Lecture before Quiz 3

for the 1d uniform plane wave:

$$\bar{E}(z,t) = \left(\bar{A}e^{i(\omega t+kz)} + \bar{B}e^{i(\omega t-kz)}\right)\hat{x}$$

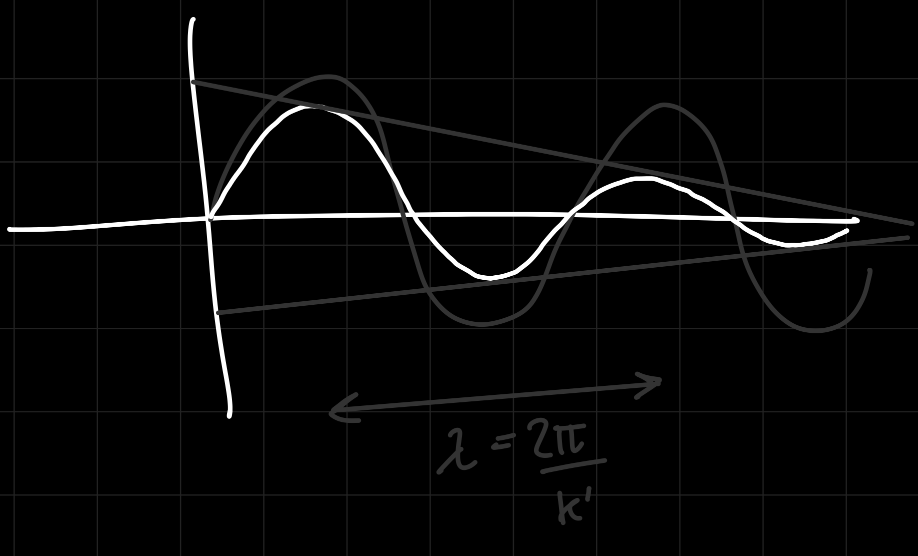
polarization of the EM wave

$$K = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \longrightarrow \text{wave number}$$

$$\lambda = \frac{2\pi}{K}$$

k'' defines the decay

k' defines the wavelength $\lambda = \frac{2\pi}{k'}$



Just like time and frequency form a fourier pair,
K and λ also form a fourier pair

also → K domain ≡ momentum space

$$\omega = \frac{2\pi}{T} \quad , \quad K = \frac{2\pi}{\lambda}$$

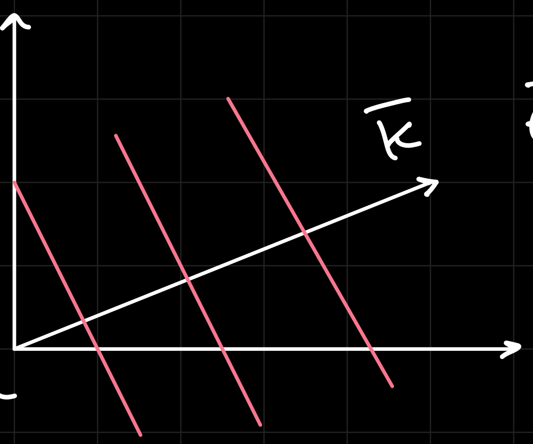
What is the physical meaning of K?
↳ represents the number of cycles in a unit distance

$$K\lambda = 2\pi$$

in a given phase of 2π, what fraction of a complete wavelength can fit?
eg → $1\lambda = 2\pi$ or 1.5λ or 2.5λ

Can we have a non uniform plane wave?

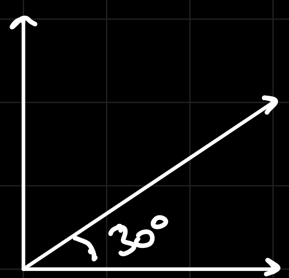
Still planar equinormal surfaces



$\bar{E}$ and $\bar{B}$ are non uniform because wave varies not in a single axis

$$\bar{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$

currently only a 2d problem so $k_z = 0$

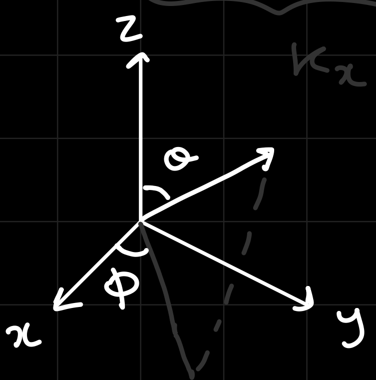


$$\bar{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$

$$= \hat{x} |k| \cos\theta + \hat{y} |k| \sin\theta$$

for a vector in sperical coordinates

$$\bar{K} = \underbrace{\hat{x} |k| \sin\theta \cos\phi}_{k_x} + \underbrace{\hat{y} |k| \sin\theta \sin\phi}_{k_y} + \underbrace{\hat{z} |k| \cos\theta}_{k_z}$$



$$\frac{k_x}{k_y} = \tan\phi$$

$$\phi = \tan^{-1}\left(\frac{k_x}{k_y}\right)$$

from $\bar{k}$, we ca find

frequency

direction of propagation

$$f = \frac{2\pi c}{\lambda} = \frac{2\pi c}{2\pi/|k|} = |k|c$$

going back:

$$\nabla^2\bar{E} + k^2 E = 0$$

↳ # of eqns =?

↳ 3 because of the 3 components

$$\left.\begin{aligned} \nabla^2\bar{E}_x + k^2E_x = 0 \\ \nabla^2\bar{E}_y + k^2E_y = 0 \\ \nabla^2\bar{E}_z + k^2E_z = 0 \end{aligned}\right\}$$

↓

also note:

$$k^2 = \left(\frac{\omega}{c}\right)^2 = \omega^2\mu\varepsilon$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + K^2 E_x = 0$$

Variable separation :

$$E_x(x, y, z) = X(x)\,Y(y)\,Z(z)$$

we proved earlier :

solution of 1d wave

↳ plane wave

now we prove,

soln of 3d is also

plane wave

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + K^2 XYZ = 0$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-K_x^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-K_y^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{-K_z^2} = -K^2$$

let → $-K_x^2$ $\quad -K_y^2 \quad -K_z^2$

$$K_x^2 + K_y^2 + K_z^2 = K^2$$

(sphere eqn

radius $= K^2 = \left(\frac{\omega}{c}\right)^2$)

we can only choose 2 variables arbitrarily (independently).
The 3rd must be a fixed value in order to satisfy

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

because $k^2 = \left(\frac{\omega}{c}\right)^2$ is const

since ω (frequency) is given

$$k_z = \pm\sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

sol^n comes out in the form →

$$X \simeq e^{\pm i k_x x}$$
$$Y \simeq e^{\pm i k_y y}$$
$$Z \simeq e^{\pm i k_z z}$$

with some constants but we will just assume that in final sol^n

$$E_x(x, y, z) = E_{x_0} e^{\pm i k_x x} e^{\pm i k_y y} e^{\pm i k_z z}$$

↳ wave of 3 planes

$$E_x(x,y,z) = E_{x_0} e^{\pm i(k_x x + k_y y + k_z z)}$$

as a dot product of 2 vectors

$$= E_{x_0} e^{\pm i(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)(\hat{x}x + \hat{y}y + \hat{z}z)}$$

$$E_x(x,y,z) = E_{x_0} e^{\pm i\vec{k}\vec{r}}$$ → radius vector

plane wave ✓
but not uniform

now back the $\bar{E}$ equation
(general)

$$\bar{E} = e^{\pm i\bar{k}\cdot\bar{r}} (\hat{x}E_{x_0} + \hat{y}E_{y_0} + \hat{z}E_z)$$

so, $\bar{k}$ determines the propagation of the wave and the general eqn of the wave is $e^{\pm i(\bar{k}\cdot\bar{r})}$

Remember Maxwells Eqn in freq domain

$$\bar{\nabla}\cdot\bar{E} = 0 \qquad \bar{\nabla}\cdot\bar{B} = 0$$
$$\bar{\nabla}\times\bar{E} = -i\omega B \qquad \bar{\nabla}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

we can achieve the time domain equivalent of a wave as the superposition of different wavelengths in the k domain.

$$g(t) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega t}\, d\omega$$

Fourier optics
↳ used for wide angle clicks in portable computing devices like VR headsets, Mobile phones etc.

$$E_x = E_{x0}\, e^{i(k_x x + k_y y + k_z z)}$$

$$\frac{\partial E_x}{\partial x} = -ik_x\left(E_{x0}\, e^{i(k_x x + k_y y + k_z z)}\right)$$
$$= -ik_x E_x$$

like fourier formulas

$\bar{K}\cdot\bar{E} = 0 \quad , \quad \bar{K}\cdot\bar{B} = 0$

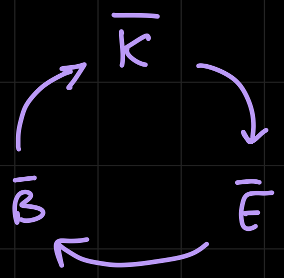
so, $\bar{K}$, $\bar{E}$ and $\bar{B}$ fields are perpendicular to each other

$$\bar{\nabla}\times\bar{E} = -i\omega\bar{B} \qquad\qquad \bar{\nabla}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

$$-i\bar{K}\times\bar{E} = -i\omega\bar{B} \qquad\qquad -i\bar{K}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

$$\bar{K}\times\bar{E} = \omega\bar{B} \qquad\qquad \bar{K}\times\bar{B} = -\omega\mu\varepsilon\bar{E}$$

not just direction, we have this relation



we know, $\dfrac{|E|}{|B|} = c$ but how?

$$\bar{K}\times\bar{E} = \omega\bar{B}$$

$$|K|\,|E| = \omega|B|$$

$$\frac{|E|}{|B|} = \frac{\omega}{|K|} = \underline{\underline{c}}$$

we knew $\bar{B} = \mu \bar{H}$

and

Voltage / Current $\quad \dfrac{\vec{E}}{\vec{H}} = \mu \dfrac{\bar{E}}{\bar{B}} = \mu c = \dfrac{\mu}{\sqrt{\mu \varepsilon}} = \sqrt{\dfrac{\mu}{\varepsilon}}$

↓

Impedance of a medium

Why not called Resistance? ⟶ because it can be complex

lecture

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = \eta : \Omega$$

$$w_e = \frac{1}{2}\varepsilon|E|^2$$

$$w_m = \frac{1}{2}\mu|E|^2$$

$$w_e + w_m = \frac{1}{2}\left(\varepsilon|E|^2 + \mu|H|^2\right)$$

$$W_e = \frac{1}{2}\varepsilon\int|E|^2 dz$$

$$W_m = \frac{1}{2}\mu\int|H|^2 dz$$

$$\bar{\nabla}\cdot(\bar{E}\times\bar{H}) = \bar{H}\cdot(\bar{\nabla}\times\bar{E}) - \bar{E}(\bar{\nabla}\times\bar{H})$$

$$= \bar{H}\left(-\mu\frac{\partial\bar{H}}{\partial t}\right) - \bar{E}(\dots)$$

$$= -\mu\,\bar{H}\cdot\frac{\partial\bar{H}}{\partial t} - \bar{E}\left(\bar{J} + \varepsilon\frac{\partial\bar{E}}{\partial t}\right)$$

$$= -\frac{\mu}{2}\frac{\partial|H|^2}{\partial t} - \bar{E}\cdot\bar{J} - \frac{\varepsilon}{2}\frac{\partial|E|^2}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\mu|\bar{H}|^2 + \varepsilon|\bar{E}|^2\right) - \bar{E}\cdot\bar{J}\right\}$$

$$-\frac{\partial}{\partial t}(W_e + W_m) = \underset{\text{(heat)}}{\bar{E}\cdot\bar{J}} + \underset{\text{(outgoing)}}{\bar{\nabla}(\bar{E}\times\bar{H})}$$

decay rate of stored energy in an electromagnetic field

we said in the last class

$\bar{E}$ = Voltage
$\bar{B}$ = current

So, $\bar{E}\cdot\bar{J} \simeq$ Power

$$dw\ \bar{F}\cdot d\bar{\ell} = q(\bar{E} + \underbrace{(\bar{v}\times\bar{B})) \cdot \bar{v}}_{0}\,dt$$

$$dw\ \bar{F}\cdot d\bar{\ell} = q\bar{E}\bar{v}\,dt$$

$$\frac{dw}{dt} = \rho\, d\tau\ \bar{E}\cdot\bar{v}$$

$$\frac{dw}{dt} = \bar{E}\cdot\bar{J}\,d\tau$$

The Energy is either used by doing some work $(\bar{E}\cdot\bar{J})$ or comes out of the surface $(\bar{\nabla}\cdot(\bar{E}\times\bar{H}))$

$$-\frac{\partial}{\partial t}\int (W_e + W_m)\, d\tau = \int \bar{E}\cdot\bar{J}\, d\tau + \int \bar{\nabla}\cdot(\bar{E}\times\bar{H})\, d\tau$$

$$= \int \bar{E}\cdot\bar{J}\, d\tau + \oint\!\!\oint (\bar{E}\times\bar{H})\, d\bar{S}$$

So, $P_{outgoing} = \oint\!\!\oint \underbrace{(\bar{E}\times\bar{H})}_{\vec{S}} d\bar{S}$

(Poynting vector)

$$\bar{S} = \bar{E}\times\bar{H}$$

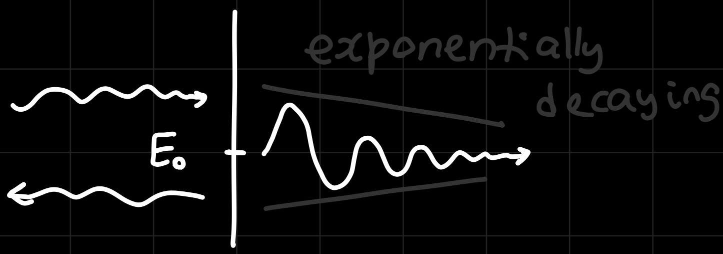
$|\bar{S}| = |E||H| = |E|\cdot\frac{|E|}{\eta} = \frac{|E|^2}{\eta}$: ≃ power per unit area (Watt/m²)

last topic for EM waves →

Next : transmission lines

SOMETHINGIDK123

$$\varepsilon_c = \varepsilon\left(1 - \frac{i\sigma}{\omega\varepsilon}\right)$$



$$K = \omega\sqrt{\mu\varepsilon_c}$$

$$K = k' - ik''$$

$$e^{-ikz} = e^{-i(k'-ik'')z} = e^{-ik'z}e^{-k''z}$$

Skin depth of a medium :

$$\delta = \frac{1}{k''}$$

$\hookrightarrow$ 0 for a PEC
$(\infty\times)$

distance for the signal to decay

for $z = \delta$,

$$\bar{E} = E_0 e^{-ik'z} e^{-k''/k''} = \frac{E_0}{e} e^{-ik'z}$$

$$K^2 = \omega^2\mu\varepsilon\left(1 - \frac{i\sigma}{\omega\varepsilon}\right)$$

$$(k' - ik'')^2 = k'^2 - k''^2 - 2ik'k'' = \omega^2\mu\varepsilon - i\omega\sigma\mu$$

$$2k'k'' = \omega\sigma\mu$$

$$k'^2 - k''^2 = \omega^2\mu\varepsilon \quad \text{— } 1$$

for simplicity, we find

$$(k'^2 + k''^2)^2 = (k'^2 - k''^2)^2 + 4(k'k'')^2$$

$$= \omega^4 \mu^2 \varepsilon^2 + \omega^2 \mu^2 \sigma^2$$

$$(k'^2 + k''^2) = \omega^2 \mu \varepsilon \left[1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right]^{1/2} \quad — \quad 2$$

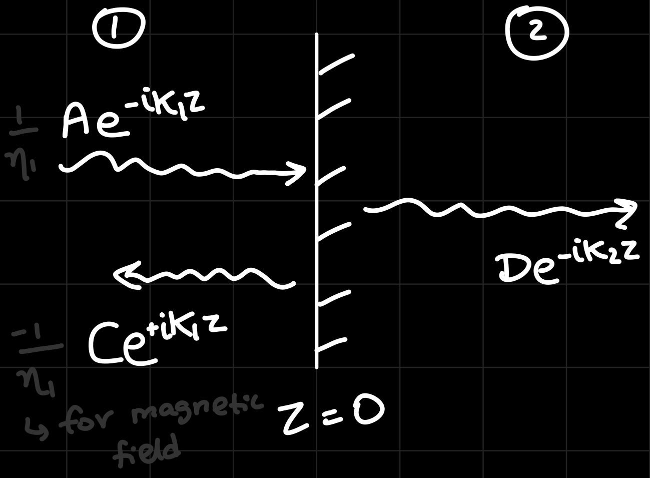
using 1 and 2 →

$$k' = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[1 + \left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} \right]^{1/2}$$

$$k'' = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} - 1 \right]^{1/2}$$

Note: Lossless medium → no decay

We figure out reflection and trasmission of the em waves



assuming SRC ≡ medium ①

transmitted ≡ medium ②

$$\Gamma(z) = \frac{Ce^{-ik_1z}}{Ae^{-ik_2z}}$$

(μ_1, ε_1) $\qquad$ (μ_2, ε_2)

$k_1 = \omega\sqrt{\mu_1\varepsilon_1}$ $\qquad$ $k_2 = \omega\sqrt{\mu_2\varepsilon_2}$

Reflection coefficient:

(gamma)

$$\Gamma(z) = \frac{C}{A} e^{i2k_1z}$$

$$\boxed{\Gamma(z) = \Gamma_0 e^{i2k_1z}}$$

at the interface → $z=0$

$$\Gamma_0 = \frac{C}{A}$$

lossless medium ≡ no conduction current (pure dielectric)

for a lossless medium → Γ_0 decreases as we move away from the interface and $|\Gamma(z)| = |\Gamma_0|$

periodicity:

$$e^{i2\pi} = 1 \rightarrow 2k_1 z = 2\pi$$

$$2 \cdot \frac{2\pi}{\lambda} \cdot z = 2\pi \rightarrow \boxed{z = \frac{\lambda}{2}}$$

Transmission coefficient : $T = \frac{D}{A}$

defined at the interface only, not all points because there exists only the transmitted wave in medium 2.

at $z=0$: $\frac{E_{t1}}{H_{t1}} = \frac{E_{t2}}{H_{t2}}$

$$\frac{A+C}{A/\eta_1 + C/(-\eta_1)} = \eta_2$$

$$\frac{1+(\Gamma_0)}{\frac{1}{\eta_1}(1-\Gamma_0)} = \eta_2$$

$$\frac{\eta_2}{\eta_1} = \frac{1+\Gamma_0}{1-\Gamma_0}$$

$$A + C = D$$

$$1 + C/A = D/A$$

$$\boxed{1 + \Gamma_0 = T}$$

$$T = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \rightarrow \boxed{T = \frac{2\eta_2}{\eta_1 + \eta_2}}$$

$$E_{incident} = E_0 e^{-ikz}$$
$$E_{reflected} = \Gamma_0 E_0 e^{ikz}$$

if Γ_0 is negative then $\bar{E}$ has changed the direction

$$\Gamma_0 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \eta_1 = \left(\frac{\mu_1}{\varepsilon_1}\right)^{1/2}, \ \eta_2 = \left(\frac{\mu_2}{\varepsilon_2}\right)^{1/2}$$

for non magnetic media

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}}, \quad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

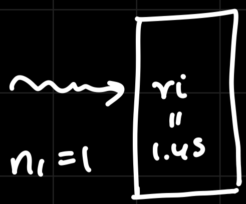
$$\Gamma_0 = \frac{\sqrt{\mu_0/\varepsilon_1} - \sqrt{\mu_0/\varepsilon_2}}{\sqrt{\mu_0/\varepsilon_1} + \sqrt{\mu_0/\varepsilon_2}} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

$$\frac{\sqrt{\varepsilon_{r_1}} - \sqrt{\varepsilon_{r_2}}}{\sqrt{\varepsilon_{r_1}} + \sqrt{\varepsilon_{r_2}}} \longleftarrow = \frac{\sqrt{\varepsilon_0 \varepsilon_{r_1}} - \sqrt{\varepsilon_0 \varepsilon_{r_2}}}{\sqrt{\varepsilon_0 \varepsilon_{r_1}} + \sqrt{\varepsilon_0 \varepsilon_{r_2}}}$$

$$\downarrow$$

$$\boxed{\frac{n_1 - n_2}{n_1 + n_2}}$$

these n are not η (eeta)
this is refractive index



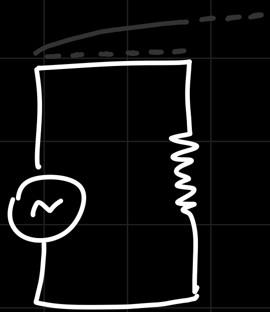
for non magnetic material

$\Gamma_o = \frac{n_1 - n_2}{n_1 + n_2} = -ve$ and so, $\bar{E_r}$ = opposite to $\bar{E_i}$

Initial KVL/KCL problem

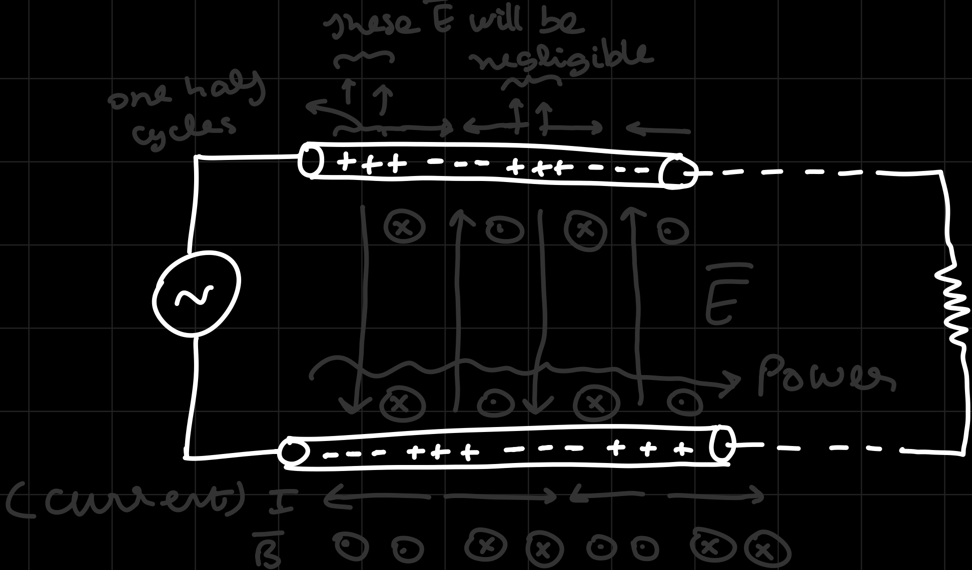
wavelength for freq = 50Hz → $\frac{3\times10^8}{50} = 6\times10^8$ cm

Since wavelength is very big, we can assume for a very small circuit that voltage remains constant along a R=0 line



but for very long ckts, we cannot assume:

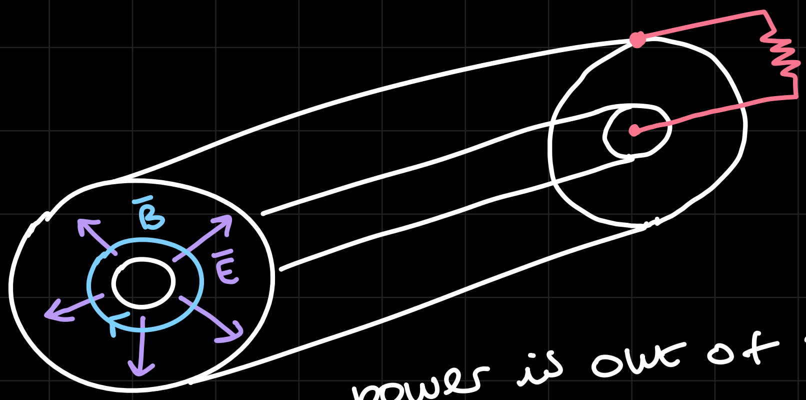
$\left.\begin{array}{l} V_{SRC} = V_{Load} \\ I_{SRC} = I_{load} \end{array}\right\} \times$



$\bar{E} \times \bar{H}$ is almost zero outside the circuit because of $\bar{E}$

$(\bar{E} \times \bar{H})$ Power is held / concentrated within the 2 conductors and being guided from SRC → Load at all points because both $\bar{E}$ and $\bar{H}$ switch directions together.

Guided wave propagation



power is out of the screen ⊙

ERROR POSSIBLE: These conductors might not remain as PEC at higher frequency

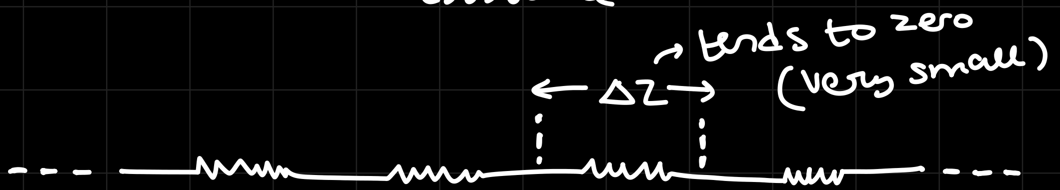
Distributed Resistance (not lump resistance ×) throughout the conductors because not PEC

Assume that distributed resistance per unit length as R_s (Ω/m)

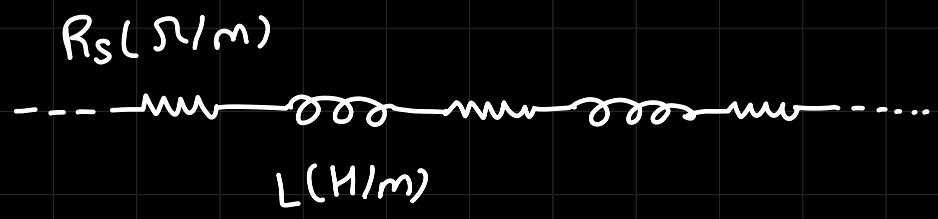
↳ not just r because distributed

↓

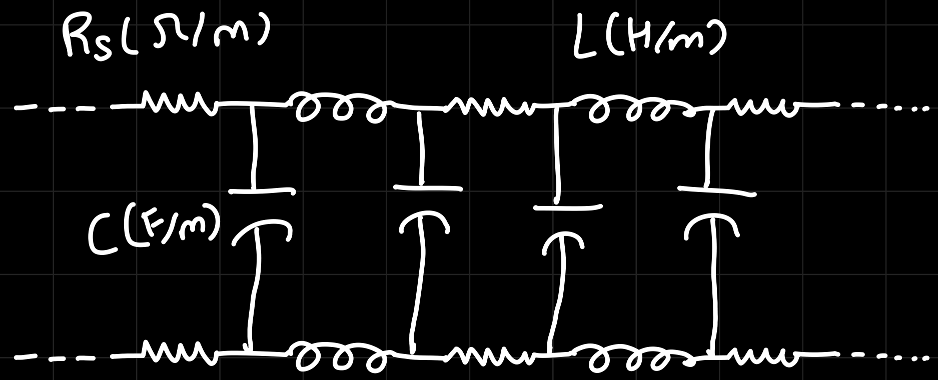
series resistances



along with this we have inductance as well because of the $\bar{B}$

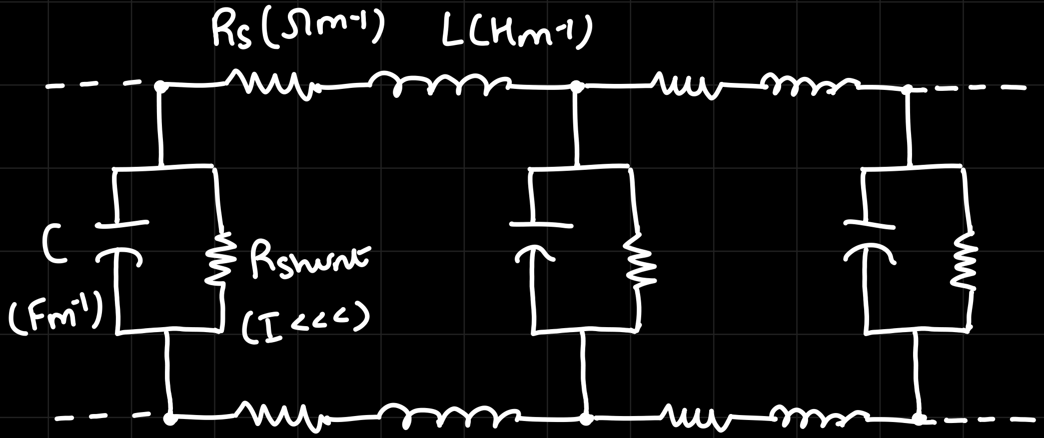


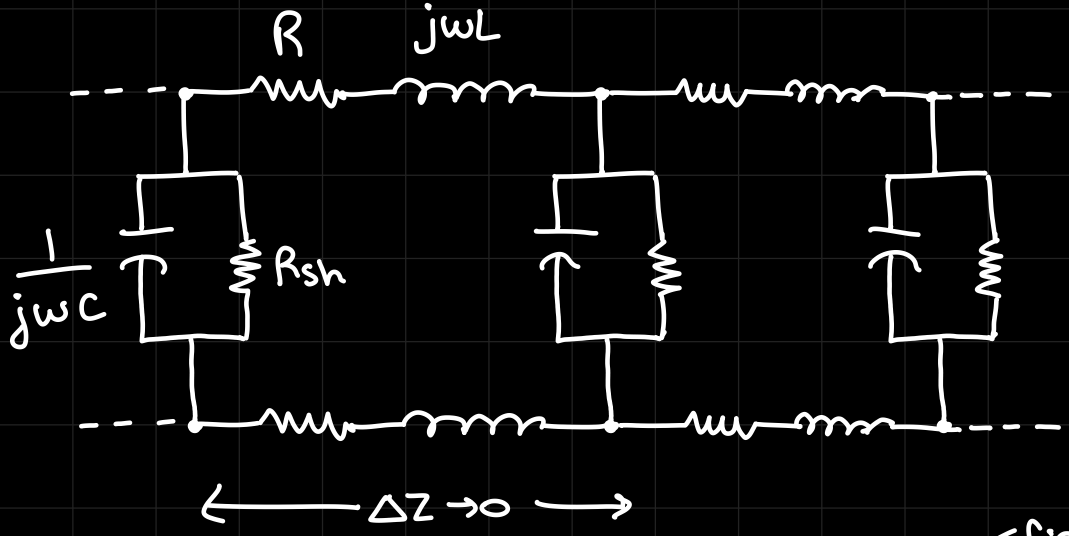
additionally, we also have distributed capacitance due to the $\bar{E}$



lastly, we have σ very small between the two conductors so that means very high resistance between them (dielectric medium)

(leakage current)





<fig L>

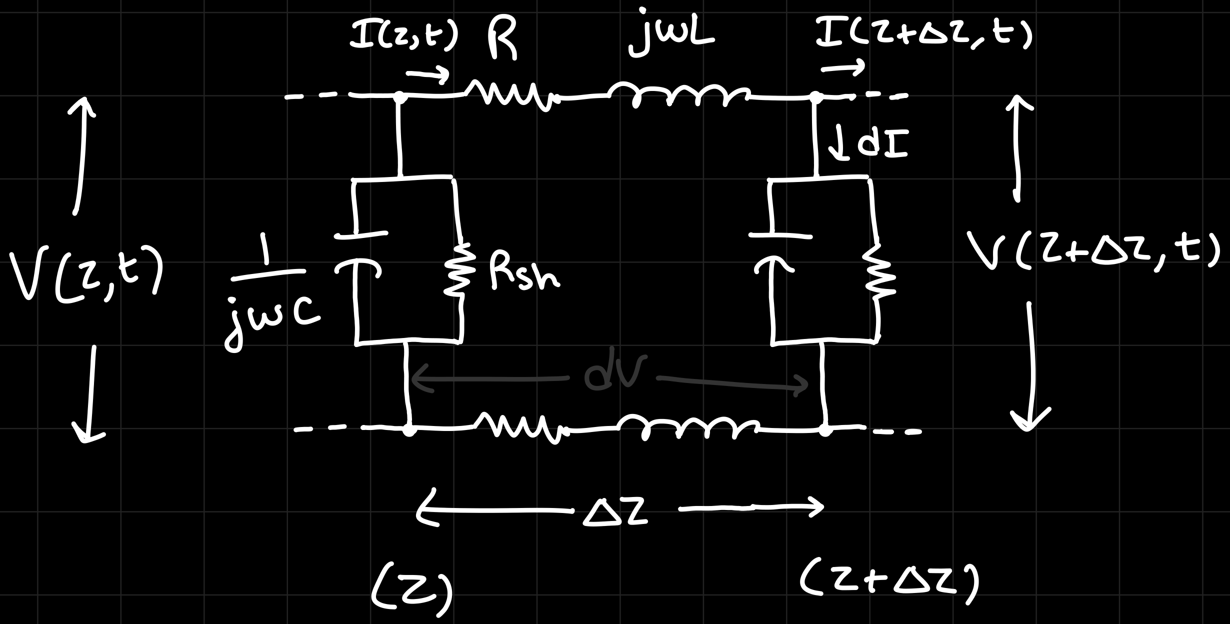
let $X = R + j\omega L$

(admittance) $Y = j\omega C + \frac{1}{R_{sh}} = (G + j\omega C)$

↳ very less because R_{sh} >>>

Since we have all these non-idealities, there must be some voltage drop and hence KVL / KCL would fail without considering these factors.

KCL/KVL fails for macroscopic circuitry but still work for <fig1> because $\Delta z \to 0$ (because distance very small)



$$V(z,t) - I(z,t)[R+j\omega L]\underbrace{\Delta z}_{} - V(z+\Delta z,t) = 0$$

because R and L are per unit values

$$-I(z,t) X \Delta z = V(z+\Delta z,t) - V(z,t)$$

$$\lim_{\Delta z \to 0} \frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -X I(z,t)$$

$$\boxed{\frac{\partial V}{\partial z} = -X I(z,t)}$$

what about the current?

$$I(z,t) = I(z+\Delta z,t) + dI$$

$$I(z+\Delta z,t) - I(z,t) = -dI$$

$$I(z+\Delta z,t) - I(z,t) = -VY\Delta z$$

$$\boxed{\frac{\partial I}{\partial z} = -VY}$$

now we find the wave equation
(requires 2nd derivation)

$$\frac{\partial^2 V}{\partial z^2} = -X\frac{\partial I}{\partial z} = -X(-VY) = V\cdot XY$$

note: $\left.\begin{aligned} X &= R+j\omega L \\ Y &= G+j\omega C \end{aligned}\right\}$ complex nums

$$\frac{\partial^2 V}{\partial z^2} - \gamma^2 V = 0 \qquad \text{where } \gamma = \sqrt{XY}$$

$$\gamma = \left((R+j\omega L)(G+j\omega C)\right)^{1/2}$$

(helmholtz eqn)

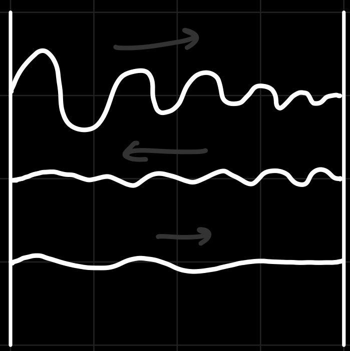
$\gamma = \alpha + j\beta$

general sol^n:

$$V = V_+ e^{-\gamma z} + V_- e^{\gamma z}$$

$$V = V_+ e^{-\alpha z} e^{-j\beta z} + V_- e^{\alpha z} e^{j\beta z}$$

$$V(z,t) = V_+ e^{-\alpha z} e^{j(\omega t - \beta) z} + V_- e^{\alpha z} e^{j(\omega t + \beta) z}$$



continuous transmission
and reflection

Now, when will have lossless transmission
i.e. no attenuation in signal?

when $e^{-\alpha z}$ and $e^{\alpha z} = 0$

↳ $\alpha = 0$ i.e. γ = purely complex

so, $R = 0$ and $G = 0$

↳ series resistance $= 0$

↳ leakage current $= 0$ (R_{shunt} ↑↑↑)

so, $\gamma = j\beta = j\omega\sqrt{LC} \rightarrow \beta = \underline{\omega\sqrt{LC}}$

wavelength : $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{1}{f\sqrt{LC}}$

and velocity → $\boxed{v = \frac{1}{\sqrt{LC}}}$

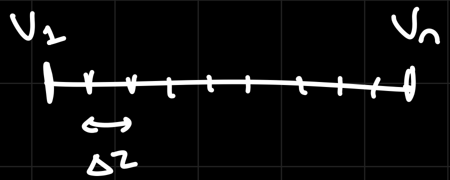
this is of the guided wave

quiz 3 and quiz 4 syllabus same
(open book) (closed book)

quiz 5 and quiz 6 syllabus : till today
(open book) (closed book)
30 mins 30 mins

PROJECT

{ finite difference method }



$\frac{V_{i+1} - V_i}{\Delta z}$ forward difference

$\frac{V_i - V_{i-1}}{\Delta z}$ backward difference

central difference (best)