Fields and waves

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	Mid sem	30%
A at 80%	End sem	40%
	Quiz × 6 (N-1)	30%

pass : 30%

grading : absolute

attendance in tutorials : atleast 75% required for attempting next quiz

Office hours:
Wed 12:30 – 1:15 pm

- Vector calc
- Electrostatics and Magnetostatics — doesn't vary with time
- Electrodynamics — varies with time
- EM waves
- Transmission Lines

Book: David J. Griffiths
Intro to Electrodynamics

M.N.O. Sadiku
Elements of electromagnetics

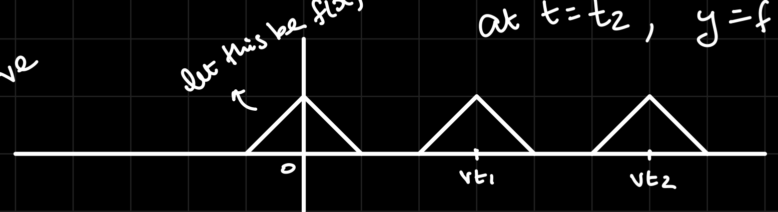
- wave

$$y = f(x \pm vt)$$

v = constant
t = time

at $t=0$, $y = f(x)$
at $t = t_1$, $y = f(x - vt_1)$
at $t = t_2$, $y = f(x - vt_2)$

let the sign here be -ve



forward travelling wave →

if sign = +ve : backward travelling wave ←

$F_E = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}$

$F_{Grav} = G \frac{m_1 m_2}{r^2}$

} with unit masses and unit charges kept at 1m distance,

$\frac{F_E}{F_{Grav}} \simeq \frac{10^{12}}{10^{-11}} = 10^{23}$

speed of free electron: $\sim 10^5$ m/s

in a conductor : electron takes 1s to move 1mm

then how does a hall lights up almost instantly when you switch on the light?

because e^- are chained together in nature and the initial e^- doesn't travel all the way from beginning to the end of the hall

CEC, FCC : radiation certification

Electric field lines have to always end at a charge : ⊕ → ⊖

When the charge has been moved, the field lines will also have a disturbance

KCL and KVL don't work with very large circuit because of signal propagation delay

direction of E × H → source to load

min time taken for propagation $= \frac{c}{x}$

$c = \lambda f$

wavelength $= \lambda = \frac{c}{f}$

let $f = 50$Hz,

$\lambda = \frac{3 \times 10^8}{50} = 6 \times 10^6$ m

KCL/KVL applicable when circuit size < wavelength

for $f = 10$GHz $= 10 \times 10^9$Hz

$\lambda = \frac{c}{f} = 3 \times 10^{-2}$m $= 0.03$m

for any circuit with length (x) greater than 3cm, KCL and KVL will not apply

- **Lecture: 2** 8/01/25

→ Gradient operator
→ Level surface

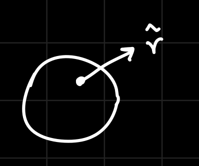
Directional derivative : $\frac{d\phi}{dr} = (\bar{\nabla}\phi)\cdot\hat{r}$

$= |\bar{\nabla}\phi| \cos\theta$

$\left|\frac{d\phi}{dr}\right|_{max} = |\bar{\nabla}\phi|$

→ for a level surface, directional derivative is along the surface normal

eg. for a sphere, the direction $\hat{r}$ at a specific point will be outwards from that point



$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$
$d\bar{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$\phi(x,y,z)$: func of x, y, z

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

- **Potential**

electrostatic potential

1 dimension : $\bar{E} = -\frac{d\Phi}{dx}\hat{x}$

3 dimension : $\bar{E} = -\left(\frac{\partial\Phi}{\partial x}\hat{x} + \frac{\partial\Phi}{\partial y}\hat{y} + \frac{\partial\Phi}{\partial z}\hat{z}\right)$

- **Cylindrical Coordinates**

ρ, ϕ, z

differential length
differential area
differential solid angle

- **Spherical Coordinates**

$0 \le r < \infty \qquad 0 \le \theta \le \pi \qquad 0 \le \phi \le 2\pi$

differential length
differential area
differential $\underbrace{\text{solid angle}}_{\frac{\text{length of arc}}{\text{area}}} = \frac{dS}{r^2} = \sin\theta\, d\theta\, d\phi$

- **Line Integral**

$$\left.\begin{array}{l} \text{WORK} : \int_A^B F\,dl \rightarrow \text{for any force} \\ \text{Electrostatic potential} : \Phi_{AB} = -\int_A^B E\,dl \end{array}\right\} \text{path dependent}$$

- Note: some are path independent

assume : $\bar{V} = \bar{\nabla}u$

$\int_A^B V\cdot dl = \int_A^B (\bar{\nabla}u)\,dl = \int_A^B du = \underbrace{U_B - U_A}$

here, path independent ← only depends on end points

- A field which can be written as the gradient of a scalar is called conservative field

- note: gravitational (represented as gradient of gravitational potential) and electric fields (represented as gradient of electrostatic potential) are conservative whereas magnetic field cannot be represented as the gradient of a scalar.

- for closed loop → $\oint_A^B \bar{V}\cdot dl = \int_A^B du + \int_B^A du$

$= U_B - U_A + U_A - U_B = 0$

so, work done by a conservative field in a closed loop is zero.

i.e. closed loop line integral = 0

- **FLUX**

- open surface : $\int_{S_{closed}} V\cdot ds$
- closed surface : $\oint_{S_{open}} V\cdot ds$

$\oint_{S_{closed}} V\cdot ds = \lim$

- **Divergence operator**

$\bar{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$

$$\text{Gradient of} \left\{\begin{array}{ll} \text{dot product of vector} & \equiv \text{Divergence operator} \\ \text{Cross product of vector} & \equiv \text{CURL} \\ \text{scalar} & \equiv \text{Conservative field} \end{array}\right.$$

if $\bar{\nabla}\times\bar{A} = 0$ it does not necessarily mean they are parallel because you cannot practically determine direction of $\frac{\partial}{\partial x} / \frac{\partial}{\partial y} / \frac{\partial}{\partial z} / \bar{\nabla}$ } Same goes for dot product as well

divergence for a very small surface $= \frac{\text{total outward flux}}{\text{volume}}$

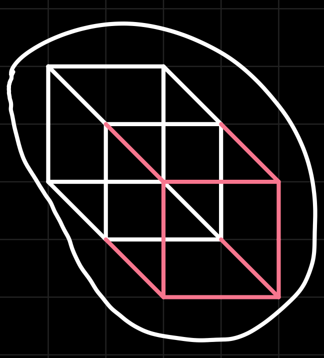
- Lecture 3

* Gauss' Divergence Theorem

divide the volume into tiny boxes and calculate the outward flux and add them up

$$\int_{V_{big}} (\bar{\nabla}\bar{v})\, d\bar{v} = \sum_{\substack{\text{all tiny} \\ \text{parallelpiped}}} \lim_{dv \to 0} (\bar{\nabla}\bar{v})\, dv$$

$$= \sum \oint_{\substack{\text{small} \\ \text{volume}}} V \cdot ds$$



outward flux from all the shared surfaces cancel out

$$\text{so, final flux} = \sum_{\substack{\text{unshared surfaces} \\ \text{coinciding with the} \\ \text{boundary of} \\ \text{the big volume}}} \bar{V} \cdot d\bar{S}$$

closed surface integral

$$= \oint_{\substack{\text{entire} \\ \text{surface}}} \bar{V} \cdot d\bar{S}$$

• LAPLACIAN OPERATOR ($\bar{\nabla}^2$)

can operate on both, scalar and vector

$$\bar{\nabla}(\bar{\nabla}U) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}\right)$$

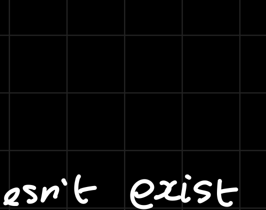
$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$(\bar{\nabla}^2 U) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U =$$

* EARNSHAW'S THEOREM

- A scalar field $\phi(x,y)$ that has $\underbrace{\nabla^2}_{\text{laplacian}}\phi = 0$, cannot have a local min/max in that region.

eg: $\phi = x^2 + y^2$
$\nabla\phi = 2x + 2y$
$\nabla^2\phi = 4$
non-zero
has a local min ✓



local min at (0,0,0)

also works with gravitational field

if $\bar{\nabla} \cdot \bar{E} = 0$
→ $\bar{\nabla}(\bar{\nabla}\phi) = 0$
→ $\bar{\nabla}^2\phi = 0$
then local min/max doesn't exist

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

if $D < 0$: saddle point

$\nabla f = f_x + f_y$
$\nabla^2 f = f_{xx} + f_{yy}$
if $\nabla^2 f = 0 \rightarrow f_{xx} + f_{yy} = 0$
and so, $D = \underbrace{-\underbrace{f_{xy}^2}_{+ve}}_{-ve \text{ always}}$

so, $D < 0$
↳ saddle point
↳ neither local min nor local max

Simplest Saddle Point

$z = x^2 - y^2$
OR
(Hyperbolic Paraboloid) $z - \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$

not a stable equilibrium

how can you stabilize a football on the saddle point?
↳ constantly rotate it
i.e. changing the electric field
≡ Paul Trap / Ion trapping in quantum computers

Earnshaw's theorem prevents stability on the saddle point

* CURL operator

$$\bar{\nabla} \times \bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

cannot say $\bar{\nabla}$ is ∥ $\bar{A}$
because operator does not has a direction

defn of curl: line integral along an infinitesimal loop
↳ loop can be in any direction

curl of a vector field ≡ closed line integral per unit area

* STOKE'S THEOREM (finite loop)

$$\int (\bar{\nabla} \times \bar{A}) \cdot d\bar{S} = \oint \bar{A} \cdot d\bar{l}$$

divide the big loop into infinitesimally small loops

• MOBIUS STRIP

cannot apply stoke's Theorem

• Irrotational Field

↳ curl : $\bar{\nabla} \times F = 0$
↳ conservative field
↳ can be written as a gradient of scalar
$F = \bar{\nabla} u$

eg: electrostatic field ✓
not electrodynamic field ✗

• SOLENOIDAL FIELD

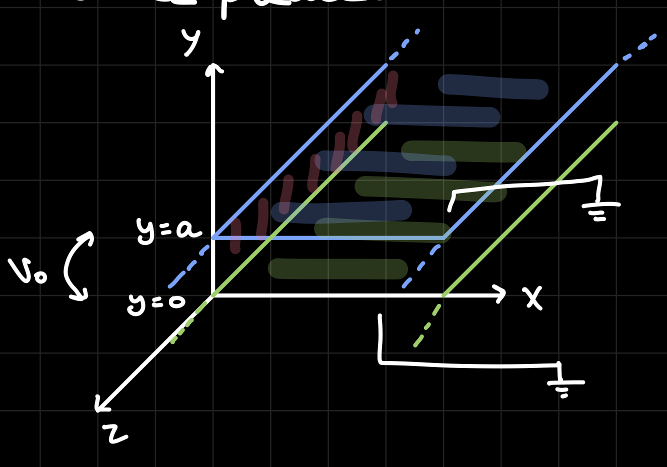
↳ divergence : $\bar{\nabla} \cdot F = 0$
↳ Solenoidal field over volume V doesn't have any source/sink in that volume
↳ $\bar{\nabla} \cdot (\bar{\nabla} \times A) = 0$
↳ A solenoidal field can be written as $\bar{\nabla} \times \bar{A}$, $\bar{A}$ ≡ vector potential

eg: magnetic field is solenoidal field
electrostatic field is solenoidal only when there is no charge

Lecture 7 :

27/01/25

Three plates



∞ along z-axis and positive x-axis

$$\nabla^2 \phi = 0$$

potential V_0 applied to the ☰ plate

Boundary conditions:

① $\phi = 0$ at $y = 0$
② $\phi = 0$ at $y = a$
③ $\phi = V_0$ at $x = 0$
④ $\phi = 0$ as $x \to \infty$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \cancel{\frac{\partial^2 \phi}{\partial z^2}}^{\,0} = 0$$

because it is ∞ along z-axis

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

we get the 2d equivalent of the equation

method of seperation of variables:

$$\phi(x,y) = X(x)\,Y(y)$$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

divided by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

Both X and Y have to be constants so as to satisfy this eqn for all points of X and Y

– Madhav 2025
:thumbs up:

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{C_1} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{C_2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

$$\frac{\partial^2 X}{\partial x^2} - k^2 X = 0 \qquad \frac{\partial^2 Y}{\partial y^2} + k^2 Y = 0$$

↳ Eigenvalue eqn ↙ $\hat{O} f(x) = c f(x)$

operator: $\frac{\partial^2}{\partial [x/y]^2}$

eigenvalue: $\pm k^2$

operand: X/Y

how to solve these DE?
assume an auxilliary equation

let $X = e^{mx}$

$$\frac{\partial^2}{\partial x^2}(e^{mx}) - k^2(e^{mx}) = 0$$

$$m^2 - k^2 = 0$$
$$m = \pm k$$

$$\text{So, } X = Ae^{-kx} + Be^{kx}$$

let $Y = e^{my}$

$$m^2 + k^2 = 0$$
$$m = \pm ik$$

$$\text{So, } Y = Pe^{ikx} + Qe^{-ikx}$$
$$= P(\cos ky + i\sin ky) + Q(\cos ky - i\sin ky)$$
$$= C\sin(ky) + D\cos(ky)$$

where $C = (P-Q)\times 2$ and $D = (P+Q)\times 2$

So, $X = Ae^{-kx} + Be^{kx}$ & $Y = C\sin(ky) + D\cos(ky)$

but due to ④ $\phi = 0$ as $x \to \infty$,

B must be zero $\forall x$

So, $X = Ae^{-kx}$

and due to ① $\phi = 0$ for $y = 0$

$$C(\sin 0) + D(\cos 0) = 0$$
$$0 + D = 0$$
$$\text{So, } D = 0 \quad \forall y$$

So, $Y = C\sin(ky)$

$$\phi = XY = ACe^{-kx}\sin(ky)$$

let $AC = G$

So, # unknown arbitrary constants = 2

$$\phi = Ge^{-kx}\sin(ky)$$

at ② $\phi = 0$ at $y = a$

$$Ge^{-kx}\sin(ka) = 0$$
$$ka = n\pi \quad : n = 1, 2, \dots$$
$$k = \frac{n\pi}{a}$$

So → $Y = C\sin\left(\frac{n\pi}{a}y\right)$

$$X = Ae^{-\frac{n\pi x}{a}}$$

$$\phi = Ge^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right) \quad : n = 1, 2, 3 \dots$$

let $\phi = \phi_1$ for $n = 1$
⋮
$\phi = \phi_n$ for $n = n$

and so, $\phi = \sum_{n=1}^{\infty} \phi_n$

http server implementation in C
network sockets
client/server
file descriptor
bind
accept
listen

$$\phi_n = G_n e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)$$

due to ③ $\phi = V_0$ at $x = 0$

$$V_0 = \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1}^{\infty}\int_0^a G_n \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right) dy$$

in this summation there will be only one term where RHS is non-zero and it is m=n

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) = \int_0^a G_m \sin\left(\frac{m\pi y}{a}\right) dy$$

$$= G_m \int_0^a \left(1 - \underbrace{\cos\left(\frac{2m\pi y}{a}\right)}_{\int \text{periodic} = 0}\right) dy$$

$$= G_m \cdot \frac{a}{2}$$

$$G_m \frac{a}{2} = V_0 \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy$$
$$= V_0 \frac{a}{m\pi}\left[-\cos\left(\frac{m\pi y}{a}\right)\right]_0^a$$
$$= \frac{V_0 a}{m\pi}\left(1 - \cos(m\pi)\right)$$
$$= \frac{V_0 a}{m\pi}\left(1 - (-1)^m\right)$$

$$G_m = \begin{cases} 0 & : m \equiv \text{even} \\ \frac{4V_0}{m\pi} & : m \equiv \text{odd} \end{cases}$$

$$\phi = \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{4V_0}{n\pi} e^{\left(-\frac{n\pi x}{a}\right)} \sin\left(\frac{n\pi y}{a}\right)$$

Charge density → $\rho = \varepsilon_0 \nabla E$

ε_0 = permittivity = 8.854×10^{-12}

$\rho = 8.854$

$\bar{\nabla}\bar{E} = \dfrac{\rho}{\varepsilon_0}$ — gauss's Law

$\bar{\nabla}(-\bar{\nabla}\phi) = \dfrac{\rho}{\varepsilon_0}$

$\bar{\nabla}^2\phi = -\dfrac{\rho}{\varepsilon_0}$ — Possoin's Eqn

if $\rho = 0 \rightarrow \nabla^2\phi = 0$ — laplacian eqn

Q1) (a) 20

(b) $\ln(3) = 1.098$

Q2) $E = Kr^3\hat{r}$ spherical coords

(a) $\rho = 8.854 \times 10^{-12} \times 3r^2 \times K$

note : $\bar{\nabla}E = \dfrac{\rho}{\varepsilon_0}$

$\rho = \bar{\nabla}\cdot\bar{D}$ note: $\bar{D} = \varepsilon\bar{E}$

↳ electric field density

$\varepsilon \rightarrow \varepsilon_0\varepsilon_r$

here $\varepsilon_r = 1$ because free space

(b) $dq = \rho\, dv$

$q = \int \rho\, dv = 3\varepsilon_0 \int_0^R r^2\, dr$

$= \varepsilon_0 R^3$

Q3) $\rho_v = \begin{cases} \dfrac{\rho_0 r}{R} & : 0 \le r \le R \\ 0 & : r > R \end{cases}$

spherical symmetry

	h_1	h_2	h_3
xyz	1	1	1
cylindrical	1	ρ	1
spherical	1	r	$r\sin\theta$

$\dfrac{1}{h_1h_2h_3} = \dfrac{1}{r^2\sin\theta}$

$\nabla\cdot D = \dfrac{1}{r^2}\left\{\dfrac{\partial(r^2 \quad r^3)}{\partial r}\right\}$

$= \dfrac{1}{\cancel{r^2}}\, 5\cancel{r^4} = 5r^2$

$\rho = 5Kr^2\varepsilon_0$

(b) $\varphi = \int \rho_v\, dv$

$= \int 5K\varepsilon_0 r^2\, dv$

$dv = 4\pi r^2 dr$

$= \int 5kr^2\varepsilon_0\, 4\pi r^2 dr$

$= 20k\pi\varepsilon_0 \int_0^R r^4 dr$

$= 4K\pi\varepsilon_0 R^5$

Q3) $\rho_v = \begin{cases} \dfrac{\rho_0 r}{R} & : 0 \le r \le R \\ 0 & : r > R \end{cases}$

$E = ?$ for both regions

Q4) $\bar{J} = \int (r^2+2)\cdot\bar{\nabla}\left(\dfrac{1}{r^2}\right)\hat{r}\, dr$

$= \int (r^2+2)(-2r^{-3})\, dr$

$= 2\int (r^{-1} + 2r^{-3})\, dr$

$= \dfrac{-1}{R^4}$

$\bar{\nabla}F = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2F_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta F_\theta)}{\partial\theta} +$

$\dfrac{1}{r\sin\theta}\dfrac{\partial F_\phi}{\partial\phi}$

Q3) (a) $Q_{enclosed} = \varepsilon_0 E \times 4\pi r^2$

for $r \le R$

$= \int_0^{2\pi}\int_0^{\pi}\int^{r} \rho_v r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= \dfrac{1}{R}\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0 r^3 \sin\theta\, d\phi\, d\theta\, dr$

$= \pi\dfrac{\rho_0 r^4}{R}$

$E = Q_{enclosed} \times \dfrac{1}{4\pi r^2\varepsilon_0}$

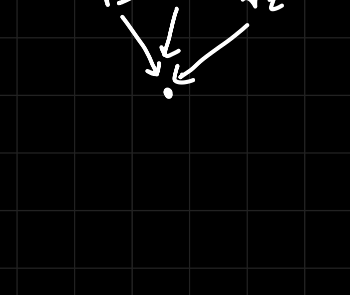
$E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{|r|^2}\hat{r} = \dfrac{1}{4\cancel{\pi}\varepsilon_0}\dfrac{\rho_0\cancel{\pi}r^2}{R}$

$= \dfrac{\rho_0 r^2}{4\varepsilon_0 R}\hat{a}_r$

(b) $E = k\dfrac{q}{r^2}$

$q = \int \rho_v \dfrac{4\pi}{3} r^3\, dr$

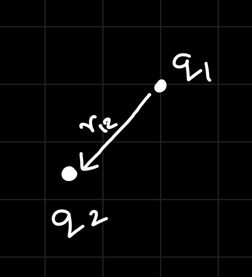
Q4) $\bar{J} = \int_r (r^2+2)\,\bar{\nabla}\left(\dfrac{\hat{r}}{r^2}\right) dr$



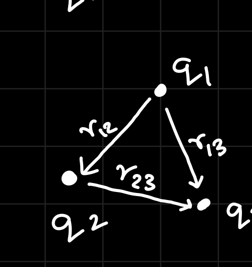
bringing charges one at a time

q_1

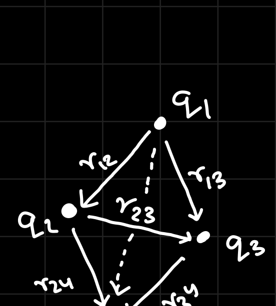
$W_1 = 0$ work done because no elec field



$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{12}}\right)$



$W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$



$W_4 = \frac{1}{4\pi\varepsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$

Total work done $= W = W_1 + W_2 + W_3 + W_4$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{q_i q_j}{r_{ij}}$$

magnitude (not vector)

$$= \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{q_i q_j}{r_{ij}}$$

took double deliberately

$$= \frac{1}{2} \sum_{i=1}^{N} q_i \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} \right\}$$

Total work done by N individual charges

$$\boxed{\frac{1}{2} \sum_{i=1}^{N} q_i V(r_i)} \quad — ①$$

potential at the position of i^{th} charge due to all other charges

What about a surface?

$q_i = \rho\, dv$

$dW = \frac{1}{2} \rho V dv$

The energy of a continuous charge distribution $\boxed{W = \frac{1}{2} \int \rho V dv}$ — ②

Vector Property

$\bar{\nabla} \cdot (f\bar{A}) = f(\bar{\nabla}\cdot\bar{A}) + \bar{A}\cdot(\bar{\nabla} f)$

$\int \bar{\nabla}\cdot(f\bar{A})\, dv = \int f(\bar{\nabla}\cdot\bar{A})\, dv + \int \bar{A}\cdot(\bar{\nabla} f)\, dv$

gauss's law: $\oint f\bar{A}\cdot d\bar{s} - \int \bar{A}\cdot(\bar{\nabla} f)\, dv = \int f(\bar{\nabla}\cdot\bar{A})\, dv$ — ③

divergence of a scalar and vector field

scalar and vector multiplication yields vector then divergence of that results in a scalar

Note: $\rho = \bar{\nabla} E \cdot \varepsilon_0$

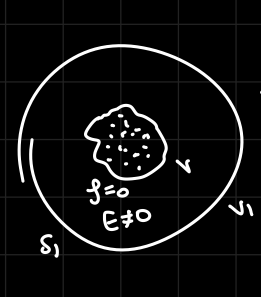
on ① → $W = \frac{1}{2} \int \rho V dv$

$W = \frac{\varepsilon_0}{2} \int (\bar{\nabla} E) V dv$

comparing with RHS of ③

$\bar{A} = \bar{E}$ and $V = f$

So, $W = \frac{\varepsilon_0}{2} \left[\oint_{S_1} V\bar{E}\, d\bar{s} - \int_{V_1} \bar{E}\cdot(\bar{\nabla} V)\, dv \right]$



→ big region finite surface/volume

we know $\bar{\nabla} V = -\bar{E}$

So, $W = \frac{\varepsilon_0}{2} \left[\oint_{S_1} V\bar{E}\, d\bar{s} + \int_{V_1} |\bar{E}|^2 dv \right]$ — ④

surface is very large
↳ r is very large
↳ r^2 grows very fast
↳ $\frac{1}{r^2} \to 0$ so, $\bar{E} \to 0$
↳ also $\frac{1}{r}$ → very small so, $V \to 0$

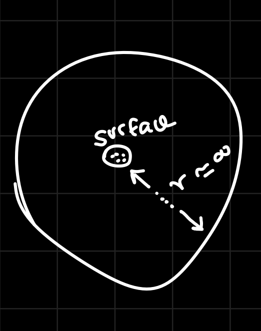
Note: ④ is the general form

but if you take the distance between the charge as very big, the surface integral tends → 0
↳ converges

So,

$W = \frac{1}{2} \varepsilon_0 \int_{\text{all space}} |E|^2 dv$ → converges?

further you are from the surface, lesser this integral becomes



$$\boxed{W = \frac{1}{2} \varepsilon_0 \int |E|^2 dv}$$

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

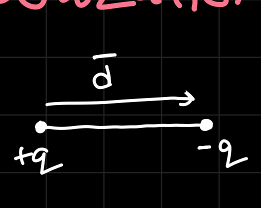
$\varepsilon_{r_{\text{water}}} = 80$

we have written Gauss's Law for vacuum.

but what about a specific medium? Next Class →

→ CHAPTER 4: Polarization

Dipole



dipole moment → $\bar{p} = q\bar{d}$

linear materials: where $\bar{p} \propto \bar{E}$

$\bar{p} = \alpha\bar{E}$
↳ atomic polarizability

at equilibrium → $\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 v$

volume of the atom

a: radius of the atom

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

polarizability matrix or polarizability tensor

So, $p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$

$p_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$

$p_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$

for isotropic elements,

diagonal α_s are α and rest are zero

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$\bar{p} = \hat{x} p_x + \hat{y} p_y + \hat{z} p_z$

$\bar{p} = \hat{x}(\alpha E_x) + \hat{y}(\alpha E_y) + \hat{z}(\alpha E_z)$

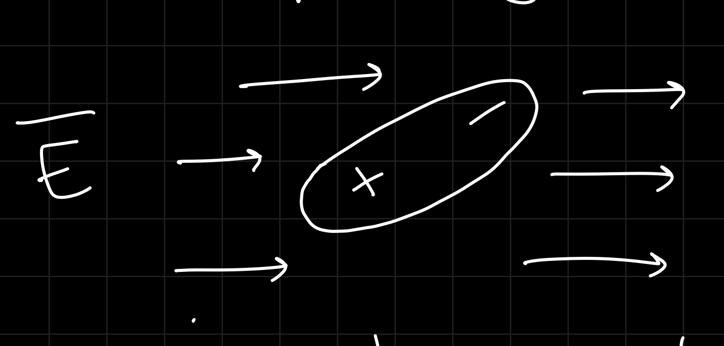
$\bar{p} = \alpha(\hat{x} E_x + \hat{y} E_y + \hat{z} E_z)$

$\bar{p} = \alpha\bar{E}$

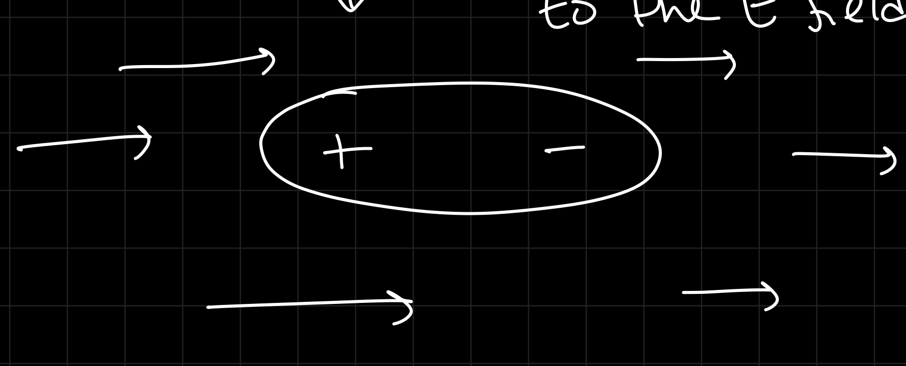
```
# include <stdio.h>
int main(int argc, char** argv){
    printf("4ΣMA 4V<K4");
    return 0;
}
```

constant thermal fluctuation throws dipole out of equilibrium

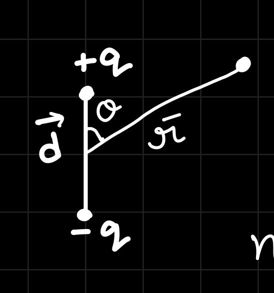
In Induced dipole moment, Electric field is mostly in the direction of the dipole only.



aligned itself acc to the $\bar{E}$ field



* LECTURE 9

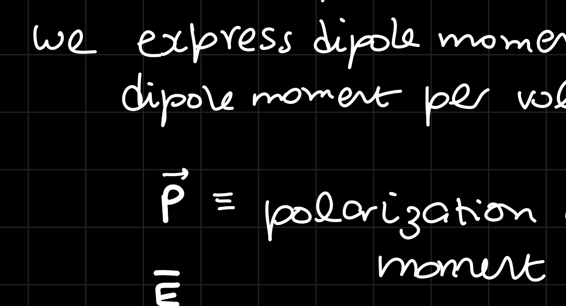


$\bar{p} = q\bar{d}$ $\qquad V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \left(\frac{p}{4\pi\varepsilon_0} \cdot \frac{\hat{r}}{r^2}\right)$

net dipole moment
due to induced and
inherent dipole moment
in the direction of electric field.

Instead of dipole moment for individual particles,
we express dipole moment as polarization
dipole moment per volume for a surface.

$\vec{P} \equiv$ polarization dipole moment per unit volume



$f(x-x', y-y', z-z')$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial(x-x')} \cdot \frac{\partial(x-x')}{\partial x}$$

$x \to$ const
$x' \to$ variable

$$= \frac{\partial f}{\partial(x-x')}$$

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial(x-x')} \cdot \frac{\partial(x-x')}{\partial x'}$$

$$= (-1)\frac{\partial f}{\partial(x-x')}$$

So, $\bar{\nabla}' f = -\bar{\nabla} f$

$$\bar{\nabla}\left(\frac{1}{r}\right) = -\bar{\nabla}'\left(\frac{1}{r}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

* remember to use the vector notation ($\rightarrow$) for quiz

$\vec{p} = \vec{P} dv'$

$$dV = \frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{P}\cdot\hat{r}}{r^2}\right) dv'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\vec{P}\cdot\hat{r}}{r^2}\, dv'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{P}\cdot\bar{\nabla}\left(\frac{1}{r}\right) dv'$$

remember last lecture $\rightarrow$

$$\int \vec{A}(\bar{\nabla} f)\, dv = \underbrace{\oint f\bar{A}\, d\bar{s}}_{\text{surface integral}} - \int f(\bar{\nabla}\bar{A})\, dv$$

here $\bar{P} = \bar{A}$ and $f = \frac{1}{r}$ } magnitude

$$V = \frac{1}{4\pi\varepsilon_0}\left[\oint_{S'} \frac{\bar{P}\cdot d\bar{s}}{\mathscr{r}} - \int \frac{1}{r}(\bar{\nabla}\bar{P}(r'))\, dv'\right]$$

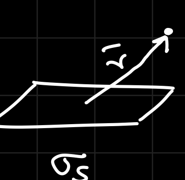
$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'} \frac{\bar{P}\cdot d\bar{s}}{\mathscr{r}} - \int_{V'} \frac{1}{\mathscr{r}}(\bar{\nabla}\bar{P}(r'))\, dv'$$

Potential due to a surface charge density:

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_s\, ds'}{r}$$

σ_s

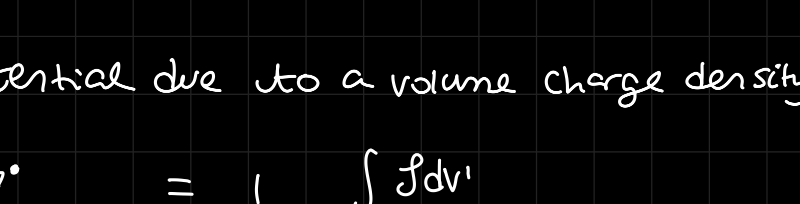
Potential due to a volume charge density



$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, dv'}{r}$$

$\rho(r)$

for a material kept in an electric field $\rightarrow$



We are assuming a uniform polarization vector

So, potential of the surface charge is $\sigma_B = \bar{P}\cdot\hat{n} = \vec{P}\cos\theta$

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'} \frac{\overbrace{(\bar{P}\cdot\hat{n})}^{\sigma_B} ds'}{\mathscr{r}} + \int_{V'} \frac{1}{\mathscr{r}}\overbrace{(-\bar{\nabla}\bar{P}(r'))}^{\rho_B}\, dv'$$

$$V(\mathscr{r}) = \frac{1}{4\pi\varepsilon_0}\oint \frac{\sigma_b\, ds'}{\mathscr{r}} + \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_B\, dv'}{\mathscr{r}}$$

[where $\underbrace{\sigma_B = \bar{P}\cdot\hat{n}}_{\text{total surface charge?}}$ and $\underbrace{\rho_B = -\bar{\nabla}\cdot\bar{P}}_{\text{total volume charge?}}$]

B: bound in σ_B, ρ_B

$$Q_{\substack{\text{volume}\\ \text{(bound)}}} = \int \rho_B(r)\, dv$$

$$= \int(-\bar{\nabla}\bar{P})\, dv$$

$$= -\oint \bar{P}\cdot d\bar{s}$$

$$= -\oint(\bar{P}\cdot\hat{n})\, d\bar{s}$$

$$= -\oint \sigma_B\, ds$$

$$= -Q_{\substack{\text{surface}\\ \text{(bounded)}}}$$

So, total surface charge is equal to negative of total volume charge.

irrespective of the type of polarization vector, whether uniform or not, the total surface and total vol charge add up to zero
Note: We are considering bounded charge
↓
none of them are mobile
They are tied to the molecules
and molecules are tied to their posn.
They are bound to the material.

Remember $\rightarrow \bar{\nabla}\cdot E = \frac{\rho}{\varepsilon_0}$

$$\bar{\nabla}\bar{E} = \frac{\rho_{free} + \rho_{bound}}{\varepsilon_0}$$

$$\bar{\nabla}(\varepsilon_0 E) = \rho_{free} - \bar{\nabla}\cdot\bar{P}$$

$$\rho_{free} = \bar{\nabla}(P + \varepsilon_0 E)$$

$\chi_e \Rightarrow$ Electric susceptibility

easier to polarize if chi/zai (χ_e) is higher

$$\bar{P} = \underbrace{\alpha}_{\substack{\text{atomic}\\ \text{polarizability}}}\bar{E} = \varepsilon_0\chi_e\bar{E}$$

So, $\rho_{free} = \bar{\nabla}(\varepsilon_0\chi_e\bar{E} + \varepsilon_0\bar{E})$

$$= \varepsilon_0(1+\chi_e)\bar{\nabla}\bar{E}$$

$$\bar{\nabla}E = \frac{\rho_{free}}{\varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}} = \frac{\rho_{free}}{\varepsilon_0\varepsilon_r} = \frac{\rho_{free}}{\varepsilon}$$

$\varepsilon_r = 1 + \chi_e$
for free space $\rightarrow \chi_e = 0$
so $\varepsilon_r = 1$

$$\boxed{\bar{\nabla}\bar{E} = \frac{\rho_{free}}{\varepsilon}}$$

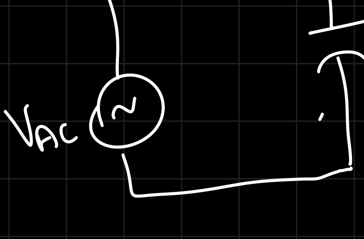
$\bar{D}$: displacement density

$\bar{I}_D$: displacement current

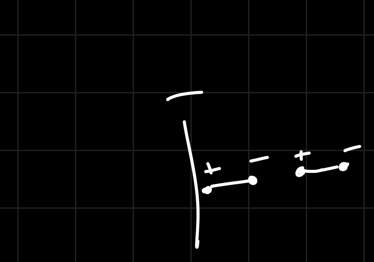
$$I_D = \varepsilon_0\frac{\partial E}{\partial t} = \frac{\partial D}{\partial t}$$

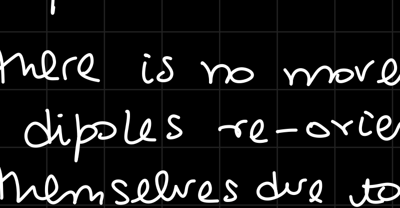
$$\vec{D} = \varepsilon_0\vec{E}$$

displacement current is the reason current flows between the plates of the capacitor.



$\rightarrow i_c = 0$
because $Z_c = \frac{1}{j\omega C} = 0$
$\because \omega \to \infty$ for DC





there is no movement of charge
dipoles re-orienting/rotating
themselves due to $\bar{E}$ and there
is a change of charge density and
so there is an illusion of
charge movement and that's
how current moves

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}, \quad \varepsilon_0 = 8.854\times10^{-12}$$

$$F = QE$$

surface
(continous)

$$E(r) = k\int \frac{1}{r^2}\hat{r}\,dq \qquad dq$$

$$E(r) = k\int \frac{\lambda}{r^2}\hat{r}\,dl' \qquad \downarrow \ \lambda\,dl'$$

$$E(r) = k\int \frac{\sigma}{r^2}\hat{r}\,da' \qquad \downarrow \ \sigma\,da'$$

$$E(r) = k\int \frac{\rho}{r^2}\hat{r}\,d\tau' \qquad \downarrow \ \rho\,d\tau'$$

$$E(r) = \frac{kq}{r^2}\hat{r}$$ single pt. charge

Flux through a surface →

$$\oint_S E\cdot da = \frac{1}{\varepsilon_0}Q_{enclosed}$$ → Gauss's Law

$$\boxed{\nabla \bar{E} = \frac{\rho}{\varepsilon_0}}$$

Remember, Stoke's Theorem

$$\oint_C \vec{A}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

$$\oint \bar{E}\cdot dl = 0 \quad \text{so, } \boxed{\bar{\nabla}\times\bar{E} = 0}$$

$$\boxed{\bar{E} = -\nabla V}$$

we know $\bar{\nabla}\bar{E} = \frac{\rho}{\varepsilon_0}$ and this

So, $$\boxed{\bar{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}}$$ → Poisson's Equation

regions with no charge → $\rho = 0$

so there, $$\boxed{\bar{\nabla}^2 V = 0}$$ → Laplace Equation

$$V(r) = k\int \frac{\rho}{r}\,d\tau, \quad \bar{E}(r) = k\int \frac{\rho}{r^2}\hat{r}\,d\tau$$

$$\boxed{W = qV(r)}$$ Point charge

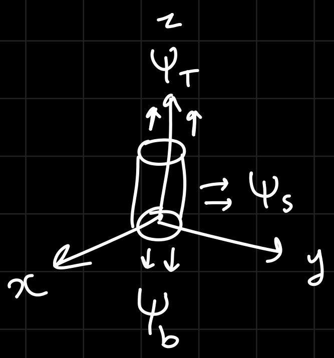
$$\boxed{W = \frac{1}{2}\int \rho V\,d\tau}$$

$$\boxed{W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\,d\tau}$$

Tut - 2

Q1) $\bar{G}(r) = 10e^{-2z}(\rho\hat{a}_\rho + \hat{a}_z)$

flux of $\bar{G}$ $= \Psi = \oint \bar{G}\cdot\bar{ds}$



$\Psi = \Psi_t + \Psi_b + \Psi_s$

for the top →

$\bar{ds} = \rho\, d\rho\, d\phi\, \hat{a}_z$

$$\Psi_t = \int \bar{G}\cdot\bar{ds} = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} 10e^{-2z}\rho\, d\rho\, d\phi$$

Cylindrical coords →

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

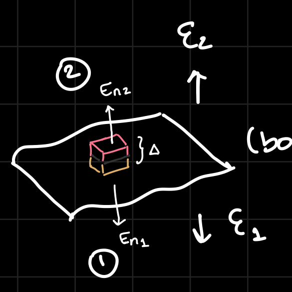
$$\bar{\nabla}\times\bar{A} = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Spherical coords →

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(V_\phi)$$

$$\bar{\nabla}\times A = \begin{vmatrix} \frac{\hat{a}_r}{r^2\sin\theta} & \frac{\hat{a}_\theta}{r\sin\theta} & \frac{\hat{a}_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

NOTE: Vec Field = Irrotational if its curl = 0 remember $\underbrace{\nabla(\bar{E}\cdot\varepsilon)}_{\bar{D}} = \rho_{free}$ } gauss's law for diff materials



(boundary between 2 mediums)

- in medium 1
- in medium 2

when $\nabla \to 0$, area and thereafter, flux $\to 0$

note: $D_{n1} = E_{n1}\varepsilon_1$
$D_{n2} = E_{n2}\varepsilon_2$

$\underbrace{\oint \bar{D}\,d\bar{s}}_{flux} = \rho_{enclosed}$ } Gauss's Law

$\Rightarrow$ $D_{n2}A - D_{n1}A = \sigma A$
↳ surface charge

$D_{n2} - D_{n1} = \sigma$

$$\boxed{\varepsilon_2 E_{n2} - \varepsilon_1 E_{n1} = \sigma}$$

if $\sigma = 0$, $E_{n2}\varepsilon_2 = E_{n1}\varepsilon_1$

or $\frac{E_{n2}}{E_{n1}} = \frac{\varepsilon_1}{\varepsilon_2}$

156 credits

if $\sigma \neq 0$, D_{n1} and D_{n2} are discontinuous
else they are continuous
but do note that potential is always continuous

NOITYN
GNUTNM

$$\oint E\cdot dl = 0$$

$$E_{t1}l - E_{t2}l = 0$$

$$\boxed{E_{t1} = E_{t2}}$$

$\underset{\text{infinite conductivity}}{\underbrace{PEC}}$ ≡ perfectly electric conductor

elec field inside PEC = 0

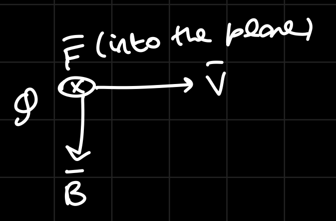
and any electric field tangential to the PEC's surface is zero

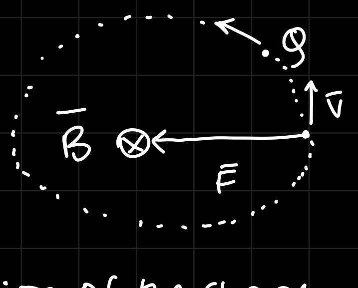
* Lorentz Force Law {magnetostatics}

$F_{mag} = \rho\cdot(\bar{v}\times\bar{B})$

$F_{total} = \rho\bar{E} + \rho(\bar{v}\times B)$

$F_{total} = \rho(\bar{E} + \bar{v}\times\bar{B})$

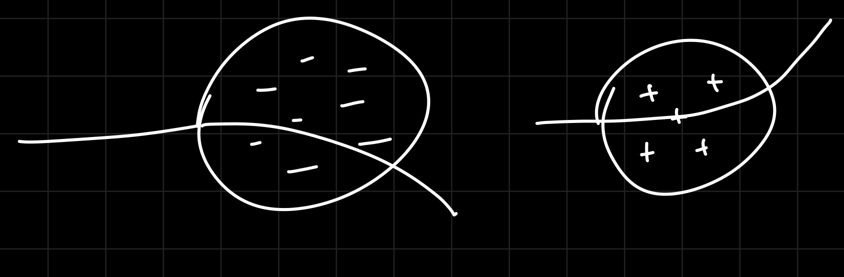




$\frac{mv^2}{r} = \rho vB$

$r = \frac{mv}{\rho B}$

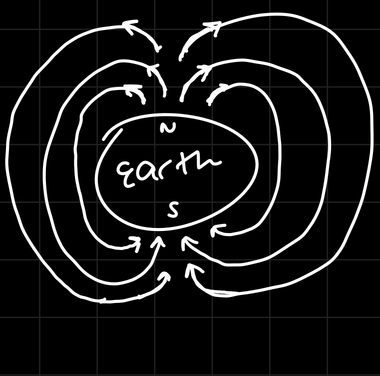
motion of the charge
≡ cyclotron motion
↳ used in mass spectrometer



magnetic field is not affected by the presence of a conductor unlike electric field

magnetic field is static in nature

magnetic bottle
↳ magnetic field is converging at the pole



collision with O_2 particles, pink aurora

green with N_2

Q1)

Ques 1. If $\vec{A}=\rho\cos\phi\,\hat{a}_\rho+\sin\phi\,\hat{a}_\phi$, Evaluate $\oint \vec{A}.dl$ around the path shown in Figure 1. Confirm this by using Stoke's theorem.

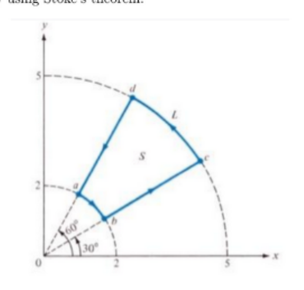


Figure 1:

$$\bar{A} = \rho\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_\phi$$

$$\oint \bar{A}\cdot\bar{dl} = \int\left[\int_a^b + \int_b^c + \int_c^d + \int_d^a\right]\bar{A}\cdot\bar{dl}$$

$\boxed{a \to b}$

$$\bar{dl} = \rho\, d\phi\, \hat{a}_\phi$$

$$\int_a^b \bar{A}\cdot\bar{dl} = \int_{\phi=30^\circ}^{\phi=60^\circ} \rho\sin\phi\, d\phi = 2\left[-\cos\theta\right]_{60}^{30^\circ}$$

$$= 2\left[\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\right]$$

$$= -(\sqrt{3}-1)$$

$\boxed{b \to c}$ $\phi = 30^\circ$

$$\bar{dl} = d\rho\, \hat{a}_\rho$$

$$\int_b^c A\cdot dl = \int_{\rho=2}^{\rho=5} \rho\cos\phi\, d\rho = \left[\frac{\rho^2}{2}\right]_2^5 \frac{\sqrt{3}}{2}$$

$$= \frac{21\sqrt{3}}{4}$$

$\boxed{c \to d}$ $\rho = 5$

$$\bar{dl} = \rho\, d\phi\, \hat{a}_\phi$$

$$\int_c^d A\cdot dl = \int_{30^\circ}^{60^\circ} \rho\sin\theta\, d\phi$$

$$= -5\left(\cos\theta\right)_{30^\circ}^{60^\circ} = \frac{-5(1-\sqrt{3})}{\sqrt{2}}$$

$$= \frac{5(\sqrt{3}-1)}{\sqrt{2}}$$

$\boxed{d \to a}$ $\phi = 60^\circ$

$$\bar{dl} = \bar{d\rho}\, \hat{a}_\rho$$

$$\int_d^a A\cdot dl = \int_5^2 \rho\cos(60^\circ) = \frac{1}{4}\left[4-25\right]$$

$$= \frac{-21}{4}$$

finally: $\oint A\cdot dl = -\sqrt{3}+1 + \frac{21\sqrt{3}}{4} + \frac{5}{\sqrt{2}}(\sqrt{3}-1) - \frac{21}{4}$

$$= \underline{4.941}$$

Stoke's Theorem

$$\oint \bar{A}\cdot\bar{dl} = \int_S (\bar{\nabla}\times\bar{A})\, d\bar{s}$$

$$\nabla\times A = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ (\rho\cos\phi) & \rho(\sin\phi) & 0 \end{vmatrix}$$

$$= 0\,\hat{a}_\rho + 0\,\hat{a}_\phi + \frac{1}{\rho}(1+\rho)\sin\phi\,\hat{a}_z$$

$$\vec{ds} = \rho\, d\phi\, d\rho\, \hat{a}_z$$

$$(\bar{\nabla}\times A)\cdot\bar{ds} = \int_{30^\circ}^{60^\circ}\int_2^5 (1+\rho)\sin\phi\, d\rho\, d\phi$$

$$= \int_{30}^{60}\left[\rho + \frac{\rho^2}{2}\right]_2^5 \sin\phi\, d\phi$$

$$= -\frac{27}{2}\left[\cos\phi\right]_{30}^{60}$$

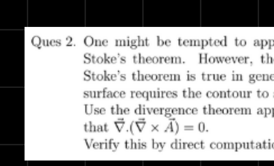
$$= \frac{27}{4}(\sqrt{3}-1) = \underline{4.941}$$

Stoke's theorem verified

Q2)

Ques 2. One might be tempted to apply the divergence theorem to the surface integral in Stoke's theorem. However, the divergence theorem requires a closed surface while Stoke's theorem is true in general for an open surface. Stoke's theorem for a closed surface requires the contour to shrink to zero giving a zero result for the line integral. Use the divergence theorem applied to the closed surface with vector $\vec{\nabla}\times\vec{A}$ to prove that $\vec{\nabla}.(\vec{\nabla}\times\vec{A}) = 0$.
Verify this by direct computation in cylindrical coordinates.

Stoke's theorem $\equiv$ only applicable for open surfaces



$$\int_{S'} (\bar{\nabla}\times\bar{A})\, d\bar{s} = \oint_C A\cdot dl$$

make this closed by putting C=0

divergence theorem $\equiv$ only applicable for closed surfaces

so, $\oint_C A\cdot dl = 0$ ①

so, $\int_{S'}(\bar{\nabla}\times\bar{A})\,d\bar{s} = 0$ for C=0

$$\int_V \bar{\nabla}\cdot A\, dv = \oint_S \bar{A}\cdot\bar{ds}$$

$$\int_V \bar{\nabla}\cdot(\bar{\nabla}\times\bar{A})\, dv = \oint_S (\nabla\times A)\, d\bar{s}$$

we know this coming to be zero from ①

Brute Force method:

$$\bar{A} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$$

$$(\bar{\nabla}\times\bar{A}) = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\frac{1}{\rho}\left[\left(\frac{\partial}{\partial\phi}A_z - \frac{\partial}{\partial z}\rho A_\phi\right)\hat{\rho} - \rho\left(\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\rho}{\partial z}\right)\hat{\phi} + \left(\frac{\partial}{\partial\rho}\rho A_\phi - \frac{\partial A_\rho}{\partial\phi}\right)\right]$$

$$(\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A})) = 0$$

Q3)

Ques 3. A sphere of radius R_1 and free space permitivity ϵ_0 has a volume charge distribution

$$\rho_f(r) = \rho_o\left(\frac{r}{R_1}\right)^4, \qquad 0 < r < R_1$$

The sphere is surrounded by free space and a perfectly conducting sphere of radius R_2 so that $\vec{E}$=0 for r>R_2. There is no surface charge on the r=R_1 surface.
(a) What is the total charge on the sphere?
(b) What is the electric field $\vec{E}$ for 0<r<R_2?
(c) What is the surface charge density on the perfectly conducting sphere of radius R_2?
(d) What is the total charge on the r=R_2 spherical surface and how is it related to your answer in (a)?

$$\rho_f(r) = \rho_0\left(\frac{r}{R_1}\right)^4 \quad : \quad 0<r<R_1$$

(a) $\rho = \frac{q}{4\pi r^3}$

$$q = \int_V \rho\, dv = \int_0^{R_1} \rho_0\left(\frac{r}{R_1}\right)^4 4\pi r^2 dr$$

$$= \frac{4\pi\rho_0}{R_1^4}\int_0^{R_1} r^6 dr$$

$$= \frac{4\pi\rho_0}{R_1^4}\left[\frac{r^7}{7}\right]_0^{R_1}$$

$$q = \frac{4\pi\rho_0 R_1^3}{7}$$

(b) $\bar{E} = ?$ for $0<r<R_2$

$$\oint E\cdot da = \frac{1}{\epsilon_0}q = \frac{1}{\epsilon_0}\int_V \rho\, dv$$

gauss's law for free space

$$4\pi r^2 E_r = \frac{1}{\epsilon_0}\int_0^r \rho_0\left(\frac{r}{R_1}\right)^4 4\pi r^2 dr +$$

for $R_1<r<R_2$ $\rightarrow$ $\frac{1}{\epsilon_0}\times\left(\frac{4\pi\rho_0 R_1^3}{7}\right)$

$$\cancel{4\pi}\epsilon_0 r^2 E_r = \frac{\cancel{4\pi}\rho_0}{7R_1^4}r^7 + \frac{4\cancel{\pi}\rho_0 R_1^3}{7}$$

$$E_r = \frac{\epsilon_0 r^2 \rho_0(r^7 + R_1^7)}{7R_1^4}$$

(c) Surface Charge density

$$\sigma_s = -\epsilon_0 E_r \quad (r=R_2)$$

$$= -\epsilon_0$$

Syllabus : lec-1 to Electrostatic boundary conditions

Reference : Griffith
↳ ch 1, 2, 4

* FORMULAS :

CH1

→ Condensed form of maxwell's equations

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\bar{\nabla}\times\bar{B} = \mu_0\left(\bar{J} + \frac{\partial\bar{D}}{\partial t}\right)$$

→ Electrostatic Potential :

1d → $\bar{E} = -\frac{d\phi_x}{dx}\hat{x}$

3d → $E = -\left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right)$

$= -\bar{\nabla}\phi$

→ Cylindrical coords

$$\bar{dl} = d\rho\,\hat{a}_\rho + \rho\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$

$$\bar{ds} = \rho\,d\phi\,dz\,a_\rho \;/\; dz\,d\rho\,\hat{a}_\phi \;/\; \rho\,d\rho\,d\phi\,a_{\hat{z}}$$

$$dv = \rho\,d\rho\,d\phi\,dz$$

→ Spherical coords

$$\bar{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

$\bar{ds} = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r$ /
$r\sin\theta\,d\phi\,dr\,\hat{a}_\theta$ /
$r\,dr\,d\theta\,\hat{a}_\phi$

differential solid angle : $d\Omega = \frac{ds_r}{r^2} = \sin\theta\,d\theta\,d\phi$

$dV = r^2\sin\,dr\,d\phi\,d\theta$

→ LINE INTEGRAL

$W = \int_A^B \bar{F}\cdot\bar{dl}$ $\qquad \phi_{AB} = -\int_A^B \bar{E}\cdot\bar{dl}$

↳ path dependent

→ CONSERVATIVE FIELD
↳ A field that can be expressed as a gradient of scalar.
↳ path independent
↳ closed loop integral = 0
↳ eg: gravitational and electric field

→ Laplacian operator

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

for a field ϕ if $\nabla^2\phi = 0$,
ϕ cannot have a local minima/maxima

* Tutorial 5

Q1) $\bar{A} = \rho\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_\phi$

$\oint A \cdot dl$

for cylindrical → $dl = d\rho\,\hat{a}_\rho + \rho\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$

$$\oint A\cdot dl = \int_{60^\circ}^{30^\circ} 2\sin\phi\,d\phi\,\hat{a}_\phi + \int_2^5 \rho\cos(30^\circ)\,d\rho\,\hat{a}_\rho + \int_{30^\circ}^{60^\circ} 5\sin\phi\,d\phi\,\hat{a}_\phi + \int_5^2 \rho\cos(60^\circ)\,d\rho\,\hat{a}_\rho$$

$$= 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot(21) - 5\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\cdot\frac{1}{2}(-21)$$

$$= 1-\sqrt{3} + \frac{21\sqrt{3}}{4} - \frac{5}{2} + \frac{5\sqrt{3}}{2} - \frac{21}{4}$$

$$= \frac{4-4\sqrt{3}+21\sqrt{3}-10+10\sqrt{3}-21}{4}$$

$$= \frac{27\sqrt{3}-27}{4} = \frac{27}{4}(\sqrt{3}-1) \checkmark$$

Using STOKE'S THEOREM :

$$\oint \bar{A}\cdot\bar{dl} = \int_s (\bar{\nabla}\times\bar{A})\,ds$$

$\bar{ds} = \rho\,d\rho\,d\phi$

$$\oint A\cdot dl = \iint (\bar{\nabla}\times A)\,\rho\,d\rho\,d\phi$$

$$\bar{\nabla}\times\bar{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$\frac{\partial}{\partial\rho}(\rho\sin\phi) - \frac{\partial}{\partial\phi}(\rho\cos\phi)$

$\sin\phi + \rho\sin\phi$

$$= \frac{1}{\rho}\left(\hat{a}_\rho\left(0 - \frac{\partial}{\partial z}(\rho\sin\phi)\right)\right) - \frac{1}{\rho}\left(\rho\,\hat{a}_\phi\left(0 - \frac{\partial}{\partial z}(\rho\cos\phi)\right)\right) + \frac{1}{\rho}\left(\hat{a}_z\left(\frac{\partial}{\partial\rho}(\rho\sin\phi) - \frac{\partial}{\partial\phi}(\rho\cos\phi)\right)\right)$$

$$= \frac{1}{\rho}\hat{a}_z\,(\sin\phi + \rho\sin\phi)$$

$$= \left(\frac{1}{\rho}\sin\phi + \sin\phi\right)\hat{a}_z$$

$$\oint \bar{A}\cdot\bar{dl} = \int_{30}^{60}\int_2^5 \sin\phi\,(\rho+1)\,d\rho\,d\phi$$

$$= \int_{30}^{60}\left[\sin\phi\left(\frac{\rho^2}{2}+\rho\right)\right]_2^5 d\phi$$

$$= \int_{30}^{60} \sin\phi\left(\frac{27}{2}\right)d\phi$$

$$= -\frac{27}{2}\left[\cos\phi\right]_{30}^{60}$$

$$= -\frac{27}{2}\left[\frac{1}{2}-\frac{\sqrt{3}}{2}\right]$$

$$= \frac{27}{4}(\sqrt{3}-1) \checkmark$$

Chapter: 2

Electric Field :

① distinct points → $\bar{E} = k \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}$

② continuous charge → $\bar{E} = k \int \frac{1}{r^2} \hat{r}\, dq$

along a line : $dq = \lambda\, dl$
on a surface : $dq = \sigma\, ds$
in a volume : $dq = \rho\, dv$

Divergence and Curl of Electrostatic fields

→ flux of $\bar{E}$ through surface S,

$$\Phi_E = \int_S E \cdot \bar{ds}$$

Gauss's Law : $\oint_S E \cdot ds = \frac{q_{enclosed}}{\varepsilon_0}$

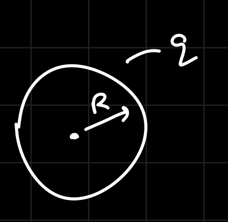
Divergence Theorem : $\oint_S E \cdot \bar{ds} = \int_V (\bar{\nabla} \cdot \bar{E})\, dv$

note : $q = \int_V \rho\, dv$

so, $\int \frac{\rho\, dv}{\varepsilon_0} = \int \bar{\nabla} \bar{E}\, dv$

$$\bar{\nabla} E = \frac{\rho}{\varepsilon_0}$$

Ex 2.2 →



field = ?

$$\oint_S E \cdot ds = \frac{1}{\varepsilon_0} q_{enclosed} = \frac{q}{\varepsilon_0}$$

$$|E| \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$|E| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

$$E = k \frac{q}{r^2} \hat{r}$$

CURL of $\bar{E}$

$$\bar{E} \cdot \bar{dl} = 0$$

→ Stoke's theorem : $\oint \bar{E} \cdot \bar{dl} = \int_S (\nabla \times E)\, ds$

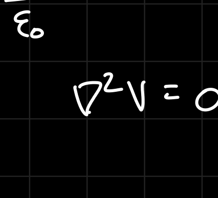
so, $\bar{\nabla} \times \bar{E} = 0$

Electric Potential

$$\bar{E} = -\bar{\nabla} \bar{V}$$

↳ electric potential

ex 2.6 →



outside ?

$\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$ $\oint E \cdot ds = \frac{q_{enclosed}}{\varepsilon_0}$

Poisson's Equation

we know → $E = -\nabla V$, $\nabla E = \frac{\rho}{\varepsilon_0}$, $\nabla \times E = 0$

so → $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

LAPLACE'S EQN → $\nabla^2 V = 0$ in regions with no charge $\therefore \rho = 0$

$V = \frac{kq}{r}$ → single charge

$V = k \sum_{i=1}^{n} \frac{q_i}{r_i}$ → multiple charge

$V = k \int \frac{1}{r} dq$ → continuous distribution

$V = k \int \frac{1}{r} \rho\, dv$ → volume charge

WORK DONE : $W = \int_a^b F \cdot dl$

$W = 0$ for single charge

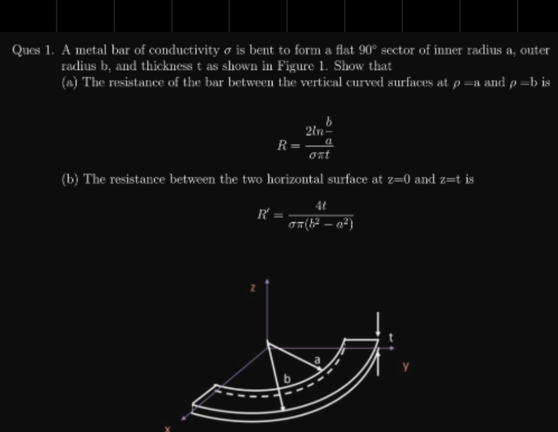
$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{12}} \right)$ for 2 charges

$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)$ for multiple charges

continuous → $W = \frac{1}{2} \int \rho V\, dv$

⇒ Tutorial 6 :

Q1)



Ques 1. A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b, and thickness t as shown in Figure 1. Show that
(a) The resistance of the bar between the vertical curved surfaces at ρ =a and ρ =b is

$$R = \frac{2\ln\frac{b}{a}}{\sigma\pi t}$$

(b) The resistance between the two horizontal surface at z=0 and z=t is

$$R' = \frac{4t}{\sigma\pi(b^2-a^2)}$$

→ cylindrical cords

(a) $R = \frac{V}{I}$ → laplacian

$$\nabla^2 V = 0$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Integrate

$$\rho \frac{\partial V}{\partial \rho} = 0 + A$$

$$\frac{\partial V}{\partial \rho} = \frac{A}{\rho}$$

$$\partial V = A \cdot \frac{1}{\rho}\, \partial \rho$$

Integrate

$$V = A \ln \rho + B$$

how to find A & B ?
boundary condition

let $V = 0$ at $\rho = a$
$V = V_0$ at $\rho = b$

$$0 = A \ln(a) + B$$
$$B = -A\ln(a)$$

at $\rho = b$,

$$V_0 = A\ln(b) + B = A\ln\left(\frac{b}{a}\right)$$

$$A = \frac{V_0}{\ln(b/a)}$$

$$V = A \ln\rho + B$$

$$A = V_0 \ln(a/b) \ , \quad B = -\frac{V_0}{\ln(b/a)} \ln(a)$$

$$V = V_0 \ln(a/b)\left(\ln\rho - \ln a\right)$$

$I = ?$

$$\bar{J} = \sigma \bar{E}$$

$$\bar{E} = -\bar{\nabla} V = -\frac{dv}{d\rho}\hat{a_\rho}$$

$$= \frac{-A}{\rho}\hat{a_\rho}$$

$$= \frac{-V_0}{\rho}\ln(a/b)\,\hat{a_\rho}$$

$$\bar{J} = -\frac{\sigma V_0}{\rho}\ln(a/b)\,\hat{a_\rho}$$

$$I = \int \bar{J}\cdot \bar{ds}$$

$$\bar{ds} = -\rho\, d\phi\, dz\, \hat{a_\rho}$$

$$I = \int_0^t\int_{0°}^{90°} \frac{V_0\sigma}{\rho}\ln(a/b)\,\rho\, d\phi\, dz$$

$$= \int_{0°}^{90°} (tV_0\sigma)\ln(a/b)\, d\phi$$

$$= \frac{\pi}{2}\cdot t\cdot V_0 \cdot \sigma \cdot \ln(a/b)$$

$$R = \frac{V}{I} = \frac{\cancel{V_0}\, \cancel{\ln}(a/b)\left(\ln\frac{\rho}{a}\right)}{\frac{\pi}{2}\cdot t\cdot \cancel{V_0}\cdot\sigma\cdot \cancel{\ln}(a/b)}$$

$$= \frac{2\ln\left(\frac{\rho}{a}\right)}{\pi t\sigma}$$

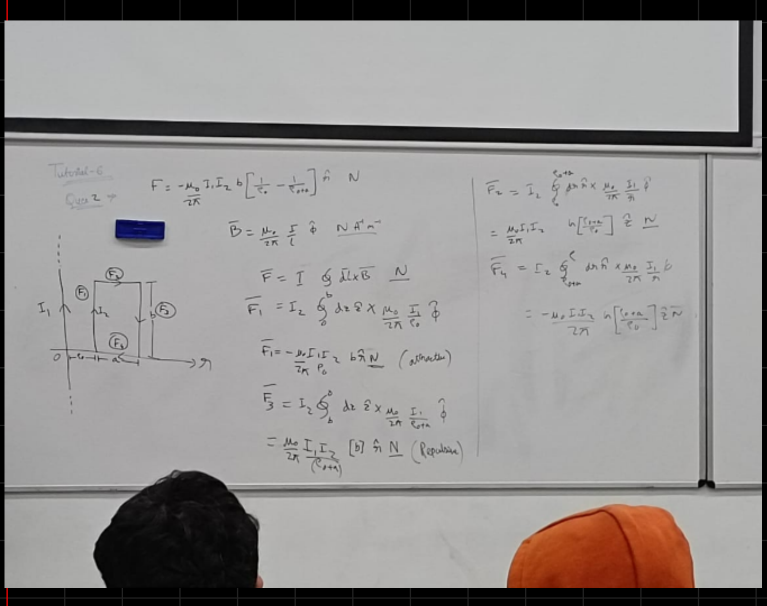
(b) Similar but just boundary conditions different

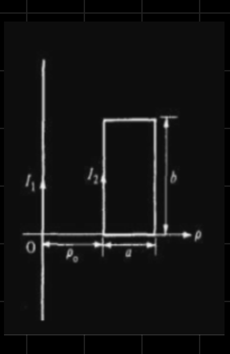
at $z = 0$, $V = 0$
$z = t$, $V = V_0$

Q2)

Ques 2. A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Figure 2. Show that the force experienced by the loop is given by

$$\vec{F} = -\frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right]\hat{a}_\rho \text{ N}$$





use biot savart

Q4)

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$r^2\frac{\partial V}{\partial r} = 0 + A$$

$$\frac{\partial V}{\partial r} = \frac{A}{r^2}$$

$$V = \frac{-A}{r} + B$$

find A, B

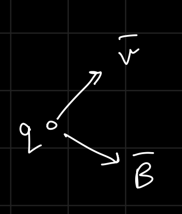
$$E = -\nabla V$$

$$Q = \int \varepsilon_0 E \cdot ds$$

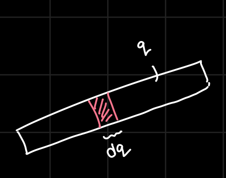
$$C = \frac{Q}{V}$$

Lorentz's Force

$$F = \underbrace{q\cdot(\bar{V}\times\bar{B})}_{\text{magnetic component}} + \underbrace{q\bar{E}}_{\text{electric field component (ignoring for now)}}$$



magnetic force on continuous charge



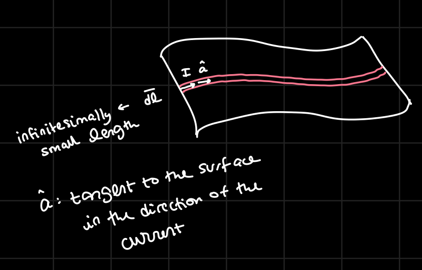
$d\bar{F}_{mag} = dq(\bar{V}\times\bar{B})$

$\downarrow$

$\bar{F}_{mag} = \int d\bar{F}_{mag}$

$$\bar{F}_{mag} = \int dq\,(\bar{V}\times\bar{B})$$
$$= \int \frac{dq}{dt}(d\bar{l}\times\bar{B})$$
$$= \boxed{\int I\cdot(d\bar{l}\times B)}$$

line charge → C/m
surface " → C/m²
volume " → C/m³



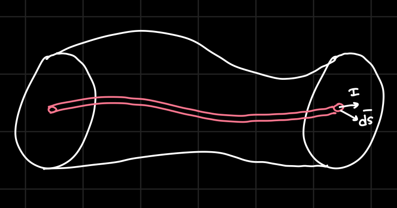
$\hat{k}$ = surface current density

$$\boxed{\hat{k} = \frac{dI}{dl_\perp}\hat{a}}$$

unit: Ampere per metre
Am^{-1}

NOTE: There is no concept of line current density

VOLUME CURRENT DENSITY : $\bar{J}$



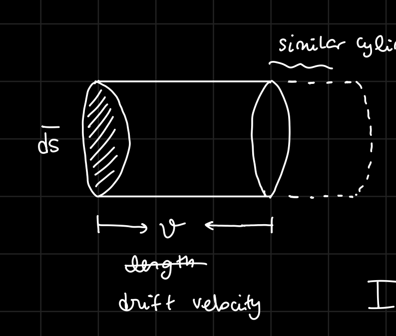
note: $\hat{n}$ ≡ direction of current
$\hat{n}$ and $\bar{ds}$ are ⊥

$$\boxed{\bar{J} = \frac{dI}{ds_\perp}\hat{n}}$$

unit: A/m^2

note: $\bar{J} = \sigma\bar{E}$ is the same $\bar{J}$

> Now we can relate current density with charge density



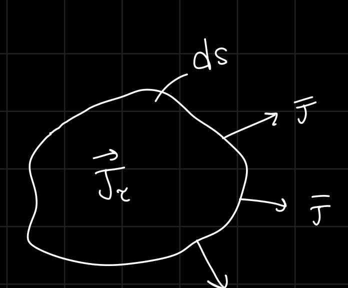
if ρ ≡ vol. charge density
now find I from ρ

$$I = \rho\,\underbrace{\bar{ds}\cdot v}_{\text{volume}}$$

$$\frac{I}{ds} = \rho v \rightarrow \boxed{\bar{J} = \rho\bar{v}}$$

$$\bar{F}_{mag} = \int dq\,(V\times B)$$
$$= \int \rho\, d\tau\,(V\times B) = \int(\rho\bar{v}\times\bar{B})\,d\tau$$

$$\boxed{\bar{F}_{mag} = (\bar{J}\times\bar{B})\,d\tau}$$



$$\boxed{\oint \bar{J}\cdot\bar{ds} = -\frac{dQ}{dt}}$$

↳ law of conservation of charge

using divergence theorem

$$\oint \bar{J}\cdot\bar{ds} = \int(\nabla\cdot J)\,d\tau = -\frac{\partial}{\partial t}\int\rho\,d\tau$$
$$= -\int\frac{\partial\rho}{\partial t}\,d\tau$$

$$\boxed{\bar{\nabla}\bar{J} + \frac{\partial\rho}{\partial t} = 0} \rightarrow \text{eq}^n \text{ of continuity}$$

also,

$$\bar{J} = \rho\bar{v}$$
$$\bar{\nabla}(\rho\bar{v}) + \frac{\partial\rho}{\partial t} = 0$$

for small Electric field, we can assume v varies with $\bar{E}$ linearly → $\bar{v} = \mu\bar{E}$

$$\bar{J} = \rho\bar{v} = (\rho\mu)\bar{E} = \sigma\bar{E}$$

↳ conductivity
also, reciprocal of resistivity

$\boxed{\vec{J} = \sigma\vec{E}}$

OHM'S LAW

from eqⁿ of continuity,

$$\sigma(\nabla\bar{E}) = -\frac{\partial\rho}{\partial t} \rightarrow \sigma\frac{\rho}{\varepsilon_0} = -\frac{\partial\rho}{\partial t}$$

$$\int\frac{\partial\rho}{\rho} = \frac{-\sigma}{\varepsilon_0}\int\partial t$$

$$\ln\rho + c = \frac{-\sigma}{\varepsilon_0}t$$

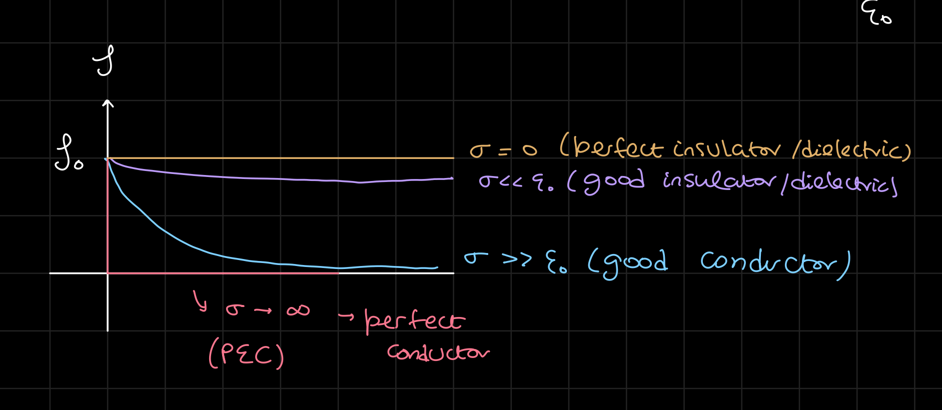
at $t = 0$, $\rho = \rho_0$

$$\ln\rho_0 = -c \rightarrow c = -\ln\rho_0$$

so,
$$\ln\rho - \ln\rho_0 = \frac{-\sigma}{\varepsilon_0}t$$
$$\ln\left(\frac{\rho}{\rho_0}\right) = -\frac{\sigma}{\varepsilon_0}t$$

$$\boxed{\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon_0}t}} \rightarrow \text{decays with time}$$

decay factor $= \frac{\sigma}{\varepsilon_0}$



$$\rho = \rho_0 e^{(-t/\tau_{relaxation})}$$

where $\tau_{relaxation} = \frac{\varepsilon_0}{\sigma}$

time variation involved

at $t = \tau$, $\rho = \rho_0 e^{-1} = \frac{\rho_0}{e}$

Significance of τ: time taken to reduce the ρ to $\frac{1}{e}$ times of ρ_0

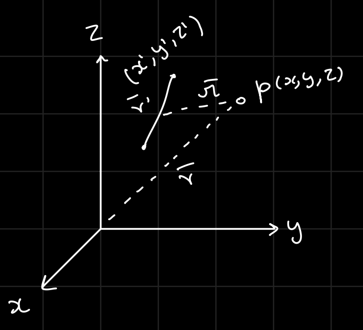
Magnetostatics

$$\bar{\nabla} \bar{J} = -\frac{\partial \rho}{\partial t}$$

let ρ = const with time
≡ steady current
so, $\frac{\partial \rho}{\partial t} = 0$ → $\boxed{\bar{\nabla} \bar{J} = 0}$

* A point charge cannot give steady current because ρ ≡ not constant because ρ is a function of space

→ BIOT SAVART'S LAW



$$\bar{B}(r) = \frac{\mu_0}{4\pi} \int \frac{I \cdot \bar{dl} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0}{4\pi} \int \frac{\bar{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} dv'$$

(x',y',z')

volume integral

$$(\bar{\nabla} \cdot \bar{B}) = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\bar{J}(r') \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \right)$$

divergence wrt (x,y,z)

$$= \frac{\mu_0}{4\pi} \int \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} (\bar{\nabla} \times \bar{J}(r')) - \bar{J}(r') \left(\bar{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \right.$$

note: $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla \cdot \left(\frac{1}{\mathscr{r}} \right)$

So, its curl is 0

also, J is dependent on $\mathscr{r}'$/ x', y', z'
but the curl is wrt $\mathscr{r}$/ x,y,z
so, that is zero as well

so, $\boxed{\bar{\nabla} \cdot \bar{B} = 0}$

$$\int (\nabla \cdot B) dv = \boxed{\oint \bar{B} \cdot \bar{ds} = 0}$$

Gauss's Law for magnetic field

Remember: This is only for steady current / pure magnetostatics

but it is true for magnetic field varying with time as well.

↑ MIDSEM SYLLABUS ↑