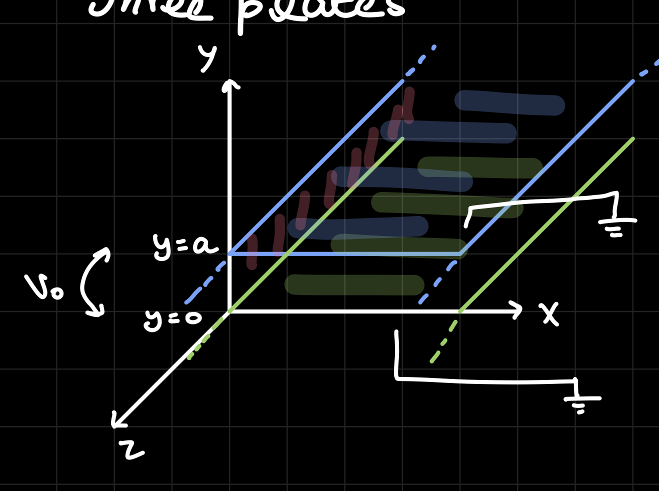
Lecture 7 :

27/01/25

Three plates



∞ along z-axis and positive x-axis

$$\nabla^2 \phi = 0$$

potential V_0 applied to the ≡ plate

Boundary conditions:

① $\phi = 0$ at $y = 0$
② $\phi = 0$ at $y = a$
③ $\phi = V_0$ at $x = 0$
④ $\phi = 0$ as $x \to \infty$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 \phi}{\partial z^2}}}_{} \overset{\nearrow 0}{} = 0$$

because it is ∞ along z-axis

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

we get the 2d equivalent of the equation

method of seperation of variables:

$$\phi(x,y) = X(x)\,Y(y)$$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

divided by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

Both X and Y have to be constants so as to satisfy this eqn for all points of X and Y

– Madhav 2025
:thumbs up:

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{C_1} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{C_2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

$$\frac{\partial^2 X}{\partial x^2} - k^2 X = 0 \qquad \frac{\partial^2 Y}{\partial y^2} + k^2 Y = 0$$

↳ Eigenvalue eqn ↙ $\hat{O} f(x) = c f(x)$

operator : $\frac{\partial^2}{\partial [x/y]^2}$

eigenvalue : $\pm k^2$

operand : X/Y

how to solve these DE?

assume an auxilliary equation

let $X = e^{mx}$

$$\frac{\partial^2}{\partial x^2}(e^{mx}) - k^2(e^{mx}) = 0$$

$$m^2 - k^2 = 0$$
$$m = \pm k$$

So, $X = Ae^{-kx} + Be^{kx}$

let $Y = e^{my}$

$$m^2 + k^2 = 0$$
$$m = \pm ik$$

$$\text{So, } Y = Pe^{ikx} + Qe^{-ikx}$$
$$= P(\cos ky + i\sin ky) + Q(\cos ky - i\sin ky)$$
$$= C\sin(ky) + D\cos(ky)$$

where $C = (P-Q) \times 2$ and $D = (P+Q) \times 2$

So, $X = Ae^{-kx} + Be^{kx}$ & $Y = C\sin(ky) + D\cos(ky)$

but due to ④ $\phi = 0$ as $x \to \infty$,

B must be zero $\forall x$ ↙

So, $X = Ae^{-kx}$

and due to ① $\phi = 0$ for $y = 0$

$$C(\sin 0) + D(\cos 0) = 0$$
$$0 + D = 0$$
$$\text{So, } D = 0 \quad \forall y$$

So, $Y = C\sin(ky)$

$$\phi = XY = ACe^{-kx}\sin(ky)$$

let $AC = G$

So, # unknown arbitrary constants = 2

$$\phi = Ge^{-kx}\sin(ky)$$

at ② $\phi = 0$ at $y = a$

$$Ge^{-kx}\sin(ka) = 0$$
$$ka = n\pi \ : \ n = 1, 2, \ldots$$
$$k = \frac{n\pi}{a}$$

So → $Y = C\sin\left(\frac{n\pi}{a} y\right)$

$$X = Ae^{-\frac{n\pi x}{a}}$$

$$\phi = Ge^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \quad : n = 1, 2, 3 \ldots$$

let $\phi = \phi_1$ for $n = 1$

⋮

$\phi = \phi_n$ for $n = n$

and so, $\phi = \sum_{n=1}^{\infty} \phi_n$

http server implementation in C ↙
network sockets
client / serve
file descriptor
bind
accept
listen

$$\phi_n = G_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

due to ③ $\phi = V_0$ at $x = 0$

$$V_0 = \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1}^{\infty} \int_0^a G_n \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy$$

in this summation there will be only one term where RHS is non-zero and it is m = n

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) = \int_0^a G_m \sin\left(\frac{m\pi y}{a}\right) dy$$

$$= G_m \int_0^a \Big(1 - \underbrace{\cos\left(\frac{2m\pi y}{a}\right)}_{\int \text{periodic} = 0}\Big) dy$$

$$= G_m \cdot \frac{a}{2}$$

$$G_m \frac{a}{2} = V_0 \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy$$
$$= V_0 \frac{a}{m\pi} \left[-\cos\left(\frac{m\pi y}{a}\right)\right]_0^a$$
$$= \frac{V_0 a}{m\pi}\left(1 - \cos(m\pi)\right)$$
$$= \frac{V_0 a}{m\pi}\left(1 - (-1)^m\right)$$

$$G_m = \begin{cases} 0 & : m \equiv \text{even} \\ \frac{4V_0}{m\pi} & : m \equiv \text{odd} \end{cases}$$

$$\phi = \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{4V_0}{n\pi} e^{\left(-\frac{n\pi x}{a}\right)} \sin\left(\frac{n\pi y}{a}\right)$$