

Dirac Delta Function

$$\text{let } \vec{v} = \frac{1}{r^2} \hat{r}$$

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot \frac{1}{r^2}) = 0$$

reason → We used the divergence formula for spherical coordinates
 ↪ not cartesian.

$$\begin{aligned} \text{sphere of radius } R &\leftarrow S \\ \oint v \cdot d\alpha &= \int \frac{1}{R^2} \hat{r} \cdot R^2 \sin\theta d\theta d\phi \hat{r} \\ &= \left(\int_0^\pi \sin\theta d\theta \right) \left(\int_0^{2\pi} d\phi \right) \\ &= [-\cos\theta]_0^\pi \cdot [\phi]_0^{2\pi} \\ &= (1+1)(2\pi) = \underline{4\pi} \end{aligned}$$

$$\text{now, } \nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(r)$$

Quiz 2 - questions →

$$\text{(a)} \quad \int_{-1}^1 (r^2 + 2) \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) dv$$

$$= \int_{-1}^1 (r^2 + 2) 4\pi \delta^3(r) dv$$

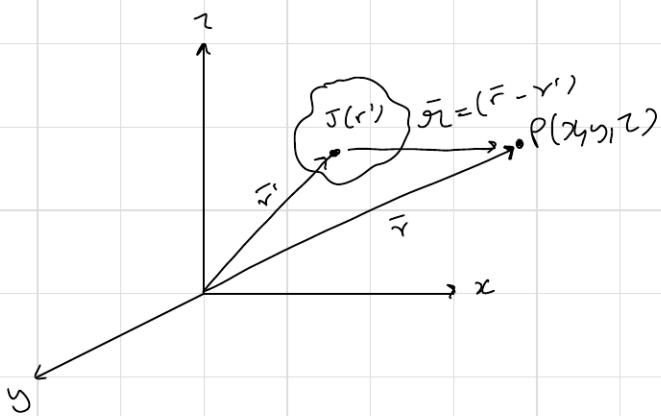
$$= (0+2)(4\pi) = \underline{8\pi}$$

$$\text{(b)} \quad \int_1^\infty (r^2 + 2) \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) dv$$

$$= \int_1^\infty (r^2 + 2) 4\pi \delta^3(r) dv = \underline{0}$$

always = 0
 ∵ $r=0$ not included
 in the integral

Lecture after midsem



$$J(\vec{r}') \rightarrow A/m^2 = \frac{dI}{da_L}$$

$$\bar{B}(r) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(r') \times \hat{r}}{r'^2}$$

→ dependent on x', y', z'

$$\bar{\nabla}_f = -\bar{\nabla}' f$$

note: assuming steady current

$$\bar{\nabla} \cdot \bar{B} = \frac{\mu_0}{4\pi} \int \nabla \left(\bar{J} \times \frac{\hat{r}}{r^2} \right) = \frac{\mu_0}{4\pi} \int \underbrace{\frac{\hat{r}}{r^2} \left(\bar{\nabla} \times \bar{J}(r) - \bar{J}(\bar{\nabla} \times \frac{\hat{r}}{r}) \right)}_{\text{zero because dir wrt } r \text{ but } J(r)}$$

$$\bar{\nabla} \cdot (\bar{P} \times \bar{Q}) = \bar{Q}(\bar{\nabla} \times \bar{P}) - \bar{P}(\bar{\nabla} \times \bar{Q})$$

and since $\frac{\hat{r}}{r^2} = \nabla \left(\frac{1}{r} \right) \rightarrow \text{curl} = 0$

So, $\boxed{\nabla \cdot \bar{B} = 0}$

Curl of magnetic field

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

↳ magnetic vector potential

(need not be unique)

$$\bar{A}_{\text{new}} = \bar{A} + \bar{\nabla} \phi \quad \text{gauge freedom}$$

$$\bar{\nabla} \times \bar{A}_{\text{new}} = \bar{\nabla} \times \bar{A} + \bar{\nabla} \times \bar{\nabla} \phi = \bar{B} \text{ always}$$

if \bar{B} given, find the magnetic vector potential

or given, \bar{J} , find the magnetic field

since $B \propto \frac{1}{r^2} \rightarrow A \propto \frac{1}{r}$ because $\bar{B} = \bar{\nabla} \times \bar{A}$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \left(\bar{J}(r') \times \frac{\hat{r}}{r'^2} \right) dV'$$

$$\text{we know } \frac{\hat{r}}{r^2} = -\nabla \left(\frac{1}{r} \right)$$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \left(\bar{J}(r') \times \underbrace{-\nabla \left(\frac{1}{r} \right)}_{(\bar{P} \times \bar{\nabla} f)} \right) dV'$$

$$(\bar{P} \times \bar{\nabla} f)$$

$$\bar{\nabla} \times (f\bar{P}) = f(\bar{\nabla} \times \bar{P}) - \bar{P} \times (\bar{\nabla} f)$$

$$\rightarrow P_x(\bar{\nabla} f) = f(\bar{\nabla} \times \bar{P}) - \bar{\nabla} \times (f\bar{P})$$

here $\bar{\nabla}$ wrt r but $J(r')$

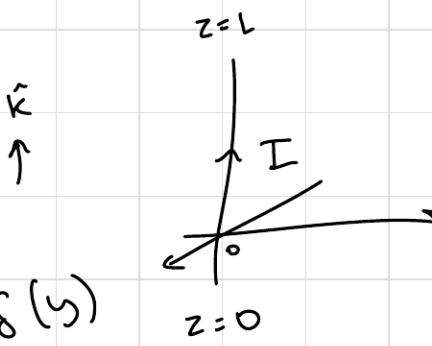
$$\text{so, } f(\bar{\nabla} \times \bar{P}) = 0$$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \left(\bar{J}(r') \times \frac{\bar{\nabla}(\perp)}{r} \right) dv'$$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \bar{\nabla} \times \left(\frac{\bar{J}(r')}{r} \right) dv'$$

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(r')}{r} dv'}$$

Current carrying filamentary conductor



$$\bar{J}(r') = ?$$

$$J = \bar{I} \delta(x) \delta(y)$$

$$\bar{J} = \frac{dI}{da_z}$$

$$\bar{A} = \frac{\mu_0}{4\pi} \iiint_{V'} \frac{\bar{J}(r')}{r} dx' dy' dz'$$

$$\bar{J}(r') \times dx' dy' = I$$

for $x, y \rightarrow$ infinitesimally small

$$\bar{A} = \frac{\mu_0}{4\pi} \iint_{-\infty}^{\infty} \delta(x) \delta(y) dx dy \cdot I \hat{z} \int_{-\infty}^L \frac{dz}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{1/2}}$$

= function of x, y, z

$\text{curl}(E) = 0$ but

$\text{div}(E) \Rightarrow$ Gauss's Law



$\text{div}(B) = 0$ but

curl of $B \Rightarrow$ Ampere's Law

$$\bar{B} = \frac{\mu_0}{4\pi} \int \bar{J}(r') \times \frac{\hat{r}}{r^2} dV'$$

$$\bar{\nabla} \times \bar{B} = ?$$

problem \Rightarrow cannot apply chain rule to $\nabla \times (\bar{P} \times \bar{G})$

$$\bar{\nabla} \times (\bar{P} \times \bar{J}) = (\bar{J} \cdot \bar{\nabla}) \bar{P} - (\bar{P} \cdot \bar{\nabla}) \bar{J} + \bar{P}(\bar{\nabla} \cdot \bar{J}) - \bar{J}(\bar{\nabla} \cdot \bar{P})$$

$$\left(J_x \frac{\partial P}{\partial y} + J_y \frac{\partial P}{\partial z} + J_z \frac{\partial P}{\partial x} \right)$$

$\bar{P} = \bar{J}$ but since \bar{J} depends on r' , not r
 so, 1st and last term = 0

non zero term:

$$\bar{\nabla} \times (\bar{P} \times \bar{J}) = \bar{P}(\bar{\nabla} \cdot \bar{J}) - (\bar{P} \cdot \bar{\nabla}) \bar{J}$$

$$\bar{\nabla} \times \bar{B} = \frac{\mu_0}{4\pi} \left[\int \bar{J} \left(\bar{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) \right) dv' + \int (\bar{J} \cdot \bar{\nabla}') \frac{\hat{r}}{r^2} dv' \right]$$

↓
not minus
because $\bar{\nabla}' f = -\bar{\nabla} f$

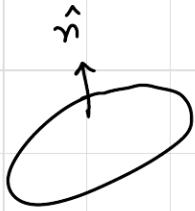
$$= \frac{\mu_0}{4\pi} \left[\int \bar{J}(\bar{r}') \cdot 4\pi \delta^3(\bar{r} - \bar{r}') dv + \int (\bar{J} \cdot \bar{\nabla}) \frac{\hat{r}}{r^2} dv \right]$$

$$= \frac{\mu_0}{4\pi} \times 4\pi \times J(\bar{r}) + \frac{\mu_0}{4\pi} \int (\bar{J} \cdot \bar{\nabla}) \frac{\hat{r}}{r^2} dv$$

$$= \mu_0 J(\bar{r}) + \frac{\mu_0}{4\pi} \int (\bar{J} \cdot \bar{\nabla}) \frac{\hat{r}}{r^2} dv$$

Assuming that 2nd term is zero

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}(\bar{r}) \rightarrow \text{differential form of Ampere's Law}$$



$$\int (\bar{\nabla} \times \bar{B}) \cdot d\bar{s} = \mu_0 \int \bar{J} \cdot d\bar{s}$$

$$\oint \bar{B} \cdot d\bar{e} = \mu_0 I_{\text{enclosed}}$$

Integral form
of Ampere's Law

Proving the previous assumption

$$\text{Steady current} \rightarrow \bar{\nabla} \bar{J}(r') = 0$$

$$\text{So, } \bar{\nabla}' \bar{J}(r') = 0 \text{ also}$$

$$\text{We need to prove} = \int (\bar{J}(r') \cdot \bar{\nabla}) \frac{\hat{n}}{r^2} dv$$

$$\frac{\hat{j}_i}{r^2} = \frac{\hat{x}\hat{x}}{r^3} = \frac{\hat{x}(x-x') + \hat{y}(y-y') + \hat{z}(z-z')}{r^3}$$

We need contribution of entire system but we will be checking for individual components

$$= \int_{V'} \left(\bar{J}(r') \cdot \bar{\nabla} \right) \frac{(x - x')}{r'^3} dv'$$

$$\bar{A} \cdot (\bar{\nabla} f) = \bar{\nabla}' \cdot (f \bar{A}) - \bar{f} (\bar{\nabla}' \cdot \bar{A})$$

$\underbrace{}_0$ because $\bar{\nabla}' \cdot \bar{J} = 0$

$$= \int_{V'} \bar{\nabla}' \cdot \left(\frac{J(r') (x - x')}{r'^3} \right) dv'$$

$$= \oint_{S'} \frac{\bar{J}(r') (x - x')}{r'^3} ds$$

Since $J(r) = 0$

outside V'

this \int holds for

$V' + D$ as well

$$\oint_{S+D} \bar{J}(r') \left| \text{surface} \right. \frac{(x - x') ds'}{r'^3}$$

≈ 0

→ Assumption correct

Magnetic field in a matter

note: $\nabla \cdot \vec{B} = 0 \rightarrow \int (\nabla \cdot \vec{B}) dV = \oint \vec{B} ds = 0$

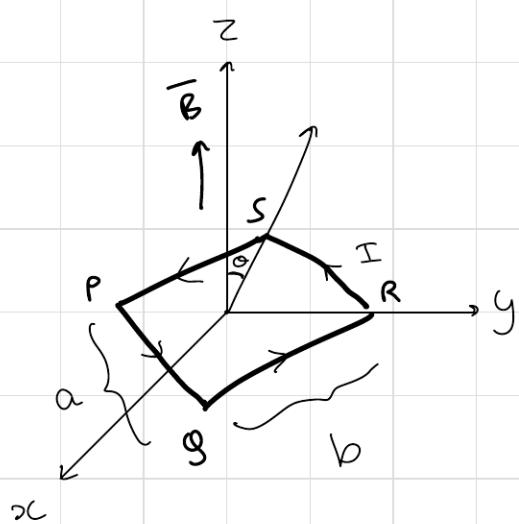
Gauss Law \uparrow

Diamagnets

Paramagnets

Ferromagnets

assuming a rectangular closed loop



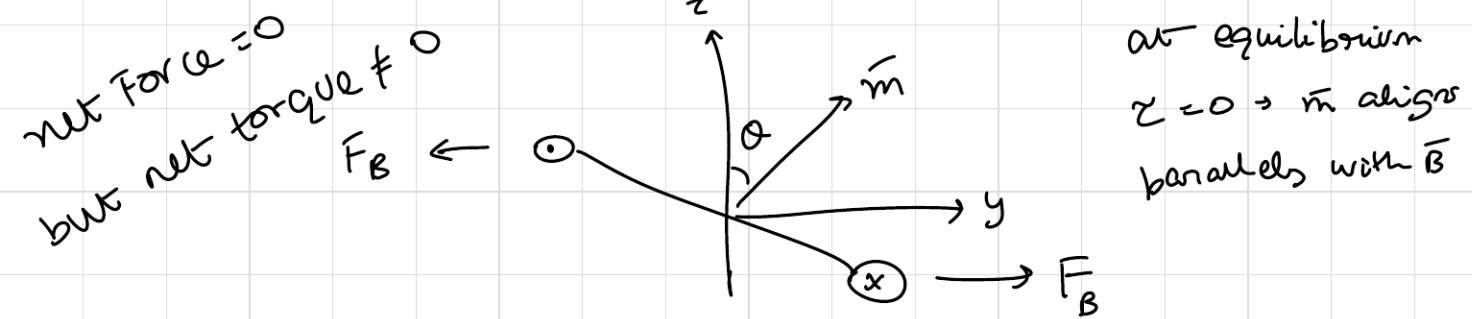
Q : angle of plane's normal
with z-axis

$$\bar{F}_B = I(\bar{l} \times \bar{B})$$

from Lorentz Law \uparrow

$$\bar{F}_B = Q(\bar{v} \times \bar{B})$$

The PQ and RS sides' force will cancel out
but the force due to the other two
sides will make the rectangle rotate



assuming the loop is kept in a uniform field

In case of dipoles →

$\left\{ \begin{array}{l} \text{constant thermal fluctuation} \\ \text{throws dipole out of equilibrium} \end{array} \right\}$

Similar reason for "out of equilibrium" state in this case as well

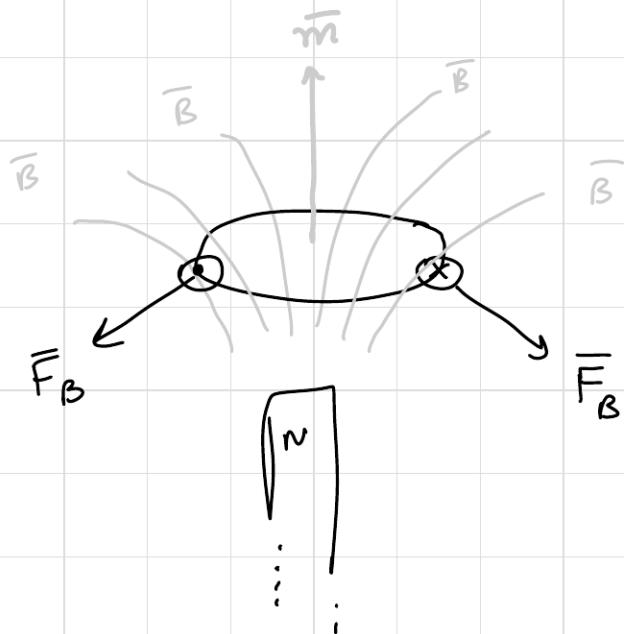
requirements:

liquid (not gas)
low temperature

or else these small loops won't respond to the magnetic field

non-uniform magnetic field = \vec{B} varying in space

e.g.:



Diamagnetic material

↳ tries to align \vec{m} in the opposite direction to \vec{B}

Paramagnetism

↳ \vec{m} and \vec{B} align in the same direction

\$ Cannot use gas because of not enough density
for it to exhibit enough magnetization

hence we go for the next lightest type
of materials \rightarrow liquid

magnetic susceptibility

\hookrightarrow determines para/ferro/diamagnetism

liquid nitrogen \rightarrow diamagnetic
doesn't hold

liquid oxygen \rightarrow paramagnetic
sticks b/w poles because of very
high magnetic susceptibility

aluminium \rightarrow paramagnetic
but doesn't hold because gravitational
pull is greater because of high mass (solid)

\bar{E} arises due to Q

\bar{B} arises due to I

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

\hookrightarrow magnetic vector potential

\bar{A} and \bar{V} vary with $\frac{1}{r}$ but for dipole $\rightarrow \frac{1}{r^2}$

| | |
|---------------------------|----------------------------|
| magnetic vec potential | electrostatic potential |
| \downarrow | \downarrow |

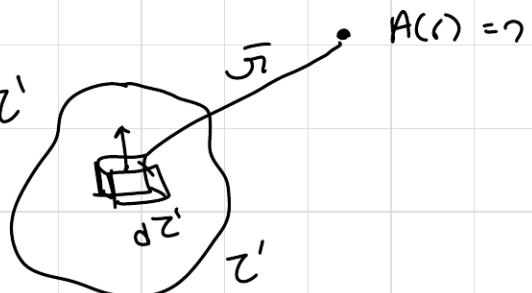
$$\bar{A}(r) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times \hat{r}}{r^2}$$

magnetic dipole moment

just like \bar{P} : polarization,
we have \bar{M} : magnetization
(\bar{m} /volume)

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int \frac{\bar{m} \times \hat{r}}{r^2} dz' \quad \text{due to continuous material}$$

$$= \frac{\mu_0}{4\pi} \int \left[\bar{m} \times \bar{\nabla}' \left(\frac{1}{r} \right) \right] dz'$$



$$(\bar{\nabla}' \times f\bar{p}) = f(\bar{\nabla}' \times \bar{p}) - \bar{p} \times (\bar{\nabla}' f)$$

$$\Rightarrow \bar{p} \times \bar{\nabla}' f = f(\bar{\nabla}' \times \bar{p}) - \bar{\nabla}' (f\bar{p})$$

$$\bar{p} = \bar{m}, f = \frac{1}{r}$$

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int \left[\frac{1}{r} (\bar{\nabla}' \times \bar{m}) - \bar{\nabla}' \times \left(\frac{\bar{m}}{r} \right) \right] dz$$

remember \rightarrow

$$\bar{J}_b(r') = \bar{\nabla}' \times \bar{m}(r')$$



bound volume current density

Gauss's divergence theorem: $\int \bar{\nabla}' \bar{P} dv = \oint \bar{P} \cdot ds$

$$\text{if } \bar{P} = \bar{\nabla}' \times \bar{C}$$

where \bar{V} varies with space and

\bar{C} doesn't vary with space

then \bar{C} \bar{P} varies with space

$$\int_v \bar{\nabla}' \cdot (\bar{V} \times \bar{C}) dv = \oint_s (\bar{V} \times \bar{C}) \cdot ds$$

$$\int \bar{C} \cdot (\bar{\nabla}' \times \bar{V}) - \bar{V} \cdot (\bar{\nabla}' \times \bar{C}) dv = \oint \bar{C} \cdot (\bar{d}\bar{s} \times \bar{V})$$

$\cancel{\text{zero because } C}$
 $\cancel{\text{doesn't vary with space}}$

$$\int C \cdot (\nabla' \times V) dv = - \oint \bar{C} \cdot (\bar{V} \times d\bar{s})$$

$$\int C \cdot (\bar{\nabla}' \times V) dv = -C \oint_s (\bar{V} \times d\bar{s})$$

$$\int (\bar{\nabla} \times \bar{V}) dV = - \oint (\bar{V} \times d\bar{s})$$

$$\begin{aligned} & \frac{\mu_0}{4\pi} \int \frac{\bar{j}(r')}{r'} d\bar{z}' \\ &= \frac{\mu_0}{4\pi} \int \left(\bar{\nabla} \times \bar{m} \right) \frac{d\bar{z}'}{r'} \\ &= - \frac{\mu_0}{4\pi} \oint \left(\frac{\bar{m}}{r} \right) ds \end{aligned}$$

$$\begin{aligned} \bar{A}(r) &= \frac{\mu_0}{4\pi} \int \left[\frac{1}{r} (\bar{\nabla}' \times \bar{m}) - \bar{\nabla} \times \left(\frac{\bar{m}}{r} \right) \right] d\bar{z}' \\ &= \frac{\mu_0}{4\pi} \int \frac{\bar{\nabla}' \times \bar{m}(r')}{r'} d\bar{z}' + \frac{\mu_0}{4\pi} \oint \frac{\bar{m} \times \hat{n}}{r} ds \end{aligned}$$

volume current density surface current density

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int \frac{J_b(r')}{r} d\bar{z}' + \frac{\mu_0}{4\pi} \oint \frac{K_b(r')}{r} ds$$

Ampere's Law

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{\text{exc}}$$

$$\bar{m} = I \bar{A} : \text{magnetic moment}$$

magnetic vector potential: $\bar{A}(r)$

due to a single magnetic dipole: $\bar{A}(r) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times \hat{r}}{r^2}$
from multiple expansion

\bar{M} : magnetic dipole moment per unit volume

for a continuous distribution:

$$\bar{A}(r) = \int \frac{\mu_0}{4\pi} \frac{\bar{m}(r') \times \hat{r}}{r'^2} dV'$$

$$= \frac{\mu_0}{4\pi} \int \frac{J_b(r')}{r'} dV' + \frac{\mu_0}{4\pi} \oint \frac{K_b(r')}{r} ds'$$

summation of old bound volume current density and bound surface current density

$$\text{Where, } J_b(r) = \vec{\nabla} \times \vec{M} , K_b(r) = \vec{M} \times \hat{n}$$

\uparrow
 Am^{-2}

\uparrow
 Am^{-1}

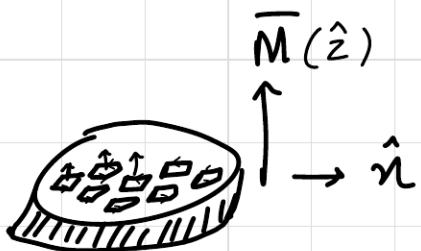
$$K = \frac{I}{l_{\perp}}$$

length perpendicular to

flow of current



Uniform magnetized material

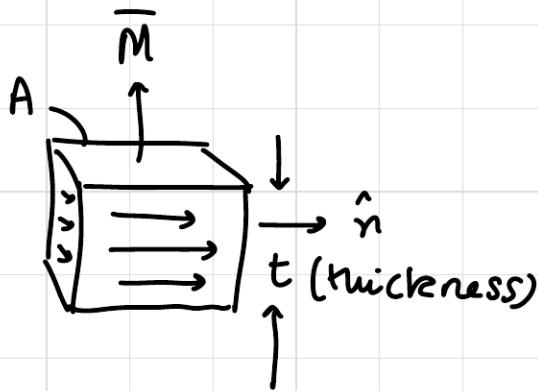


$$\bar{m} = I \bar{A}$$

internal currents are
cancelled out



called bound surface current
like a ribbon wrapped around the
material carrying the current



$$\bar{m} = \bar{M}\bar{A}t$$

$$I\bar{A} = \bar{M}\bar{A}t$$

$$\bar{M} = \frac{\bar{I}}{t} = K_b$$

$$\text{So, } \boxed{\bar{K}_b = \bar{M} \times \hat{n}}$$

because $\bar{m}, \hat{n}, \bar{I}$
are mutually
perpendicular
 K_b is = dir
of current

NOTE: volume current density is zero for uniform magnetized material

NON - UNIFORM MAGNETIZATION

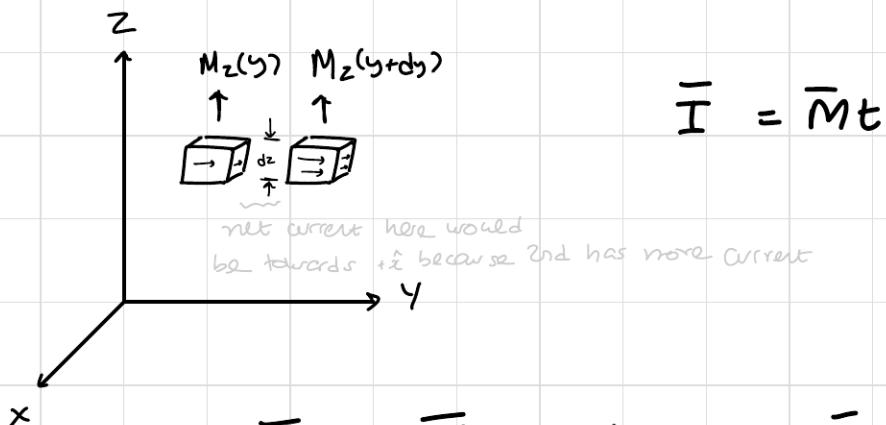
\bar{M} will not be same in 2 different dipoles
and hence, same goes for current

\bar{M} varies with posⁿ

from above $\rightarrow \bar{I} = \bar{M}t$

$$\bar{J}_b = \bar{J}_{b-x}\hat{i} + \bar{J}_{b-y}\hat{j} + \bar{J}_{b-z}\hat{z}$$

Consider slabs



$$\bar{I}_x = \bar{M}_z(y+dy)dz - \bar{M}_z(y)dz \\ = [\bar{M}_z(y+dy) - \bar{M}_z(y)]dz$$

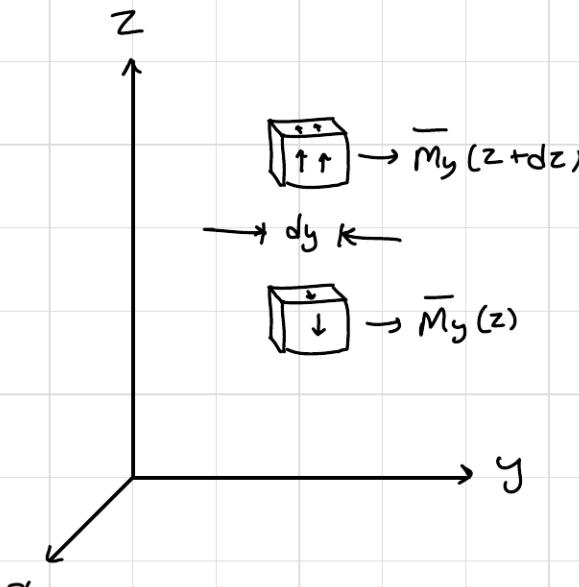
taylor series expansion

$$\bar{I}_x = [M_z(y) + \frac{\partial M_z}{\partial y} dy]dz - M_z(y)dz \\ = \frac{\partial M_z}{\partial y} dy dz$$

so,

$J_{b-x} = \frac{\partial M_z}{\partial y}$

but magnetization can have a
x and y component as well



$$I_x = -\bar{M}_y(z+dz)dy + \bar{M}_y(z)dy$$

again Taylor series expansion

$$= -\frac{\partial M_y}{\partial z} dy dz$$

$$J_{b-x} = -\frac{\partial M_y}{\partial z}$$

Total volume : current density

$$J_{b_x} = \left(\frac{\partial \bar{M}_z}{\partial y} - \frac{\partial \bar{M}_y}{\partial z} \right)$$

which is equal to $\nabla \times \bar{M}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{M}_x & \bar{M}_y & \bar{M}_z \end{vmatrix}$$

$$\text{So, } J_{b_x} = \hat{x} \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \right)$$

again Ampere's Law $\rightarrow \nabla \times \bar{B} = \mu_0 J_{\text{total}}$

$$\frac{\nabla \times \bar{B}}{\mu_0} = \bar{J}_b + \bar{J}_{\text{free}}$$

$$\bar{J}_{\text{free}} = \nabla \times \left(\frac{\bar{B}}{\mu_0} - \bar{M} \right)$$

let $\bar{H} = \frac{\bar{B} - \bar{M}}{\mu_0}$

So, $\bar{J}_f = \nabla \times \bar{H}$

and $\oint \bar{H} \cdot d\ell = I_{\text{free}}$

$$\bar{B} = \mu_0 \bar{H} + \bar{M}$$

$$\bar{M} = \chi_m \bar{H}$$

↳ magnetic susceptibility

$$\bar{B} = \mu_0 (\bar{H} + \chi_m \bar{H}) = \underbrace{\mu_0 (\chi_m + 1) \bar{H}}_{\mu}$$

$$\mu = \mu_0 (1 + \chi_m) \rightarrow \bar{B} = \mu \bar{H}$$

magnetic materials

linear
 $(\bar{M} = \chi_m \bar{H})$

Para-
-magnetic
 $(\chi_m > 0)$

Dia-
-magnetic
 $(\chi_m < 0)$

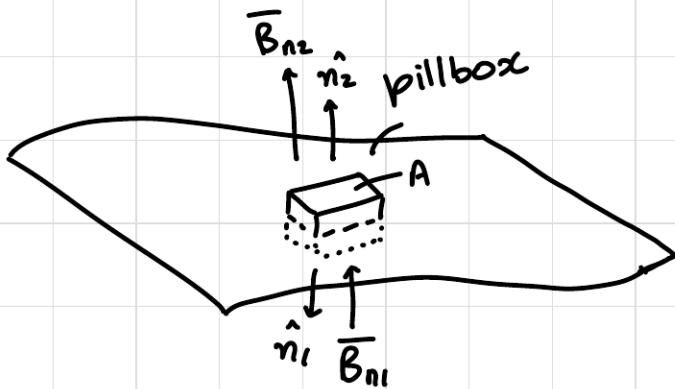
non-linear
(ferromagnetic)
 $\chi_m \sim 10^5$

$$\downarrow M = M_0(1 + \frac{\chi_m}{\epsilon_0} H)$$

$$M \approx M_0$$

Very low magnetization (at room temp)
in general for both
linear elements

Boundary Conditions (magnetostatics)



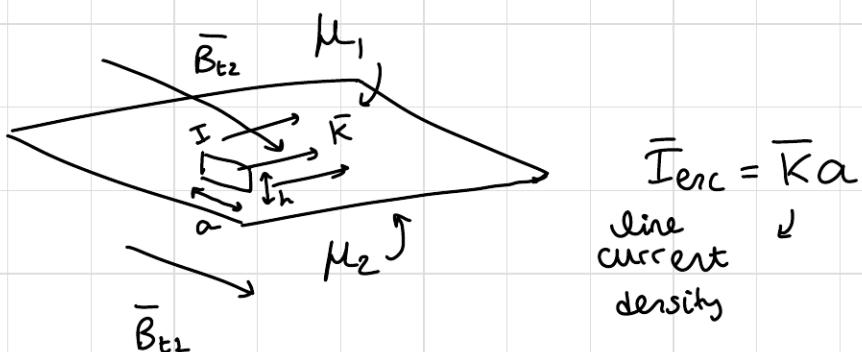
flux through sides = 0
because area of sides $\rightarrow 0$

Gauss's Law: $\oint \bar{B} \cdot d\bar{s} = 0$

$$B_{n2}A - B_{n1}A = 0$$

$$B_{n1} = B_{n2}$$

normal component: equal



$$\bar{I}_{enc} = \bar{K}a$$

↓
line current density

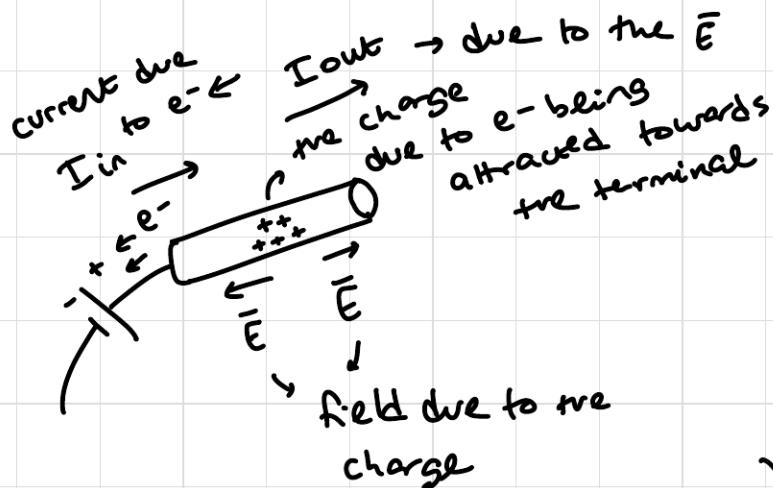
Ampere's Law $\Rightarrow \oint \bar{H} \cdot d\bar{l} = I_{enc} = \bar{K}a \Rightarrow H_{t1}a - H_{t2}a = \bar{K}a$

$$H_{t1} - H_{t2} = K$$

tangential component

* Electrodynamics

EMF: Voltage across a cell without the internal resistance.



This process is repeated throughout

I_{in} is stopped and

I_{out} is allowed due to \bar{E}

$$\leftarrow \bar{f} = \bar{f}_s + \bar{E} \rightarrow \text{drives the current}$$

↓

total force

effect of this force
is confined to the
close vicinity of the
source

) taking
closed loop
integral

$$\oint \bar{f} \cdot d\bar{l} = \oint \bar{f}_s \cdot d\bar{l} + \underbrace{\oint \bar{E} \cdot d\bar{l}}_0$$

assumption:
because DL source { electrostatic
field

for a circuit with no resistance, we don't need any force to push the current

So, for zero resistance,

$$\bar{f}_s + \bar{E} = 0$$

$$\Rightarrow \oint \bar{f}_s \cdot d\ell = - \oint \bar{E} \cdot d\ell = 0$$

but for a non closed loop

$$\underbrace{\int_A^B \bar{f}_s \cdot d\ell}_{\text{EMF } (\epsilon)} = - \int_A^B \bar{E} \cdot d\ell = V_{BA}$$

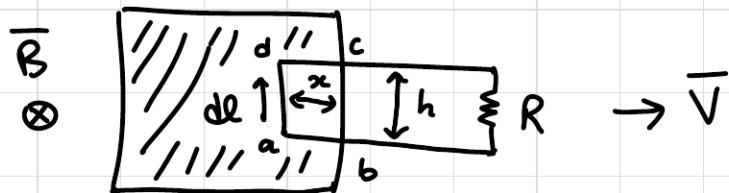
So, EMF is the voltage when there is no internal resistance

now, we can also say

$$\oint \bar{F}_s \cdot d\bar{\ell} = \text{EMF} = 0$$

closed loop EMF

→ Proving Faraday's Law using an example



$$\bar{f}_{\text{mag}} = qvB = vb$$

$$\oint \bar{f}_{\text{mag}} \cdot d\bar{l} = vbh = \text{EMF} \quad \text{assuming } q=1$$

note: only ad contribute to the closed loop integral because

ab and cd have $\vec{B} \perp dl$

and so dot product = 0 and

rest of the ckt is considered

outside the magnetic field \vec{B}

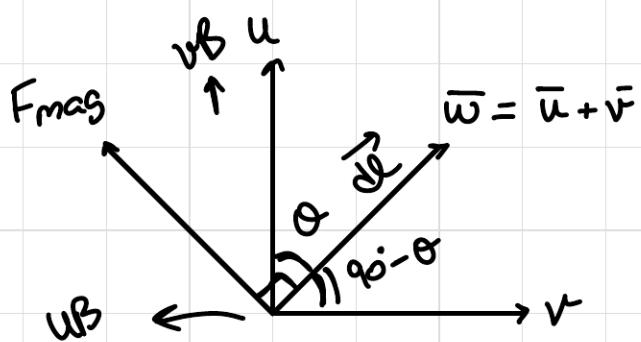
magnetic force doesn't do any work

current will flow through the resistor ad
there is EMF induced

but f_{mag} can do any work
→ f_{mag} is not the source of
the EMF then

the current flowing through R is not due to
 f_{mag} but due to the ckt being pulled to right
with velocity v

Let the force due to the motion be equal to \vec{f}_{pull}



$\otimes \vec{B}$

f_{mag} would now be \perp to $\vec{\omega}$ (motion)

$$\begin{aligned}\vec{f}_{\text{mag}} &= (\vec{\omega} \times \vec{B}) \quad \text{assuming } q=1 \\ &= (\vec{u} + \vec{v}) \times \vec{B}\end{aligned}$$

$$\text{Work done} = \int \vec{f}_{\text{pull}} \cdot d\vec{e}$$

$$= \int uB \sin\theta \, de$$

$$= uB \sin\theta \frac{h}{\cos\theta}$$

$$= Utan\theta Bh = vBh = \underline{\underline{\text{EMF}}}$$

$$\phi = B \tilde{x} h \leftarrow \int \vec{B} \cdot d\vec{s}$$

$$\begin{aligned}\text{Rate of decrease of flux} \rightarrow \frac{d\phi}{dt} &= B \frac{dx}{dt} h = Bvh = \text{EMF} \\ &\text{(x is decreasing)} \rightarrow \text{rate of change} = -\text{EMF}\end{aligned}$$

$$\oint \bar{E} \cdot d\bar{l} = -\frac{d\phi}{dt} = -\frac{d}{dt}(\int \bar{B} \cdot d\bar{s}) = -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

if \bar{B} is not varying with time $\rightarrow \oint \bar{E} \cdot d\bar{l} = 0$

electrostatics

$$\oint \bar{E} \cdot d\bar{l} = \int (\nabla \times \bar{E}) d\bar{s} = - \int \frac{\partial \bar{B}}{\partial t} d\bar{s}$$

$$\boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

So, Change in magnetic field induces an electric field

$$\begin{aligned} \nabla \times \bar{B} &= \mu_0 J \\ \text{so, } \nabla \cdot \bar{J} &= 0 \end{aligned} \quad \left. \begin{array}{l} + (...) \text{ for electrodynamics} \\ \text{only valid in case of} \\ \text{- statics} \\ \hookrightarrow \text{electro/magneto} \end{array} \right\}$$

$$\text{but } \nabla \cdot \bar{J} = -\frac{\partial \phi}{\partial t} \rightarrow \text{conservation of charge}$$

now we need a modified ampere's Law for electrodynamics

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\oint E \cdot d\ell = -\frac{\partial}{\partial t} \int B \cdot ds$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{J}) = \mu_0 \bar{\nabla} \cdot \bar{J} = 0$$

$$so, \bar{\nabla} \cdot \bar{J} = 0$$

conservation of charge \rightarrow

$$\bar{\nabla} \cdot \bar{J} = -\frac{\partial \phi}{\partial t}$$

$$\bar{\nabla} \cdot \bar{J} + \frac{\partial \phi}{\partial t} = 0$$

$$\bar{\nabla} \cdot \bar{J} + \bar{\nabla} \cdot \frac{\partial \bar{D}}{\partial t} = 0 \quad \text{where } \bar{D} = \epsilon \bar{E}$$

$$\nabla \cdot (\bar{J} + \frac{\partial \bar{D}}{\partial t}) = 0$$

$$so, \bar{\nabla} \times \bar{B} = \mu_0 \left(J + \frac{\partial D}{\partial t} \right)$$

so, we just need a time varying electric field for magnetic field.

No need for even a conduction current

Maxwell's modification:

time varying \bar{E} gives rise to \bar{B}

Faraday's Law of Induction:

time varying \bar{B} gives rise to \bar{E}

remember: Faraday's Law

$$\bar{\nabla} \times \bar{E} = -\frac{d\bar{B}}{dt}$$

$$\oint \bar{E} \cdot d\ell = -\frac{\partial}{\partial t} \iint_S \bar{B} \cdot d\bar{s}$$

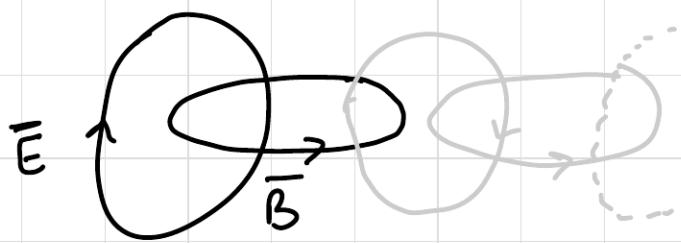
So, yes, \bar{E} can be generated with just a time varying \bar{B} even without charge / conductor.

How will the \bar{E} field lines look like without a conductor / charges

$$\begin{aligned}\bar{\nabla} \times \bar{B} &= \mu_0 \bar{J} + \mu_0 \epsilon \frac{\partial \bar{E}}{\partial t} \\ &= \mu_0 (\bar{J}_D + \bar{J}_C) + \mu_0 \epsilon \frac{\partial \bar{E}}{\partial t}\end{aligned}$$

displacement current conduction current

the \vec{B} field lines will always loop and
be \perp to \vec{E} field lines



Self reproducing
 \vec{E} and \vec{B} fields from just
a single initial \vec{E} or \vec{B}
on all sides

These imaginary chains are
Electromagnetic waves

Static Case (no time variation)

$$\oint \vec{E} \cdot d\vec{s} = \frac{\rho_{enc}}{\epsilon_0}$$

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0}$$

$$\oint E \cdot d\ell = 0$$

$$\bar{\nabla} \times \bar{E} = 0 \Leftrightarrow \bar{E} = -\bar{\nabla} \phi$$

$$\oint \bar{B} \cdot d\bar{s} = 0$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\oint \bar{B} \cdot d\bar{e} = \mu_0 \bar{I}_{\text{enclosed}}$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

Dynamic case

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho(t)}{\epsilon_0}$$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \left(\frac{\partial \bar{E}}{\partial t} \right) \epsilon \quad - \text{maxwell}$$

 EM waves

Potential in Electrodynamics

We can still write $\bar{B} = \bar{\nabla} \times \bar{A}$

but not $\bar{E} = -\bar{\nabla} \phi$

$$\text{from (2)} \rightarrow \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\frac{\partial (\bar{\nabla} \times \bar{A})}{\partial t}$$

$$\Rightarrow \nabla \times (E + \frac{\partial A}{\partial t}) = 0$$

from ② statics $\rightarrow \nabla \times E = 0$

where $E = -\nabla \phi$

$$\text{so, } \bar{E} + \frac{d\bar{A}}{dt} = -\nabla \phi$$

$$\text{so, } \bar{E} = -\left(\bar{\nabla} \phi + \frac{d\bar{A}}{dt}\right)$$

Everything defined till now in dynamics
is in time domain. We will be moving to
the frequency domain now.

square $\hat{\equiv}$ sinc

if $x(t)$ given as square wave and
System's freq response given,
we find $y(t)$ by \rightarrow

$x(t) \rightarrow X(\omega) \quad \{ \text{sinc} \}$
multiply with $H(\omega)$
then IFT to get $y(t)$

This works for a linear system

Time \rightarrow Freq (maxwell's equations)

$$\bar{\nabla} \bar{E}(t) = \frac{\mathcal{J}(t)}{\epsilon_0}$$

$$\bar{\nabla}_x \bar{E}(t) = -\frac{\partial \bar{B}(t)}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B}(t) = 0$$

$$\bar{\nabla}_x \bar{B}(t) = \mu_0 \left(\mathcal{J}(t) + \frac{\partial \bar{D}(t)}{\partial t} \right)$$

Just take Fourier Transform

notation : $\bar{E}(t) \rightleftharpoons \bar{E}(\omega)$

$$\bar{\nabla} \cdot \bar{E}(\omega) = \frac{j\omega}{\epsilon}$$

$$\bar{\nabla} \times \bar{E}(\omega) = -i\omega \bar{B}(\omega)$$

$$\bar{\nabla} \bar{B}(\omega) = 0$$

$$\bar{\nabla} \times \bar{B}(\omega) = \mu (\bar{\sigma}(\omega) + i\omega \epsilon \bar{E}(\omega))$$

We know $\bar{\sigma} = \sigma \bar{E}$

$$\boxed{\bar{\nabla} \times \bar{B} = \mu \bar{E}(\sigma + i\omega \epsilon)}$$

In a source free region \rightarrow

| | | |
|------|---|--|
| Time | $\left\{ \begin{array}{l} \bar{\nabla} \cdot \bar{E} = 0 \\ \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \end{array} \right.$ | $\begin{aligned} \bar{\nabla} \cdot \bar{B} &= 0 \\ \bar{\nabla} \times \bar{B} &= \mu \epsilon \frac{\partial \bar{E}}{\partial t} \end{aligned}$ |
| freq | $\left\{ \begin{array}{l} \bar{\nabla} \cdot \bar{E} = 0 \\ \bar{\nabla} \times \bar{E} = -i\omega \bar{B} \end{array} \right.$ | $\begin{aligned} \bar{\nabla} \cdot \bar{B} &= 0 \\ \bar{\nabla} \times \bar{B} &= -i\omega \mu \epsilon \bar{E} \end{aligned}$ |

wave eqn in 1d

$$\hookrightarrow f(x-vt)$$

$$\frac{\partial^2 f}{\partial x^2} = f''(x-vt) = \frac{\partial^2 f(x-vt)}{\partial (x-vt)^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial t}$$

$$= f'(x-vt) \times (-v)$$

$$\frac{\partial^2 f}{\partial t^2} = v^2 f''(x-vt)$$

so, $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$

for 3d \rightarrow

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \nabla^2 \bar{f} = \frac{1}{v^2} \frac{\partial^2 \bar{f}}{\partial t^2}$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times (\nabla \times E) = - \frac{\partial}{\partial t} (\nabla \times B)$$

$$= - \frac{\partial}{\partial t} \left(\mu \epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

considering source
free region

$$\cancel{\nabla \cdot (\nabla \times E)} - \vec{\nabla}^2 \vec{E} = - \mu \epsilon \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

$$\Rightarrow \vec{\nabla}^2 \vec{E} = \mu \epsilon \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

$$\text{So, the velocity} \Rightarrow \frac{1}{v^2} = \mu \epsilon$$

$$v = \frac{1}{\sqrt{\mu \epsilon}}$$

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu\epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$\vec{\nabla}(\vec{\nabla} \times \vec{B}) - \nabla^2 \vec{B} = -\mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

velocity: $\frac{1}{\sqrt{\mu\epsilon}}$

no material involved
this wave still propagates
even through vacuum

we just need time varying
magnetic field and electric field
for EM waves

$$\nabla^2 E = \mu \epsilon \left(\frac{\partial^2 \bar{E}}{\partial t^2} \right)$$

$$\nabla^2 B = \mu \epsilon \frac{\partial^2 \bar{B}}{\partial t^2}$$

in frequency domain:

$$\bar{\nabla}^2 \bar{E} = \mu \epsilon (-\omega^2 \bar{E})$$

$$\nabla^2 \bar{E} + (\mu \epsilon \omega^2) \bar{E} = 0$$

let $K = \omega \sqrt{\mu \epsilon} = \frac{\omega}{v}$ or $\frac{\omega}{c}$ \rightarrow velocity

$$\nabla^2 \bar{E} + K^2 \bar{E} = 0 \rightarrow \text{Helmholtz eqn}$$

$$\nabla^2 \bar{E} = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}$$

let $K = \omega \sqrt{\mu \epsilon} = \frac{\omega}{v}$ or $\frac{\omega}{c}$ → velocity

$$\nabla^2 \bar{E} + K^2 \bar{E} = 0 \rightarrow \text{Helmholtz eqn}$$

freq domain → time?

IFT

$$\bar{E}(r, t) = \int_{-\infty}^{\infty} \bar{E}(r, \omega) e^{i\omega t} d\omega$$

(for multiple freq)

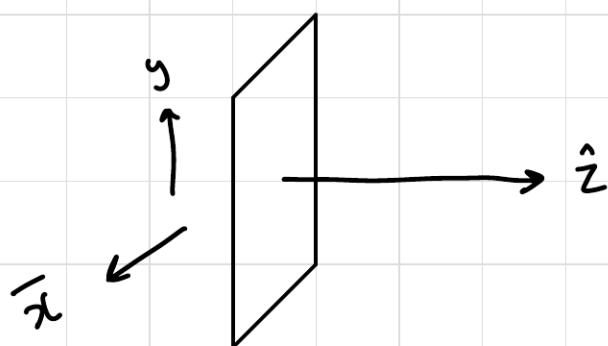
$$f(x) \xrightarrow{\text{IFT}} 1$$

for single frequency

$$E_0(r) S(\omega - \omega_0) \approx E_0(r) e^{i\omega t}$$

↳ only a func of space

1D \rightarrow Uniform Plane wave



\bar{E} and \bar{B}/\bar{H} only

vary with z
they are uniform in
the xy plane

considering a monochromatic wave

i.e. only one frequency

$$\text{So, } \bar{E}(z, t) \geq \bar{E}(z) e^{i\omega t}$$

Maxwell's equation \rightarrow

$$\frac{\partial^2 \bar{E}(z)}{\partial z^2} + k^2 \bar{E}(z) = 0$$

$$\text{assume } \bar{E}(z) \approx e^{mz}$$

$$m^2 e^{mz} + k^2 e^{mz} = 0$$

$$m^2 + k^2 = 0$$

$$m = \pm ik$$

$$E(z) = e^{\pm ikz}$$

$$\bar{E}(z, \omega) = \bar{A} e^{ikz} + \bar{B} e^{-ikz}$$

$$(-\hat{z}) \quad (\hat{z})$$

$$\bar{E}(z, t) = \bar{A} e^{i(\omega t + kz)} + \bar{B} e^{i(\omega t - kz)}$$

Let's take the case of forward travelling wave

$$\bar{E}(z, t) = \bar{E}_0 e^{i(\omega t - kz)}$$

(Amplitude) \rightarrow (phase)

direction of EM wave = polarization
 (\bar{E}) of the wave

$t=t_1$ locus of the points at the same phase = ?
 we will look at it now.

$$\omega t_1 - kz = M$$

\hookrightarrow same phase M at $t=t_1$.
 Locus = ?

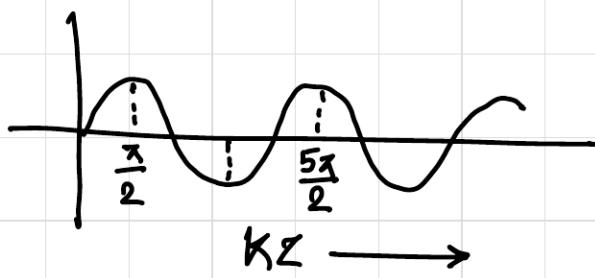
$$Z = \frac{\omega t_1 - M}{k} = 0$$

\hookrightarrow totally a constant

Locus $\Rightarrow Z = \text{constant}$

Plane parallel to xy -plane

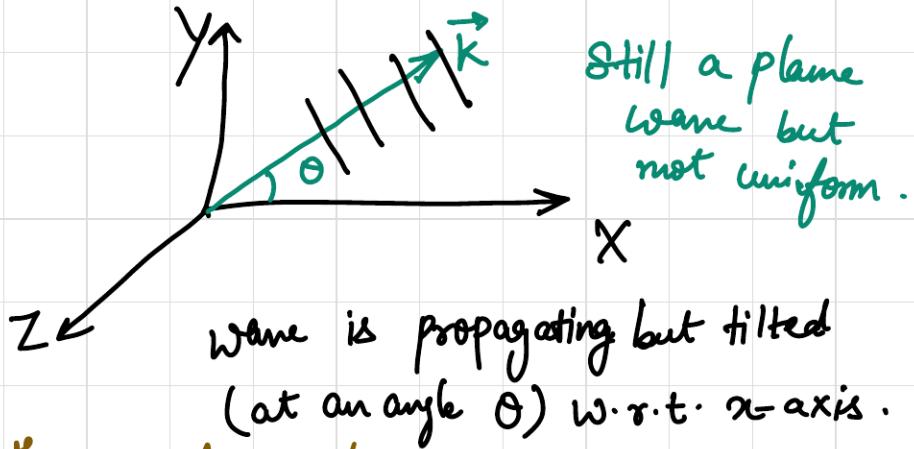
$$\bar{E}(z, t) = \bar{E}_0 e^{i(\omega t - kz)} e^{i2\pi z}$$



$2n\pi$ gap in k_z axis.

* wave = planar but not uniform. Possible?

YES



wave is propagating but tilted
(at an angle θ) w.r.t. x -axis.

There can be other examples ←
but this is simplest one.

Basic Helm h. eqn we studied till now,
we assumed $\sigma = 0$ (source-free)

↳ No conduction current.

If not this,
it will be very
tough to analyze.

$$\begin{aligned}
 \nabla \times (\nabla \times \vec{B}) &= i\omega \mu \epsilon (\nabla \times \vec{E}) \\
 &= i\omega \mu \epsilon (-i\omega \vec{B}) \\
 &= -i^2 \omega^2 \mu \epsilon \vec{B}
 \end{aligned}$$

$$\vec{\nabla} \times \vec{B} = i\omega \mu \epsilon_c \vec{E} = i\omega \mu \epsilon \vec{E} \quad (\text{written in simpler})$$

$$i\omega \mu \epsilon_c = \mu \nabla + i\omega \mu \epsilon \quad (\text{equating both } \vec{\nabla} \times \vec{B} \text{ eqn.})$$

$$\epsilon_c = \epsilon + \frac{\nabla}{i\omega}$$

$$\boxed{\epsilon_c = \epsilon - i \frac{\nabla}{\omega}}$$

Permittivity of a medium becomes complex no.

$$\epsilon_c = \epsilon \left(1 - i \frac{\nabla}{\omega \epsilon} \right)$$

$$* K = \omega \sqrt{\mu \epsilon_c} = K' - i \frac{K''}{\omega}$$

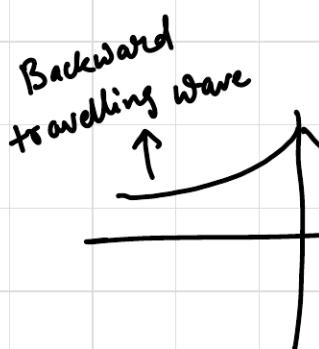
Initially, K was simpler. ($K = \omega \sqrt{\mu \epsilon}$)

Now, it is complex. ($\epsilon \rightarrow \epsilon_c$)

Just ϵ changed to a complex no. (ϵ_c).

$$\begin{aligned} * \vec{E}_0 e^{i(\omega t - kz)} &= \vec{E}_0 e^{i\omega t} \cdot e^{-i(K' - iK'')z} \\ &= E_0 e^{i\omega t} e^{-ik' z} e^{-K'' z} \end{aligned}$$

Nature of this wave = ?



↳ Exponentially decaying wave moving forward

↳ forward travelling wave

Both decaying in diff. distns.

If we had taken +ve from $k' + ik''$, we would have an exp. increasing wave and at an ∞ amplitude, Energy will become ∞ (which is not possible obv.)



* When to assume $\epsilon_c = \text{real}$ and $\epsilon_c = \text{complex}$

$\tau \neq 0$, current $\neq 0$

Lost energy will be released in form of heat because of conduction current.

What if $\tau \ll \omega\epsilon$

↳ Perfect dielectric
 $\epsilon_c = \epsilon$ (real ϵ)

$\tau \sim \omega\epsilon$

↳ Lot of heat will be generated.

medium with decreasing constant

$\tau \gg \omega_E$

EM wave will die inside a P.E.C.

It will decay very fast.

* Static \vec{E}
Time-varying \vec{E} } inside P.E.C. = 0

* Static \vec{B} can exist inside P.E.C.

Q: Can Time-varying \vec{B} exist inside P.E.C?

NO, as $\nabla \times \vec{B}$ will generate \vec{E} but
 $\vec{E} = 0$ inside P.E.C.

Ag, Cu → good conductors at microwave frequencies
not good conductors at optical frequencies