Q1)

Ques 1. If $\vec{A} = \rho\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_\phi$, Evaluate $\oint \vec{A}.dl$ around the path shown in Figure 1. Confirm this by using Stoke's theorem.

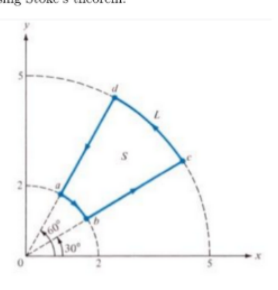

Figure 1:

$$\bar{A} = \rho\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_\phi$$

$$\oint \bar{A}\cdot\bar{dl} = \int\left[\int_a^b + \int_b^c + \int_c^d + \int_d^a\right]\bar{A}\cdot\bar{dl}$$

$\boxed{a \to b}$

$$\bar{dl} = \rho\, d\phi\,\hat{a}_\phi$$

$$\int_a^b \bar{A}\cdot\bar{dl} = \int_{\phi=30^\circ}^{\phi=60^\circ} \rho\sin\phi\, d\phi = 2\left[-\cos\theta\right]_{60}^{30^\circ}$$

$$= 2\left[\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\right]$$

$$= -(\sqrt{3}-1)$$

$\boxed{b \to c}$ $\phi = 30^\circ$

$$\bar{dl} = d\rho\,\hat{a}_\rho$$

$$\int_b^c A\cdot dl = \int_{\rho=2}^{\rho=5} \rho\cos\phi\, d\rho = \left[\frac{\rho^2}{2}\right]_2^5 \frac{\sqrt{3}}{2}$$

$$= \frac{21\sqrt{3}}{4}$$

$\boxed{c \to d}$ $\rho = 5$

$$\bar{dl} = \rho\, d\phi\,\hat{a}_\phi$$

$$\int_c^d A\cdot dl = \int_{30^\circ}^{60^\circ} \rho\sin\theta\, d\phi$$

$$= -5\left(\cos\theta\right)_{30^\circ}^{60^\circ} = \frac{-5(1-\sqrt{3})}{\sqrt{2}}$$

$$= \frac{5(\sqrt{3}-1)}{\sqrt{2}}$$

$\boxed{d \to a}$ $\phi = 60^\circ$

$$\bar{dl} = \bar{d\rho}\,\hat{a}_\rho$$

$$\int_d^a A\cdot dl = \int_5^2 \rho\cos(60^\circ) = \frac{1}{4}\left[4-25\right]$$

$$= \frac{-21}{4}$$

finally : $\oint A\cdot dl = -\sqrt{3}+1 + \frac{21\sqrt{3}}{4} + \frac{5}{\sqrt{2}}(\sqrt{3}-1) - \frac{21}{4}$

$$= \underline{4.941}$$

Stoke's Theorem

$$\oint \bar{A}\cdot\bar{dl} = \int_S (\bar{\nabla}\times\bar{A})\, d\bar{s}$$

$$\nabla\times A = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ (\rho\cos\phi) & \rho(\sin\phi) & 0 \end{vmatrix}$$

$$= 0\,\hat{a}_\rho + 0\,\hat{a}_\phi + \frac{1}{\rho}(1+\rho)\sin\phi\,\hat{a}_z$$

$$\vec{ds} = \rho\, d\phi\, d\rho\,\hat{a}_z$$

$$(\bar{\nabla}\times A)\cdot\bar{ds} = \int_{30^\circ}^{60^\circ}\int_2^5 (1+\rho)\sin\phi\, d\rho\, d\phi$$

$$= \int_{30}^{60}\left[\rho + \frac{\rho^2}{2}\right]_2^5 \sin\phi\, d\phi$$

$$= -\frac{27}{2}\left[\cos\phi\right]_{30}^{60}$$

$$= \frac{27}{4}(\sqrt{3}-1) = \underline{4.941}$$

Stoke's theorem verified

Q2)

Ques 2. One might be tempted to apply the divergence theorem to the surface integral in Stoke's theorem. However, the divergence theorem requires a closed surface while Stoke's theorem is true in general for an open surface. Stoke's theorem for a closed surface requires the contour to shrink to zero giving a zero result for the line integral. Use the divergence theorem applied to the closed surface with vector $\vec{\nabla}\times\vec{A}$ to prove that $\vec{\nabla}.(\vec{\nabla}\times\vec{A}) = 0$.
Verify this by direct computation in cylindrical coordinates.

Stoke's theorem ≡ only applicable for open surfaces

$$\int_{S'} (\bar{\nabla}\times\bar{A})\, d\bar{s} = \oint_C A\cdot dl$$

make this closed by putting c=0

so, $\oint_C A\cdot dl = 0$ ①

so, $\int_{S'} (\bar{\nabla}\times\bar{A})\, d\bar{s} = 0$ for c=0

divergence theorem ≡ only applicable for closed surfaces

$$\int_V \bar{\nabla}\cdot A\, dv = \oint_S \bar{A}\cdot\bar{ds}$$

$$\int_V \bar{\nabla}\cdot(\bar{\nabla}\times\bar{A})\, dv = \oint_S (\nabla\times A)\, d\bar{s}$$

we know this coming to be zero from ①

Brute Force method:

$$\bar{A} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$$

$$(\bar{\nabla}\times\bar{A}) = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\frac{1}{\rho}\left[\left(\frac{\partial}{\partial\phi}A_z - \frac{\partial}{\partial z}\rho A_\phi\right)\hat{\rho} - \rho\left(\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\rho}{\partial z}\right)\hat{\phi} + \left(\frac{\partial}{\partial\rho}\rho A_\phi - \frac{\partial A_\rho}{\partial\phi}\right)\right]$$

$$(\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A})) = 0$$

Q3)

Ques 3. A sphere of radius R_1 and free space permitivity ϵ_0 has a volume charge distribution

$$\rho_f(r) = \rho_0\left(\frac{r}{R_1}\right)^4, \quad 0 < r < R_1$$

The sphere is surrounded by free space and a perfectly conducting sphere of radius R_2 so that $\vec{E}=0$ for $r>R_2$. There is no surface charge on the $r=R_1$ surface.
(a) What is the total charge on the sphere?
(b) What is the electric field $\vec{E}$ for $0<r<R_2$?
(c) What is the surface charge density on the perfectly conducting sphere of radius R_2?
(d) What is the total charge on the $r=R_2$ spherical surface and how is it related to your answer in (a)?

$$\rho_f(r) = \rho_0\left(\frac{r}{R_1}\right)^4 \quad : \quad 0<r<R_1$$

(a) $\rho = \dfrac{q}{4\pi r^3}$

$$q = \int_V \rho\, dv = \int_0^{R_1} \rho_0\left(\frac{r}{R_1}\right)^4 4\pi r^2 dr$$

$$= \frac{4\pi\rho_0}{R_1^4}\int_0^{R_1} r^6 dr$$

$$= \frac{4\pi\rho_0}{R_1^4}\left[\frac{r^7}{7}\right]_0^{R_1}$$

$$q = \frac{4\pi\rho_0 R_1^3}{7}$$

(b) $\bar{E} = ?$ for $0<r<R_2$

$$\oiint E\cdot da = \frac{1}{\varepsilon_0}q = \frac{1}{\varepsilon_0}\int_V \rho\, dv$$

gauss's law for free space

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^r \rho_0\left(\frac{r}{R_1}\right)^4 4\pi r^2 dr +$$

for $R_1<r<R_2$ →

$$\frac{1}{\varepsilon_0}\times\left(\frac{4\pi\rho_0 R_1^3}{7}\right)$$

$$\cancel{4\pi}\varepsilon_0 r^2 E_r = \frac{\cancel{4\pi}\rho_0 r^7}{7R_1^4} + \frac{4\cancel{\pi}\rho_0 R_1^3}{7}$$

$$E_r = \frac{\varepsilon_0 r^2\rho_0(r^7 + R_1^7)}{7R_1^4}$$

(c) Surface charge density

$$\sigma_s = -\varepsilon_0 E_r \quad (r=R_2)$$

$$= -\varepsilon_0$$