14/08/24 # Signals And Systems 11:00AM

Evaluation : (relative grading)

yoooooooooooo :fire emoji:

Attendance	10%
Quiz	16%
Assignments	15%
mid sem	24%
end sem	35%

Book: Signals & System 2nd edition by Alan V. oppenheimer

Lecture : 1 14/08/24 : 11:00AM

* Linear Algebra Revision

⇒ GROUP : consists of ....

- A set G
- A rule / binary operation "$*$"
  - a. associative
    $x*(y*z) = (x*y)*z \quad \forall x,y,z \in G$
  - b. There exists an element "e" called the identity of group G such that
    $e*x = x*e = x \quad \forall x \in G$
  - c. $\forall x \in G, \exists x^{-1}$ such that
    $x*x^{-1} = e = x^{-1}*x \quad \forall x \in G$
  - d. if $x*y = y*x \quad \forall x,y \in G$, this group is called

Commutative / Abelian Group

Note: we only need (1), (2.a), (2.b), (2.c), for a set to be a group.

a

eg: A set of invertible $n \times n$ matrices with binary operation = matrix multiplicati
But it is not an Abelian Group

eg: set of continuous time periodic signals with time period T with "$*$" = "$+$"

⇒ FIELD : consists of the following

- A set F
- Two binary operations "+" and "."
  such that ....
  - $(F,+)$ is an abelian group
  - define $F^* = F - \{0\} \Rightarrow (F^*, \cdot)$ is an abelian group
  - multiplication operation distributes over addition
    - left distributive
      $x\cdot(y+z) = xy + xz \quad \forall x,y,z \in F$
    - Right distributive
      $(x+y)\cdot z = xz + yz \quad \forall x,y,z \in F$

eg: F = Real Numbers $\mathbb{R}$

* Vector space : A set V with a map...

- '+' : $V \times V \to V$
  $(v_1, v_2) \to (v_1 + v_2)$ called vector addition
- '.' : $F \times V \to V$
  $(a, v_1) \to (a \times v_1)$ called scalar multiplication

... is called a F-vector space or vector space over the field F if the following are satisfied:

- $(V,+)$ is an abelian group
- $a\cdot(v_1+v_2) = av_1 + av_2$
- $(a_1+a_2)\cdot v = a_1 v + a_2 v$
- $(\alpha\beta)v = \alpha(\beta v) = \beta(\alpha v)$
- $1 \cdot v = v$
- $a \cdot 0 = 0$
- $0 \cdot v = 0$
- if $v \neq 0$, then $a \cdot v = 0$ implies $a = 0$
- if V is a vector space over field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V

* METRIC SPACE : metric is a map

$$d : X \times X \to R$$

satisfies the following: $(\forall x,y,z \in X)$

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

This map is called a metric and a set equipped with this map is called a metric space and is denoted by (X,d)

eg: Euclidean distance

Note: metric is merely a generalization of the notion of distance

* NORM : let V be a F-vector space.

A map $\|\cdot\| : V \to R$ is called a norm if it satisfies ...

- $\|\bar{v}\| \geq 0$ and $\|\bar{v}\| = 0$ if $\bar{v} = 0$
- $\|a\bar{v}\| = |a| \, \|v\|$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

A vector space equipped with a norm is called a normed vector space m

eg: let V be a F-vector space with a nor

prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric

Lecture : 2

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* Inner Product :

let V be a F-vector space
A map,

$$<,> : V \times V \to F$$

is called an inner product if ...

- $<v,v> \geq 0$ and $<v,v> = 0$ iff $v = 0$
- $<v_1, v_2> = \overline{<v_2, v_1>}$ " $-$ " : complex conjugate operation
- it is linear in the first co-ordinate
  $$<a_1 v_1 + a_2 v_2, w> = a_1 <v_1, w> + a_2 <v_2, w>$$
  $\forall\ v_1, v_2, w \in V$ and $a_1, a_2 \in F$
- it is conjugate linear in the second co-ordinate
  $$<v, a_1 w_1 + a_2 w_2> = \bar{a}_1 <v, w_1> + \bar{a}_2 <v, w_2>$$
  $\forall\ w_1, w_2, v \in V$ and $a_1, a_2 \in F$

measures cosine similarity
" norm(x) . norm(y) . cos θ

eg: dot product

$$<x,y> = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

complex inner Product

$$<x,y> = \sum_{i=1}^{n} x_i \cdot \overbrace{\bar{y}_i}^{\text{complex conjugate}}$$

* Linear Independence of Vectors

A set of vectors $v_1, v_2, \dots v_n$ in $V_n(F)$ is called linearly independent (LI) if

$$\sum_{i=1}^{n} a_i v_i = 0 \quad \text{implies } a_i = 0\ \forall i$$

Note: In a 2d space, we represent all non-colinear vectors using 2 vectors

Definition : The number of maximal LI vectors in a vector space V is called the DIMENSION of the vector space and the maximal LI vectors is called a BASIS for V.

if $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ is a basis for $V_n(F)$, then $\forall\ v_i \in V$ & $a_i \in F$

weighted linear combination of vector V

$$V = \sum_{i=1}^{n} a_i e_i$$

$V = a_1 e_1 + a_2 e_2 + \dots a_n e_n$

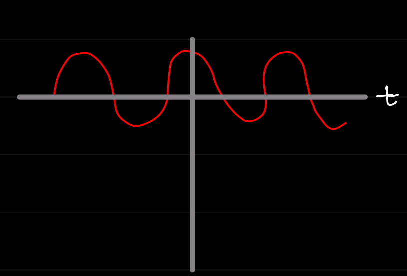
* ORTHOGONAL & ORTHONORMAL Basis

A set of basis vectors $(v_1, v_2 \dots v_n)$ spanning an inner product space V if :

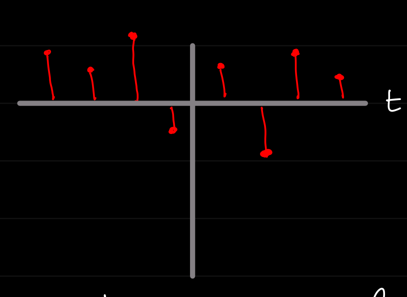
$$v_i \neq 0 \quad \forall\ i$$
$$<v_i, v_j> \neq 0 \quad \forall\ i \,\&\, j$$

SIGNALS (CTS & DTS)

- continuous time signal



- Discrete time signal



- Even Signal : symmetric about vertical axis
  $x(t) = x(-t)\ \forall t$
  eg: cosine function

- Odd Signal : non-symmetric about vertical axis
  $x(-t) = -x(t)\ \forall t$
  eg: sine function

signal notation: $x(t)$ continuous time signal
$x[n]$ discrete time signal

images are two dimensional signal
videos are 3d
videos with audio : 4d

* Even/odd component of a signal:

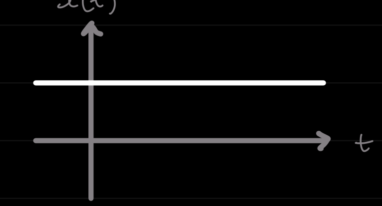
$$\text{even}\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$\text{odd}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

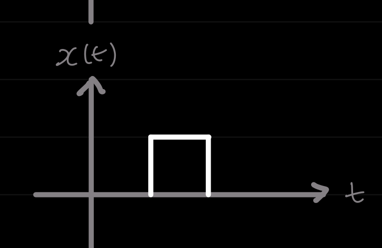
* PERIODIC : if $x(t) = x(t+T)\ \forall t$ then T is the time period

The smallest time period T defined for this property is called the fundamental time period

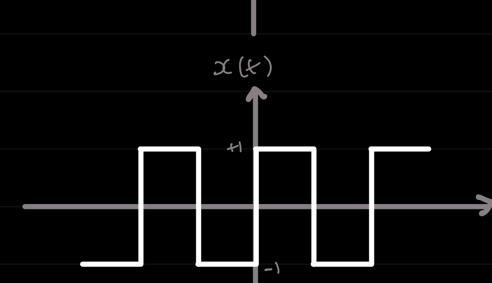
eg:



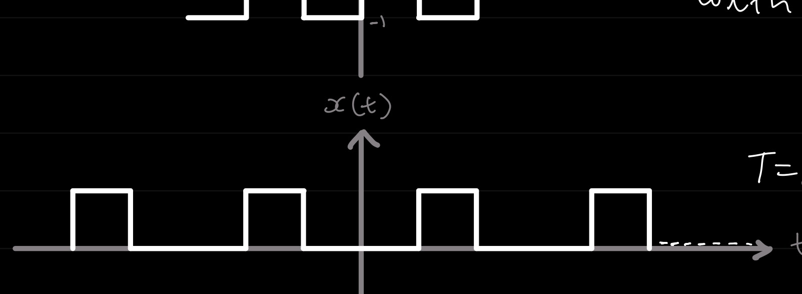
Not periodic



$T \to \infty$
Aperiodic



Periodic with T=2



T=3

T = 8

classification { Probabilistic
Dataministic

* ENERGY OF SIGNAL

Continuous Time signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Discrete Time signal

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

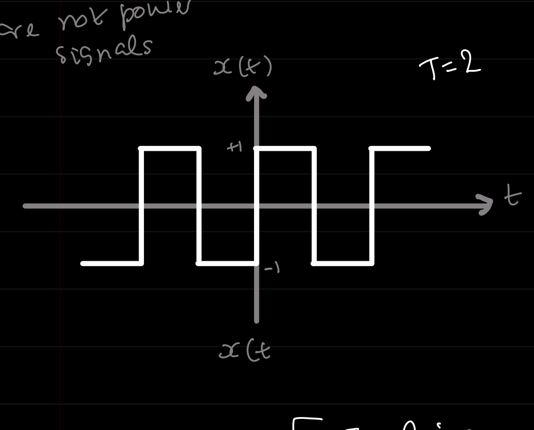
Note: Periodic Signals are power signals ✓
Aperiodic signals are not power signals ✗

T = ∞
so, power = not defined
hence, Aperiodic signals are not power signals

avg. energy in
" a time duration

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

T=2



compute power?

$$P = \lim_{T \to \infty} \frac{1}{2T} \int$$

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{x=-N}^{N} |x[n]|^2$$

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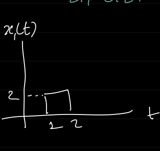
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$



$$= \int_{1}^{2} |2|^2 dt = 4\int_0^2 dt$$

$$= 4 < \infty$$

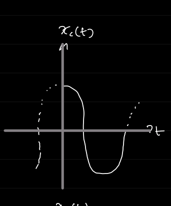
↑ energy signal

$x_1(t) \equiv$ energy signal, continuous signal, aperiodic, deterministic

$x_1(t) \neq x_1(t+T) \Rightarrow$ aperiodic signal

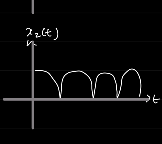
$x_2(t) =$ continuous signal, periodic signal, deterministic, power signal

we know its value at every point t → not probabilistic



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

after mod ⇒



$$E = \underbrace{\lim_{T_0 \to \infty} \int_{-T_0}^{T_0} |x(t)|^2 dt}_{\infty}$$

Since $E \to \infty$ and not $E < \infty$, this signal is not an energy signal

$$P = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} |x(t)|^2 dt$$

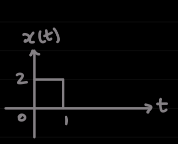
intuition $\left\{ \approx \frac{\text{infinite block}}{\text{infinite block}} \right.$ = finite

$\boxed{0 < P < \infty} \Rightarrow$ Power Signal ✓

* TRANSFORMATION OF SIGNAL

① Translation

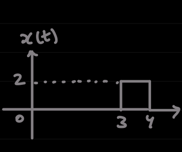
energy of signal doesn't change



(a) DELAY

$y(t) = x(t - t_0)$

let $t_0 = 3$

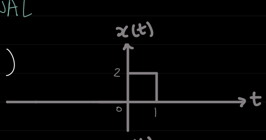


$y(t) = x(t-3)$

$y(0) = x(-3) = 0$
$y(3) = x(0) = 2$
$y(4) = x(1) = 2$
$y(5) = x(2) = 0$

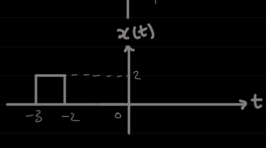
} verification of transformation

(b) ADVANCE A SIGNAL



$y(t) = x(t + t_0)$

let $t_0 = 3$

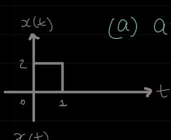


$y(-3) = x(0) = 2$
$y(-2) = x(1) = 2$
$y(-1) = x(2) = 0$
verified

energy of scale changes in accordance with area under curve

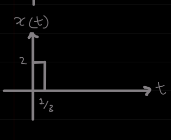
② Scaling

$y(t) = x(at)$



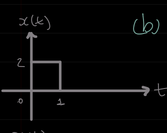
(a) $a > 1$: compression of the signal on the t-axis

let $a = 3$



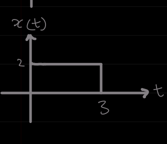
$y(t) = x(3t)$

$y(0) = x(0) = 2$
$y(\frac{1}{3}) = x(1) = 2$
$y(\frac{2}{3}) = x(2) = 0$



(b) $0 < a < 1$: Dilation on t-axis

let $a = \frac{1}{3}$

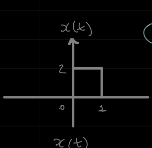


$y(t) = x(at) = x(t/3)$

$y(0) = x(0) = 2$

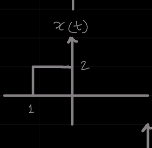
$y(3) = x(1) = 2$

$y(6) = x(2) = 0$

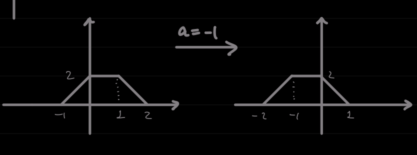


ⓒ SPECIAL CASE: $a = -1$

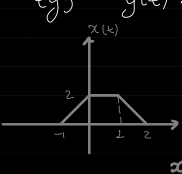
$y(t) = x(-t)$

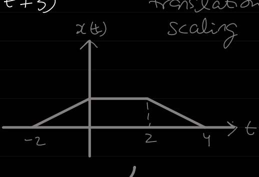


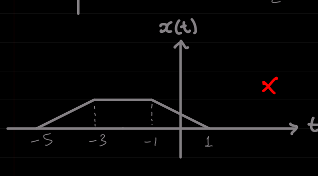
flipped on the t-axis ≡ time reversal



eg} $y(t) = x(2t+3)$ translation + scaling



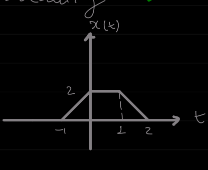




scaling then translation because otherwise we get $2t+6$ ✗

here,
scaling → translation ✗
translation → scaling ✓

$x(t)$
↓ translate
$x(t+3)$
↓ scale
$x(2t+3)$



* FUNDAMENTAL SIGNALS

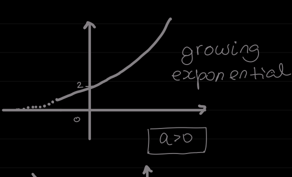
• Continuous Time Signals

$$x(t) = ce^{at}$$

• CASE: 1 $c \equiv$ real constant let $c = 2$
$a \equiv$ real constant

$x(t) = 2e^{at}$

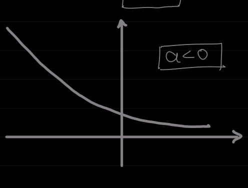
① let $a = 1$



growing exponential

$\boxed{a>0}$

② let $a = -1$



$\boxed{a<0}$

③ let $a = 0$

• CASE: 2 $c \equiv$ real constant
$a \equiv$ purely imaginary $= j\omega$

$x(t) = e^{j\omega t}$ $\forall t$

periodic if $x(t) = x(t+T)$

$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t} \cdot e^{j\omega T} = e^{j\omega t} = x(t)$
↳ if $e^{j\omega T} = 1$

$e^{j2\pi k}$

$e^{j\omega T} = 1$ if
$\omega T = 2\pi k$

since we need to find the fundamental time period (smallest interval), let $K = 1$

So, $\omega T = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$ fundamental time period

$$E = \int_{-\infty}^{\infty} |e^{j\omega t}|^2 dt = \int_{-\infty}^{\infty} 1\, dt = \infty$$

• CASE: 3 $C \equiv$ complex number
$a \equiv$ real number

$x(t) = |c| e^{j\theta} \cdot e^{at} = |c| e^{at + j\theta}$

e^{at} will define if it is growing / decaying exponential

we can also say $|c|e^{j\theta} = c\cos(\theta) + c \cdot j\sin(\theta)$

$Re\{x(t)\} = |c| \cdot \cos\theta \cdot e^{at}$

$Im\{x(t)\} = |c| \cdot \sin\theta \cdot e^{at}$

• CASE: 4 $C \equiv$ complex number $= |c|e^{j\theta}$
$a \equiv$ complex number $= r + j\omega$

$x(t) = ce^{at} = |c|e^{j\theta} \cdot e^{(r+j\omega)t}$

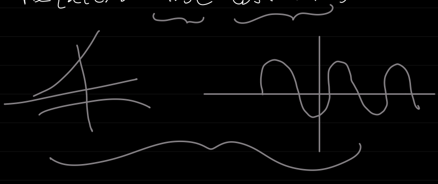
$$= \underbrace{|c|e^{rt}}_{\substack{\text{envelope with either} \\ \text{increasing or decaying} \\ \text{exponential}}} \cdot \underbrace{e^{j(\omega t + \theta)}}_{\substack{\cos(\omega t+\theta) + j\sin(\omega t+\theta) \\ \text{cosinosoidal signal} \quad \text{sinosoidal signal}}}$$

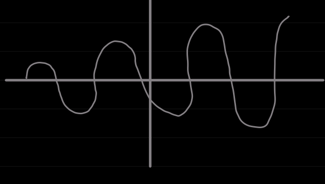
$Re\{x(t)\} = |c|e^{rt} \cdot \cos(\omega t + \theta)$

$Im\{x(t)\} = |c|e^{rt} \cdot \sin(\omega t + \theta)$

let $r = 2$, $|c| = 1.5$

$Re\{x(t)\} = 1.5e^{2t} \cos(\omega t + \theta)$





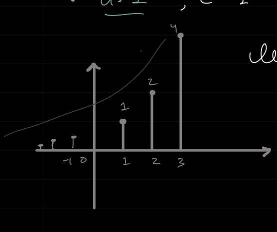
• Discrete Time Signals

$$x[n] = c\alpha^n$$

• Case 1: $C \equiv$ real constant
$\alpha \equiv$ real

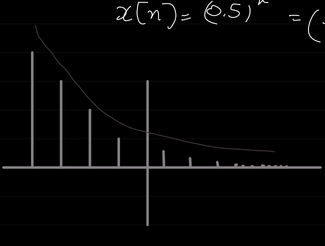
a. $\alpha > 1$, $c = 1$ $x[n] = \alpha^n$

let $\alpha = 2$, $x[n] = 2^n$



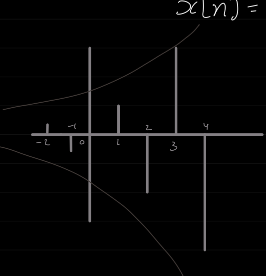
b. $\alpha < 1$ let $\alpha = 0.5$

$x[n] = (0.5)^n = (\frac{1}{2})^n$



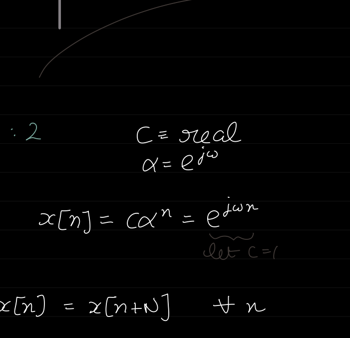
c. $\alpha < -1$ let $\alpha = -2$

$x[n] = (-2)^n$



d. $-1 < \alpha < 0$ let $\alpha = \frac{-1}{2}$

$x[n] = (\frac{-1}{2})^n$



• Case : 2 $C \equiv$ real
$\alpha = e^{j\omega}$

$x[n] = c\alpha^n = \underbrace{e^{j\omega n}}_{\text{let } c=1}$

if $x[n] = x[n+N]$ $\forall n$

then $x[n]$ is periodic

$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N} = e^{j\omega n}$
if $e^{j\omega N} = 1$

$e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi K$

here we cannot say $K = 1$ for smallest interval necessarily because we want N to be an integer

$N = \frac{2\pi K}{\omega}$ ≡ has to be integer

$\frac{N}{K} = \frac{2\pi}{\omega}$ ≡ has to be rational number

Note: ω has to be a multiple of π

* Lecture: 4

28/08/24

- for continuous time signals
  → frequency is unique $(0 \to \infty)$

- for discrete time signal
  → frequency $\in [0, 2\pi]$ and then loops

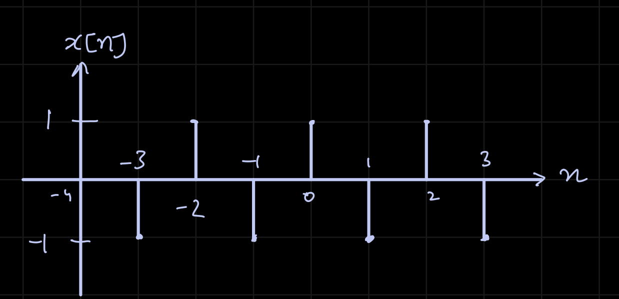
DTS

$$x[n] = e^{j\omega_0 n}$$

let $\omega_0 = 0 \Rightarrow e^{j0} = 1\angle 0 = \cos 0 + j\sin 0 = 1$

let $\omega_0 = 2\pi \Rightarrow e^{j \cdot 2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$

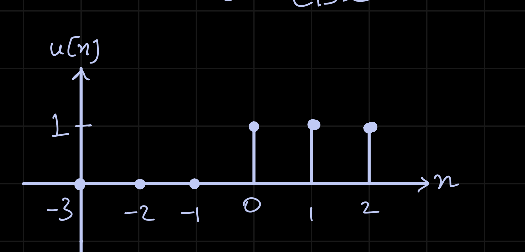
let $\omega_0 = \pi \Rightarrow e^{j\cdot \pi n} = \cos(\pi n) + j\sin(\pi n) = (-1)^n$



* Discrete Time Signals

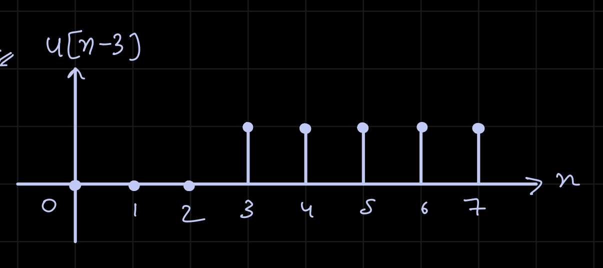
unit step signal $u[n]$ | unit impulse function $\delta[n]$

$$u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : \text{else} \end{cases} \qquad \delta[n] = \begin{cases} 1 : n = 0 \\ 0 : \text{else} \end{cases}$$



$1 : n-3 \geq 0$
$n \geq 3$

note:



conversion btw u[ ] and δ[ ]

$$\boxed{\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^{n} \delta[k] \end{aligned}}$$

note: $u[n] = 0 \quad \forall\, n < 0$

so, $u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$

$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$

$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$

$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1$ → 2?

- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

⇒ for $n = -2$, $\delta[\,]$ will be 1 only when
$n - k = 0$ i.e. $k = -2$
but k is never negative
since $k \in [0, \infty)$

hence $u[-2] = 0$

⇒ for $n = 0$, $n - k = 0$ to get a 1 from δ fn
$k = 0$ ✓
$k \in [0, \infty)$

⇒ for $n = 1$, $n - k = 0$ to get $\delta[0] = 1$
$k = 1$ ✓

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underset{0}{\delta[1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[-1]} + \ldots = \underline{1} \checkmark$$

- **SIFTING / SAMPLING**

$x[n] \cdot \delta[n] = ?$ $\qquad x[n]\,\delta[n] = \begin{cases} x[0] & : n = 0 \\ 0 & : \text{otherwise} \end{cases}$

we can also say $x[n]\,\delta[n] = x[0]\,\delta[n]$

we cannot directly say $x[n]\,\delta[n] = x[0]$ since LHS: function and RHS: scalar

↳ gives only the value of $x[n]$ at $n = 0$
$\equiv x[0]$

eg) $x[n] \cdot \delta[n - n_0] = ?$ $\qquad x[n] \cdot \delta[n-n_0] = \begin{cases} x[n_0] & : n = n_0 \\ 0 & : \text{otherwise} \end{cases}$

↳ gives just the value of $x[n]$ at $n = n_0$
$\equiv x[n_0]$

* Continuous Time Signal

- unit step function

$$u(t) = \begin{cases} 1 & : t \geq 0 \\ 0 & : \text{else} \end{cases}$$

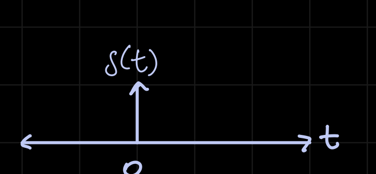
OR

$$u(t) = \begin{cases} 1 & : t > 0 \\ 1/2 & : t = 0 \\ 0 & : \text{else} \end{cases}$$

- Unit Impulse function $\equiv$ singularity function

$$\delta(t) = 0 \quad : t \neq 0$$

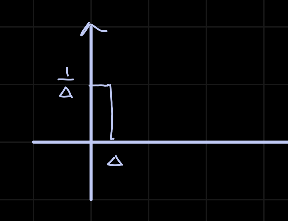
and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

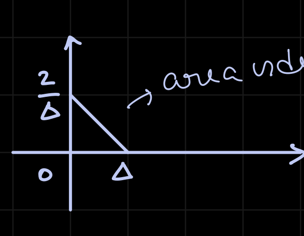


$\delta_\Delta(t)$:

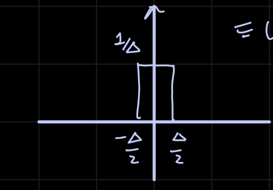
infinitesimally small

$$\delta_\Delta(t) = \begin{cases} 0 : t \geq \Delta \ \& \ t < 0 \\ \frac{1}{\Delta} : \text{otherwise} \end{cases}$$





→ area under the curve $= \frac{1}{2} \times \Delta \times \frac{2}{\Delta} = 1$
$\equiv$ unit impulse func ✓



$\equiv$ unit imp. ✓

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t - \tau)\,d\tau$$

$$\boxed{\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

* Tutorial : 1 30/08/24

(7) $even\{x[t]\} = \frac{x(t)+x(-t)}{2}$

$odd\{x[t]\} = \frac{x(t)-x(-t)}{2}$

8) (a) $x_1[n] = \sin(0.02\pi n)$

$\omega = 0.02\pi$

$T = \frac{2\pi}{\omega} =$

(3) $p(x,y) = a^2x_1y_1 + abx_1y_2 + b^2x_2y_2 + ab\,x_2y_1$

$A = \begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

$|A| = a^2b^2 - a^2b^2 = 0$

atleast one eigenvalue is zero

So, since not all eigenvalues are +ve

(d) for no values of a,b

eigenvalues are values of λ such that

$|A - \lambda I| = 0$

eigenvalues / eigenvectors

2/) $B = \{V_1, V_2, \ldots V_n\}$

B is span of V

$\bar{x} \in V$

$\bar{x} = K_1V_1 + K_2V_2 + \cdots K_nV_n$

$-\ \bar{x} = C_1V_1 + C_2V_2 + \cdots C_nV_n$

$0 = (K_1-C_1)V_1 + (K_2-C_2)V_2 + \cdots (K_n-C_n)V_n$

since $\bar{V_i} \neq 0$, $(K_i - C_i)$ need to be 0

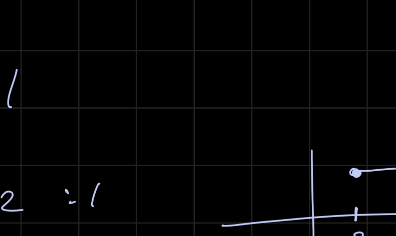
i.e. $K_i = C_i$

$\frac{2\pi}{\omega} = \frac{2\pi}{2\times10^{-2}\pi} = 100 \in T$

(8) $x_1[n] = \sin(0.02\pi n)$

Periodic ✓

(11) $y[n] = \sum_{n=-5}^{5} \sin(2n)\,\delta[n+7]$

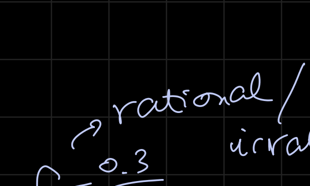


but range is $-5 \to 5$, so ans = 0

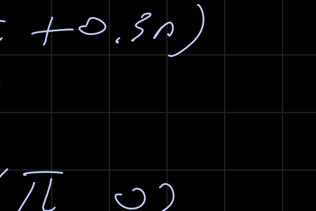
(12) $x(t) = u(t-2) - u(t-4)$

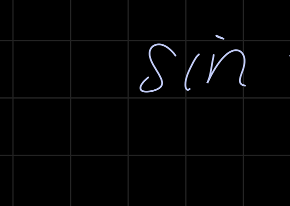
$u(t) \Rightarrow t \geq 0 : 1$

$u(t-2) \Rightarrow t \geq 2 : 1$



$u(t-4) \Rightarrow t \geq 4 : 1$



$\Rightarrow$ 

but since $\delta(t) = 1$ at $t = 0$

So, ans = 0

rational / irrational ?

$f = \frac{0.3}{2\pi}$

(8) (b) $x_2[n] = \cos(\frac{\pi}{2} + 0.3n)$

$\sin = \cos(\frac{\pi}{2} - \theta)$

(c) $x_3[n] = 1 - e^{j2\pi n/5} + e^{j4\pi n/7}$

note: $e^{j\theta} = \cos\theta + j\sin\theta$

$\Rightarrow 1 - \cos(\frac{2\pi n}{5}) - j\sin(\frac{2\pi n}{5})$

$+ \cos(\frac{4\pi n}{7}) + j\sin(\frac{4\pi n}{7})$

5 5 7/2 7/2

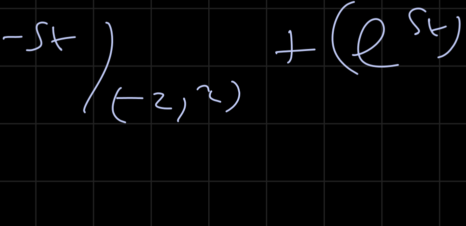
$\Rightarrow \mathrm{LCM}(5, \frac{7}{2}) \Rightarrow$

(a) $\frac{x(t) + x(-t)}{2} = 0$

$x(t) = -x(-t)$

$\forall t$

(b) $e^{-5t}u(t+2)$



$e^{-5t}u(t+2) = -[e^{5t}u(-t+2)]$

$(e^{-5t})_{(-2,\infty)} + (e^{5t})_{(-\infty,2)}$

$(e^{-5t})_{(-2,2)} + (e^{5t})_{(2,2)}$

(5) $P = tu(t)$

$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |t\,u(t)|^2 dt$

$\lim_{T\to\infty} \frac{1}{2T}\int_0^T t^2 dt$

$\lim_{T\to\infty} \frac{1}{2T}\left[\frac{t^3}{3}\right]_0^T$

$\lim_{T\to\infty} \frac{1}{2T} \times \frac{T^3}{3} \Rightarrow \frac{T^2}{6}$

$\to \infty$

neither energy nor power signal

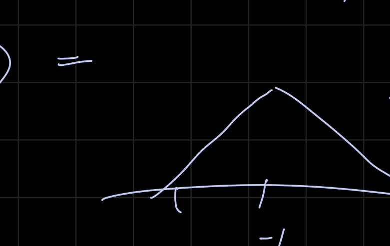
(9) (c) $\frac{u[n] - u[n-4] + u[-n] - u[-n-4]}{2} = 0$

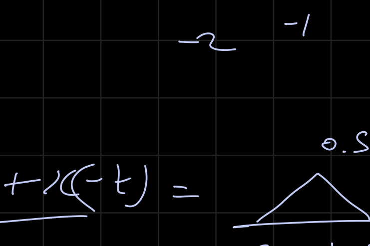
$u[n] + u[-n] = u[n-4] + u[-n-4]$

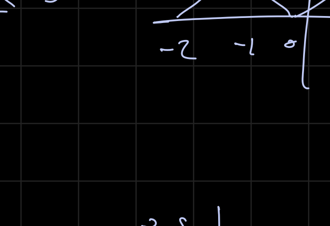
$1 : n \geq 0$ $1 : n \leq 0$ $n \geq 4$ $n \leq 4$

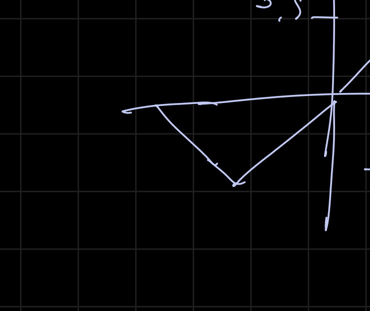
$-3 < n < 3$

$\boxed{|n| < 3}$ ✓

(7) $x(t) =$ 

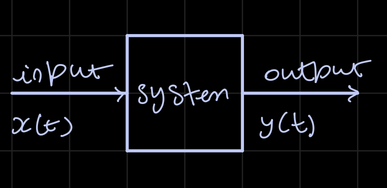
$x(-t) =$ 

$even = \frac{x(t) + x(-t)}{2} =$ 

$odd =$ 

* Lecture: 5

⇒ Systems and their properties



① System: with memory or without memory

$x[n] \rightarrow y[n]$

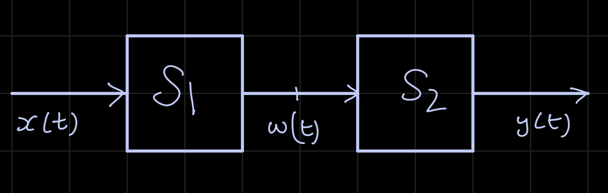
$S_1: y(t) = x(t)$ no memory
$S_2: y(t) = x^2(t)$ no memory
$S_3: y(t) = x(t-2)$ memory ✓ require past value
$S_4: y(t) = \frac{1}{2}\{x(t) + x(t-1)\}$ mem ✓

output depends on just present inputs → no memory required

output needs to store previous outputs → requires memory

$S_5: y(t) = x(t+2)$: requires memory

• Invertible / Non-invertible Systems



$y(t) = x(t)$ iff S_2 = inverse(S_1)

ⓐ let $S_1: w(t) = 2x(t)$
$S_2: y(t) = \frac{1}{2}w(t) = \frac{1}{2} \times 2x(t) = x(t)$

S_1 is invertible and S_2 is the inverse system

ⓑ let $S_1: w[n] = \sum_{K=-\infty}^{n} x[n]$

then $S_2: y[n] = w[n] - w[n-1]$ invertible ✓

$= \{x[\cancel{-\infty}] + \dots \cancel{x[n-1]} + x[n]\} - \{x[\cancel{-\infty}] + \dots \cancel{x[n-1]}\}$

$= x[n]$

ⓒ let $S_1: w[n] = x[n] - x[n-1]$

$S_2: y[n] = \sum_{K=-\infty}^{n} w[K] = x[n]$ invertible ✓

$\cancel{x[-\infty]} - x[-\infty-1] + \dots \cancel{x[1]} - \cancel{x[2]} + \cancel{x[2]} - x[3] + x[3] - x[4] + \dots - x[n-1] + x[n-1] - x[n]$

ⓔ $S_1: w(t) = x^2(t)$
$S_2: y(t) = \sqrt{w(t)}$ } S_1 is not invertible

* System is invertible if and only if distinct inputs lead to distinct outputs

(very casual)

• Causal Systems

$S_1: y(t) = x(t - t_0)$: t_0 is a +ve real number

$S_2: y[n] = x[n+3]$

System output is depending on past system output

$S_1 \equiv$ causal system
$S_2 \equiv$ non-causal system

Causality: unless there is no cause, there should be no effect.

OR unless there is a non-zero input, there should not be non-zero output

OR a causal system depends on current and past input

eg} causal system: $y[n] = \sum_{K=-\infty}^{n} x[K]$

image processing: $S_4: y[n,m] = \frac{1}{5}\{x[n,m] + x[n-1,m] + x[n+1,m] + x[n, m-1] + x[n, m+1]\}$

non-causal system

• Stability System

A bounded input $\equiv |x(t)| < M_x < \infty$
finite value ∀ t

not on the duration (y input) → but on the amplitude (x axis) ↓

OR $|x[n]| < M_x < \infty$

This bounded input is fed to a system. If the system generates an output which is also bounded, we call it a stable system.

eg} $x(t) = u(t)$; $x[n] = u[n]$
$[0,1]$

unbounded: $x(t) = tu(t)$ OR $x[n] = tu[n]$

if $|y(t)| < M_y < \infty \ \forall t$, for bounded input,
↳ finite number

then the system is stable

eg} $S_1: y[n] = \frac{1}{3}\{x[n] - x[n-1] - x[n+1]\}$

$|y[n]| = |\frac{1}{3}x[n] - \frac{1}{3}x[n-1] - \frac{1}{3}x[n+1]|$

since $|x[n]| < M_x < \infty \ \forall n$

and the sum & difference of 3 finite terms is finite, so,

$|y[n]| < M_y < \infty \ \forall n$

$S_3: y[n] = nx[n]$

$|x[n]| < M_x < \infty \ \forall n$
bounded input assumed

$\Rightarrow |y[n]| = |nx[n]| \rightarrow \infty$ as $n \rightarrow \infty$

unstable system

$S_4: y(t) = x(2t)$

no manipulation on the y axis
2t means manipulation on the x-axis

• Time Invariant System

if $x(t) \xrightarrow{\text{system}} y(t)$

then $x(t - t_0) \rightarrow y(t - t_0)$

and $x(t + t_0) \rightarrow y(t + t_0)$

system output does not change if there is a change in time

if $x_1(t) \rightarrow y_1(t)$
and $x_2(t) \rightarrow y_2(t)$
and $x_2(t) = x_1(t - t_0)$

then $y_2(t)$ must be equal to $y_1(t - t_0)$

if it is a time invariant system

eg} $S: y(t) = x(2t)$

⇒ let $t_0 = 4$: $y(t - t_0) = x(2t - 8)$

⇒ time variant system

$x_1(t) \rightarrow y_1(t) = x_1(2t)$
$x_2(t) \rightarrow y_2(t) = x_2(2t)$

$y_1(t - t_0) = x_1(2t - 2t_0)$
first scaling : $x_1(2t)$
then delay : $x_1(2(t - t_0))$ ($t \rightarrow t - t_0$)

$y_2(t - t_0) = x(2t - t_0)$
first delay : $x(t - t_0)$
then scaling : $x(2t - t_0)$

$S_2: y[n] = nx[n]$

$x_1[n] \rightarrow y_1[n] = nx_1[n]$
↓ delayed output
$y_1[n - n_0] = (n - n_0)x[n - n_0]$

$x_2[n] \rightarrow y_2[n] = nx_2[n]$

let $x_2[n] = x_1[n - n_0]$
be delayed input

$y_2[n] = (n)x[n - n_0]$

since $y_1[n] \neq y_2[n]$, system is time varying

note: 1st system had delayed output and we gave delayed input

• Linearity

: A system is linear if the properties of additivity & homogeneity holds true

$x_1(t) \rightarrow y_1(t)$
$x_2(t) \rightarrow y_2(t)$

⇒ $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
then system is linear

eg) $y(t) = 2x(t) + 3$

$x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$

$x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$

$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) = 2(x_1 + x_2) + 2(3)$

but $y(x_1 + x_2) = 2(x_1 + x_2) + 3$

Since, $y_1(t) + y_2(t) \neq y(x_1 + x_2)$

not a linear system

• A system which is linear and time invariant ≡ LTI
↓
linear & time invariant system

* Lecture: SNS 6 4/9/24

Q) $x[n] = \cos(\frac{\pi}{8} n^2)$

periodic if $x(n) = x[n+N] \ \forall n$

$\cos(\frac{\pi}{8}(n^2 + N^2 + 2nN)) = \cos(\frac{\pi}{8} n^2)$

$\cos(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{4} nN) = \cos(\frac{\pi}{8} n^2)$

we know $\cos(2\pi\theta + K) = \cos\theta \quad \forall K \in \mathbb{Z}$

$\frac{\pi}{8} N^2 \Rightarrow$ can it be a multiple of 2π?

if $N = 4 \Rightarrow 2\pi = K2\pi$ ✓ ($K=1$)

true for $N=8$ as well

but not for $N=2$

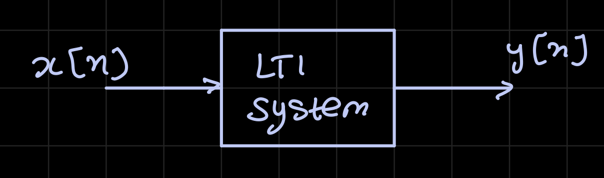
$\frac{\pi}{4} nN \rightarrow K2\pi$?

$N=4 \Rightarrow \pi 4n \rightarrow$ ✗

$N=8 \Rightarrow$ ✓ $2\pi nN$ = ✓

---

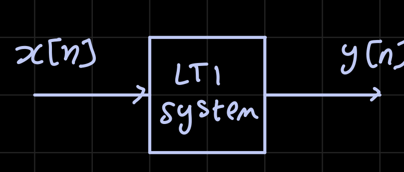
* Discrete Time convolution



$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\, \delta[n-K]$$

weighted linear combination of delayed signals

* Impulse Response of an LTI System



$\delta[n] \rightarrow y[n] = h[n]$

= Impulse response of the LTI System

$\delta[n-1] \rightarrow h[n-1]$

$x[1]\,\delta[n-1] \rightarrow x[1]\, h[n-1]$

$$x(n) = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K] \longrightarrow \sum_{K=-\infty}^{\infty} x[K]\, h[n-K] = y[n]$$

discrete time convolution of

$x[n] \times h[n] = y[n]$

Valid for LTI systems only

convolution → linear & commutative

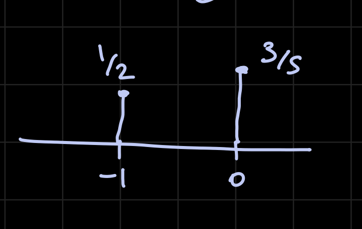
$x[n] \times h[n] = h[n] \times x[n]$

$(x_1[n] + x_2[n])\, h[n] = x_1[n]\, h[n] + x_2[n]\, h[n]$

eg) given an LTI System whose impulse response is

$h[n] = \begin{cases} 1 & n = 0,1,2 \\ 0 & \end{cases}$

and $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$



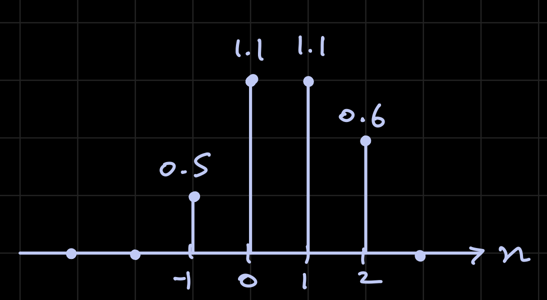
compute $y[n]$

Ans) $y[n] = h[n] * x[n]$

$y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1]$

because for $x[n] \rightarrow y[n]$,
$\delta[n] \rightarrow h[n]$

$$y[n] = \frac{3}{5}[\delta(n) + \delta(n-1) + \delta(n-2)] + \frac{1}{2}[\delta[n+1] + \delta[n] + \delta[n-1]]$$



eg) LTI System (← given)

$h[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$

$x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

Ans) $y(n) = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$

OR $\equiv \sum_{-\infty}^{\infty} h[K]\, x[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} x[n-K] = \sum_{K=0}^{\infty} \alpha^{(n-K)} u[n-K]$

$\Rightarrow \sum_{K=0}^{\infty} \frac{\alpha^n}{\alpha^K} u[n-K]$ ($1 : n \geq K$)

$y(n) = \sum_{K=0}^{n} \frac{\alpha^n}{\alpha^K}$ ✓

$\frac{a(1-r^n)}{1-r}$

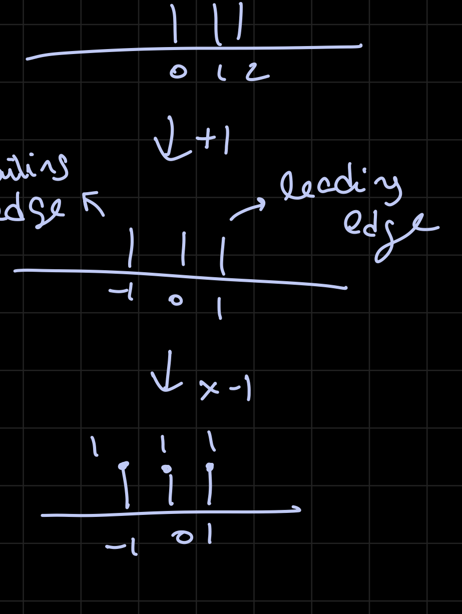
$\times -1$: $-K \xrightarrow{+1} -K+1$

$K \rightarrow (1-K)$

$\downarrow +1$

$K+1$

$\times -1$: $-K+1$



* LECTURE: 7

06/09/24

Note: $y[n] \rightarrow$ function $\equiv$ variable
$y[10] \rightarrow$ scalar $\equiv$ constant

$h(n) \equiv$ characteristic impulse response of the system

$$y[n] = x[n]\, h[n]$$

An LTI system is uniquely charaterized by its $h[n]$ - hence two systems are same if they are LTI and their $h[n]$ are same

bonus question} $\int_{-\infty}^{t} \cos(\tau)\, x(\tau)\, d\tau$

let $x(\tau) = \cos(\tau)$

$$\Rightarrow \int_{-\infty}^{t} \cos^2(\tau)\, d\tau \Rightarrow \int_{-\infty}^{t} \frac{1+\cos(2\tau)}{2}\, d\tau$$

$$\Rightarrow \Big[\underbrace{\frac{t}{2}}_{\text{becomes unbounded as } t\to\infty\ (\equiv \text{unstable})} + \underbrace{\overbrace{\frac{\sin(2t)}{4}}^{\in[-1,1]}}_{\text{bounded}}\Big]_{-\infty}^{t} \Rightarrow$$

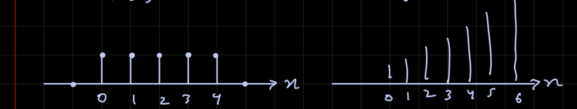
Q1) $\delta(8t-2) = 1$ at $t = \frac{1}{4}$ ✗

$\frac{1}{8}\,\delta\!\left(t - \frac{1}{4}\right) = 1$ at $t = \frac{1}{4}$ ✓

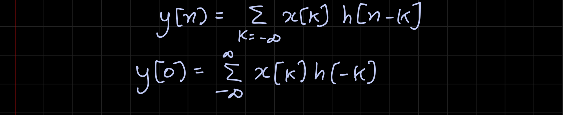
eg} $x[n] = \begin{cases} 1 : & 0 \le n \le 4 \\ 0 : & \text{else} \end{cases}$

$x[n] = u[n] - u[n-4]$

$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{else} \end{cases}$ & $\alpha > 1$



$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$
$$y[0] = \sum_{-\infty}^{\infty} x[K]\, h(-K)$$



only common point : $k = 0$

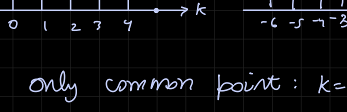
$$y(n) = x[0]\, h[-0] = h(n) = \alpha^0 = \underline{1}$$

$$y[1] = \sum_{-\infty}^{\infty} x[K]\, h[-1-K]$$

$h(K) \xrightarrow{-1} h(K-1) \rightarrow h(-K-1)$

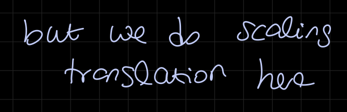
shifting
↓
scaling
↓
reversing

✗



$h[1-K) = 1$ at $1-K = 0 \Rightarrow K = 1$

but we do scaling then translation here



Check: $h[1-K) = 1$ at $1-K = 0$ because $\alpha^K = \alpha^0 = 1$

$K = 1$

$\Rightarrow y(K) = x[K]\, h[1-K)$

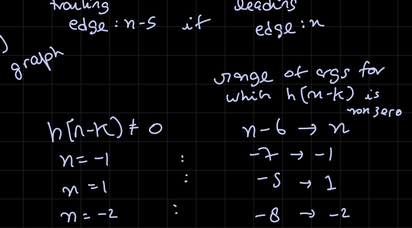
now, we shifted $h[-K]$ 1 unit to the right and we have overlap with $x[K]$ on $K = 0, 1$

Similarly for $x[K-n]$, we have overlap with $K = 0, 1, \ldots\ldots n-1, n$

404: Brain not found
2nd attempt initiated ...

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

we are not shifting $x(K)$ in this case but we could



verification:
$h(0)$ at $1-K = 0$
$K = 1$
$\equiv$ same as $h[-0]$ in previous graph
(argument is zero)

range of args for which $h(n-K)$ is non zero

$h[n-K] \neq 0$		
	:	$n-6 \rightarrow n$
$n = -1$	:	$-7 \rightarrow -1$
$n = 1$	:	$-5 \rightarrow 1$
$n = -2$	:	$-8 \rightarrow -2$

① when $n < 0$, there is no overlap with $x[n]$ hence $y[n] = 0$

② $n = 11$: 5 to 11
$n = 10$: 4 to 10

So, no overlap with $x(n)$ when $n > 10$

hence, $y[n] = 0$: $n < 0$ and $n > 10$

③ $0 \le n < 4$: $y[n] = \sum_{K=0}^{n} x[K]\, h(n-K]$

overlap	n	range
$K=0$	0	$-6 \rightarrow 0$
$K=0,1$	1	$-5 \rightarrow 1$
$K=0,1,2$	2	$-4 \rightarrow 2$
$K=0,1,2,3$	3	$-3 \rightarrow 3$
$K=0,1,2,3,4$	4	$-2 \rightarrow 4$
$K=0,1,2,3,4$	5	$-1 \rightarrow 5$
$K=0,1,2,3,4$	6	$0 \rightarrow 6$
$K=1,2,3,4$	7	$1 \rightarrow 7$
$K=2,3,4$	8	$2 \rightarrow 8$
$K=3,4$	9	$3 \rightarrow 9$
$K=4$	10	$4 \rightarrow 10$

of $h[$ leading edge left $x(n)$

So, $y(n)$ is defined for $0 \le n \le 10$

② $0 \le n \le 4 \Rightarrow y[n] = \sum_{K=0}^{n} x[K]\, h[n-K]$

③ $4 < n \le 6 \Rightarrow y[n] = \sum_{K=0}^{4} x(K)\, h[n-K]$

④ $6 < n \le 10 \Rightarrow y[n] = \sum_{\underbrace{K=n-6}} ^{4} x[K]\, h[n-K]$

overlap: $1,2,3,4 \rightarrow 2,3,4 \rightarrow 3,4 \rightarrow 4$
n: $7 \rightarrow 8 \rightarrow 9 \rightarrow 10$

final $\rightarrow y[n] = \sum_{K=0}^{n} \alpha^{n-K}$: $0 \le n \le 4$
$+ \sum_{K=0}^{4} \alpha^{n-K}$: $4 < n \le 6$
$+ \sum_{K=n-6}^{4} \alpha^{n-K}$: $6 < n \le 10$

convolution for CTS = DTS

$$\sum \alpha^{n-K} = \alpha^n \sum \frac{1}{\alpha^K} = \alpha^n \{GP\}$$
$$\frac{a(1-r^n)}{1-r}$$

$\boxed{\text{CTS}}$

$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

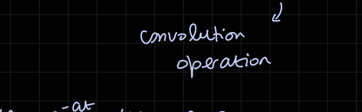
useful when $h(t) = u(t)$

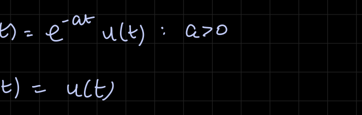
$y(t) = x(t) * h(t)$

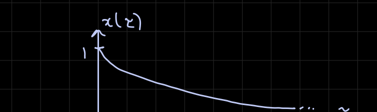
convolution operation

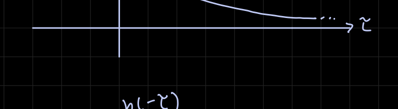
eg) $x(t) = e^{-at}\, u(t)$: $a > 0$

$h(t) = u(t)$









leading edge : $\tau = 0$

no overlap when we shift $h(-\tau)$ to left, there will be no overlap and hence that integral = 0 ✗

shift to right ✓

① $t < 0 \Rightarrow y(t) = 0$

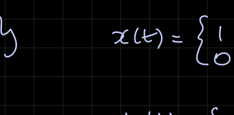
② $t > 0 \Rightarrow y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$ ✓

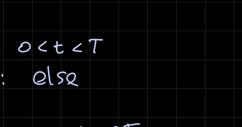
integral $-\frac{e^{-at}}{a}$

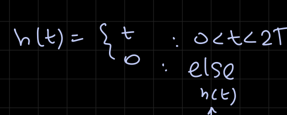
starts at 0 then saturates as $t \rightarrow \infty$

eg} $x(t) = \begin{cases} 1 & : 0 < t < T \\ 0 & : \text{else} \end{cases}$

$h(t) = \begin{cases} t & : 0 < t < 2T \\ 0 & : \text{else} \end{cases}$







t = shift

① $t < 0$: $y(t) = 0$ no overlap

② $0 \le t < T$: $y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$

③ $T \le t \le 2T$: $y(t) = \int_{0}^{T} x(\tau)\, h(t-\tau)\, d\tau$

④ $2T < t \le 3T$ $y(t) = \int_{t-2T}^{T} x(\tau)\, h(t-\tau)\, d\tau$

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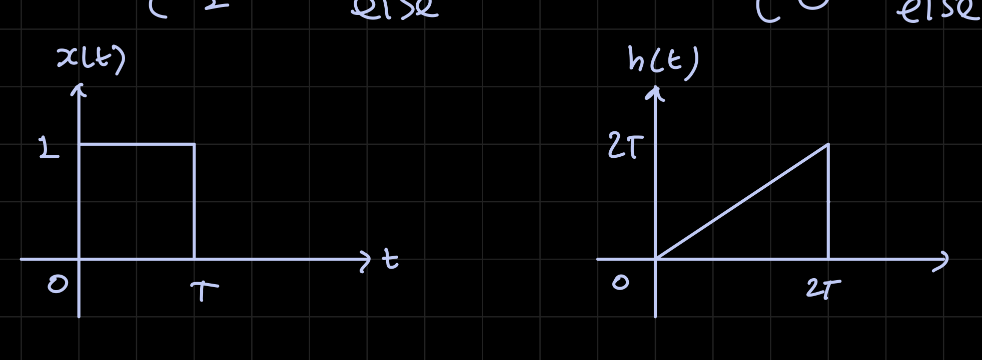
⇒ convolution

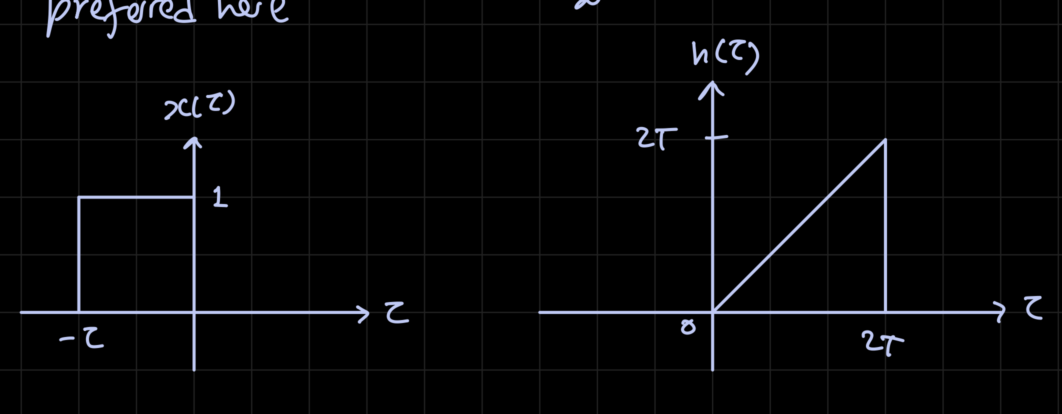
$x(t) = \begin{cases} 0 & 0 \le t \le T \\ 1 & \text{else} \end{cases}$ $h(t) = \begin{cases} t & 0 \le t \le 2t \\ 0 & \text{else} \end{cases}$



easier to reverse $x(t)$

so, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

this one preferred here ⇐ $= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$



at $y(0) \Rightarrow$ overlap = none

so, $y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

moving forward, $y(1) \cdots y(n)$
if we shift $x(-\tau)$ to left
i.e. $t<0$, it will never overlap

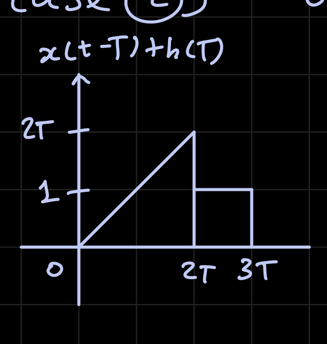
So, we move to right,
$t>0$
when plotting $x(t-\tau)$

case ① } $t<0$

$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = 0$

case ② } $t > 3T$ again no overlap

x(t−T)+h(T)



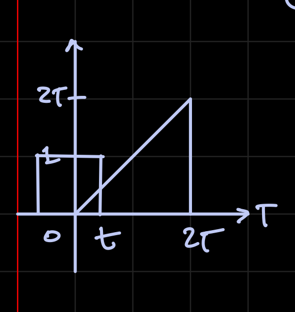
$y(t) = 0$

no non-zero overlap

Case ③ } $0 < t < T$

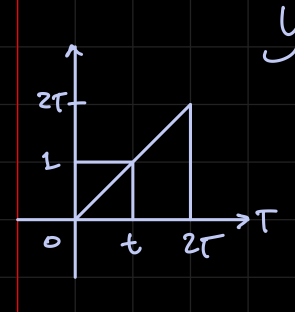
$y(t) = \int_0^t h(\tau)\, d\tau = \int_0^t \tau\, d\tau = \frac{\tau^2}{2}\Big|_0^t$

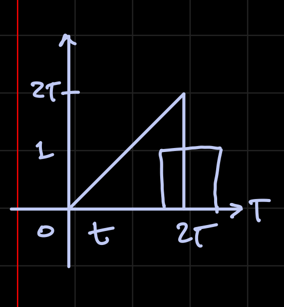
$= \frac{t^2}{2}$



Case ④ : $T \le t \le 2T$

$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{-T^2}{2} + tT$





case ⑤ } $2T \le t \le 3T$

$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$

$= \frac{4T^2}{2} - \frac{(t-T)^2}{2}$

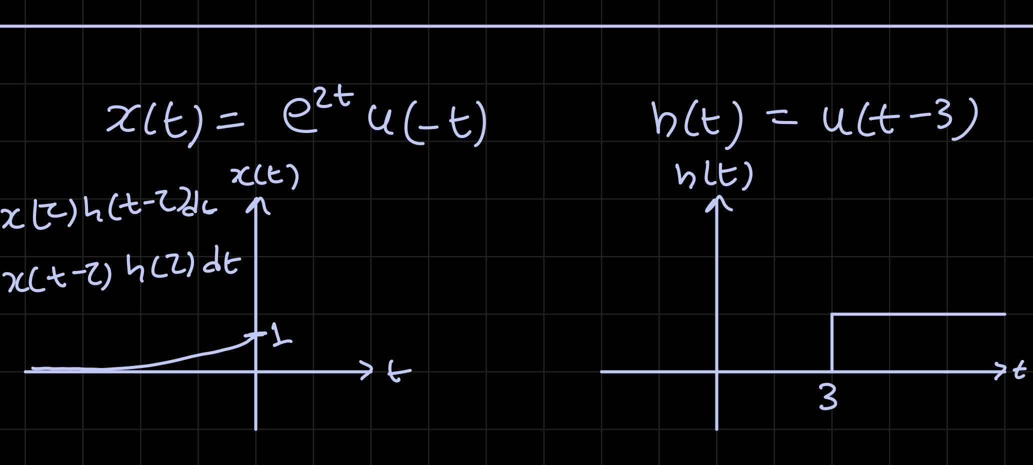
$= \frac{4T^2 - t^2 - T^2 + 2Tt}{2}$

$= \frac{3T^2 + 2tT - t^2}{2}$

---

eg } $x(t) = e^{2t} u(-t)$ $h(t) = u(t-3)$

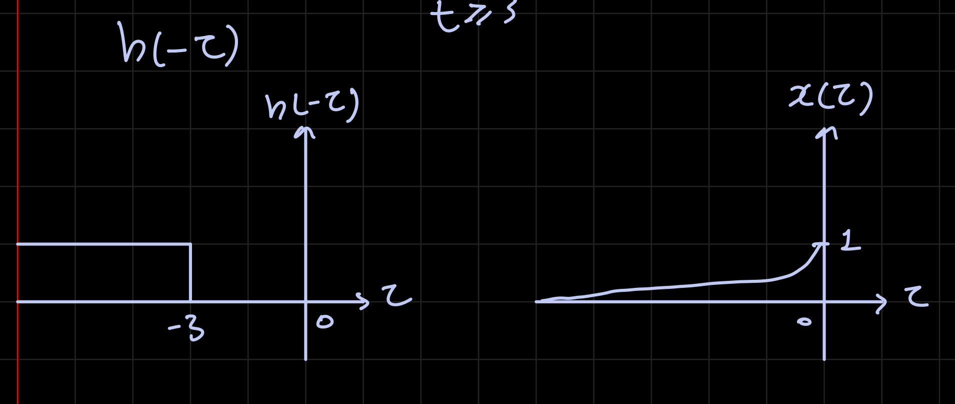
$y(t) = \int x(\tau) h(t-\tau) d\tau$
$= \int x(t-\tau)\, h(\tau)\, dt$



Which one to reverse?
we can reverse
both but which one is easier?

$t \ge 3$

$h(-\tau)$

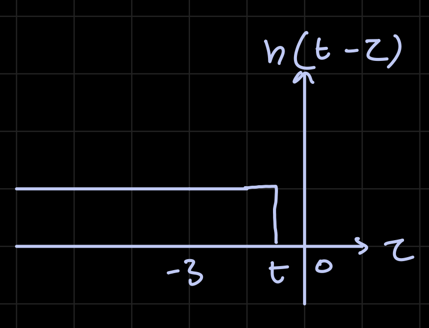


case 1: $t \le 3$ in $h(t-\tau)$

moves to right if $t>0$
left if $t<0$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{t-3} x(\tau)\, d\tau$



$\int_{-\infty}^{t-3} e^{2\tau}\, d\tau$

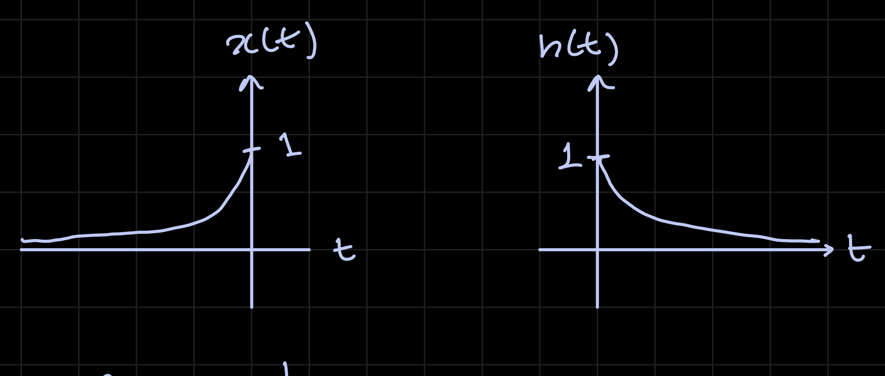
$\Rightarrow \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t-3} = \frac{e^{2t-6}}{2} - 0 = \frac{e^{2t-6}}{2} = \frac{e^{2t}}{2e^6}$

case ② } $t>3$: constant: $e^{2\tau}$

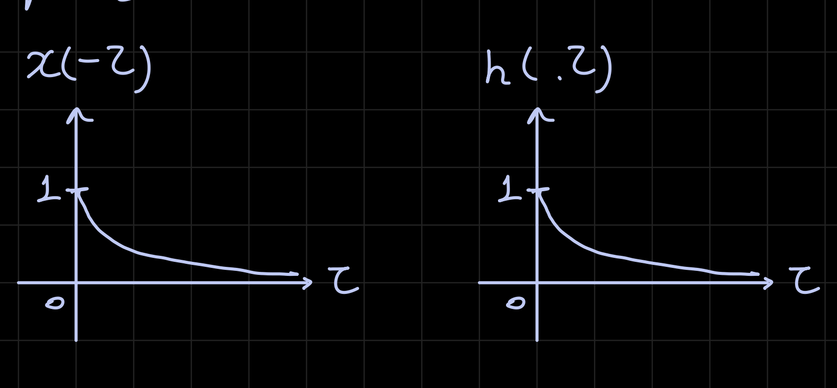
$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \int_{-\infty}^{0} \frac{e^{2\tau}}{2} = \frac{1}{2}$ ✓

---

eg } $x(t) = e^{at} u(-t)$: $a>0$
$h(t) = e^{-at} u(t)$



flip any here are similar



Shifting by t Ranges: $t \ge 0$, $t \le 0$
OR $t \ge 0$, $t < 0$

case ① : $t=0$

$y(0) = \int_0^{\infty} x(-\tau)\, h(\tau) = \int_0^{\infty} e^{-2a\tau}\, d\tau$

$\Rightarrow \frac{e^{-2a\tau}}{-2a}\Big|_0^{\infty}$

$\Rightarrow 0 + \frac{1}{2a} = \frac{1}{2a}$

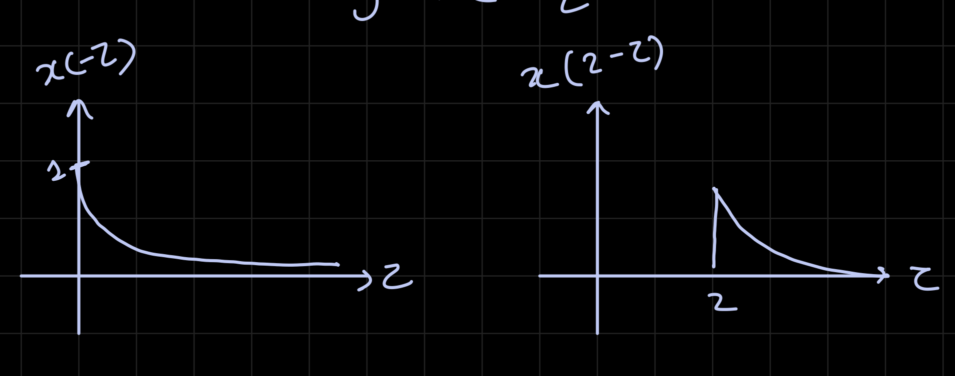
case ③: $t<0$

$y(t) = \int_0^{\tau - t} x(t-\tau)\, h(\tau)\, d\tau$

$t - \tau \to \infty$

$= \int_0^{\tau - t}$

left edge of $x(-\tau)$ will always be equal to t



for $t \ge 0$: $t \to \infty$

for $t<0$: $0 \to \infty$

case ① : $t \ge 0$ $t \to \infty$

$y(t) = \int_t^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$\int_t^{\infty} e^{-at}\, e^{a(t-\tau)}\, d\tau$

case ② : $t<0$ $0 \to \infty$

$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_0^{\infty} e^{-at} \cdot e^{a(t-\tau)}\, d\tau$

* SYSTEM PROPERTIES

given an LTI system and

$h[n] = \begin{cases} 1 & : n = 0,1 \\ 0 & : \text{else} \end{cases}$

defaults: input : $x[\,]$
output : $y[\,]$
impulse response : $h[\,]$

S: $y[n] = ? \Rightarrow$ System output = ?

apply convolution

$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

from the given defn, $h[n] = \delta[n] + \delta[n-1]$

$\Rightarrow h[k] = \delta[k] + \delta[k-1]$

so, $y[n] = \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1])\, x[n-k]$

$= \sum_{-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{-\infty}^{\infty} \delta[k-1]\, x[n-k]$

$= x[n] + x[n-1]$

eg } find impulse response of systems

S_2 : $y[n] = (x[n] + x[n-1])^2$

S_3 : $y[n] = \max\{x[n], x[n-1]\}$

we know: $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

There is only one LTI system that gives the same impulse response

these systems S_2, S_3 give same response because they are not LTI

* Lecture: 9 13/09/24

- **SYSTEM PROPERTIES**

② (i) Commutativity

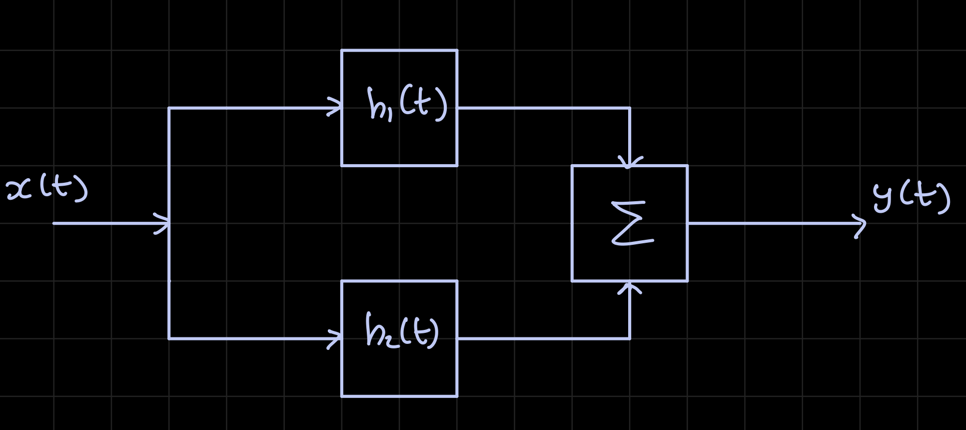
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Prove

(ii) Convolution distributes over addition

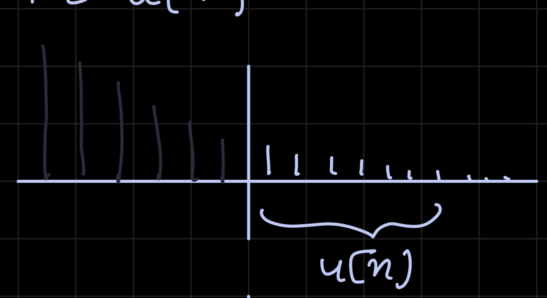
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



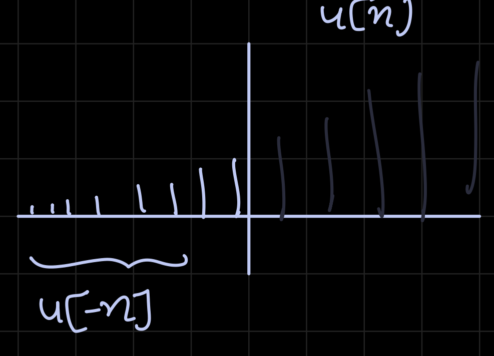
holds true for both CTS and DTS

eg3 $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$

$x_1[n] = 2^n u[n]$



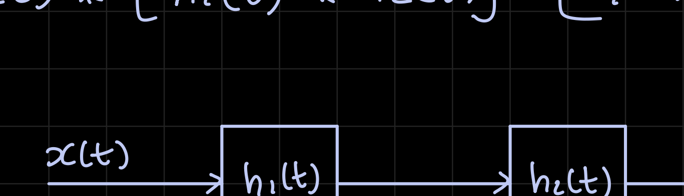
$x_2[n] = 2^n u[-n]$



easier : $y[n] = x_1[n] * h[n] + x_2[n] * h[n]$

(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



* only LTI

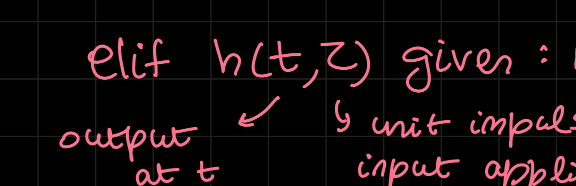
for LTI systems, in cascading, order of these systems can be swapped

③ Invertibility :

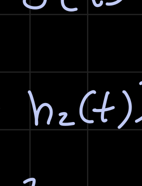
Note: if $h(t)$ given: LTI system ✓

elif $h(t,\tau)$ given : Non LTI system ✓

output at t ↳ unit impulse input applied at τ



eg3 if $h_1(t) * h_2(t) = \delta(t)$ → continuous delta(t)



$$y(t) = x(t) * (h_1(t) * h_2(t))$$

$= x(t)\,\delta(t)$ } operational definition of $\delta(t)$

$= x(t)$

↓ we do not write

✗ $\delta(t) = 1$ at $t=0$

↓

✓ $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

↳ not a scalar signal?

↓ can we have a signal with ∞ value at a single point?

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, dt}$$

Discrete Time

$$x[n] = \sum_{-\infty}^{\infty} x[k]\, \delta[n-k]$$

$$= x[n] * \delta[n]$$

↓ for this we can say $\delta[n] = 1 : n = 0$

④ Causality

claim: if a system is causal,

$h(t) = 0$ for $t<0$ (CTS)
$h[n] = 0$ for $n<0$ (DTS)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

because x cannot have future values so $k>0$

hence, for causal systems,

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

OR

$$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

} if $h[n] = 0$ at $n<0$

integrator : S if $h(n) = u(n)$ CTS
accumulator : S if $h[n] = u[n]$ DTS

⑤ memory / memoryless system

$S_1 : y(t) = Kx(t) \Rightarrow h(t) = K\delta(t)$
$S_2 : y[n] = Kx[n] \Rightarrow h[n] = K\delta[n]$

⑥ stability (BIBO Stability Criteria)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

let input signal is bounded and $|x(t)| < M_x \ \forall t$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)\, d\tau| \le M_x \times \left(\text{some finite number}\right) < \infty \ \forall t$$

if the system is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)\, d\tau| < \infty$$

DTS : $|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

if the system holds absolute summability,

then the system is stable

Note: $h[n] = u[n]$, $h(t) = u(t)$ } LTI, unstable, causal, with memory

but still we use these for integrators & accumulators because we use clipper and clamper since we know the upper and lower bound

Identity : system with $h(t) = \delta(t)$

* Constant Coefficient difference eqⁿ for causal LTI systems

$$\sum_{k=0}^{N} b_k\, y[n-k] = \sum_{k=0}^{M} a_k\, x[n-k]$$

a_k, b_k : not functions of time ..x constant coeff

① N ≡ order of the system

② The system is at rest if :

if, $x[n] = 0 : n < n_0$
then, $y[n] = 0 : n < n_0$
$y[n-1] = 0 : n < n_0$
⋮
$y[n-N] = 0 : n < n_0$

} auxiliary conditions

note: $N > 0$

MA

→ case : 1 ⇒ $N = 0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots a_m x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_1} x[n-1] + \dots \frac{a_m}{b_m} x[n-M]$$

causal system ✓ ≡ moving average system → weighted sum

$n \to n-1 \to n-2 \to n-3 \cdots$

system is called finite impulse response system

≡ FIR system

IIR / AR

→ case 2: $N = 1$ and $M = 0$ (simplicity)

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] - \alpha_1 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$ and $\alpha_1 = \frac{b_1}{b_0}$

if $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$h[0] = \alpha_0$
$h[1] = -\alpha_1 \alpha_0$
$h[2] = +\alpha_1^2 \alpha_0$
⋮

} FIR? ✗

Infinite impulse response system ✓ (IIR)

* **Auto Regressive System**
≡ AR system

output depends on regresses of its previous/past inputs

ARMA

→ Case : 3 $N > 0$, $M > 0$

* Lecture : 10 18/09/24

⇒ constant coeff differential eqⁿ representation of causal LTI systems

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k x(t)}{dt^k}$$

$N \equiv$ order of the system

let $N = 2$ & $M = 3$

$$b_2 \frac{d^2 y(t)}{dt^2} + b_1 \frac{dy(t)}{dt} + b_0 y(t) =$$
$$a_0 x(t) + a_1 \frac{d\,x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a^3 \frac{d^3 x(t)}{dt^3}$$

Initial condition

if $x(t) = 0 \quad \forall \ t < t_0$

then $\frac{d^k y(t)}{dt^k} = 0 \quad \forall \ t < t_0$

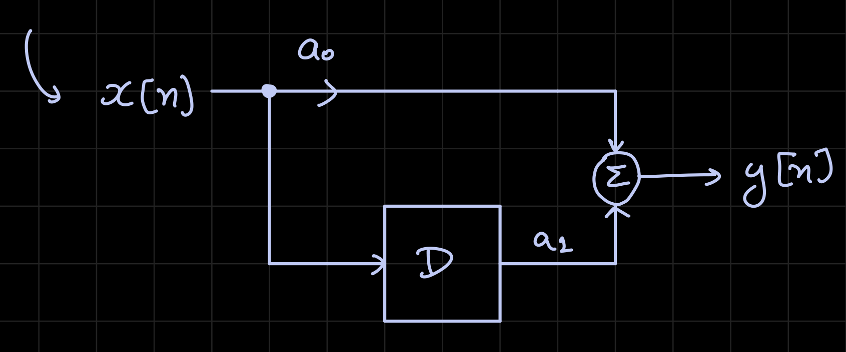
$\forall \ k \in [0, N]$

* Block diagram representation
useful for DTS

let $a_0 x[n] \equiv$ scalar multiplier

then $x[n-1] \equiv$ delay element

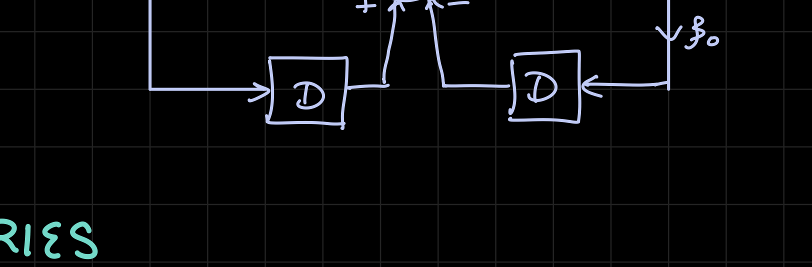
let S_1 : $y[n] = a_0 x[n] + a_1 x[n-1]$



S_2 : $b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$
$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

let $\alpha_0 = \frac{a_0}{b_0}$
and $\beta_0 = \frac{b_1}{b_0}$



* FOURIER SERIES

we know, $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

and $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)$

$x[k]$ are scalars / constants
we can write them as a_k

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k]$$

$\boxed{CT}$ $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

weighted sum of frequencies

Synthesis equation $\left\{ \ x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \right.$ finite basis

weighted linear combination of vectors

$\phi_k(t) = e^{jk\omega_0 t}$
(freq. of oscillation)

$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T}$$

fundamental time period of $x(t)$

$f = \omega_0$ rad/s
freq $\neq$ -ve
Time Period = T
counter clockwise $\phi_1(t) = e^{j\omega_0 t}$ $\qquad \phi_2(t) = e^{2j\omega_0 t}$

completes 2 cycles in one time period ($\frac{T}{2}$)
freq = $2\omega_0$ rad/s

Time Period = T
clockwise $\phi_{-1}(t) = e^{-j\omega_0 t}$ $\qquad \phi_{-2}(t) = e^{-2j\omega_0 t}$
freq $\neq$ -ve
$f = \omega_0$ rad/s

$\phi_0(t) = e^{j0} = e^0 = 1 = \cos 0 + j\sin 0$ ✓

DC component of signal
no fluctuation

can we write $\phi_2(t)$ in terms of $\phi_1(t)$?

No

they are independent of each other

inner product $\langle \phi_k(t), \phi_L(t) \rangle_T = \frac{1}{T}\int_0^T \phi_k(t)\,\phi_L^*(t)\,dt$

$$= \frac{1}{T}\int_0^T e^{jk\omega_0 t} \cdot e^{-jL\omega_0 t}\,dt$$

integers; let = m

$$= \frac{1}{T}\int_0^T e^{j(k-L)\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt$$

$$= \begin{cases} 0 & : k \neq L \\ 1 & : k = L \end{cases}$$

defining the inner product

$$\langle \phi_k(t), \phi_L(t) \rangle_T = \frac{1}{T}\int_T \phi_k(t)\,\phi_L^*(t)\,dt$$

(continuous time periodic signals)

⇒ Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

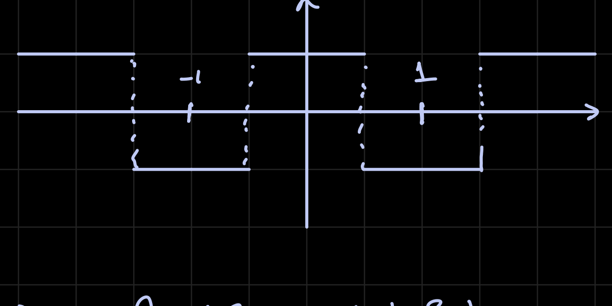
$\{\phi_k(t)\}$ = span a vector space from an orthonormal basis

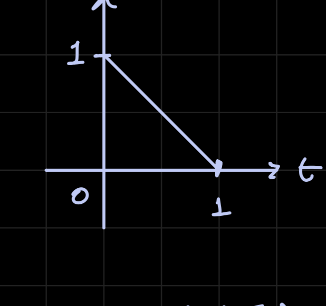
⇒ Analysis : $a_k = \langle x(t), \phi_k(t) \rangle_T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_L(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_L(t) \right\rangle_T$$
$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\langle \phi_k(t), \phi_L(t) \rangle_T}_{\delta[k-L]}$$

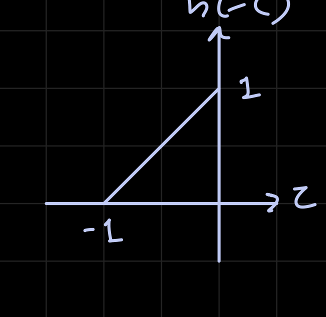
eg3



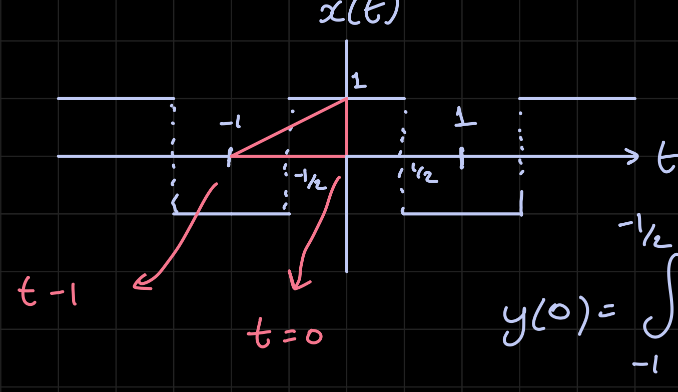


convolute $x(t)$ & $h(t)$

Ans) $h(\tau) = 1 - \tau$
$h(-\tau) = 1 + \tau$ reflect



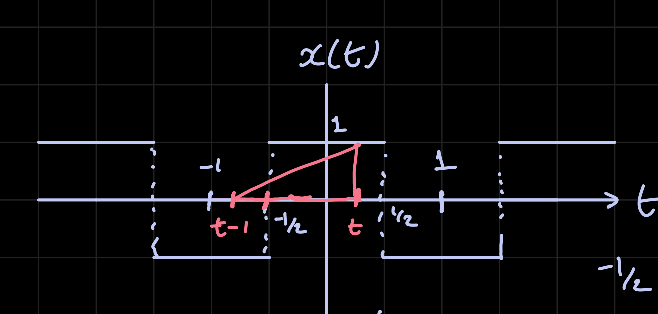
overlap



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(0) = \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

case : 1 : $0 \le t \le \frac{1}{2}$



$$y(t) = \int_{t-1}^{-1/2} (1+\tau+t)\,d\tau + \int_{-1/2}^{t} (1+\tau+t)\,d\tau$$

case : 2 ⇒

* Lecture: 11

⇒ Continuous time periodic signals FOURIER SERIES

$e^{jk\omega_0 t} \to e^{j2k\omega_0 t}$

time period is decreasing

synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

constant

→ pure imaginary exponential (no real part) (in power)

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$ rad/s

$f_0 = \frac{1}{T}$ per sec or Hz

(DC) 3, amplitude 2, 1

eg: $x(t) = \underbrace{3}_{} + \underbrace{2}_{}e^{j\omega_0 t} + \underbrace{1}_{}e^{j2\omega_0 t}$ synthesis equation

$$= 3 + 2(\cos\omega_0 t + j\sin\omega_0 t) + \cos 2\omega_0 t + j\sin 2\omega_0 t$$

will keep oscillating till ∞ and pattern holds and since it is not decaying ⇒ periodic

$$= \sum_{k=0}^{2} a_k e^{jk\omega_0 t}$$

$a_0 = 3$, $a_1 = 2$, $a_2 = 1$ } in this case

all other a_k ∀ $k \neq 0,1,2$ are zero

analysis equation

→ here we take conjugate

→ 2nd coordinate

$$a_k = \langle x(t), \phi_k(t) \rangle_T$$

$$= \frac{1}{T}\int_T x(t)\,\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,dt$$

$\{\phi_k(t)\}_{k\in\mathbb{Z}}$ ≡ orthogonal basis ⇒ $\int_T \phi_m(t)\phi_k^*(t)\,dt = T\delta[m-k]$

$$a_k = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m\left(\int_T \phi_m(t)\phi_k^*(t)\,dt\right)\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \int_T e^{j(m-k)\omega_0 t}\,dt\right]$$

$$= \frac{1}{\cancel{T}}\sum_{m=-\infty}^{\infty} a_m \cdot \cancel{T} \cdot \delta[m-k] = a_k = \underline{LHS}$$

$\phi_1(t), \phi_{-1}(t)$ ⇒ 1st harmonic

$\phi_2(t), \phi_{-2}(t)$ ⇒ 2nd harmonic

and so on...

* Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

* Analysis: $a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$

$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

let's assume $x(t) = e^{st}$ where $s = j\omega_0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,e^{st}\cdot e^{-s\tau}\,d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau \Big\}\ \text{let } H(S)$$

constant

$$\boxed{y(t) = e^{st}\cdot H(S)}$$

e^{st} ≡ act as eigenfunctions of an LTI system

$H(S)$ = corresponding eigenvalue $A\bar{v} = \lambda\bar{v}$

$e^{st} \longrightarrow \boxed{LTI} \longrightarrow e^{st}\cdot H(S)$

↳ here the LTI system is acting as a linear operator

if i feed an LTI system with a continuous signal with some frequencies, then, it outputs a signal with same frequencies but with different amplitudes of the frequencies

output ≡ **Line Spectrum** of $y(t)$

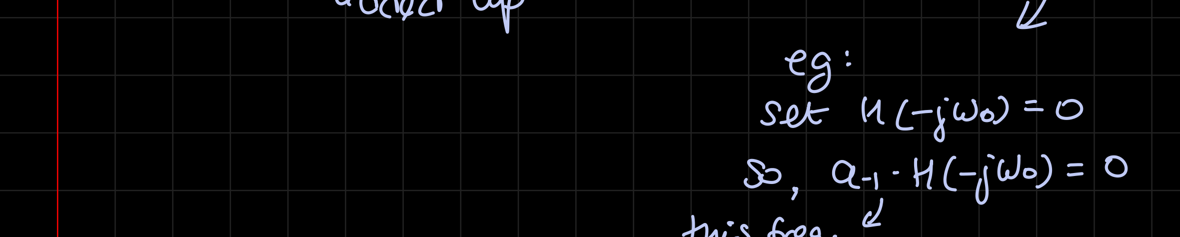
existing frequencies can disappear in output but a different frequency will not get added up

eg: set $H(-j\omega_0) = 0$

so, $a_{-1}\cdot H(-j\omega_0) = 0$

this freq. will disappear in output

eg) given signal with 4 frequencies



freq domain representation

eg) S: $y(t) = x(t-2)$ and $x(t) = e^{j2t}$

Ans) only delaying system observed from eqn

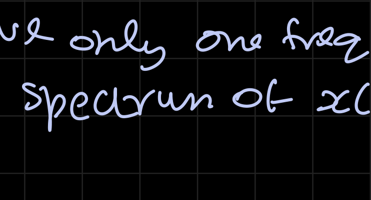
will it alter the signal frequencies? Intuitively, No

$x(t) = e^{j2t}$ ↳ $\omega_0 = 2$ rad/s

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$ $\boxed{T = \pi}$

we have only one freq. component in the line spectrum of $x(t)$

$x(t) = 1\cdot e^{j2t}$ (1 ↳ amplitude)

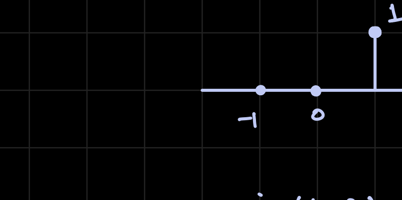


$$y(t) = e^{j2(t-2)} = e^{j2t}e^{-j4} = e^{j2t}\cdot H(j2)$$

$\boxed{H(j2) = e^{-j4}}$

note: in $H(S)$, $s = j\omega$, so, $H(j2) = e^{-j4}$

line spectrum of $y(t)$



no more freq. will be added

same freq. comes out with altered amplitude

---

eg} what is $h(t)$? $h(t) = \delta(t-2)$

$$H(S) = \int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau = \int_{-\infty}^{\infty}\delta(\tau-2)\,e^{-j2\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty}\delta(\tau-2)\,e^{-j2\cdot 2}\,d\tau \Rightarrow H(j2) = e^{2}$$

NOTE: $x(t) * \delta(t) = x(t)$

$x(t) * \delta(t-t_0) = x(t-t_0)$

shifts the signal by t_0 units

---

eg} $x(t) = \sin(\omega_0 t)$

find and plot the line spectrum of $x(t)$

$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

$e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$

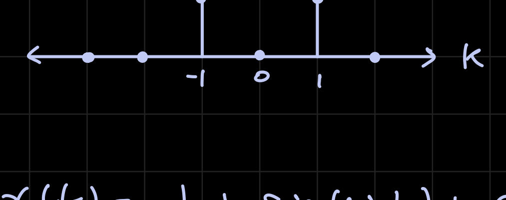
$e^{j\omega_0 t} - e^{-j\omega_0 t} = j2\sin\omega_0 t$

so, $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2} = \frac{-j}{2}\cdot e^{j\omega_0 t} + \frac{j}{2}e^{-j\omega_0 t}$

2 frequencies with amplitude $\frac{-j}{2}$ and $\frac{j}{2}$

we have form $a_k e^{jk\omega_0 t}$

so we have 2 frequencies at $k = 1$ and -1



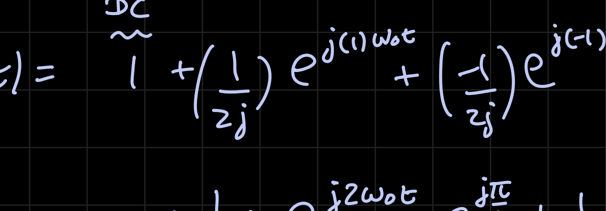
eg} $x(t) = 1 + \sin(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

Ans} $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$\cos\left(2\omega_0 t + \frac{\pi}{4}\right) = \frac{e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{j(-2\omega_0 t + \frac{\pi}{4})}}{2}$$

$$x(t) = \underbrace{1}_{DC} + \left(\frac{1}{2j}\right)e^{j(1)\omega_0 t} + \left(\frac{-1}{2j}\right)e^{j(-1)\omega_0 t} + \left(\frac{1}{2}\right)e^{j2\omega_0 t}\cdot e^{j\frac{\pi}{4}} + \left(\frac{1}{2}\right)e^{j(-2)\omega_0 t}e^{j\frac{\pi}{4}}$$

line spectrum of $x(t)$

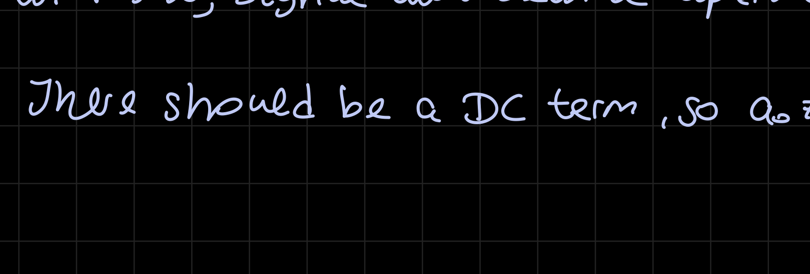


5 frequencies

if the question asks for absolute value line spectrum, then just write the mod of the complex amplitudes

---

eg} $x(t) = \begin{cases} 1 & : |t| < T_1 \\ 0 & : T_1 < |t| < \frac{T}{2} \end{cases}$



① if $T \to \infty$, signal will become aperiodic

② There should be a DC term, so $a_0 \neq 0$

HW

$$a_0 = \frac{1}{T}\int_T x(t)\,e^{0}\,dt = \frac{1}{T}\int_T x(t)\,dt$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} = \underline{a_0} = \frac{2T_1}{4T_1} = \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{1}{T}\left[\frac{-e^{-jk\omega_0 t}}{jk\omega_0}\right]_{-T_1}^{T_1}$$

euler

$a_k = \frac{1}{k\pi}\sin(k\omega_0 T_1)$: $k \neq 0$

$a_k = \frac{1}{k\pi}\sin\left(k\frac{2\pi}{4T_1}T_1\right)$

$a_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$

line spectrum has becomes dense more frequency points created

* lecture: 12

25/09/24

• LINEARITY

• Prove the linearity property of signals

if $x(t) \overset{\text{fourier series coefficients}}{\rightleftharpoons} a_k$

and $y(t) \rightleftharpoons b_k$

$x(t)$ and $y(t)$ are periodic with fundamental time period $= T$

prove $z(t) = \alpha x(t) \beta y(t) \rightleftharpoons \alpha a_k + \beta b_k$

and $z(t)$ = some fundamental time period T

sol^n)

let $z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ where $\omega_0 = \frac{2\pi}{T}$

$$C_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T (\alpha x(t) + \beta y(t)) e^{-jk\omega_0 t} dt$$

$$= \alpha \underbrace{\frac{1}{T}\int_T x(t) e^{jk\omega_0 t} dt}_{a_k} + \beta \cdot \underbrace{\frac{1}{T}\int_T y(t) e^{jk\omega_0 t} dt}_{b_k}$$

$= \alpha a_k + \beta b_k$ hence proved

• TIME SHIFTING

$$x(t) \xrightarrow[T]{F.S.} a_k$$

find fourier series coefficients for $x(t - \underset{\text{constant}}{t_0})$

$$z(t) = x(t-t_0) \xrightarrow{F.S.} C_k$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

$$C_k = \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt$$

let $t - t_0 = z$

$dt = dz$

let limits: $-\frac{T}{2}$ to $\frac{T}{2}$

now: $-\frac{T}{2} - t_0$ to $\frac{T}{2} - t_0$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}-t_0}^{\frac{T}{2}-t_0} x(z) e^{-jk\omega_0 (t_0+z)} dz$$

$$C_k = e^{-jk\omega_0 t_0} \underbrace{\frac{1}{T}\int_T x(z) e^{-jk\omega_0 z} dz}_{a_k}$$

$$C_k = a_k \cdot e^{-jk\omega_0 t_0}$$

There is a shift in frequency but the magnitude remains the same

• TIME REVERSAL

$$x(t) \xrightarrow[T]{FS} a_k$$

$z(t) = x(-t) \xrightarrow[T]{FS} C_k = ?$

note: same time period

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(-t) e^{jk\omega_0 t} dt$$

let $z = -t$ and $-\frac{T}{2} \to \frac{T}{2}$

$dz = -dt$ to $\frac{T}{2} \to -\frac{T}{2}$

$$C_k = \frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} x(z) e^{jk\omega_0 z} \, \underset{\text{reversed the limits instead of } -dz}{-dz}$$

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(z) e^{jk\omega_0 t} dz$$

$$C_{-k} = \frac{1}{T}\int_T x(z) e^{-jk\omega_0 t} dz$$

$C_{-k} = a_k$

$$\boxed{C_k = a_{-k}}$$

no change in frequencies only change in magnitude

• COMPLEX CONJUGATE

$x(t) \xrightarrow[T]{F.S.} a_k$ time period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$z(t) = x^*(t) \xrightarrow{F.S.} ?$$

$$z(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$\downarrow k \to -k$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$\Rightarrow$ so,

$$z(t) = x^*(t) \xrightarrow[T]{F.S.} a_{-k}^*$$

pattern changes but timeperiod & ω_0: same

• **CASE: 1** let $x(t)$ is real

$x(t) = x^*(t)$

$\downarrow \qquad \downarrow$

$a_k = a_{-k}^*$

so, $\boxed{a_k^* = a_{-k}}$

magnitude will be symmetric about y-axis

eg: $a_0^* = a_0$
$a_1^* = a_{-1}$
$a_2^* = a_{-2}$ } provided $x(t)$ is real

• **CASE: 2** let $x(t)$ is real and even

real $\to a_k^* = a_{-k}$

even $\to x(t) = x(-t) \to a_k = a_{-k}$

$\boxed{a_k^* = a_k}$ $x(t)$ is real and even $\to$ a_k are also real and even

$\Rightarrow a_k$ are even and real

• **CASE: 3** let $x(t)$ is real and odd

real $\to a_k^* = a_{-k}$

odd $\to x(t) = -x(-t)$

$\boxed{a_k^* = -a_k}$

let $a_k = \alpha + j\beta \Rightarrow a_k^* = \alpha - j\beta$

but $a_k^* = -a_k$ only when real part doesn't exist in a_k

$\Rightarrow a_k$ are purely complex

• TIME SCALING

$$x(t) \xrightarrow[T]{F.S.} a_k$$

$z(t) = x(\alpha t) \xrightarrow[\frac{T}{\alpha}]{F.S.} C_k = ?$: $\alpha \neq -1$ (time reversal done already)

$T \to \frac{T}{\alpha}$

$\omega_0 \to \alpha\omega_0$

$$z(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\alpha\omega_0 t}$$

$$C_k = \frac{1}{T/\alpha}\int_T x(\alpha t) e^{-jk\alpha\omega_0 t} dt$$

let $z = \alpha t$

$dz = \alpha dt$

and $-\frac{T}{2} \to \frac{T}{2}$ to $-\frac{T}{2\alpha} \to \frac{T}{2\alpha}$

$$C_k = \frac{1}{\alpha} \cdot \frac{1}{T/\alpha} \cdot \int_{-T/2\alpha}^{T/2\alpha} x(z) e^{-jk\omega_0 z} dz = a_k$$

Coefficients are same fundamental time period and fundamental freq. changes

• multiplication

$$x(t) \xrightarrow[T]{FS} a_k$$

$$y(t) \xrightarrow[T]{FS} b_k$$

$$z(t) = x(t) \cdot y(t) \xrightarrow[T]{FS} C_k = ?$$

$$C_k = \frac{1}{T}\int_T x(t) y(t) e^{-jk\omega_0 t} dt$$

$$x(t) = \sum_{l=-\infty}^{\infty} a_l e^{jk\omega_0 t} dt$$

$$y(t) = \sum_{m=-\infty}^{\infty} a_m e^{jk\omega_0 t} dt$$

$$C_k = \frac{1}{T}\int_T \sum_l \sum_m a_l b_m e^{j(l+m)\omega_0 t} \cdot e^{-jk\omega_0 t} dt$$

$$C_k = \sum_l \sum_m a_l b_m \underbrace{\left(\frac{1}{T}\int_T e^{-j(k-l-m)\omega_0 t} dt\right)}_{=1}$$

when $k - l - m = 0$
$k = l + m$
OR
$l = k - m$

$$C_k = \sum_m a_{k-m} b_m$$

$$= \sum_l a_l b_{k-l}$$

multiplication in the time domain leads to convolution in the frequency domain and VICE VERSA

• PARSEVALS RELATION

$$x(t) \xrightarrow[T]{FS} a_k$$

$$\underbrace{\frac{1}{T}\int |x(t)|^2 dt}_{\text{Avg. Power in time domain}} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

avg. power of a signal is partitioned in its fourier coefficients

* DISCRETE TIME FOURIER SERIES

given a discrete time periodic signal $x[n]$

SYNTHESIS $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$, $\omega_0 = \frac{2\pi}{N}$

fundamental time period $= N$

same contiguous time period

$0\omega_0 \to 1\omega_0 \to 2\omega_0 \to (N-1)\omega_0 \to N\omega_0$

$0 \to \frac{2\pi}{N} \to \frac{4\pi}{N} \to \frac{(N-1)2\pi}{N} \to 2\pi$

frequencies are also periodic

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n]$$

where $\phi_k[n] = e^{jk\omega_0 n}$

$$\phi_{k+N}[n] = \phi_k[n] = e^{jk\omega_0 n}$$

$$\langle x[n], \phi_l[n] \rangle_N = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jl\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N \rangle}\left(\sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}\right) e^{-jl\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N \rangle}\sum_{k=\langle N \rangle} a_k e^{j(k-l)\omega_0 n}$$

$$= \sum_{k=\langle N \rangle} a_k \underbrace{\left(\frac{1}{N}\sum_{n=\langle N \rangle} e^{j(k-l)\omega_0 n}\right)}_{=1}$$

when $k - l = 0 \Rightarrow k = l$

$$\boxed{a_l = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] \phi_l^*[n]}$$

* lecture :13 27/09/24

eg3) $x[n] = \sin \omega_0 n$ $\omega_0 = \frac{2\pi}{N}$

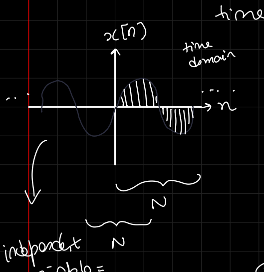
$a_k = ?$

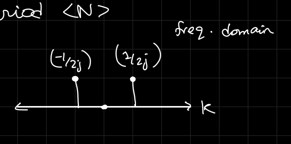
$x[n] = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]$

$= \left(\frac{1}{2j}\right)e^{j\omega_0 n} + \left(\frac{-1}{2j}\right)e^{-j\omega_0 n}$

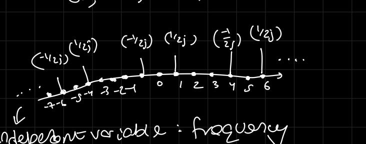
$\Rightarrow a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}$

a_1, a_{-1} are non-zero
rest are zero in the contiguous time period $\langle N \rangle$





let $N = 5$, we need to
so, $a_5 = a_0$, $a_4 = a_{-1}$, $a_6 = a_1$



independent variable : frequency

periodicity in both domains ✓
(dots required to show periodicity)

eg3) $x[n] = 1 + \sin \frac{2\pi}{N} + 3\cos \frac{2\pi}{N}n + \cos\left(\frac{4\pi}{N}n + 2\pi\right)$

DC component $= 1$
so, a_0 exists as non-zero

$\omega = \frac{2\pi}{N}$

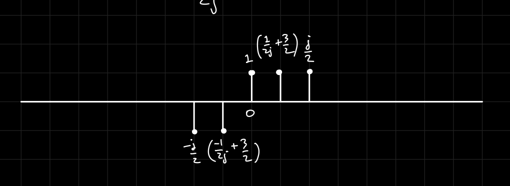
also, every trig component here is periodic with period $= N$
except last which is periodic with $\frac{N}{2}$ but we can also say it is periodic with N

$x[n] = 1 + \frac{e^{j\omega n} - e^{-j\omega n}}{2j} + 3\left[\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right] - \left[\frac{e^{j2\omega n} - e^{-j2\omega n}}{2j}\right]$

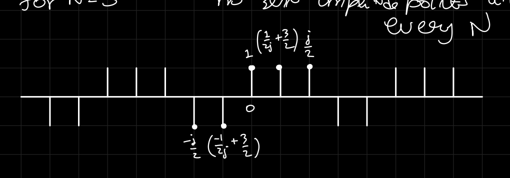
note: $\cos(\theta + \frac{\pi}{2}) = -\sin(\theta)$

cases: $N = 5, 7, 11, 3$

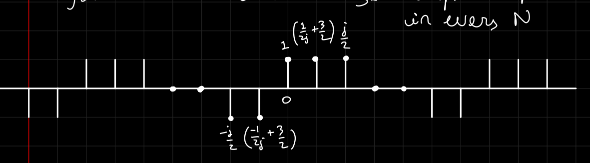
$a_0 = 1$
$a_1 = \frac{1}{2j} + \frac{3}{2}$
$a_{-1} = \frac{-1}{2j} + \frac{3}{2}$
$a_2 = \frac{-1}{2j}$
$a_{-2} = \frac{1}{2j}$



for $N = 5$: no zero amplitude points in every N

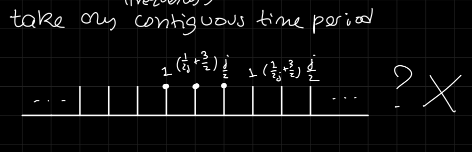


for $N = 7$: two zero amplitude points in every N



for $N = 11$: we have $11 - 5 = 6$ points which zero amplitude

$\Rightarrow$ for $N = 3$: $\ldots, -\frac{2\pi}{3}, 0, \frac{2\pi}{3}, 2\left(\frac{2\pi}{3}\right), 3\left(\frac{2\pi}{3}\right), \ldots$ ($3\left(\frac{2\pi}{3}\right) \equiv 0$)
ω_0 (frequency)
take any contiguous time period



is $\sin\left(\frac{2\pi n}{3}\right)$ periodic?
$x(t) = x(t + t_0)$

period: $N = 3$, $\omega = \frac{2\pi}{3}$

but $\sin\left(\frac{2}{3}n\right)$ is not periodic

$3\cos\left(\frac{2\pi}{3}n\right)$ is also periodic with $N = 3$, $\omega_0 = \frac{2\pi}{3}$

$\cos\left(\frac{4\pi}{3}n + \frac{\pi}{2}\right) = -\sin\left(\frac{4}{3}\pi n\right)$

$\Rightarrow \omega_0 = \frac{4\pi}{3}, N = 3$

all three periodic
hence compatible with $N = 3$ ✓

coeff:

$x[n] = 1 + \frac{1}{2j}\left(e^{j\frac{2\pi n}{3}} - e^{-j\cdot\frac{2\pi}{3}\cdot n}\right) + \frac{3}{2}\left(e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n}\right) - \frac{\left(e^{j\frac{4\pi}{3}n} - e^{-j\frac{4\pi}{3}n}\right)}{2j}$

$N/2 \to N$

note: $e^{j\frac{4\pi}{3}n}$ is periodic with ω_0 doubling and time period halves $(N/2)$
but we can say it is periodic with N as well

$x[n] = 1 + \left(\frac{1}{2j} + \frac{3}{2} + \frac{1}{2j}\right)e^{j\frac{2\pi}{3}n} + \left(\frac{-1}{2j} + \frac{3}{2} + \frac{1}{2j}\right)e^{-j\frac{2\pi}{3}n}$

three terms ✓

we have a_0, a_1, a_2 ✓

---

eg) $x[n] \equiv$ periodic with period $= N$

Show that F.S. coeffs of the periodic signal:

$$g(t) = \sum_{k=-\infty}^{\infty} x[k]\,\delta(t - kT)$$

are periodic with period N

$g(k) = x[0]\delta(t) + x[1]\delta[t - T] + x[2]\delta[t - 2T] + \cdots x[N-1]\delta[t - \ldots]$
$0, 1, 2, \ldots N-1$

so, these terms will keep on repeating
so,
Time periodic with NT

$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{NT}t}$

$a_0 = \frac{1}{NT}\int_{NT} g(t)\,dt$

let $N = 3$

$g(t)$ in one period $= x[0] + x[1] + x[2]$

$a_0 = \frac{1}{NT}\int_{NT} g(t)\,dt = \frac{1}{NT}\left[x[0] + x[1] + \cdots x[N-1]\right]$

$x[n] \xrightarrow{FS} b_k$

$a_0 = \frac{b_0}{T}$ summation $\equiv$ divided by time period

$a_l = \frac{1}{NT}\int_{NT} g(t)\,e^{-jl\frac{2\pi}{NT}t}\,dt$

$a_l = \frac{1}{NT}\int_0^{NT}\sum_{k=0}^{N-1} x[k]\,\delta(t - kT)\,e^{-jl\frac{2\pi}{NT}t}\,dt$

$t = kT$ for impulse functions $= 1$

$= \frac{1}{NT}\sum_{k=0}^{N-1} x[k]\int_0^{NT} e^{-jkl\frac{2\pi}{N}}\,\delta(t - kt)\,dt$

$a_l = \frac{1}{NT}\sum_{k=0}^{N-1} x[k]\,e^{-jk\frac{2\pi}{N}\cdot l}$

$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{x[n]}{T}\right)e^{-jl\frac{2\pi}{N}n}$

---

eg) ① $x(t)$ is a real signal
② $x(t)$ is periodic with $T = 4$
③ $a_k = 0$ for $|k| > 1$
④ A signal with F.S. coeff $b_k = e^{-j\frac{\pi}{2}k}a_{-k}$ is odd (signal is odd, not the coeff)
⑤ $\frac{1}{4}\int_4 |x(t)|^2\,dt = \frac{1}{2}$

Ans) from ③ we get $a_k = 0$ for $-1 < k < 1$
so, only a_0, a_1, a_{-1} exist as nonzero amp coefficients

⑤ $\to$ avg. power in one time period
using parseval's eqn

$\frac{1}{T}\int |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty}|a_k|^2$

$\Rightarrow |a_{-1}|^2 + |a_0|^2 + |a_1|^2 = \frac{1}{2}$

① $\to$ real signal:

$a_k^* = a_{-k}$

so, $a_1^* = a_{-1}$
and $a_{-1}^* = a_1$

④ $\to$ if $x'(t) = x(-t)$
then coeff of $x'(t)$ are a_{-k} if a_k : coeff of $x(t)$

$x(t) \xrightarrow[T]{FS} a_k$

$x(-t) \xrightarrow[T]{FS} a_{-k}$

$z(t) = x(-t+1) \xrightarrow[T]{FS} ?$

$c_k = \frac{1}{T}\int_T z(t)\,e^{-jk\omega_0 t}\,dt$

$\frac{1}{T}\int_T x(-t+1)\,e^{-jk\omega_0(-t+1)}\,dt$

let $\tau = -t + 1 \Rightarrow d\tau = -dt$

$= \frac{1}{T}\int_T x(\tau)\,e^{-jk\omega_0\tau} + d\tau$

$\Rightarrow a_k$

if signal = real + odd
$\Rightarrow$ coefficients are purely imaginary

answer: $b_0 = 0$
$|b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$

and b_1, b_{-1} are purely imaginary

let $b_1 = \frac{j}{2}$ then $b_{-1} = \frac{-j}{2}$

or general equation:

$b_1 = ja, \quad b_{-1} = -ja$

so, $|ja|^2 + |-ja|^2 = \frac{1}{2}$

④ we have a signal which is not $x(t)$
and $b_k = e^{-j\frac{\pi k}{2}}a_{-k}$
also signal is odd

so, we found $x(-t+1)$ to be a matching function

energy remains same in time shifting so parseval's eqn holds for $y(t)$

also if $b_0 = 0$ then
$a_0 = 0 \Rightarrow$ because $y(t)$ is constructed by only shifting & flipping so, no DC component
so, we have only 2 coeffs left

eg) ① $x[N]$ is periodic with $N = 6$

② $\sum_{n=0}^{5} x[n] = 2$

③ $\sum_{n=2}^{7} (-1)^n x[n] = 1$ ($(-1) = e^{j\pi}$)

④ $x[n]$ has the min. power per period among the set of signals satisfying the proceeding 3 conditions

$a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n]\,e^{-jk\frac{2\pi}{6}n}$

$= \frac{1}{6} \times 2 \times e^{-jk\frac{\pi}{3}n}$

so, $a_0 = \frac{1}{3}$

$a_3 = \frac{1}{3} \times \cos(-\pi n) + j\sin(-\pi n)$

$a_3 = \frac{1}{3} \times \cos(\pi n) - j\sin(\pi n)$

$a_3 = \frac{1}{3} \times \cos(\pi n)$

$a_3 = \frac{1}{6}$?

③ $\sum_{n=2}^{7} (-1)^n x[n] = 1$

$-1 = e^{-j\pi} \to e^{-j \times \frac{2}{2} \times \frac{2\pi}{3}}$

$= \cos(\pi) - j\sin(\pi)$

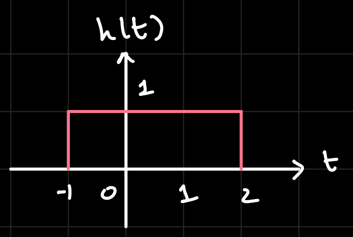
$\sum (e^{-j\pi})^n\, x[n]$ $e^{-j\frac{2\pi}{6} \times 3}$

$= \sum e^{-j\omega 3n}\, x[n]$

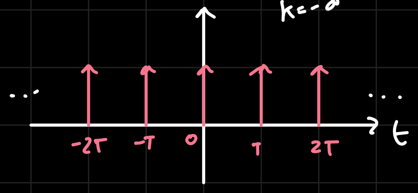
$\frac{1}{6}(1)$

* SNS : midsem 2023

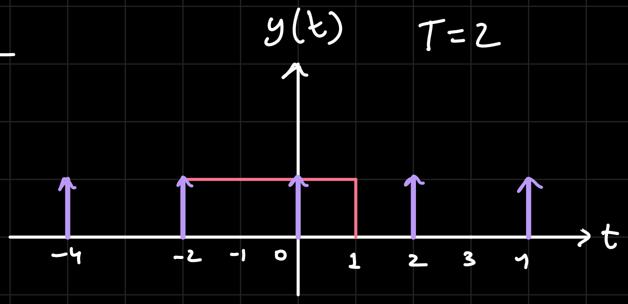
①



$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$



(a) $T = 2$ $\quad y(t) \quad T=2$



for one period:

$t=0$

$y(0) = \begin{cases} 1 & : -2 \\ 1 & : 0 \end{cases}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

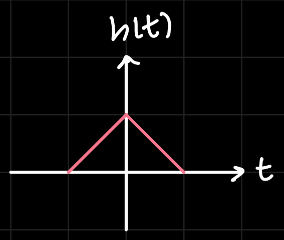
$$y(t) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(\tau - 2k)\, h(t-\tau)\, d\tau$$

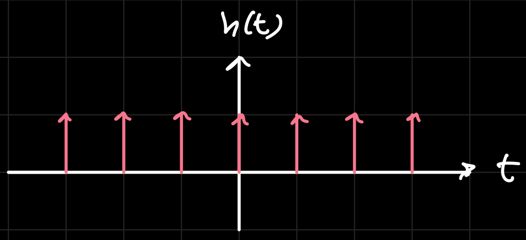
$$= \int_{-2-t}^{1-t} \sum_{k=-\infty}^{\infty} \delta(\tau-2k)\, d\tau$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \int$$

similar question from book:

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$





$$y(t) = x(t) * h(t)$$

$$x(t) * \delta(t) = \int_{\tau} x(\tau)\delta(t-\tau)\, d\tau$$

since $\delta(t-\tau) = 0$ for $t \neq \tau$

so, $x(t) * \delta(t) = x(t)$

similarly,

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

given : $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

* Lecture : 14

Filter : $e^{st} \rightarrow e^{st} \cdot H(s)$

$\hookrightarrow s = j\omega$

$\hookrightarrow$ attenuates or passes amplitude of certain frequencies

High pass filter
$\downarrow$
image's edge's freq. attenuated hence feels blurred

for CTS
$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

DTS: $x[n] = z^n \longrightarrow \boxed{LTI}_{h[n]} \longrightarrow y[n] = ?$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$
$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$
$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}_{H(z)}$$

$$y[n] = z^n H(z)$$

let $z = e^{j\omega_0}$

$$x[n] = e^{j\omega_0 n} \longrightarrow e^{j\omega_0 n} \cdot H(e^{j\omega_0})$$

$\downarrow$

Z Transform (last chp)

DT FS

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$h[n]$ $\downarrow$ LTI system

$$y[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \cdot H(e^{jk\omega_0})$$

Given: $x[n] = \sum_{k=\langle 5 \rangle} a_k e^{jk\frac{2\pi}{5}n}$

$a_0 = 1$

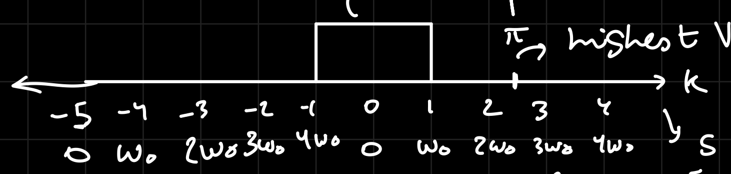
$a_1 = a_{-1} = \frac{1}{2}$

$a_2 = a_{-2} = \frac{3}{4}$

$$x[n] = 1 + \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n} + \frac{3}{4} e^{j2\omega_0 n} + \frac{3}{4} e^{-j2\omega_0 n}$$

let this be the low pass filter

$\pi \rightarrow$ highest value at $k = 2.5$ so it will be passed in a high pass filter



$-\frac{4\pi}{5}$ $-\frac{2\pi}{5}$ 0 $\frac{2\pi}{5}$ $\frac{4\pi}{5}$ $\frac{6\pi}{5}$

$\frac{6\pi}{5} \rightarrow 2\pi - \frac{4\pi}{5}$, $2\pi \rightarrow 0$

$= -\frac{4\pi}{5} = \frac{6\pi}{5}$

$5 \rightarrow 5\omega_0 \rightarrow 5 \cdot \frac{2\pi}{5} \rightarrow = 0$

freq. response of filter

$H(0) = x$

$H(e^{j\omega_0}) = x$

$H(e^{-j\omega_0}) = x$

$H(e^{j2\omega_0}) = 0$

$H(e^{-j2\omega_0}) = 0$

let $x = 1$ for this case