot \delta[n - n_0] = ?$

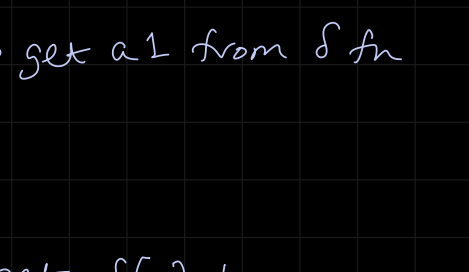
$$x[n] \cdot \delta(n - n_0) = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- unit step function

$$u(t) = \begin{cases} 1 & : t \geq 0 \\ 0 & : \text{else} \end{cases}$$



OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function ≡ singularity function ≡

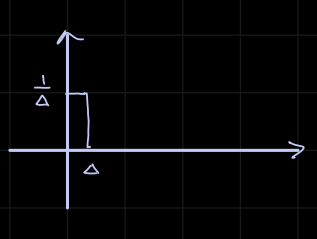
$$\delta(t) = 0 \quad : t \neq 0$$

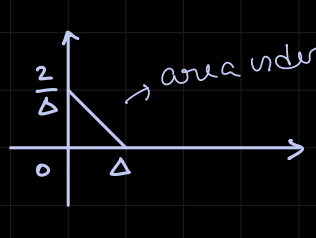
and $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$\delta_\Delta(t)$:

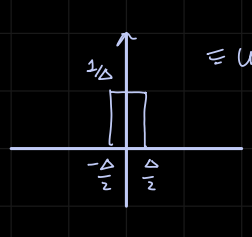
$$\delta_\Delta(t) = \begin{cases} 0 & : t \geq \Delta \ \&\ t < 0 \\ \frac{1}{\Delta} & : \text{otherwise} \end{cases}$$

infinitesimally small





area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
≡ unit imp use func ✓



≡ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t - \tau)\,d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

* Tutorial: 1 30/08/24

(7) even $\{x(t)\} = \dfrac{x(t)+x(-t)}{2}$

odd $\{x(t)\} = \dfrac{x(t)-x(-t)}{2}$

8) (a) $x[n] = \sin(0.02\pi n)$

$\omega = 0.02\pi$

$T = \frac{2\pi}{\omega} =$

(b) $p(x,y) = a^2x_1y_1 + abx_1y_2 + b^2x_2y_2 +$ $ab\ x_2y_1$

$$A = \begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$$

$|A| = a^2b^2 - a^2b^2 = 0$

atleast one eigenvalue is zero

So, sina not all eigenvalues are +ve

(c) for no values of a, b

eigenvalues are values of λ such that

$|A - \lambda I| = 0$

eigenvalues / eigenvectors

8(7) $B = \{v_1, v_2, \dots v_n\}$

B is span of V

$z \in V$

$z = k_1v_1 + k_2v_2 + \dots k_nv_n$

$\bar{x} = c_1v_1 + c_2v_2 + \dots c_nv_n$

$0 = (k_1 - c_1)v_1 + (k_2 - c_2)v_2 + \dots (k_n - c_n)v_n$

Since $\bar{v}_i \neq 0$, $(k_i - c_i)$ need to be 0

i.e. $k_i = c_i$

(8) $x_1[n] = \sin(0.02\pi n)$ $\quad \frac{2\pi}{\omega} = \frac{2\pi}{0.02\pi} = 100 \in \mathbb{Z}$

Periodic ✓

(11) $y[n] = \sum_{n=-5}^{5} \sin(n)\ \delta[n+7]$

but $\sin(n)$ is $-5 \to 5$, so ans = 0

(14) $x(t) = u(t-2) - u(t-4)$

$u(t) \to t > 0$:1

$u(t-2) \to t \geq 2$:1

$u(t-4) \to t \geq 4$:1

but since $\delta(t) = 1$ at $t = 0$

So, ans = 0

(2) (b) $x_1[n] = \cos\left(\frac{\pi}{2} + 0.5n\right)$ $\quad f = \frac{0.5}{2\pi}$ irrational / irrational ?

$\sin = \cos\left(\frac{\pi}{2} - \omega\right)$

(1) $x_1(n) = 1 - e^{j2\pi n/5} + e^{j4\pi n/7}$

note: $e^{j\theta} = \cos\theta + j\sin\theta$

$\Rightarrow 1 - \cos\left(\frac{2\pi n}{5}\right) - j\sin\left(\frac{2\pi n}{5}\right)$

$+ \cos\left(\frac{4\pi n}{7}\right) + j\sin\left(\frac{4\pi n}{7}\right)$

$\Rightarrow \text{LCM}\left(5, \frac{7}{2}\right) \Rightarrow$

(a) $\dfrac{x(t) + x(-t)}{2} = 0$

$x(t) = -x(-t)$

$\forall t$

(b) $e^{-5t}u(t+2)$

$e^{-5t}u(t+2) = -\left[e^{5t}u(-t+2)\right]$

$\left(e^{-5t}\right)_{(t,\omega)} + \left(e^{5t}\right)_{(\omega,2)}$

$\left(e^{-5t}\right)_{(t,\omega)} + \left(e^{5\omega}\right)_{(t,\omega)}$

(5) $P = tu(t)$

$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (t\,u(t))^2 dt$

$\lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} t^2 dt$

$\lim_{T\to\infty} \frac{1}{2T}\left[\frac{t^3}{3}\right]_0^T$

$\lim_{T\to\infty} \frac{1}{2T} \times \frac{T^3}{3} \Rightarrow \frac{T^2}{6}$

$\Rightarrow \infty$

neither energy nor power signal

(9) (c) $\dfrac{u[n] - u[n-7] + u[-n] - u[-n-4]}{2} = 0$

$u[n] + u[-n] = u[n-4] + u[-n-4]$

$2 \geq n \geq 0$ $\quad 1 \geq n \geq 0$ $\quad n \geq 4$ $\quad n \geq 4$

$-2 \leq n \leq 3$

$\boxed{|n| \leq 3}$ ✓

(7) $x(t) =$

$x(-t) =$

$\text{even} = \dfrac{x(t) + x(-t)}{2} =$

odd =

* Lecture : 5

⊙ Systems and their properties

① System with memory or without memory

$x(n) \rightarrow y(n)$

$S_1 : y(n) = 2(n)$ no memory

$S_2 : y(n) = 2^n(n)$ no memory

$S_3 : y(n) = x(n-2)$ memory (depends on past input)

$S_4 : y(n) = \frac{1}{3}\{x(n) + x(n-1)\}$ memory

output depends on past-present inputs + no memory required

output needs to store previous output + requires memory

$S_5 : y(t) = z(t+2)$: requires memory

• Invertible / Non-invertible Systems

$y(n) = x(n)$ iff S_2 = inverse$\{S_1\}$

① let $S_1 : w(n) = 2x(n)$

$S_2 : y(n) = \frac{1}{2} w(n) = \frac{1}{2} \times 2x(n) = x(n)$

S_1 is invertible and S_2 is the inverse system

② let $S_1 : w(n) = \sum_{k=-\infty}^{n} x[n]$

then $S_2 : y[n] = w[n] - w[n-1]$ invertible ✓

$= \{x[-\infty] + \dots x[n-1]\} - \{x[-\infty] + \dots x[n-2]\}$

$= x[n]$

③ let $S_1 : w[n] = x(n) - x[n-1]$

$S_2 : y[n] = \sum_{k=-\infty}^{n} w[k] = x[n]$ invertible ✓

$x(n) - x(n-1) + \dots x(-1) - x(-2) + x(-2) - x(-3) + \dots - x(-\infty) + x(-\infty)$

$x(n) - x(n-1) + \dots x(-1) - x(-2) + x(-2) - x(-3) + \dots - x(-\infty) + x(-\infty)$

$x(n) - x(n-1)$

④ $S_1 : w(n) = x^2(t)$
$S_2 : y(n) = \sqrt{w(n)}$
} S_1 is not invertible

* System is invertible if and only if distinct inputs lead to distinct outputs

(one input)

• Causal Systems

$S_1 : y(n) = x(t-t_0)$: t_0 is a +ve real number

$S_2 : y[n] = x[n+3]$

System output at depending on past system output

S_1 = causal system
S_2 = non-causal system

Causality : unless there is a non-zero input, there should be no effect.

OR unless there is a non-zero input, there should not be non-zero output

OR a causal system depends on current and past input

eg) Causal system : $y(n) = \sum_{k=-\infty}^{n} x[k]$

average processing : $S_1 : y[n,m] = \frac{1}{2}\{x(n,m) + x(n-1,m) + x(n,m-1) + x(n-1,m-1)\}$ non-causal system

$x(n,m)$

• Stability System

A bounded input = $|x(n)| < M_x < \infty$ (finite value ✓)

not on the average (output) → or and t

but on the amplitude OR $|x(n)| < M_x < \infty$

This bounded input is fed to a system. If the system generates an output which is also bounded, we call it a stable system.

eg) $x(t) = u(t)$; $x[n] = u[n]$

$[0,1]$

unbounded : $x(t) = tu(t)$ or $x[n] = tu[n]$

if $|y(t)| < M_y < \infty \forall t$, for bounded finite number input,

then the system is stable

eg) $S_1 : y(n) = \frac{1}{3}\{x(n) - x(n-1) - x(n+1)\}$

$|y[n]| = \left|\frac{1}{3}x(n) - \frac{1}{3}x(n-1) - \frac{1}{3}x(n+1)\right|$

since $|x[n]| < M_x < \infty \forall n$

and the sum & difference of 3 finite terms is finite, so,

$|y[n]| < M_y < \infty \quad \forall n$

$S_3 : y[n] = nx[n]$

$|x[n]| < M_x < \infty \forall n$

bounded input stored

$\Rightarrow |y[n]| = |nx[n]| \rightarrow \infty$ as $n \rightarrow \infty$

unstable system

$S_4 : y(t) = z(2t)$

no manipulation on the years
2t means manipulation on the x-axis

• Time Invariant System

if $x(t) \xrightarrow{system} y(t)$

then $x(t-t_0) \rightarrow y(t-t_0)$

and $x(t+t_0) \rightarrow y(t+t_0)$

System output does not change if there is a change in time

if $x_1(t) \rightarrow y_1(t)$
and $x_2(t) \rightarrow y_2(t)$
and $x_3(t) = x_1(t-t_0)$

then $y_3(t)$ must be equal to $y(t-t_0)$

if it is a time invariant system

eg) $S : y(t) = z(2t)$

$\Rightarrow$ let $t_0 = 4$: $y(t-4) = z(2t-8)$

$\Rightarrow$ time variant system

$x_1(t) \rightarrow y_1(t) = z(2t)$
$x_2(t) \rightarrow y_2(t) = z(2(t-t_0))$

$y_2(t-t_0) = z(2(t-t_0))$
first delay : $x(t-t_0)$
then scaling : $x(2(t-t_0))$

$y_2(t-t_0) = z(2t-4)$
first scaling : $x(2t)$
then delay : $x(2(t-t_0))$

$S_2 : y[n] = nx[n]$

$x_1[n] \rightarrow y_1[n] = nx_1[n]$

let $x_2[n] = x_1[n-n_0]$
(w/ delayed input)

$y_2[n] = x[n-n_0]$

since $y_1[n] \neq y_2[n]$, system is time varying

note: 1st system had delayed output and we gave delayed input

• Linearity

A system is linear if the properties of additivity & homogeneity holds true

$x_1(t) \rightarrow y_1(t)$
$x_2(t) \rightarrow y_2(t)$

$\Rightarrow ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

then system is linear

eg) $y(t) = 2x(t) + 3$

$x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$

$x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$

$x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t) = 2(x_1 + x_2) + 2(3)$

but $y(x_1 + x_2) = 2(x_1 + x_2) + 3$

since, $y_1(t) + y_2(t) \neq y(x_1 + x_2)$

not a linear system

* A system which is linear and time invariant = LTI

linear & time invariant system

* Lecture: SNS 6 — 4/9/24

q) $x[n] = \cos\left(\frac{\pi}{8} n^2\right)$

periodic if $x[n] = x[n+N] \ \forall n$

$$\cos\left(\frac{\pi}{8}(n^2 + N^2 + 2nN)\right) = \cos\left(\frac{\pi}{8} n^2\right)$$

$$\cos\left(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{4} nN\right) = \cos\left(\frac{\pi}{8} n^2\right)$$

we know $\cos(2\pi\theta + k) = \cos\theta \ \forall k \in \mathbb{Z}$

$\frac{\pi}{8} N^2 \Rightarrow$ can it be a multiple of 2π?

if $N=4 \Rightarrow 2\pi = k2\pi$ ✓ ($k=1$)

true for $N=8$ as well

but not for $N=2$

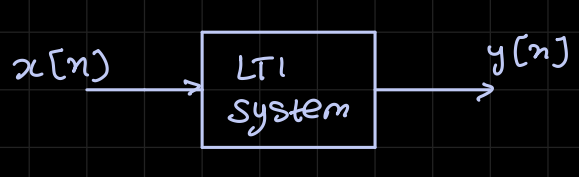
$\frac{\pi}{4} nN \to k2\pi$?

$N=4 \Rightarrow \pi 4n \to \times$

$N=8 \Rightarrow$ ✓ $2\pi nN =$ ✓

---

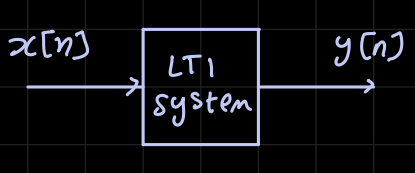
* Discrete Time convolution



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

weighted linear combination of delayed signals

* Impulse Response of an LTI System



$\delta[n] \to y[n] = h[n]$

≡ Impulse response of the LTI system

$\delta[n-1] \to h[n-1]$

$x[1]\,\delta[n-1] \to x[1]\,h[n-1]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = y[n]$$

discrete time convolution of $x[n] \times h[n] = y[n]$

Valid for LTI systems only

convolution → linear & commutative

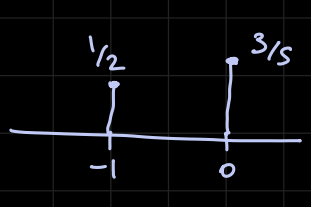
$$x[n] \times h[n] = h[n] \times x[n]$$

$$(x_1[n] + x_2[n])\,h[n] = x_1[n]\,h[n] + x_2[n]\,h[n]$$

eg) given an LTI system whose impulse response is

$$h[n] = \begin{cases} 1 & n = 0,1,2 \\ 0 & \end{cases}$$

and $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$



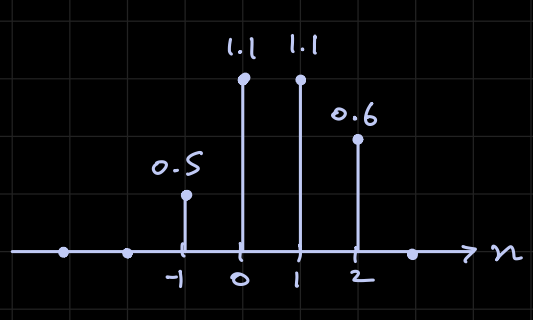
Compute $y[n]$

Ans) $y[n] = h[n] * x[n]$

$y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1]$

because for $x[n] \to y[n]$, $\delta[n] \to h[n]$

$$y[n] = \frac{3}{5}\left[\delta(n) + \delta(n-1) + \delta(n-2)\right] + \frac{1}{2}\left[\delta[n+1] + \delta[n] + \delta[n-1]\right]$$



eg) LTI System (← given)

$$h[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$$

$x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

Ans) $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

OR ≡ $\sum_{-\infty}^{\infty} h[k]\,x[n-k]$

$$\Rightarrow \sum_{k=0}^{\infty} x[n-k] = \sum_{k=0}^{\infty} \alpha^{(n-k)} u[n-k]$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{\alpha^n}{\alpha^k} u[n-k] \quad (1 : n \geq k)$$

$$y[n] = \sum_{k=0}^{n} \frac{\alpha^n}{\alpha^k} \ ✓ \qquad \frac{a(1-r^n)}{1-r}$$

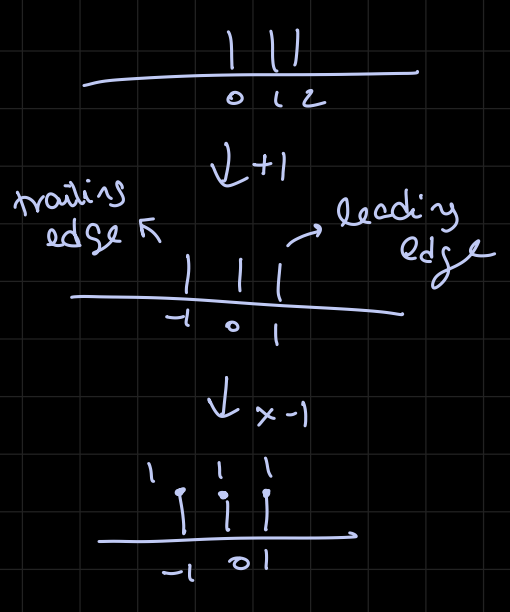
$x-1$: $-k \xrightarrow{+1} -k+1$

$k \to (1-k)$

$\downarrow +1$

$k+1$

$x-1$: $-k+1$



LECTURE: 7 04/01/24

Note: $y(n) = \text{output}$, $x(n) = \text{input}$, $y(t) = \text{output}$, $x(t) = \text{input}$

For a continuous input output system of the form

$$y(t) = x(t) * h(t)$$

for CT system, it is widely characterized by its $h(t)$. Here, two systems are said alike, each LTI and their $h(t)$ are same.

Extra question: $\int_{-\infty}^{\infty} |x(t)|^2 \, |x(2t)|^2 \, dt$

Ans $x(t) = \cos(t)$

$$= \int_{-\infty}^{\infty} |\cos(t)|^2 \, |\cos 2t|^2 \, dt = \int_{-\infty}^{\infty} \left|\frac{1+\cos 2t}{2}\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{4} + \frac{\cos(2t)}{2}\right) dt$$

Because integrated $\cos 2t = 0$ (symmetric)

$$= \int_{-\infty}^{\infty} \frac{1}{4} \, dt = \infty$$

eg) $x[n] = \begin{cases} 1 : & 0 \le n \le 4 \\ 0 : & \text{else} \end{cases}$

$x[n] = u[n] - u[n-4]$

$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \quad \alpha > 2 \\ 0 & \text{else} \end{cases}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \, h[n-k]$$

$$y[0] = \sum_{-\infty}^{\infty} x[k] \, h[-k]$$

only common point: $k=0$

$$y[0] = x[0] \, h[0] = h[0] = \alpha^0 = 1$$

but we do scaling then translation here

$h[0]^2 = h[k_1] \rightarrow h[-k_1]$

Flipping & Scaling

$h[-k] = 1$ at $-k = 0 \Rightarrow k = 0$

but we do scaling then translation here

Check: $h[-k] = 1$ at $-k+4 = 0$ because $\alpha^0 = 1$

$k = 4$

$\Rightarrow y[4] = x[k] \, h[-k]$

here, we started $h[-k]$ 1 with to the origin. So we have overlap with $x[k]$ on $k = 0, 1$

Similarly for $x[n-1]$, we have overlap with $k = 0, 1 \ldots\ldots n, n-1$

you can find the 2nd example yourself...

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \, h[n-k]$$

we use the same flipping $x[k]$ & $h[-k]$ but we scale

Verification:
$h[n] \text{ at } -k+n$
$k=1$
$\Rightarrow 1 - \frac{1}{2} = 0$
$\Rightarrow h\left(-\frac{1}{2}\right)$

Change of origin for which $(n-k) = 0$

(original)
$-2 \le n \le 0$

$h[n-k] \neq 0$: $n - k \rightarrow n$
$n - 4 \rightarrow -2$
$n - 1 \rightarrow 0$

$n - 6 \rightarrow -2$
$n - 0 \rightarrow 0$

① when $n=0$, there is no overlap with $x[n]$ hence $y[n] = 0$

② $n = 1$: 5 to 11
$n = 10$: 4 to 10

So, no overlap with $x[n]$ when $n > 10$

hence, $y[n] = 0$: $n < 0$ and $n > 10$

③ $0 \le n \le 4$: $y[n] = \sum_{k=0}^{n} x[k] \, h[n-k]$

overlap	n	range
$k=0$	0	$-6 \rightarrow 0$
$k=1$	1	$-5 \rightarrow 1$
$k=0,1,2$	2	$-4 \rightarrow 2$
$k=0,1,2,3$	3	$-3 \rightarrow 3$
$k=0,1,2,3,4$	4	$-2 \rightarrow 4$
$k=0,1,2,3$	5	$0 \rightarrow 5$
$k=1,2,3,4$	6	$1 \rightarrow 6$
$k=2,3,4$	7	$2 \rightarrow 7$
$k=3,4$	8	$3 \rightarrow 8$
$k=4$	9	$4 \rightarrow 9$

$\Rightarrow y[n]$ is defined for $0 \le n \le 10$

④ $0 \le n \le 4 \Rightarrow y[n] = \sum_{k=0}^{n} x[k] \, h[n-k]$

⑤ $4 \le n \le 6 \Rightarrow y[n] = \sum_{k=n-4}^{n} x[k] \, h[n-k]$

⑥ $6 \le n \le 10 \Rightarrow y[n] = \sum_{k=n-4}^{4} x[k] \, h[n-k]$

final $\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{-k}$: $0 \le n \le 4$

$+ \sum_{k=n-4}^{4} \alpha^{-k}$: $4 \le n \le 6$

$+ \sum_{k=n-4}^{4} \alpha^{-k}$: $6 \le n \le 10$

Convolution for CT = DT

$\sum \alpha^{-k} = \alpha^{k} \sum \frac{1}{\alpha^{k}} \rightarrow \alpha^{n} \left(\frac{1}{1-\frac{1}{\alpha}} \right)$

CTS

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \, h(t-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \, x(t-\tau) \, d\tau$$

useful when $u(t) * u(t)$

$y(t) = x(t) * h(t)$

convolution operation

eg) $x(t) = e^{-at} u(t)$: $a > 0$

$h(t) = u(t)$

leading edge $= 0$

no overlap when we shift $h(-\tau)$ on left, there will be no overlap and hence result comes out $= 0$

shift so range ✓

① $t < 0 \Rightarrow y(t) = 0$

② $t > 0 \Rightarrow y(t) = \int_0^t x(\tau) \, h(t-\tau) \, d\tau$

integral $= \frac{-e^{-at}}{a}$

starts at 0
has zero at ∞

eg) $x(t) = \begin{cases} 1 & : 0 < t < T \\ 0 & : \text{else} \end{cases}$

$h(t) = \begin{cases} t & : 0 < t < 2T \\ 0 & : \text{else} \end{cases}$

no shift

① $t < 0$: $y(t) = 0$ no overlap

② $0 \le t \le T$: $y(t) = \int_0^t x(\tau) \, h(t-\tau) \, d\tau$

③ $T \le t \le 2T$: $y(t) = \int_0^T x(\tau) \, h(t-\tau) \, d\tau$

④ $2T \le t \le 3T$: $y(t) = \int_{t-2T}^{T} x(\tau) \, h(t-\tau) \, d\tau$

* LECTURE : 8 11/6/23

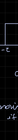
→ convolution

$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{else} \end{cases}$ $h(t) = \begin{cases} t & 0 \le t \le T \\ 0 & \text{else} \end{cases}$

easier to reverse $x(t)$

so, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

this our prefered way $\Leftarrow$ $y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$



ok $y(0) \Rightarrow$ overlap = want

so, $y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(\tau - 0)\, d\tau = 0$

moving forward, $y(1) \ldots y(n)$

i.e. we shift $x(-\tau)$ to left

i.e. $t<0$, it will never overlap

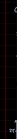
so, we must imagine $t \ge 0$

when plotting $x(t-\tau)$

case ①: $t<0$

$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

case ②: $t \ge 3T$ again no overlap



$y(t) = 0$

no non-zero overlap

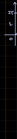
case ③: $0 < t < T$

$y(t) = \int_{0}^{t} h(\tau)\, d\tau = \int_{0}^{t} \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t$

$= \frac{t^2}{2}$

case ④: $T \le t \le 2T$

$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{t^2}{2} - \frac{(t-T)^2}{2}$



$= \frac{\cancel{t^2} - \cancel{t^2} + 2tT - T^2}{2}$

$= \frac{2tT - T^2}{2}$

case ⑤: $2T \le t \le 3T$

$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$

$= \frac{4T^2 - (t-T)^2}{2}$

$= \frac{4T^2 - t^2 + 2tT - T^2}{2}$

$= \frac{3T^2 + 2tT - t^2}{2}$

---

eg: $x(t) = e^{2t}\, u(-t)$ $h(t) = u(t-3)$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$

which one to reverse?

we can reverse both but which one is easier?

$t \ge 3$

case 2: $t \le 3$ in $h(t-\tau)$

moves to right if $t>0$

left if $t<0$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{t-3} x(\tau)\, d\tau$

$\int_{-\infty}^{t-3} e^{2\tau}\, d\tau$

$= \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t-6}}{2}$

$\left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t}}{2e^6}$

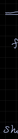
case ②: $t \ge 3$: constant: e^{2t}

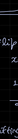
$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2}$ ✓

---

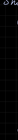
eg: $x(t) = e^{2t}\, u(-t)$: $a>0$

$h(t) = e^{-at}\, u(t)$





flip any one as similar



Shifting by t Ranges: $t>0$ $t<0$

case ①: $t<0$ case ②: $t>0$

$y(0) = \int_{0}^{\infty} x(\tau)\, h(-\tau) = \int_{0}^{\infty} e^{-2\tau}\, d\tau$

$\Rightarrow \frac{e^{-2\tau}}{-2a}\Big|_0^{\infty}$

$\Rightarrow 0 + \frac{1}{2a} = \frac{1}{2a}$

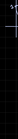
case ②: $t<0$

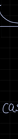
$y(t) = \int_{0}^{t-0} x(t-\tau)\, h(\tau)\, d\tau$

$t-\tau \Rightarrow$

$= \int_{0}^{t-0}$

left edge of $x(-\tau)$ will always be equal to t





for $t>0$: $t \to \infty$

for $t<0$: $0 \to \infty$

case ①: $t \ge 0$ $t \to \infty$

$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_{0}^{\infty} e^{-a\tau}\, e^{2(t-\tau)}\, d\tau$

case ②: $t<0$ $0 \to \infty$

$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_{0}^{\infty} e^{-a\tau}\, e^{2(t-\tau)}\, d\tau$

* SYSTEM PROPERTIES

given an LTI system and

$h[n] = \begin{cases} 1 & n=0, \\ 0 & \text{else} \end{cases}$

determine: input : $x[]$

output: $y[]$

impulse: $h[]$

response.

S: $y[n] = ?$ → System output = ?

apply convolution

$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

from the given def^n, $h[n] = \delta[n] + \delta[n-1]$

$\Rightarrow h[k] = \delta[k] + \delta[k-1]$

so, $y[n] = \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1])\, x[n-k]$

$= \sum_{-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{-\infty}^{\infty} \delta[k-1]\, x[n-k]$

$= x[n] + x[n-1]$

eg: find impulse response of systems

S_2: $y[n] = (x[n] + x[n-1])^2$

S_3: $y[n] = \max\{x[n], x[n-1]\}$

we know: $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

There is only one LTI system that give the same impulse response

Note system S_2, S_3 give same impulse response because they are not LTI

* lecture: 9 13/01/24

• SYSTEM PROPERTIES

① (i) Commutativity

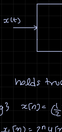
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Prove

(ii) Convolution distributes over addition

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

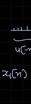


holds true for both CTS and DTS

eg: $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$

$x_1[n] = 2^n u[n]$

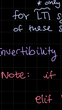
$x_2[n] = 2^n u[-n]$



easier: $y[n] = x_1[n] * h[n] + x_2[n] * h[n]$

(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



if only LTI

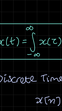
for LTI systems in cascading, order of these systems can be swapped

③ Invertibility:

Note: if $h(t)$ gives: LTI system ✓

Elif $h(t, \tau)$ gives: Non LTI system ✓

output at t — ↳ next impulse input applied at τ



eg: if $h_1(t) * h_2(t) = \delta(t)$ } contiguous deltas(t)

$$y(t) = x(t) * (h_1(t) * h_2(t)) \quad \text{— } \delta_{t_0}$$

$$= x(t) * \delta(t) \quad \text{} \}\ \text{operational definition of } \delta(t)$$

$$= x(t)$$

we do not want $\times$ $\delta(t) = 1$ at $t=0$

$\Downarrow$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau}$$ ✓

} not a scalar signal?

Em ya know, what signal with a value we a single point?

Discrete Time

$$x[n] = \sum_{-\infty}^{\infty} x[k]\, \delta[n-k]$$

$$= x[n] * \delta[n]$$

for this we can say $\delta[n] = 1 : n=0$

④ Causality

Claim: if a system is causal,

$h(t) = 0$ for $t<0$ (CTS)

$h[n] = 0$ for $n<0$ (DTS)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$ because x count is not dating values to $k<0$

hence, for causal systems,

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

OR $$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

} with $h[n] = 0$ at $n<0$

integrator: S if $h(n) = u(n)$ CTS

accumulator: S if $h[n] = u[n]$ DTS

⑤ memory / memoryless system

S_1: $y(t) = Kx(t) \Rightarrow h(t) = K\delta(t)$

S_2: $y[n] = Kx[n] \Rightarrow h[n] = K\delta[n]$

⑥ Stability (BIBO Stability Criteria)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

let input signal is bounded and $|x(t)| < M_x \ \forall t$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right|$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \le M_x \times \left(\text{some finite value, say } K\right)$$

$< \infty$ $\forall t$

if the system is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

DTS: $|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

if the system holds absolute summability,

then the system is stable

Note: $h[n] = u[n]$ } LTI, unstable, $h(t) = u(t)$ } causal, with memory

but still we use these for integrators & accumulators because we use clipper and clamper since we know the upper and lower bounds

Identity: System with $h(t) = \delta(t)$

* Constant Coefficient Difference eq for causal LTI systems

$$\sum_{k=0}^{N} b_k\, y[n-k] = \sum_{k=0}^{M} a_k\, x[n-k]$$

a_0, b_0: not dependent on time, & cannot exist

① N = order of the system

② The system is at rest if:

if $x[n] = 0 : n<n_0$

then, $y[n] = 0 : n<n_0$

$y[n-1] = 0 : n<n_0$

$y[n-N] = 0 : n<n_0$

} causality conditions

note: $N>0$

MA

→ case: 1 ⇒ $N=0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots a_M x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_1} x[n-1] + \dots \frac{a_M}{b_M} x[n-M]$$

causal system ✓ ≡ moving average system

System is called finite impulse response system

≡ FIR System

IIR/AR

→ case 2: $N=1$ and $M=0$ (simplicity)

$$b_0 y[n-1] + b_1 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0\, x[n] - \alpha_1 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\alpha_1 = \frac{b_1}{b_0}$

if $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$h[0] = \alpha_0$

$h[1] = -\alpha_1 \alpha_0$

$h[2] = +\alpha_1^2 \alpha_0$

$\vdots$

} FIR? ✗

Infinite impulse response system ✓ (IIR)

* Auto Regressive System

≡ AR system

output depends on regresses of its previous/past inputs

ARMA

→ case: 3 $N>0$, $M>0$

* Lecture : 10 18/09/24

⇒ Constant coeff differential eqn representation of causal LTI systems

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k x(t)}{dt^k}$$

N = order of the system

let N = 2 & M = 3

$$b_0 \frac{d^2 y(t)}{dt^2} + b_1 \frac{dy(t)}{dt} + b_0 y(t) =$$

$$a_0 x(t) + a_1 \frac{d\,x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a^3 \frac{d^3 x(t)}{dt^3}$$

Initial condition :

if $x(t) = 0 \quad \forall \quad t < t_0$

then $\frac{d^k}{dt^k} y(t) = 0 \quad \forall \quad t < t_0$

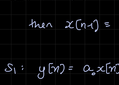
$\forall \quad k \in [0, N]$

* Block diagram representation
Useful for DTS

let $a_0 x[n]$ = scalar multiplier

then $x[n-1]$ = delay element

let S_1 : $y[n] = a_0 x[n] + a_1 x[n-1]$



S_2 : $b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$

and $\beta_0 = \frac{b_1}{b_0}$

* FOURIER SERIES

we know, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

and $x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau)$

x[k] are scalars / coefficients
we can write them as a_k

$$x[n] = \sum_{k=-\infty}^{\infty} a_k\,\delta[n-k]$$

CT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

weighted sum of frequencies
finite basis
$\phi_k(t) = e^{jk\omega_0 t}$
(deg. of oscillation)

weighted sum of vectors
combination of vectors

$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$

$\omega_0 = \frac{2\pi}{T}$

↳ fundamental time period of x(t)

f is vector
freq of f is
time period T
Cosine component
time period T
frequency
freq = 2πk / T
time period T
Sin deg. of osc
time period T
non freq component
$\phi_1(t) = e^{j\omega_0 t}$ $\quad \phi_2(t) = e^{-j\omega_0 t}$
freq of osc
fr. vector
vector
$\phi_1(t) = e^{-j\omega_0 t}$ $\quad \phi_2(t) = e^{-2j\omega_0 t}$

$\phi_0(t) = e^{j0} = e^0 \cdot 1 = \cos 0 + j\sin 0$ ✓

↳ DC component of signal
no fluctuation

can we write $\phi_2(t)$ in terms of $\phi_1(t)$?

No

they are independent of each other

inner product $\langle \phi_k(t), \phi_l(t) \rangle_T = \frac{1}{T}\int_0^T \phi_k(t)\,\phi_l^*(t)\,dt$

$$= \frac{1}{T}\int_0^T e^{jk\omega_0 t} \cdot e^{-jl\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T e^{j(k-l)\omega_0 t}\,dt$$

integral: $k\omega = m$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt$$

$$= \begin{cases} 0 & : k \neq l \\ 1 & : k = l \end{cases}$$

defining the inner product

$$\langle \phi_k(t), \phi_l(t) \rangle_T = \frac{1}{T}\int \phi_k(t)\,\phi_l^*(t)\,dt$$

(continuous time periodic signals)

⇒ Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$\{\phi_k(t)\}$ = span a vector space from an orthonormal basis

⇒ Analysis : $a_k = \langle x(t), \phi_k(t) \rangle_T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,\phi_k(t)$$

$$\langle x(t), \phi_l(t) \rangle_T = \langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_l(t) \rangle_T$$

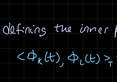
$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\langle \phi_k(t), \phi_l(t) \rangle_T}_{\delta[k-l]}$$

eg:

convolve x(t) & h(t)

firstly $h(\tau) = 1 - \tau$
$h(-\tau) = 1 + \tau$ ↻ reflect

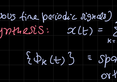
overlap



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(0) = \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

case : 1. $\quad 0 \leq t \leq \frac{1}{2}$



$$y(t) = \int_{t-1}^{-1/2} (1+\tau+t)\,d\tau + \int_{-1/2}^{t} (1+\tau+t)\,d\tau$$

case : 2 ⇒

* Lecture: 11

→ Continuous time periodic signals FOURIER SERIES

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

synthesis equation

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

eg: $x(t) = 3 + 2e^{j\omega t} + e^{j2\omega t}$ synthesis equation

$$= 3 + 2(\cos\omega t + j\sin\omega t) + \cos 2\omega t + j\sin 2\omega t$$

will keep indefinitely
will be real positive beats
and $\sin\omega t$ not disappear → periodic

$$= \sum_{k=0}^{\infty} a_k e^{jk\omega t} \qquad \left.\begin{array}{l} a_0 = 3 \\ a_1 = 2 \\ a_2 = 1 \end{array}\right\} \text{all real coeff}$$

$$\text{coefficient equation} \quad a_k = \langle x(t), \phi_k(t) \rangle \qquad \text{add other } a_k \text{ of } k = 0, 1, 2 \text{ are zero}$$

$$= \frac{1}{T}\int_T x(t)\, \phi_k^*(t)\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-jk\omega t}\, dt$$

$$\{\phi_k(t)\}_{k=\pm}^{\infty} = \text{orthogonal basis} \Rightarrow \int_T \phi_k(t)\phi_k^*(t)\, dt = T\delta[k-k]$$

$$a_k = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\, dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T \phi_m(t)\phi_k^*(t)\, dt\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T e^{j(m-k)\omega t}\, dt\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} a_m \cdot T \cdot \delta[m-k] = a_k = \text{LHS}$$

$\phi_1(t), \phi_{-1}(t) \Rightarrow$ 1st harmonic

$\phi_2(t), \phi_{-2}(t) \Rightarrow$ 2nd harmonic

and so on...

* Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$

* Analysis: $a_k = \frac{1}{T}\int_T x(t) e^{jk\omega t}\, dt$

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let's assume $x(t) = e^{st}$ where $s = j\omega$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \quad \text{let } H(s) \text{ complex}$$

$$\boxed{y(t) = e^{st} \cdot H(s)}$$

$e^{st} \rightarrow$ act as eigenfunctions of an LTI system
$H(s) =$ corresponding eigenvalue

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow e^{st} H(s)$$

↳ here the LTI system is acting as a linear operator

if I feed an LTI system with a continuous signal with some frequencies, then, it outputs a signal with same frequencies but with different amplitudes of the frequencies

output = line spectrum of $y(t)$

existing frequencies can disappear in output but a different frequency will not get added up

eg: set $H(-j\omega) = 0$
So, $a_k = H(-jk\omega) = 0$
this freq. ω will disappear in output

eg) given signal with 4 frequencies

freq domain representation

eg) S: $y(t) = x(t-2)$ and $x(t) = e^{jt}$

Ans) only delays system
observed from eq.

will it alter the signal frequencies? Intuitively, No

$$x(t) = e^{jt}$$

$$\text{Now, } \omega = 2\pi f$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = \pi \qquad \boxed{T = \pi}$$

We have only one freq. component in the line spectrum of $x(t)$

$x(t) = 1 \cdot e^{jt}$
$\frac{T}{\text{exponent}}$

$$y(t) = e^{j(t-2)} = e^{jt} e^{-j2} = e^{jt} \cdot H(j2)$$

$$\boxed{H(j2) = e^{-j2}}$$

where
ω = 2, $s = j\omega$
So, $H(s) = e^{-s}$

line spectrum of $y(t)$

no more freq. will be added

Same freq. comes out with altered amplitude

---

eg) What is $h(t)$? $\qquad h(t) = \delta(t-2)$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \delta(t-2)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-2)\, e^{-s2}\, dt \quad \Rightarrow H(j2) = e^{-s}$$

NOTE: $x(t) * \delta(t) = x(t)$

$x(t) * \delta(t-t_0) = x(t-t_0)$
shifts the signal by t_0 units

---

eg) $x(t) = \sin(\omega t)$

find and plot the line spectrum of $x(t)$

$$\begin{cases} e^{j\omega t} = \cos\omega t + j\sin\omega t \\ e^{-j\omega t} = \cos\omega t - j\sin\omega t \end{cases}$$

$$e^{j\omega t} - e^{-j\omega t} = j2\sin\omega t$$

So, $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{j2} = \frac{-1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$

2 frequencies with amplitude $\frac{-1}{2}$ and $\frac{1}{2}$

has have form $a_k e^{jk\omega t}$

So we have 2 frequencies at $k = 1$ and -1

eg) $x(t) = 1 + \sin(\omega t) + \cos(2\omega t + \frac{\pi}{4})$

Ans) $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$

$$\cos\left(2\omega t + \frac{\pi}{4}\right) = \frac{e^{j(2\omega t + \frac{\pi}{4})} + e^{-j(2\omega t + \frac{\pi}{4})}}{2}$$

$$x(t) = \overset{DC}{1} + j\left(\frac{1}{2j}\right)e^{j\omega t} + \left(\frac{-1}{2j}\right)e^{-j\omega t} + \left(\frac{1}{2}\right)e^{j(2\omega t + \frac{\pi}{4})} + \left(\frac{1}{2}\right)e^{-j(2\omega t + \frac{\pi}{4})}$$

line spectrum $x(t)$
line spectrum of $x(t)$

5 frequencies

if the question asks for absolute value line spectrum then just write the mod of the complex amplitudes

---

eg) $x(t) = \begin{cases} 1 & : |t| < T_1 \\ 0 & : T_1 < |t| < \frac{T}{2} \end{cases}$

① if $T \to \infty$, signal will become aperiodic

② There should be a DC term, so $a_0 \neq 0$

Ans

$$a_0 = \frac{1}{T}\int_T x(t)\, e^{0}\, dt = \frac{1}{T}\int_T x(t)\, dt$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} \quad \text{as } \omega = \frac{2\pi}{T} = \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{jk\omega t}\, dt = \frac{1}{T}\left[\frac{e^{jk\omega t}}{jk\omega}\right]_{-T_1}^{T_1}$$

euler

$$a_k = \frac{1}{k\pi}\sin(k\omega T_1) \qquad : k \neq 0$$

$$a_k = \frac{1}{k\pi}\sin\left(k\frac{2\pi}{4T_1}T_1\right)$$

$$a_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

line spectrum has become more frequency peaks created

LECTURE: 12 — 25/09/21

• LINEARITY

• Prove the linearity property of signals

if $x(t) \xrightarrow{\text{Fourier series coefficient}} a_k$ $\quad$ $x(t)$ and $y(t)$ are periodic with fundamental time period $=T$

and $y(t) \rightleftharpoons b_k$

prove $z(t) = \alpha x(t) + \beta y(t) \rightleftharpoons \alpha a_k + \beta b_k$

and $z(t)$ is same fundamental time period T

sol) let $z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t}$ where $\omega = \frac{2\pi}{T}$

$$C_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{T}\int_T (\alpha x(t) + \beta y(t))\, e^{-jk\omega t}\, dt$$

$$= \underbrace{\alpha \frac{1}{T}\int_T x(t) e^{-jk\omega t}\, dt}_{a_k} + \underbrace{\beta \frac{1}{T}\int_T y(t) e^{-jk\omega t}\, dt}_{b_k}$$

$$= \alpha a_k + \beta b_k \quad \text{hence proved}$$

• TIME SHIFTING

$$x(t) \xrightarrow{FS} a_k$$

find fourier series coefficients for $z(t-t_0)$ (shifted)

$$z(t) = x(t-t_0) \xrightarrow{F.S} C_k$$

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t-t_0)\, e^{-jk\omega t}\, dt$$

$$C_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega t}\, dt$$

let $\tau = t - t_0 \Rightarrow \tau$
$d\tau = dt$

let limits: $-\frac{T}{2}$ to $\frac{T}{2}$

now: $-\frac{T}{2} - t_0$ to $\frac{T}{2} - t_0$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}-t_0}^{\frac{T}{2}-t_0} x(\tau)\, e^{-jk\omega(\tau+t_0)}\, d\tau$$

$$C_k = e^{-jk\omega t_0} \underbrace{\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega\tau}\, d\tau}_{a_k}$$

$$C_k = a_k \cdot e^{-jk\omega t_0}$$

there is a shift in frequency but the magnitude remains the same

• TIME REVERSAL

$$x(t) \xrightarrow{FS} a_k$$

$$z(t) = x(-t) \xrightarrow{FS} C_k = ?$$

same time period

$$C_k = \frac{1}{T}\int_T x(-t)\, e^{-jk\omega t}\, dt$$

let $\tau = -t$ and $-\frac{T}{2}$ to $\frac{T}{2}$
$d\tau = -dt$ $\quad$ to $\frac{T}{2}$ to $-\frac{T}{2}$

$$C_k = \frac{1}{T}\int_T x(\tau)\, e^{jk\omega\tau}\, d\tau$$

reverse the limits without of $-d\tau$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)\, e^{jk\omega\tau}\, d\tau$$

$$C_{-k} = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega\tau}\, d\tau$$

$$C_k = a_{-k}$$

$$\boxed{C_k = a_{-k}}$$

no change in frequencies
only change in magnitude

• COMPLEX CONJUGATE

$$x(t) \xrightarrow{FS} a_k \qquad \text{time period } T$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

$$x(t)^* = x^*(t) \xrightarrow{FS} ?$$

$$z(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{jk\omega t}$$

$\Rightarrow$ so,

$$x(t)^* \xrightarrow{FS} a_{-k}^*$$

period changes but time period & ω = same

• CASE: 1 $\quad$ let $x(t)$ is real

$$x(t) = x^*(t)$$
$$\downarrow$$
$$a_k = a_{-k}^*$$

So, $\quad$ magnitude will be symmetric about y-axis

$\boxed{a_k = a_{-k}}$

eg $a_0^* = a_0$
$a_1^* = a_{-1}$ provided $x(t)$ is real
$a_{-1}^* = a_{+1}$

• CASE: 2 $\quad$ let $x(t)$ is real and even

$x(t) = x(-t)$
$\downarrow$
$a_k = a_{-k}$

$a_k^* = a_{-k}$

$\boxed{a_k^* = a_k}$

$\Rightarrow$ a_k are even and real

$\Rightarrow$ a_k are also real and even

• CASE: 3 $\quad$ let $x(t)$ is real and odd

$x(t) = -x(-t)$

$a_k^* = -a_{-k}$

$\boxed{a_k^* = -a_k}$

let $a_k = \alpha + j\beta \Rightarrow a_k^* = \alpha - j\beta$
but $a_k^* = -a_k$ only when real part doesn't exist in a_k

$\Rightarrow$ a_k are purely complex

• TIME SCALING

$$x(t) \xrightarrow{FS} a_k$$

$$z(t) = x(\alpha t) \xrightarrow{FS} C_k ? \qquad ; \alpha \neq -1$$

(real number (not density))

$T \to \frac{T}{\alpha}$
$\omega \to \alpha\omega$

$$z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega\alpha t}$$

$$C_k = \frac{1}{T_\alpha}\int_{T_\alpha} x(\alpha t)\, e^{-jk\omega\alpha t}\, dt$$

let $\tau = \alpha t$
$d\tau = \alpha\, dt$
and $-\frac{T}{2}\alpha$ to $\frac{T}{2}\alpha$ $\quad$ to $-\frac{T}{2\alpha}$ to $\frac{T}{2\alpha}$

$$C_k = \frac{1}{\alpha}\frac{1}{T_\alpha}\int_{T_\alpha} x(\tau)\, e^{-jk\omega\tau}\, d\tau = a_k$$

Coefficients are same
fundamental time period
and fundamental freq. changes

• multiplication

$$x(t) \xrightarrow{FS} a_k$$

$$y(t) \xrightarrow{FS} b_k$$

$$z(t) = x(t) \cdot y(t) \xrightarrow{FS} C_k = ?$$

$$C_k = \frac{1}{T}\int_T z(t) y(t)\, e^{-jk\omega t}\, dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}\, dt$$

$$y(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega t}\, dt$$

$$C_k = \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty} a_l b_m e^{j(l+m)\omega t} \cdot e^{-jk\omega t}\, dt$$

$$C_k = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_l b_m \underbrace{\left(\frac{1}{T}\int_T e^{j(k-l-m)\omega t}\, dt\right)}_{=1}$$

$$C_k = \sum_{l=-\infty}^{\infty} a_l b_{m} \qquad \begin{cases} \text{when } k-l-m=0 \\ k = l+m \\ \text{or} \\ l = k-m \end{cases}$$

$$= \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

■ multiplication
in the time domain leads to convolution in the frequency domain and VICE VERSA

• PARSEVALS RELATION

$$x(t) \xrightarrow{FS} a_k$$

$$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Parseval's relation states that the total power of a signal is partitioned in the frequency domain and vice versa

* DISCRETE TIME FOURIER SERIES

given a discrete time periodic signal $x[n]$

SYNTHESIS

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega n} \qquad \omega = \frac{2\pi}{N}$$

fundamental time period $= N$

Some arbitrary time period

$$\text{Domain} = 2\pi \to 2\pi - \omega \to 2\pi - 2\omega \to \dots \to N\omega$$

$$\to 0 \to \frac{2\pi}{N} \to \frac{4\pi}{N} \to \frac{(N-1)2\pi}{N} \to 2\pi$$

frequencies are also periodic

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n]$$

where $\phi_k[n] = e^{jk\omega n}$

$$\phi_{k+N}[n] = \phi_k[n] = e^{jk\omega n}$$

$$\langle x[n], \phi_l[n]\rangle_N = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jl\omega n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\left(\sum_{k=\langle N\rangle} a_k e^{jk\omega n}\right) e^{-jl\omega n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-l)\omega n}$$

$$= \sum_{k=\langle N\rangle} a_k \underbrace{\left(\frac{1}{N}\sum_{n=\langle N\rangle} e^{j(k-l)\omega n}\right)}_{=1}$$

$$\text{when } k-l=0 \Rightarrow k=l$$

$$\boxed{a_l = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, \phi_l^*[n]}$$

Q) $x(t) = \cos\omega_0 t \qquad \omega_0 = \frac{2\pi}{3}$

$a_k = ?$

$= \frac{1}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$

$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

$\Rightarrow a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{1}{2j}$

a_k is non zero only for the complex exponentials present in $x(t)$

Note: NT = 3, we need to see $a_1, a_{-1}, a_2, a_{-2}, a_3, a_{-3}$

$\Rightarrow a_0 = a_3 = (1)\frac{1}{2j} - \frac{1}{2j} = 0$

Conclusion on coeff. : frequency

periodicity in both domains

(both required for true periodicity)

eg3) $x(t) = 1 + \sin\frac{2\pi}{3}t + 1\cos\frac{2\pi}{3}t = 1 + \frac{1}{2}\left(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}\right)$

DC component = 1

So, $a_0 = 1$ and so over $\frac{2\pi}{3}$

also, every AC component has its periodic part related to unique slots which it periodic with but we can adjust slots

so we periodic with N

$x(t) = 1 + \frac{e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}}{2j} + \frac{1}{2}\left(\frac{e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}}{2}\right)$

CASE 3: N = 3, 6

$a_0 = 1$

$a_1 = \frac{1}{2} + \frac{1}{4}$

$a_2 = \frac{1}{2j} + \frac{1}{4}$

$a_3 = \frac{1}{2j}$

$a_4 = \frac{1}{4}$

for N=3 : no sum overlapping occurs in every N

for N=6 : two sum overlapping occurs in every N

for N=12 : two, three, four, five, six, seven, eight, nine, ten, eleven, twelve

$\Rightarrow$ for N = 3, $\frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}$

same as original function

$\frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}, \frac{1}{2j}, 0, \frac{1}{2j}, \frac{1}{2}$

is $E(x(\frac{2\pi}{3}t))$ periodic?

$z(t) = x(t) + x(-t)$

periodic N=3, $\omega_0 = \frac{2\pi}{3}$

but $E(x(\frac{2\pi}{3}t))$ is not periodic

$3\cos(\frac{2\pi}{3}t) + 3\cos(\frac{2\pi}{3}t)$ is also periodic with N=3, $\omega_0 = \frac{2\pi}{3}$

$\cos\left(\frac{4\pi}{3}t + \frac{\pi}{3}\right) = 2\cos\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)$

$\frac{3}{2}\cos\left(\frac{4\pi}{3}t\right), N=3$

all are periodic

hence conjugate was not

coeff.

$z(t) = 1 + \frac{1}{2j}\left(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}\right) + \frac{1}{2}\left(\frac{e^{j\frac{4\pi}{3}t} + e^{-j\frac{4\pi}{3}t}}{2}\right) - \left(\frac{e^{j\frac{4\pi}{3}t} - e^{-j\frac{4\pi}{3}t}}{2j}\right)$

$N_{new} = 6$

note: $e^{j\frac{4\pi}{3}t}$ is periodic with N=3 but the period before N_{new} has to be 6 as periodic with N as well.

$z(t) = 1 + \left(\frac{1}{2j} + \frac{1}{2} + \frac{1}{2}\right)e^{j\frac{2\pi}{3}t} + \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2}\right)e^{j\frac{4\pi}{3}t}$

new zeros

we have a_0, a_1, a_2

eg4) $x(t)$ = periodic with period N

Some real F.S. coeffs of an periodic signal.

$g(t) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(t-kT)$

and periodic with period N

$g(t) = x(0)\delta(t) + x(1)\delta(t-T) + \dots$

$0, 1, \dots, N-1 \longrightarrow N, N+1, \dots, 2N-1$

So, there know will keep on repeating

so, time periodic with NT

$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\frac{2\pi}{NT}t}$

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt$

like $N=5$

$g(t)$ is not $x(0) + x(1) + x(2) + x(3) + x(4)$ period

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt = \frac{1}{NT}\int_{0}^{NT} x(0)\delta(t) + x(1)\delta(t-T) + \dots + x(N-1)\delta(t-(N-1))\,dt$

So, there know will keep on repeating

so, time periodic with NT

$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\frac{2\pi}{NT}t}$

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt$

like $N=5$

$g(t)$ is not $x(0) + x(1) + x(2) + x(3) + x(4)$ period

$a_n = \frac{1}{NT}\int_{NT} g(t)\,dt = \frac{1}{NT}\int_{0}^{NT} x(0)\delta(t) + x(1)\delta(t-T) + \dots + x(N-1)\delta(t-(N-1))\,dt$

$x(n) \xrightarrow{DTFT} b_n$

$a_n = \frac{b_n}{NT}$ Summation is divided by time period

$a_n = \frac{1}{NT}\int_{NT} g(t)\,e^{-jn\frac{2\pi}{NT}t}\,dt$

$a_n = \frac{1}{NT}\int_{NT} \sum_{k=0}^{N-1} x(k)\,\delta(t-kT)\,e^{-jn\frac{2\pi}{NT}t}\,dt$

$= \frac{1}{NT}\int_{NT} \sum_{k=0}^{N-1} x(k)\,e^{-jn\frac{2\pi}{NT}k}\,\delta(t-k)\,dt$

$a_n = \frac{1}{NT}\sum_{k=0}^{N-1} x(k)\,e^{-jn\frac{2\pi}{NT}k}$

$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{x(k)}{T}\right)e^{-jn\frac{2\pi}{N}k}$

eg5) a) $x(t)$ is a real signal

b) $x(t)$ is periodic with T=4

c) $a_{k+3} = 0$ for $|k| \geq 1$

d) A signal with F.S coeff $b_k = e^{jk}\,a_k$

e) $\frac{1}{T}\int_{T} |x(t)|^2\,dt = \frac{1}{2}$

(find?) from (a) we get $a_k = a_{-k}$ (cc)

so, $b_k = e^{jk}\,a_k$ exists as conjugate symmetry

(b) $\rightarrow$ same power in one time period

(using parseval) eq^n

$\frac{1}{T}\int_{T} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

$\Rightarrow |a_{-1}|^2 + |a_0|^2 + |a_1|^2 = \frac{1}{2}$

(c) $\rightarrow$ real signal :

$a_k^* = a_{-k}$

so, $a_k^* = a_k$

and $a_k = a_{-k}$

(d) $\rightarrow$ if $x(t) = x(-t)$

then coeff of $x(t)$ are real

so, if $x(t) = x(-t)$ and $x(t)$

$x(t) \xrightarrow{FS} a_k$

$x(-t) \xrightarrow{FS} a_{-k}$

$x(-t) = x(t) \xrightarrow{FS} ?$

$x(-t) = \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt$

$= \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt$

like $t \to -t$ and $dt \to -dt$

$= \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt$

$= a_k$

so stage 2 = real + odd

$\Rightarrow$ coefficients are purely imaginary

answer : $\operatorname{Im} = 0$

$\operatorname{Im}(a_k) = \frac{1}{2j}$

and b_1, b_2 are purely imaginary

like $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$

so general equation :

$b_1 = \frac{1}{2}j, \quad b_2 = \frac{1}{2}j$

So, $|b_k|^2 = |j|^2 = \frac{1}{4}$

so we have a signal which (2 poles) and $b_k = e^{-jk}\,a_k$

also signal is odd

So we found $z(t+1)$ to be a reversing function

except reversing lower is not showing so positioned opp half for $z(t)$

after $t = 0$ we

$a_{k+3} = 0$ because $|k| \geq 1$ (only 1 is allowed)

So we have only z coeff. Plugging $\frac{1}{2}j$

$\frac{1}{2}j$

(e) we have a signal which (2 poles) and $b_k = e^{-jk}\,a_k$

also signal is odd

So we found $z(t+1)$ to be a reversing function

except reversing lower is not showing so positioned opp half for $z(t)$

after $t = 0$ we

$a_{k+3} = 0$ because $|k| \geq 1$ (only 1 is allowed)

So we have only z coeff. Plugging $\frac{1}{2}j$

$\frac{1}{2}j$

eg9) D $x(t)$ is periodic with $N=6$

1) $\sum_{n=0}^{5} x(n) = 2$

2) $\sum_{n=0}^{5} (-1)^n\,x(n) = 1$

3) $x(n)$ has the max power per period among all $x(t)$ of the $x(t)$ satisfying the frequency & time shift

$a_{k+3} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,e^{-jk\frac{2\pi}{N}n}$

$= \frac{1}{6} + \frac{1}{6}e^{-jk\pi}$

$a_k = \frac{1}{6}$

$a_1 = \frac{1}{6} + \cos(-\pi n) + j\sin(-\pi n)$

$a_1 = \frac{1}{6} + \cos(\pi n) - j\sin(\pi n)$

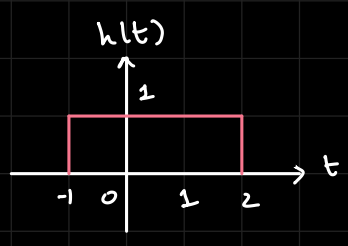
$a_2 = \frac{1}{6} + \cos(2\pi n)$

$a_3 = ?$

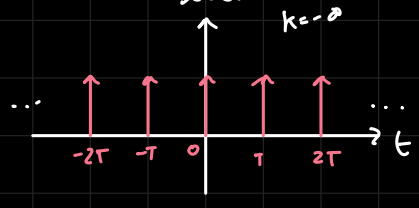
2) $\sum_{n=0}^{5} (-1)^n\,x(n) = 1$

$\Rightarrow \

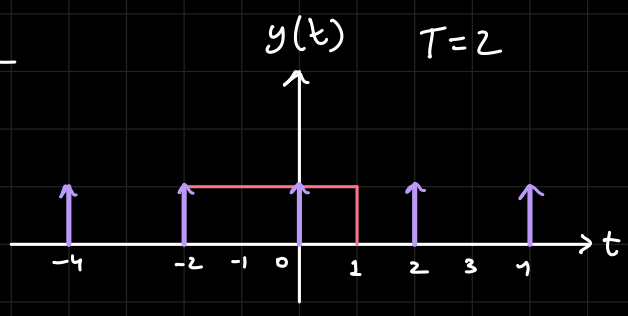
* SNS : midsem 2023

① 

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$



(a) $T = 2$



for one period:

$t = 0$

$y(0) = \begin{cases} 1 & : -2 \\ 1 & : 0 \end{cases}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

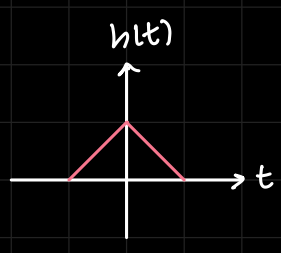
$$y(t) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(\tau-2k)\, h(t-\tau)\, d\tau$$

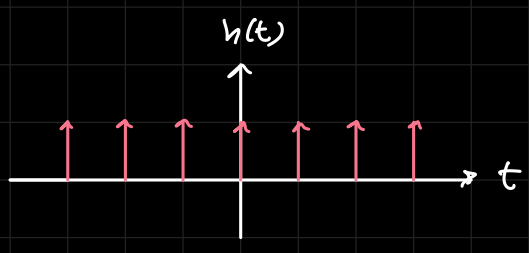
$$= \int_{-2-t}^{1-t} \sum_{k=-\infty}^{\infty} \delta(\tau-2k)\, d\tau$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \int$$

similar question from book:

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$





$$y(t) = x(t) * h(t)$$

$$x(t) * \delta(t) = \int_{\tau} x(\tau)\, \delta(t-\tau)\, d\tau$$

since $\delta(t-\tau) = 0$ for $t \neq \tau$

so, $x(t) * \delta(t) = x(t)$

similarly,

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

given : $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

* Lecture : 14

Filter : $e^{st} \rightarrow e^{st} \cdot H(s)$

$s = j\omega$

attenuates or passes amplitude of certain frequencies

High pass filter
↓
image's edges
freq. attenuated
hence feels blurred

for CTS
$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

DTS:

$$x[n] = z^n \longrightarrow \boxed{LTI} \longrightarrow y[n] = ?$$
$h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}_{H(z)}$$

$$y[n] = z^n H(z)$$

let $z = e^{j\omega_0}$

$$x[n] = e^{j\omega_0 n} \longrightarrow e^{j\omega_0 n} \cdot H(e^{j\omega_0})$$
↓
Z Transform (last chp)

DT FS

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$h[n]$ | LTI system
↓

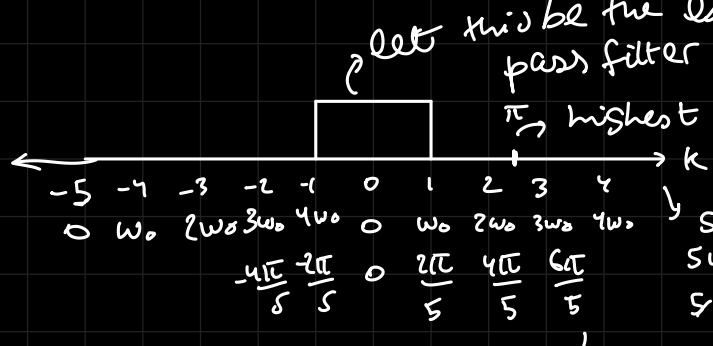
$$y[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \cdot H(e^{jk\omega_0})$$

---

given : $x[n] = \sum_{k=\langle 5 \rangle} a_k e^{jk\frac{2\pi}{5}n}$

$a_0 = 1$
$a_1 = a_{-1} = \frac{1}{2}$
$a_2 = a_{-2} = \frac{3}{4}$

$$x[n] = 1 + \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n} + \frac{3}{4} e^{j2\omega_0 n} + \frac{3}{4} e^{-j2\omega_0 n}$$



$2\pi - \frac{4\pi}{5}$
↓
0

$= -\frac{4\pi}{5} = \frac{6\pi}{5}$

freq. response of filter

let $x = 1$ for this case

$H(0) = x$
$H(e^{j\omega_0}) = x$
$H(e^{-j\omega_0}) = x$
$H(e^{j2\omega_0}) = 0$
$H(e^{-j2\omega_0}) = 0$

Midterm Review

Name:
Period:

Ch 1 Quiz 1 ~ Ch 1 Quiz 2

1) $y = \frac{1}{x}$

2) $y = \frac{1}{x^2}$

3) $y = \frac{1}{x^3}$

4) $y = \frac{1}{x^4}$

5) $y = \frac{1}{x^{-1}} = x$

6) $y = \frac{1}{x^{-2}} = x^2$

7) $y = \frac{1}{x^{-3}} = x^3$

8) $y = \frac{1}{x^{-4}} = x^4$

9) $y = \frac{1}{x^{-5}} = x^5$

10) $y = \frac{1}{x^{-6}} = x^6$

Ch 2 Quiz 1 ~ Ch 2 Quiz 2

11) $y = \frac{1}{x+1}$

12) $y = \frac{1}{x^2+1}$

13) $y = \frac{1}{x^3+1}$

14) $y = \frac{1}{x^4+1}$

15) $y = \frac{1}{x^5+1}$

16) $y = \frac{1}{x^6+1}$

Ch 2 Quiz 3

17) $y = \frac{1}{x^2+2x+1}$

$y = \frac{1}{(x+1)^2}$

18) $y = \frac{1}{x^2+2x+2}$

$y = \frac{1}{(x+1)^2+1}$

19) $y = \frac{1}{x^2+2x+3}$

$y = \frac{1}{(x+1)^2+2}$

20) $y = \frac{1}{x^2+2x+4}$

$y = \frac{1}{(x+1)^2+3}$

Ch 2 Quiz 4

21) $y = \frac{1}{x^2-2x+1}$

$y = \frac{1}{(x-1)^2}$

22) $y = \frac{1}{x^2-2x+2}$

$y = \frac{1}{(x-1)^2+1}$

23) $y = \frac{1}{x^2-2x+3}$

$y = \frac{1}{(x-1)^2+2}$

24) $y = \frac{1}{x^2-2x+4}$

$y = \frac{1}{(x-1)^2+3}$

Ch 2 Quiz 5

25) $y = \frac{1}{x^2-4}$

$y = \frac{1}{(x-2)(x+2)}$

$y = \frac{1}{x^2-4}$

$y = \frac{1}{x^2-4}$

$y = \frac{1}{x^2-4

- Formulas

odd : $x(-t) = -x(t)$
even : $x(-t) = x(t)$

$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

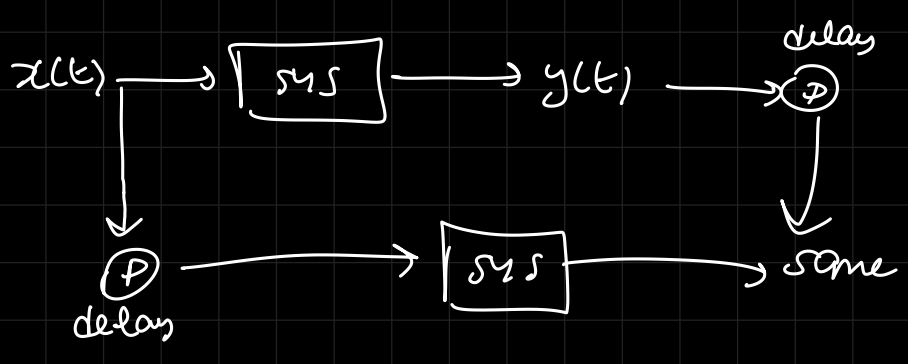
$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

Energy of signal = $\int_{-\infty}^{\infty} |x(t)|^2 dt$ CTS

$\sum_{n=-\infty}^{\infty} |x[n]|^2$ DTS

Power of signal : $\lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x(t)|^2 dt$ CTS

$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} |x[n]|^2$ DTS



time invariance

linear : no time scaling
coeff of i/p & o/p should be constant

time invariance : no added/subtracted term besides input and output

- convolution

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

for memoryless systems: $h[n] = k\delta[n]$
i.e. $k = h[0]$
and $y(t) = kx(t)$

for invertible systems :

if $h_2(t)$ is the imp. response of system which is inverse of sys with $h_1(t)$ as the impulse response then,

$$h_1(t) * h_2(t) = \delta(t)$$

BIBO stability: $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$ ✓

unit step response : $s(t)$

$$s(t) = \int h(\tau)\,d\tau$$

$$h(t) = \frac{d}{dt} s(t)$$