Fields and waves

lignagn

A at 80%.

Mid sem	30%
End sem	40%
Quiz × 6 (N-1)	30%

pass : 30%

grading : absolute

attendance in tutorials : atleast 75% required for attempting next quiz

Office hours:
wed 12:30 – 1:15 pm

- Vector calc
- Electrostatics and Magnetostatics — doesn't vary with time
- Electrodynamics — varies with time
- EM waves
- Transmission Lines

Book: David J. Griffiths
Intro to Electrodynamics

M.N.O. Sadiku
Elements of electromagnetics

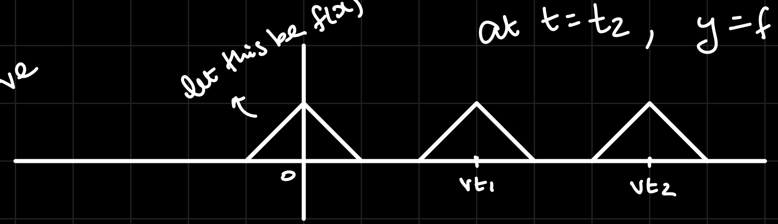
- wave

$$y = f(x \pm vt)$$

v = constant
t = time

at $t=0$, $y = f(x)$
at $t = t_1$, $y = f(x - vt_1)$
at $t = t_2$, $y = f(x - vt_2)$

let the sign here be -ve



forward travelling wave →

if sign = +ve : backward travelling wave ←

$$F_E = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}$$

$$F_{Grav} = G \frac{m_1 m_2}{r^2}$$

} with unit masses and unit charges kept at 1m distance,

$$\frac{F_E}{F_{Grav}} \simeq \frac{10^{12}}{10^{-11}} = 10^{23}$$

speed of free electron: $\sim 10^5$ m/s

in a conductor : electron takes 1s to move 1mm

then how does a hall lights up almost instantly when you switch on the light?

because e^- are chained together in nature and the initial e^- doesn't travel all the way from beginning to the end of the hall

CEE, FCC : radiation certification

Electric field lines have to always end at a charge : ⊕ → ⊖

When the charge has been moved, the field lines will also have a disturbance

KCL and KVL don't work with very large circuits because of signal propagation delay

direction of E × H → source to load

min time taken for propagation $= \frac{c}{x}$

$$c = \lambda f$$

wavelength $= \lambda = \frac{c}{f}$

let $f = 50$ kHz,

$$\lambda = \frac{3 \times 10^8}{50} = 6 \times 10^6 \text{ m}$$

KCL/KVL applicable when circuit size < wavelength

for $f = 10$ GHz $= 10 \times 10^9$ Hz

$$\lambda = \frac{c}{f} = 3 \times 10^{-2} \text{ m} = 0.03 \text{ m}$$

for any circuit with length (x) greater than 3cm, KCL and KVL will not apply

Lecture: 2

8/01/25

→ Gradient operator
→ Level surface

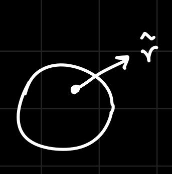
Directional derivative : $\frac{d\phi}{dr} = (\bar{\nabla}\phi)\cdot\hat{r}$

$= |\bar{\nabla}\phi| \cos\theta$

$\left|\frac{d\phi}{dr}\right|_{max} = |\bar{\nabla}\phi|$

→ for a level surface, directional derivative is along the surface normal

eg. for a sphere, the direction $\hat{r}$ at a specific point will be outwards from that point



$\bar{r} = x\hat{i} + y\hat{j} + z\hat{z}$
$d\bar{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$\phi(x,y,z)$: func of x,y,z

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

Potential

1 dimension : $\bar{E} = -\frac{d\Phi}{dx}\hat{x}$ (→ electrostatic potential)

3 dimension : $\bar{E} = -\left(\frac{\partial\Phi}{\partial x}\hat{x} + \frac{\partial\Phi}{\partial y}\hat{y} + \frac{\partial\Phi}{\partial z}\hat{z}\right)$

Cylindrical Coordinates

ρ, ϕ, z

differential length
differential area
differential solid angle

Spherical coordinates

$0 \le r < \infty \qquad 0 \le \theta \le \pi \qquad 0 \le \phi \le 2\pi$

differential length
differential area
differential $\underbrace{\text{solid angle}}_{\frac{\text{length of arc}}{\text{area}}} = \frac{dS}{r^2} = \sin\theta\, d\theta\, d\phi$

Line Integral

WORK : $\int_A^B F\, dl$ → for any force
Electrostatic potential : $\Phi_{AB} = -\int_A^B E\, dl$
} path dependent

- Note: some are path independent

assume : $\bar{V} = \bar{\nabla}u$

$$\int_A^B V\cdot dl = \int_A^B (\bar{\nabla}u)\,dl = \int_A^B du = \underbrace{U_B - U_A}$$

here, path independent ← only depends on end points

- A field which can be written as the gradient of a scalar is called conservative field

- note: gravitational (→ represented as gradient of gravitational potential) and electric fields (→ represented as gradient of electrostatic potential) are conservative whereas magnetic field cannot be represented as the gradient of a scalar.

- for closed loop → $\oint_A^B \bar{V}\cdot dl = \int_A^B du + \int_B^A du$

$= U_B - U_A + U_A - U_B = 0$

so, work done by a conservative field in a closed loop is zero.

i.e. closed loop line integral = 0

FLUX

- open surface : $\int_{S_{closed}} V\cdot ds$
- closed surface : $\oint_{S_{open}} V\cdot ds$

$\oint_{S_{closed}} V\cdot ds = \lim$

Divergence operator

$$\bar{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

Gradient of {
dot product of vector ≡ Divergence operator
Cross product of vector ≡ CURL
scalar ≡ Conservative field
}

if $\bar{\nabla}\times\bar{A} = 0$ it does not necessarily mean they are parallel because you cannot practically determine direction of $\frac{\partial}{\partial x} / \frac{\partial}{\partial y} / \frac{\partial}{\partial z} / \bar{\nabla}$ } Same goes for dot product as well

divergence for a very small surface $= \frac{\text{total outward flux}}{\text{volume}}$

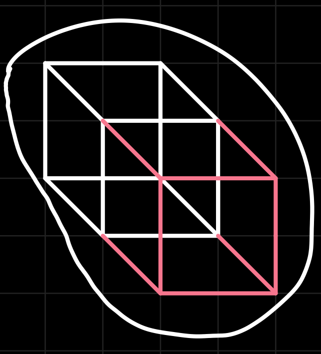
- Lecture 3

* Gauss' Divergence Theorem

divide the volume into tiny boxes and calculate the outward flux and add them up

$$\int_{V_{big}} (\bar{\nabla}\bar{v})\, d\bar{v} = \sum_{\substack{\text{all tiny} \\ \text{parallelpiped}}} \lim_{dv \to 0} (\bar{\nabla}\bar{v})\, dv$$

$$= \sum \oint_{\substack{\text{small} \\ \text{volume}}} V \cdot ds$$



outward flux from all the shared surfaces cancel out

so, final flux $= \sum_{\substack{\text{unshared} \\ \text{surfaces} \\ \text{coinciding} \\ \text{with the} \\ \text{boundary of} \\ \text{the big volume}}} \bar{V} \cdot d\bar{S}$

closed surface integral

$$= \oint_{\substack{\text{entire} \\ \text{surface}}} \bar{V} \cdot d\bar{S}$$

• LAPLACIAN OPERATOR ($\bar{\nabla}^2$)

can operate on both, scalar and vector

$$\bar{\nabla}(\bar{\nabla}U) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}\right)$$

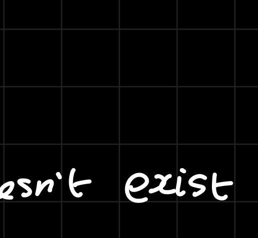
$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$(\bar{\nabla}^2 U) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U =$$

* EARNSHAW'S THEOREM

- A scalar field $\phi(x,y)$ that has $\underbrace{\nabla^2}_{\text{laplacian}}\phi = 0$, cannot have a local min/max in that region.

eg: $\phi = x^2 + y^2$
$\nabla\phi = 2x + 2y$
$\nabla^2\phi = 4$
non-zero
has a local min ✓



local min at $(0,0,0)$

also works with gravitational field

if $\bar{\nabla} \cdot \bar{E} = 0$
→ $\bar{\nabla}(\bar{\nabla}\phi) = 0$
→ $\bar{\nabla}^2\phi = 0$
then local min/max doesn't exist

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

if $D < 0$: saddle point

$\nabla f = f_x + f_y$
$\nabla^2 f = f_{xx} + f_{yy}$
if $\nabla^2 f = 0 \rightarrow f_{xx} + f_{yy} = 0$
and so, $D = \underbrace{-\underbrace{f_{xy}^2}_{+ve}}_{-ve \text{ always}}$

so, $D < 0$
↳ saddle point
↳ neither local min nor local max

Simplest Saddle Point

$z = x^2 - y^2$
OR
(Hyperbolic Paraboloid) $z - \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$

not a stable equilibrium

how can you stabilize a football on the saddle point?
↳ constantly rotate it
i.e. changing the electric field
≡ Paul Trap / Ion trapping in quantum computers

Earnshaw's theorem prevents stability on the saddle point

* CURL operator

$$\underbrace{\bar{\nabla} \times \bar{A}}_{} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

cannot say $\bar{\nabla}$ is $\parallel \bar{A}$
because operator does not has a direction

defn of curl: line integral along an infinitesimal loop
↓
loop can be in any direction

curl of a vector field ≡ closed line integral per unit area

* STOKE'S THEOREM (finite loop)

$$\int (\bar{\nabla} \times \bar{A}) \cdot d\bar{S} = \oint \bar{A} \cdot dl$$

divide the big loop into infinitesimally small loops

• MOBIUS STRIP

cannot apply stoke's Theorem

• Irrotational Field

↳ curl : $\bar{\nabla} \times F = 0$
↳ conservative field
↳ can be written as a gradient of scalar
$\bar{F} = \bar{\nabla}u$

eg: electrostatic field ✓
not electrodynamic field ✗

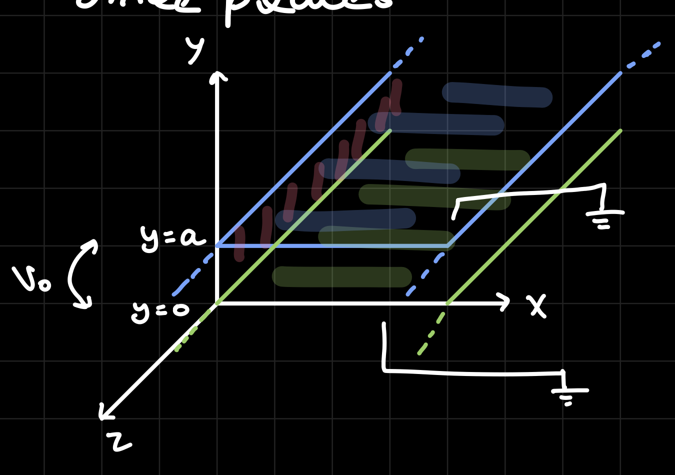
• SOLENOIDAL FIELD

↳ divergence : $\bar{\nabla} \cdot F = 0$
↳ solenoidal field over volume V doesn't have any source/sink in that volume
↳ $\bar{\nabla} \cdot (\bar{\nabla} \times A) = 0$
↳ A solenoidal field can be written as $\bar{\nabla} \times \bar{A}$, $\bar{A}$ ≡ vector potential

eg: magnetic field is solenoidal field
electrostatic field is solenoidal only when there is no charge

Lecture 7 : 27/01/25

Three plates



∞ along z-axis and positive x-axis

$\nabla^2 \phi = 0$

potential V_0 applied to the ≡ plate

Boundary conditions:

① $\phi = 0$ at $y = 0$
② $\phi = 0$ at $y = a$
③ $\phi = V_0$ at $x = 0$
④ $\phi = 0$ as $x \to \infty$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 \phi}{\partial z^2}}}_{} \xrightarrow{} 0 = 0$$

because it is ∞ along z-axis

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

we get the 2d equivalent of the equation

method of seperation of variables:

$$\phi(x,y) = X(x)\,Y(y)$$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

divided by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

Both X and Y have to be constants so as to satisfy this eqn for all points of X and Y

– Madhav 2025 :thumbs up:

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{C1} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{C2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

$$\frac{\partial^2 X}{\partial x^2} - k^2 X = 0 \qquad \frac{\partial^2 Y}{\partial y^2} + k^2 Y = 0$$

↳ Eigenvalue eqn ↙ $\hat{O} f(x) = c f(x)$

operator : $\frac{\partial^2}{\partial [x/y]^2}$

eigenvalue : $\pm k^2$

operand : X/Y

how to solve these DE?
assume an auxilliary equation

let $X = e^{mx}$

$$\frac{\partial^2}{\partial x^2}(e^{mx}) - k^2(e^{mx}) = 0$$

$$m^2 - k^2 = 0$$
$$m = \pm k$$

So, $X = Ae^{-kx} + Be^{kx}$

let $Y = e^{my}$

$$m^2 + k^2 = 0$$
$$m = \pm ik$$

So, $Y = Pe^{ikx} + Qe^{-ikx}$
$= P(\cos ky + i\sin ky) + Q(\cos ky - i\sin ky)$
$= C\sin(ky) + D\cos(ky)$

where $C = (P-Q)\times 2$ and $D = (P+Q)\times 2$

So, $X = Ae^{-kx} + Be^{kx}$ & $Y = C\sin(ky) + D\cos(ky)$

but due to ④ $\phi = 0$ as $x \to \infty$,

B must be zero $\forall x$

So, $X = Ae^{-kx}$

and due to ① $\phi = 0$ for $y = 0$

$$C(\sin 0) + D(\cos 0) = 0$$
$$0 + D = 0$$
So, $D = 0 \quad \forall y$

So, $Y = C\sin(ky)$

$$\phi = XY = ACe^{-kx}\sin(ky)$$

let $AC = G$

So, # unknown arbitrary constants = 2

$$\phi = Ge^{-kx}\sin(ky)$$

at ② $\phi = 0$ at $y = a$

$$Ge^{-kx}\sin(ka) = 0$$
$$ka = n\pi \ : \ n = 1, 2, \dots$$
$$k = \frac{n\pi}{a}$$

So → $Y = C\sin\left(\frac{n\pi}{a}y\right)$

$X = Ae^{-\frac{n\pi}{a}x}$

$$\phi = Ge^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right) \ : \ n = 1, 2, 3 \dots$$

let $\phi = \phi_1$ for $n = 1$
⋮
$\phi = \phi_n$ for $n = n$

and so, $\phi = \sum_{n=1}^{\infty} \phi_n$

http server implementation in C

network sockets
client / serve
file descriptor
bind
accept
listen

$$\phi_n = G_n e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)$$

due to ③ $\phi = V_0$ at $x = 0$

$$V_0 = \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right)dy = \sum_{n=1}^{\infty}\int_0^a G_n \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy$$

in this summation there will be only one term where RHS is non-zero and it is m = n

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) = \int_0^a G_m \sin\left(\frac{m\pi y}{a}\right)dy$$

$$= G_m \int_0^a \left(1 - \underbrace{\cos\left(\frac{2m\pi y}{a}\right)}_{\int \text{periodic} = 0}\right)dy$$

$$= G_m \cdot \frac{a}{2}$$

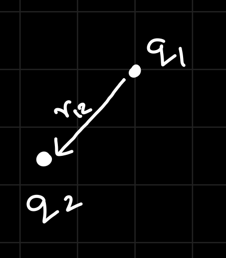
$$G_m \frac{a}{2} = V_0 \int_0^a \sin\left(\frac{m\pi y}{a}\right)dy$$
$$= V_0 \frac{a}{m\pi}\left[-\cos\left(\frac{m\pi y}{a}\right)\right]_0^a$$
$$= \frac{V_0 a}{m\pi}\left(1 - \cos(m\pi)\right)$$
$$= \frac{V_0 a}{m\pi}\left(1 - (-1)^m\right)$$

$$G_m = \begin{cases} 0 & : m \equiv \text{even} \\ \frac{4V_0}{m\pi} & : m \equiv \text{odd} \end{cases}$$

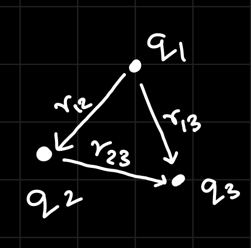
$$\phi = \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{4V_0}{n\pi} e^{\left(-\frac{n\pi x}{a}\right)} \sin\left(\frac{n\pi y}{a}\right)$$

bringing charges one at a time

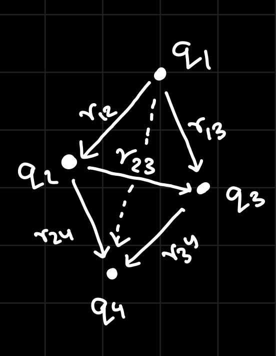
q_1 — $W_1 = 0$ work done because no elec field



$$W_2 = \frac{1}{4\pi\varepsilon_0} q_2\left(\frac{q_1}{r_{12}}\right)$$



$$W_3 = \frac{1}{4\pi\varepsilon_0} q_3\left(\frac{q_2}{r_{23}} + \frac{q_1}{r_{13}}\right)$$



$$W_4 = \frac{1}{4\pi\varepsilon_0} q_4\left(\frac{q_3}{r_3 r_4} + \frac{q_2}{r_2 r_4} + \frac{q_1}{r_1 r_4}\right)$$

Total work done $= W = W_1 + W_2 + W_3 + W_4$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i \sum_{j<i} \frac{q_j}{r_{ij}}$$

r_{ij} → magnitude (not vector)

$$= \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N} q_i \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{q_i q_j}{r_{ij}}$$

eg ⇒ $W_3 = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0} q_3\left\{\frac{q_2}{r_{23}} + \frac{q_1}{r_{13}}\right\} + \left\{\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right\}\right)$ took double deliberately

$= \frac{1}{2}\left(2kq_3\left\{\frac{q_2}{r_{23}} + \frac{q_1}{r_{13}}\right\}\right)$

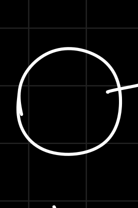
same

$$= \frac{1}{2}\sum_{i=1}^{N} q_i \left\{\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}\right\}$$

Total work done by N individual charges $= \boxed{\frac{1}{2}\sum_{i=1}^{N} q_i V(r_i)}$ — ①

$V(r_i)$ → potential at the position of i^{th} charge due to all other charges

What about a surface?



$q_i = \rho\, dv$

$$dW = \frac{1}{2}\rho V dv\ dV$$

The energy of a continuous charge distribution

$$\boxed{W = \frac{1}{2}\int \rho V dV} \quad \text{— ②}$$

Vector Property

$$\bar{\nabla}(f\bar{A}) = f(\bar{\nabla}\cdot\bar{A}) + \bar{A}\cdot(\bar{\nabla}f)$$

$$\int \bar{\nabla}(f\bar{A})\, dv = \int f(\bar{\nabla}\cdot\bar{A})\, dv + \int \bar{A}(\bar{\nabla}f)\, dv$$

gauss's law

$$\oint f\cdot\bar{A}\, d\bar{s} - \int \bar{A}\cdot(\bar{\nabla}f)\, dv = \int f(\bar{\nabla}\cdot\bar{A})\, dv$$ ③

divergence of a scalar and vector field

scalar and vector multiplication yields vector then divergence of that results in a scalar

Note: $\rho = \bar{\nabla}E\cdot\varepsilon_0$

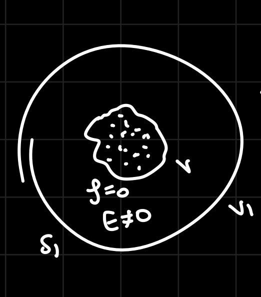
on ① → $W = \frac{1}{2}\int \rho V dv$

$$W = \frac{\varepsilon_0}{2}\int(\bar{\nabla}E) V dv$$

comparing with RHS of ③

$\bar{A} = \bar{E}$ and $V = f$

$$\text{So, } W = \frac{\varepsilon_0}{2}\left[\oint_{S_1} V\bar{E}\, d\bar{s} - \int_{V_1} \bar{E}\cdot(\bar{\nabla}V)\, dv\right]$$



→ big region
finite surface/volume

we know $\bar{\nabla}V = -\bar{E}$

$$\text{So, } W = \frac{\varepsilon_0}{2}\left[\oint_{S_1} V\bar{E}\, d\bar{s} + \int_{V_1} |\bar{E}|^2 dv\right] \quad \text{— ④}$$

surface is very large
↳ r is very large
↳ r^2 grows very fast
↳ $\frac{1}{r^2} \to 0$ so, $\bar{E} \to 0$
↳ also $\frac{1}{r}$ → very small so, $V \to 0$

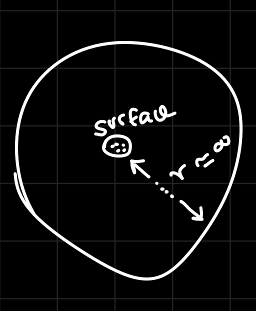
Note: ④ is the general form

but if you take the distance between the charge as very big, the surface integral tends → 0
↳ converges

So,

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} |E|^2 dv \ \to \text{converges?}$$

further you are from the surface, lesser this integral becomes



$$\boxed{W = \frac{1}{2}\varepsilon_0 \int |E|^2 dv}$$

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

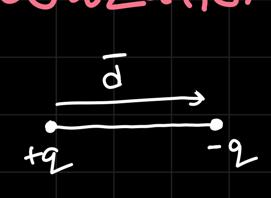
$\varepsilon_{r\,\text{water}} = 80$

We have written Gauss's Law for vacuum.

but what about a specific medium? Next Class →

→ CHAPTER 4: Polarization

Dipole



dipole moment → $\bar{p} = q\bar{d}$

linear materials: where $\bar{p} \propto \bar{E}$

$\bar{p} = \alpha\bar{E}$
↳ atomic polarizability

at equilibrium → $\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 v$ ← volume of the atom

a: radius of the atom

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

polarizability matrix or polarizability tensor

So, $p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$

$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$

$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$

for isotropic elements,

diagonal α_s are α and rest are zero

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$\bar{p} = \hat{x}p_x + \hat{y}p_y + \hat{z}p_z$

$\bar{p} = \hat{x}(\alpha E_x) + \hat{y}(\alpha E_y) + \hat{z}(\alpha E_z)$

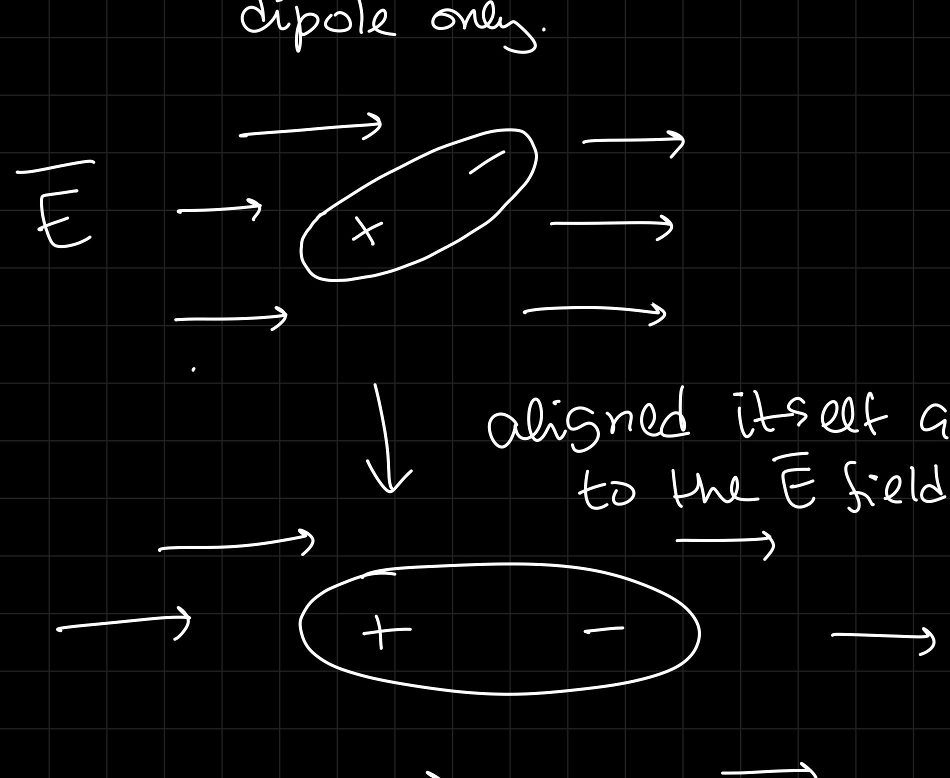
$\bar{p} = \alpha(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z)$

$\bar{p} = \alpha\bar{E}$

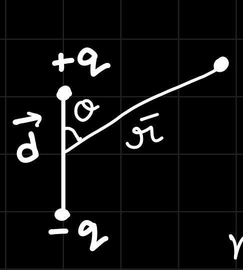
```
# include <stdio.h>
int main(int argc, char** argv){
    printf("9EM4 4V<K4");
    return 0;
}
```

constant thermal fluctuation throws dipole out of equilibrium

In Induced dipole moment, Electric field is mostly in the direction of the dipole only.



* LECTURE 9

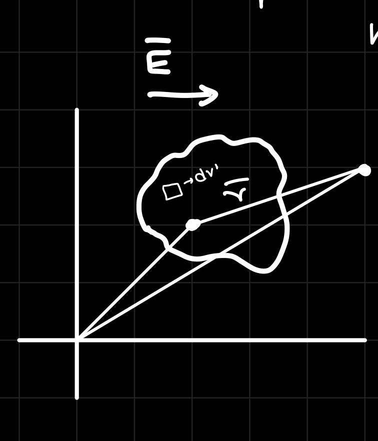


$\bar{p} = q\bar{d}$ $\quad V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \left(\frac{p}{4\pi\varepsilon_0} \cdot \frac{\hat{r}}{r^2}\right)$

net dipole moment due to induced and inherent dipole moment in the direction of electric field.

Instead of dipole moment for individual particles, we express dipole moment as polarization dipole moment per volume for a surface.

$\vec{P} \equiv$ polarization dipole moment per unit volume



$f(x-x', y-y', z-z')$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial(x-x')} \cdot \frac{\partial(x-x')}{\partial x}$$

$x \to$ const
$x' \to$ variable

$$= \frac{\partial f}{\partial(x-x')}$$

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial(x-x')} \cdot \frac{\partial(x-x')}{\partial x'}$$

$$= (-1)\frac{\partial f}{\partial(x-x')}$$

So, $\bar{\nabla}' f = -\bar{\nabla} f$

$$\bar{\nabla}\left(\frac{1}{r}\right) = -\bar{\nabla}'\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$$

* remember to use the vector notation ($\rightarrow$) for quiz

$\vec{p} = \vec{P}dv'$

$$dV = \frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{P}\cdot\hat{r}}{r^2}\right)dv'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\vec{P}\cdot\hat{r}}{r^2}\,dv'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{P}\cdot\bar{\nabla}\left(\frac{1}{r}\right)dv'$$

remember last lecture $\rightarrow$

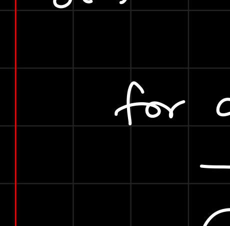
$$\int \vec{A}(\bar{\nabla} f)dv = \underbrace{\oint f\bar{A}\,d\bar{s}}_{\text{surface integral}} - \int f(\bar{\nabla}\bar{A})dv$$

here $\bar{P} = \bar{A}$ and $f = \frac{1}{r}$ } magnitude

$$V = \frac{1}{4\pi\varepsilon_0}\left[\oint_{S'} \frac{\bar{P}\cdot d\bar{s}}{r} - \int \frac{1}{r}(\bar{\nabla}\bar{P}(r'))\,dv'\right]$$

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'} \frac{\bar{P}\cdot d\bar{s}}{r} - \int_{V'} \frac{1}{r}(\bar{\nabla}\bar{P}(r'))\,dv'$$

Potential due to a surface charge density:

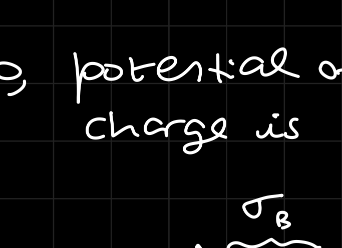


$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_s\,ds'}{r}$$

Potential due to a volume charge density

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\,dv'}{r}$$

for a material kept in an electric field $\rightarrow$



We are assuming a uniform polarization vector

So, potential of the surface charge is $\sigma_B = \bar{P}\cdot\hat{n} = \vec{P}\cos\theta$

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'} \frac{\overbrace{(\bar{P}\cdot\hat{n})}^{\sigma_B}ds'}{r} + \int_{V'} \frac{1}{r}\overbrace{(-\bar{\nabla}\bar{P}(r'))}^{\rho_B}\,dv'$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\oint \frac{\sigma_b}{r}ds' + \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_B\,dv'}{r}$$

[where $\underbrace{\sigma_B = \bar{P}\cdot\hat{n}}_{\text{total surface charge?}}$ and $\underbrace{\rho_B = -\bar{\nabla}\cdot\bar{P}}_{\text{total volume charge?}}$]

B: bound in σ_B, ρ_B

$$Q_{\substack{\text{volume}\\ \text{(bound)}}} = \int \rho_B(r)dv$$

$$= \int(-\bar{\nabla}\bar{P})dv$$

$$= -\oint \bar{P}\cdot d\bar{s}$$

$$= -\oint(\bar{P}\cdot\hat{n})\,d\bar{s}$$

$$= -\oint \sigma_B\,ds$$

$$= -Q_{\substack{\text{surface}\\ \text{(bounded)}}}$$

So, total surface charge is equal to negative of total volume charge.

irrespective of the type of polarization vector, whether uniform or not, the total surface and total vol charge add up to zero
Note: We are considering bounded charge
↓
none of them are mobile
They are tied to the molecules and molecules are tied to their posn.
They are bound to the material.

Remember $\rightarrow$ $\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$

$$\bar{\nabla}\bar{E} = \frac{\rho_{free} + \rho_{bound}}{\varepsilon_0}$$

$$\bar{\nabla}(\varepsilon_0 E) = \rho_{free} - \bar{\nabla}\cdot\bar{P}$$

$$\rho_{free} = \bar{\nabla}(P + \varepsilon_0 E)$$

$\chi_e \Rightarrow$ Electric susceptibility

easier to polarize if chi/zai (χ_e) is higher

$$\bar{P} = \underbrace{\alpha}_{\substack{\text{atomic}\\ \text{polarizability}}}\bar{E} = \varepsilon_0\chi_e\bar{E}$$

So, $\rho_{free} = \bar{\nabla}(\varepsilon_0\chi_e\bar{E} + \varepsilon_0\bar{E})$

$$= \varepsilon_0(1+\chi_e)\bar{\nabla}\bar{E}$$

$$\bar{\nabla}E = \frac{\rho_{free}}{\varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}} = \frac{\rho_{free}}{\varepsilon_0\varepsilon_r} = \frac{\rho_{free}}{\varepsilon}$$

$\varepsilon_r = 1 + \chi_e$
for free space $\rightarrow$ $\chi_e = 0$
so $\varepsilon_r = 1$

$$\boxed{\bar{\nabla}\bar{E} = \frac{\rho_{free}}{\varepsilon}}$$

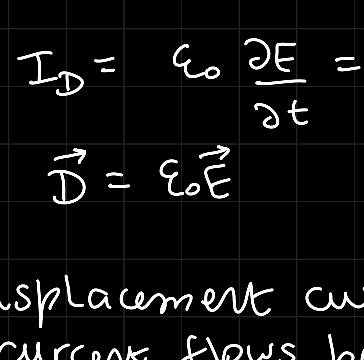
$\bar{D}$: displacement density

$\bar{I}_D$: displacement current

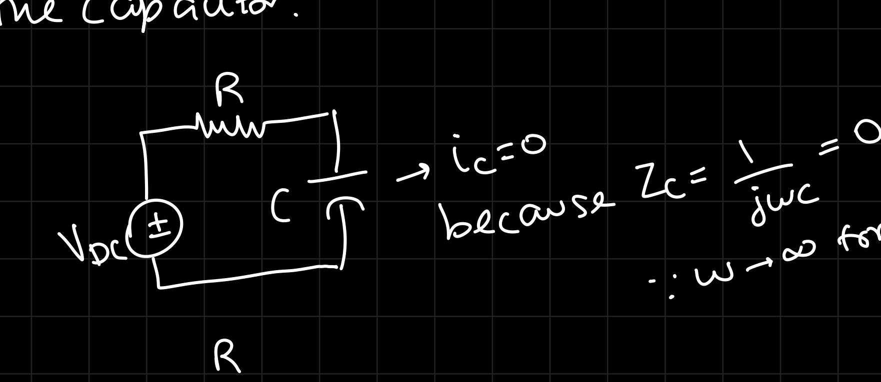
$$I_D = \varepsilon_0\frac{\partial E}{\partial t} = \frac{\partial D}{\partial t}$$

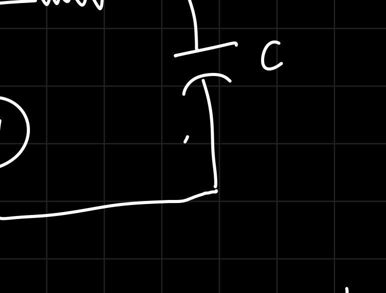
$$\vec{D} = \varepsilon_0\vec{E}$$

displacement current is the reason current flows between the plates of the capacitor.



$\rightarrow i_c = 0$ because $Z_c = \frac{1}{j\omega c} = 0$ $\because \omega \to \infty$ for DC





there is no movement of charge dipoles re-orienting/rotating themselves due to $\bar{E}$ and there is a change of charge density and so there is an illusion of charge movement and that's how current moves

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} , \quad \varepsilon_0 = 8.854\times10^{-12}$$

$$F = QE$$

surface
(continous)

$$E(r) = k\int \frac{1}{r^2}\hat{r}\,dq \qquad dq$$
$$\downarrow$$
$$E(r) = k\int \frac{\lambda}{r^2}\hat{r}\,dl' \qquad \lambda\,dl'$$
$$\downarrow$$
$$E(r) = k\int \frac{\sigma}{r^2}\hat{r}\,da' \qquad \sigma\,da'$$
$$\downarrow$$
$$E(r) = k\int \frac{\rho}{r^2}\hat{r}\,d\tau' \qquad \rho\,d\tau'$$

$$E(r) = \frac{kq}{r^2}\hat{r}$$ single pt. charge

Flux through a surface →

$$\oint_S E\cdot da = \frac{1}{\varepsilon_0} Q_{enclosed}$$ → Gauss's Law

$$\boxed{\bar{\nabla}\bar{E} = \frac{\rho}{\varepsilon_0}}$$

Remember, Stoke's Theorem

$$\oint_C \vec{A}.\vec{dl} = \iint_S (\vec{\nabla}\times\vec{A})\,.\vec{dS}$$

$$\oint \bar{E}\cdot dl = 0 \quad \text{so,} \quad \boxed{\bar{\nabla}\times\bar{E} = 0}$$

$$\boxed{\bar{E} = -\nabla V}$$

we know $\bar{\nabla}\bar{E} = \frac{\rho}{\varepsilon_0}$ and this

So, $\boxed{\bar{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}}$ → Poisson's Equation

regions with no charge → $\rho = 0$

so there, $\boxed{\bar{\nabla}^2 V = 0}$ → Laplace Equation

$$V(r) = k\int \frac{\rho}{r}\,d\tau , \quad \bar{E}(r) = k\int \frac{\rho}{r^2}\hat{r}\,d\tau$$

$$\boxed{W = qV(r)}$$ Point charge

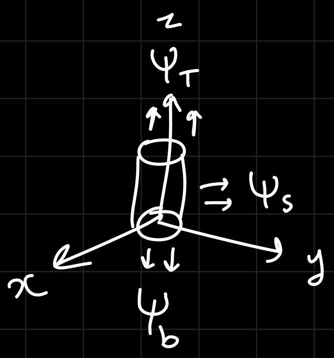
$$\boxed{W = \frac{1}{2}\int \rho V\,d\tau}$$

$$\boxed{W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\,d\tau}$$

Tut - 2

Q1) $\bar{G}(r) = 10e^{-2z}(\rho\hat{a}_\rho + \hat{a}_z)$

flux of $\bar{G}$ $= \psi = \oint \bar{G}\cdot\bar{ds}$



$\psi = \psi_t + \psi_b + \psi_s$

for the top →

$\bar{ds} = \rho\, d\rho\, d\phi\, \hat{a}_z$

$$\psi_t = \int \bar{G}\cdot\bar{ds} = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} 10e^{-2z}\rho\, d\rho\, d\phi$$

Cylindrical coords →

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

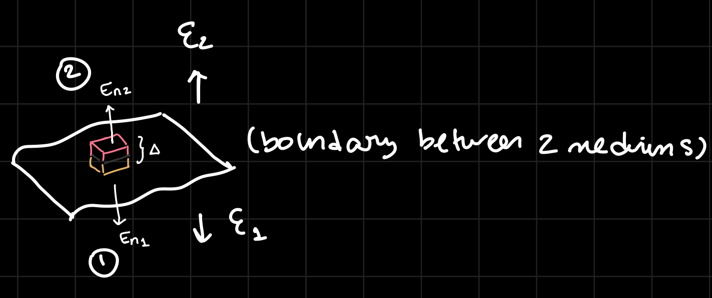
$$\bar{\nabla}\times\bar{A} = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Spherical coords →

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(V_\phi)$$

$$\bar{\nabla}\times A = \begin{vmatrix} \frac{\hat{a}_r}{r^2\sin\theta} & \frac{\hat{a}_\theta}{r\sin\theta} & \frac{\hat{a}_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

NOTE: Vec Field = Irrotational if its curl = 0 remember $\underbrace{\bar{\nabla}(\bar{E}\cdot\varepsilon)}_{\bar{D}} = \rho_{free}$ } gauss's law for diff materials



- in medium 1
- in medium 2

when $\Delta \to 0$, area and thereafter, flux $\to 0$

note: $D_{n1} = E_{n1}\varepsilon_1$
$D_{n2} = E_{n2}\varepsilon_2$

$\underbrace{\oint \bar{D}\,\bar{ds}}_{\text{flux}} = Q_{enclosed}$ } Gauss's Law

$\Rightarrow$ $D_{n2}A - D_{n1}A = \sigma A$ ↳ surface charge

$D_{n2} - D_{n1} = \sigma$

$$\boxed{\varepsilon_2 E_{n2} - \varepsilon_1 E_{n1} = \sigma}$$

if $\sigma = 0$, $E_{n2}\varepsilon_2 = E_{n1}\varepsilon_1$

or $\dfrac{E_{n2}}{E_{n1}} = \dfrac{\varepsilon_1}{\varepsilon_2}$

156 credits

if $\sigma \neq 0$, D_{n1} and D_{n2} are discontinuous
else they are continuous
but do note that potential is always continuous

NOITYN GNUTNM

$\oint E\cdot dl = 0$

$E_{t1}l - E_{t2}l = 0$

$$\boxed{E_{t1} = E_{t2}}$$

PEC ≡ perfectly electric conductor
↓
infinite conductivity

elec field inside PEC = 0

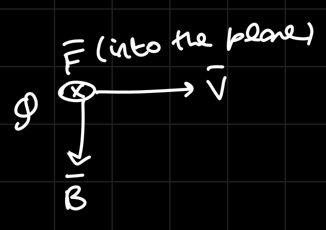
and any electric field tangential to the PEC's surface is zero

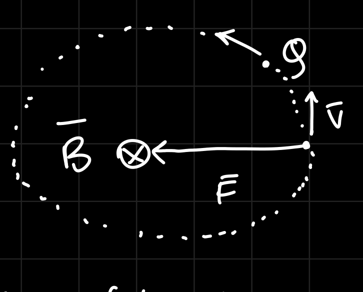
* Lorentz Force Law {magnetostatics}

$F_{mag} = q\cdot(\bar{v}\times\bar{B})$

$F_{total} = q\bar{E} + q(\bar{v}\times B)$

$F_{total} = q(\bar{E} + \bar{v}\times\bar{B})$

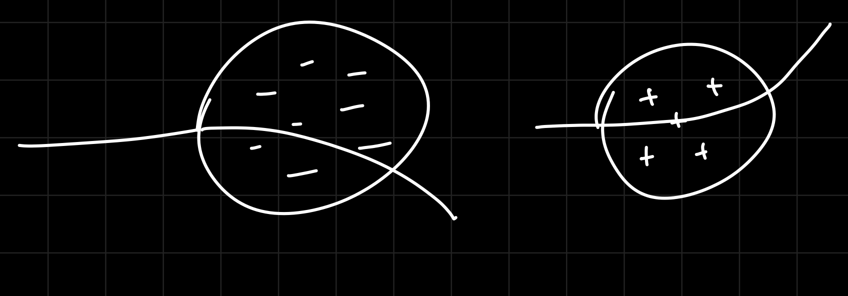




$\dfrac{mv^2}{r} = qvB$

$r = \dfrac{mv}{qB}$

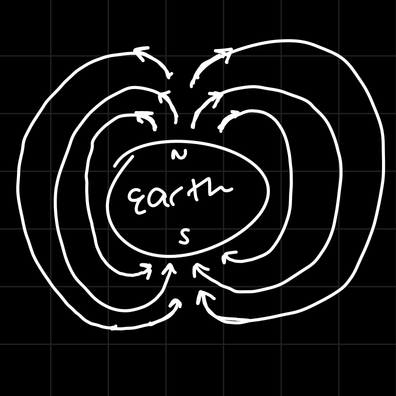
motion of the charge
≡ cyclotron motion
↳ used in mass spectrometer



magnetic field is not affected by the presence of a conductor unlike electric field

magnetic field is static in nature

magnetic bottle
↳ magnetic field is converging at the pole



collision with O_2 particles, pink aurora

green with N_2

Syllabus : lec-1 to Electrostatic boundary conditions

Reference : Griffith
↳ ch 1, 2, 4

* FORMULAS :

CH1

→ condensed form of maxwell's equations

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla}\times\bar{B} = \mu_0\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)$$

→ Electrostatic Potential :

1d → $\bar{E} = -\frac{d\phi_x}{dx}\hat{x}$

3d → $E = -\left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right)$

$= -\bar{\nabla}\phi$

→ cylindrical coords

$\bar{dl} = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$

$\bar{ds} = \rho\, d\phi\, dz\, a_\rho \,/\, dz\, d\rho\,\hat{a}_\phi \,/\, \rho\, d\rho\, d\phi\, a_{\hat{z}}$

$dv = \rho\, d\rho\, d\phi\, dz$

→ Spherical coords

$\bar{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi$

$\bar{ds} = r^2\sin\theta\, d\theta\, d\phi\,\hat{a}_r \,/$
$r\sin\theta\, d\phi\, dr\,\hat{a}_\theta \,/$
$r\,dr\,d\theta\,\hat{a}_\phi$

differential solid angle : $d\Omega = \frac{ds_r}{r^2} = \sin\theta\, d\theta\, d\phi$

$dV = r^2\sin\, dr\, d\phi\, d\theta$

→ LINE INTEGRAL

$W = \int_A^B \bar{F}\cdot\bar{dl}$ $\qquad \phi_{AB} = -\int_A^B \bar{E}\cdot\bar{dl}$

↳ path dependent

→ CONSERVATIVE FIELD
↳ A field that can be expressed as a gradient of scalar.
↳ path independent
↳ closed loop integral = 0
↳ eg: gravitational and electric field

→ Laplacian Operator

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

for a field ϕ if $\nabla^2\phi = 0$,
ϕ cannot have a local minima/maxima

* Tutorial 5

Q1) $\bar{A} = \rho\cos\phi\, \hat{a}_\rho + \sin\phi\, \hat{a}_\phi$

$\oint A \cdot dl$

for cylindrical → $dl = d\rho\, \hat{a}_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z$

$$\oint A \cdot dl = \int_{60^\circ}^{30^\circ} 2\sin\phi\, d\phi\, \hat{a}_\phi + \int_2^5 \rho\cos(30^\circ)\, d\rho\, \hat{a}_\rho +$$

$$\int_{30^\circ}^{60^\circ} 5\sin\phi\, d\phi\, \hat{a}_\phi + \int_5^2 \rho\cos(60^\circ)\, d\rho\, \hat{a}_\rho$$

$$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot (21) - 5\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

$$+ \frac{1}{2} \cdot \frac{1}{2}(-21)$$

$$= 1 - \sqrt{3} + \frac{21\sqrt{3}}{4} - \frac{5}{2} + \frac{5\sqrt{3}}{2} - \frac{21}{4}$$

$$= \frac{4 - 4\sqrt{3} + 21\sqrt{3} - 10 + 10\sqrt{3} - 21}{4}$$

$$= \frac{27\sqrt{3} - 27}{4} = \frac{27}{4}(\sqrt{3} - 1) \checkmark$$

Using STOKE'S THEOREM:

$$\oint \bar{A} \cdot \bar{dl} = \int_s (\nabla \times \bar{A})\, ds$$

$\bar{ds} = \rho\, d\rho\, d\phi$

$$\oint A \cdot dl = \iint (\bar{\nabla} \times A)\, \rho\, d\rho\, d\phi$$

$$\bar{\nabla} \times \bar{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$\frac{\partial}{\partial \rho}(\rho\sin\phi)$

$-\frac{\partial}{\partial \phi}(\rho\cos\phi)$

$\sin\phi + \rho\sin\phi$

$$= \frac{1}{\rho}\left(\hat{a}_\rho \left(0 - \frac{\partial}{\partial z}(\rho\sin\phi)\right)\right)$$

$$-\frac{1}{\rho}\left(\rho\,\hat{a}_\phi \left(0 - \frac{\partial}{\partial z}(\rho\cos\phi)\right)\right)$$

$$+ \frac{1}{\rho}\left(\hat{a}_z \left(\frac{\partial}{\partial \rho}(\rho\sin\phi) - \frac{\partial}{\partial \phi}(\rho\cos\phi)\right)\right)$$

$$= \frac{1}{\rho}\hat{a}_z\ (\sin\phi + \rho\sin\phi)$$

$$= \left(\frac{1}{\rho}\sin\phi + \sin\phi\right)\hat{a}_z$$

$$\oint \bar{A} \cdot \bar{dl} = \int_{30}^{60}\int_2^5 \sin\phi\,(\rho + 1)\, d\rho\, d\phi$$

$$= \int_{30}^{60} \left[\sin\phi\left(\frac{\rho^2}{2} + \rho\right)\right]_2^5 d\phi$$

$$= \int_{30}^{60} \sin\phi\left(\frac{27}{2}\right) d\phi$$

$$= -\frac{27}{2}\left[\cos\phi\right]_{30}^{60}$$

$$= -\frac{27}{2}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right]$$

$$= \frac{27}{4}(\sqrt{3} - 1) \checkmark$$

Chapter : 2

Electric Field :

① distinct points → $\bar{E} = k \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}$

② continuous charge → $\bar{E} = k \int \frac{1}{r^2} \hat{r} \, dq$

↳ along a line : $dq = \lambda \, dl$
on a surface : $dq = \sigma \, ds$
in a volume : $dq = \rho \, dv$

Divergence and Curl of Electrostatic fields

→ flux of $\bar{E}$ through surface S,

$$\Phi_E = \int_S E \cdot d\bar{s}$$

Gauss's Law : $\oint_S E \cdot ds = \frac{q_{enclosed}}{\varepsilon_0}$

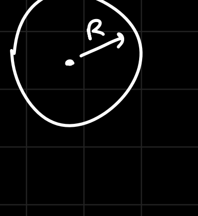
Divergence Theorem : $\oint_S E \cdot d\bar{s} = \int_V (\bar{\nabla} \cdot \bar{E}) \, dv$

note : $q = \int_V \rho \, dv$

so, $\int \frac{\rho \, dv}{\varepsilon_0} = \int \bar{\nabla} \bar{E} \, dv$

$$\bar{\nabla} E = \frac{\rho}{\varepsilon_0}$$

Ex 2.2 →



field = ?

$$\oint_S E \cdot ds = \frac{1}{\varepsilon_0} q_{enclosed} = \frac{q}{\varepsilon_0}$$

$$|E| \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$|E| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

$$E = k \frac{q}{r^2} \hat{r}$$

CURL of $\bar{E}$

$$\bar{E} \cdot d\bar{l} = 0$$

→ Stoke's theorem : $\oint \bar{E} \cdot d\bar{l} = \int_S (\nabla \times E) \, ds$

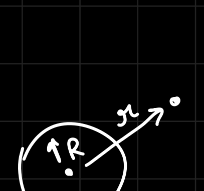
so, $\bar{\nabla} \times \bar{E} = 0$

Electric Potential

$$\bar{E} = -\bar{\nabla} \bar{V}$$

↳ electric potential

ex 2.6 →



outside ↴

$\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$ $\oint E \cdot ds = \frac{q_{enclosed}}{\varepsilon_0}$

Poisson's Equation

we know → $E = -\nabla V$, $\nabla E = \frac{\rho}{\varepsilon_0}$, $\nabla \times E = 0$

so → $\nabla^2 V = \frac{-\rho}{\varepsilon_0}$

LAPLACE'S EQN → $\nabla^2 V = 0$ in regions with no charge $\because \rho = 0$

$V = \frac{kq}{r}$ → single charge

$V = k \sum_{i=1}^{n} \frac{q_i}{r_i}$ → multiple charge

$V = k \int \frac{1}{r} dq$ → continuous distribution

$V = k \int \frac{1}{r} \rho \, dv$ → volume charge

WORK DONE : $W = \int_a^b F \cdot dl$

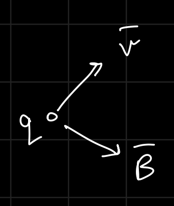
$W = 0$ for single charge

$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{12}} \right)$ for 2 charges

$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)$ for multiple charges

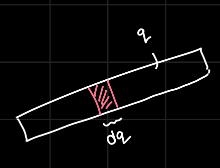
Continuous → $W = \frac{1}{2} \int \rho V \, dv$

Lorentz's Force



$$F = \underbrace{q\cdot(\bar{V}\times\bar{B})}_{\text{magnetic component}} + \underbrace{q\bar{E}}_{\text{electric field component (ignoring for now)}}$$

\# magnetic force on continuous charge



$d\bar{F}_{mag} = dq(\bar{V}\times\bar{B})$

$\downarrow$

$\bar{F}_{mag} = \int d\bar{F}_{mag}$

$$\bar{F}_{mag} = \int dq\,(\bar{v}\times\bar{B})$$

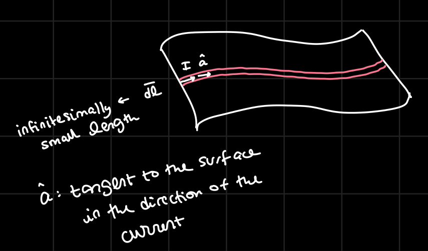
$$= \int \frac{dq}{dt}(d\bar{l}\times\bar{B})$$

$$= \boxed{\int I\cdot(d\bar{l}\times B)}$$

line charge $\rightarrow$ C/m

surface " $\rightarrow$ C/m^2

volume " $\rightarrow$ C/m^3



$\hat{k}$ = surface current density

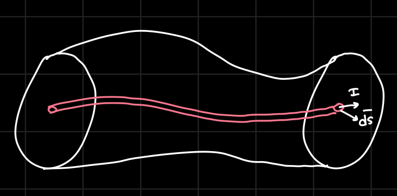
$$\boxed{\hat{k} = \frac{dI}{dl_\perp}\hat{a}}$$

unit: Ampere per metre

Am^{-1}

\# NOTE: There is no concept of line current density

VOLUME CURRENT DENSITY : $\bar{J}$



note: $\hat{n}$ ≡ direction of current

$\hat{n}$ and $\bar{ds}$ are $\perp$

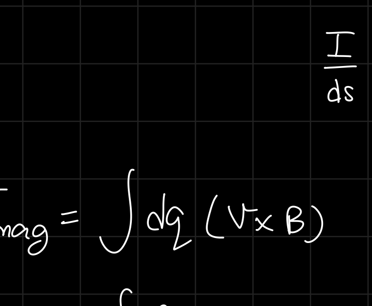
$$\boxed{\bar{J} = \frac{dI}{ds_\perp}\hat{n}}$$

unit: A/m^2

\# note: $\bar{J} = \sigma\bar{E}$ is the same $\bar{J}$

> Now we can relate current density with charge density

similar cylinder → only charges that are v dist far can cross



if ρ ≡ vol. charge density

now find I from ρ

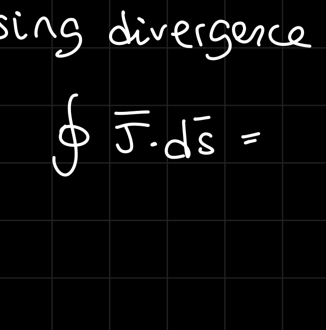
$$I = \rho\,\underbrace{\bar{ds}\cdot v}_{\text{volume}}$$

$$\frac{I}{ds} = \rho v \quad\rightarrow\quad \boxed{\bar{J} = \rho\bar{v}}$$

$$\bar{F}_{mag} = \int dq\,(v\times B)$$

$$= \int \rho\, d\tau\,(v\times B) = \int(\rho\bar{v}\times\bar{B})\,d\tau$$

$$\boxed{\bar{F}_{mag} = (\bar{J}\times\bar{B})\,d\tau}$$



$$\boxed{\oint \bar{J}\cdot\bar{ds} = -\frac{dQ}{dt}}$$

↳ law of conservation of charge

using divergence theorem

$$\oint \bar{J}\cdot\bar{ds} = \int(\nabla\cdot J)\,d\tau = -\frac{\partial}{\partial t}\int\rho\,d\tau$$

$$= -\int\frac{\partial\rho}{\partial t}d\tau$$

$$\boxed{\bar{\nabla}\bar{J} + \frac{\partial\rho}{\partial t} = 0} \rightarrow \text{eq}^n \text{ of continuity}$$

\# also,

$$\bar{J} = \rho\bar{v}$$

$$\bar{\nabla}(\rho\bar{v}) + \frac{\partial\rho}{\partial t} = 0$$

for small Electric field, we can assume v varies with $\bar{E}$ linearly $\rightarrow$ $\bar{v} = \mu\bar{E}$

$$\bar{J} = \rho\bar{v} = (\rho\mu)\bar{E} = \sigma\bar{E}$$

↳ conductivity

also, reciprocal of resistivity

$\boxed{\vec{J} = \sigma\vec{E}}$

OHM'S LAW

from eq^n of continuity,

$$\sigma(\nabla\bar{E}) = -\frac{\partial\rho}{\partial t} \quad\rightarrow\quad \sigma\frac{\rho}{\varepsilon_0} = -\frac{\partial\rho}{\partial t}$$

$$\int\frac{\partial\rho}{\rho} = \frac{-\sigma}{\varepsilon_0}\int\partial t$$

$$\ln\rho + c = \frac{-\sigma}{\varepsilon_0}t$$

at $t = 0$, $\rho = \rho_0$

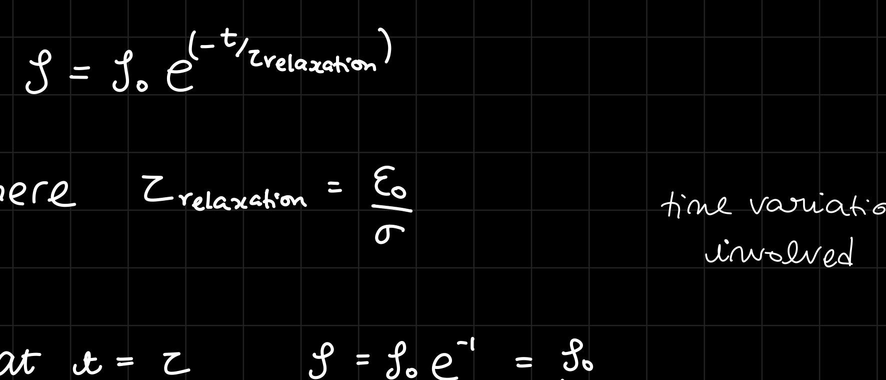
$$\ln\rho_0 = -c \quad\rightarrow\quad c = -\ln\rho_0$$

so, $$\ln\rho - \ln\rho_0 = \frac{-\sigma}{\varepsilon_0}t$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -\frac{\sigma}{\varepsilon_0}t$$

$$\boxed{\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon_0}t}} \rightarrow \text{decays with time}$$

decay factor $= \frac{\sigma}{\varepsilon_0}$



$$\rho = \rho_0 e^{(-t/\tau_{relaxation})}$$

where $\tau_{relaxation} = \frac{\varepsilon_0}{\sigma}$

time variation involved

at $t = \tau$, $\rho = \rho_0 e^{-1} = \frac{\rho_0}{e}$

significance of τ: time taken to reduce the ρ to $\frac{1}{e}$ times of ρ_0

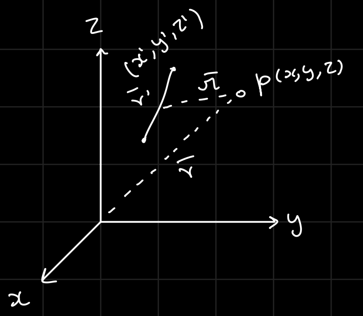
Magnetostatics

$$\bar{\nabla}\bar{J} = -\frac{\partial \rho}{\partial t}$$

let ρ = const with time
$\equiv$ steady current
so, $\frac{\partial \rho}{\partial t} = 0 \quad \rightarrow \quad \boxed{\bar{\nabla}\bar{J} = 0}$

* A point charge cannot give steady current
because $\rho \equiv$ not constant
because ρ is a function of space

→ **BIOT SAVART'S LAW**



$$\bar{B}(r) = \frac{\mu_0}{4\pi}\int \frac{I \cdot d\bar{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\bar{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dv'$$

(x',y',z')

volume integral

$$(\bar{\nabla}\cdot\bar{B}) = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\bar{J}(r') \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)$$

divergence wrt (x,y,z)

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}(\bar{\nabla}\times\bar{J}(r')) - \bar{J}(r')\left(\bar{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right.$$

note: $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla\cdot\left(\frac{1}{\mathscr{r}}\right)$

so, its curl is 0

also, J is dependent on $\mathscr{r}'/x',y',z'$
but the curl is wrt $\mathscr{r}/x,y,z$
so, that is zero as well

so, $\boxed{\bar{\nabla}\cdot\bar{B} = 0}$

$$\int(\nabla\cdot B)\,dv = \boxed{\oint \bar{B}\cdot d\bar{s} = 0}$$

Gauss's Law for magnetic field

Remember : This is only for steady current / pure magnetostatics

but it is true for magnetic field varying with time as well.

↑ MIDSEM SYLLABUS ↑

Non cartesian coordinates

→ <u>Spherical coords</u> → r, θ, ϕ

angle with z axis

projection's angle with x-axis

$dl_r = dr$, $dl_\theta = r\,d\theta$, $dl_\phi = r\sin\theta\, d\phi$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

Curl:

$$\begin{aligned}\nabla\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}.\end{aligned}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}.$$

→ <u>Cylindrical coords</u> → s, ϕ, z

$dl_s = ds$, $dl_\phi = s\,d\phi$, $dl_z = dz$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}.$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}.$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

Dirac Delta Function

let $\bar{v} = \frac{1}{r^2}\hat{r}$

$$\bar{\nabla}\cdot\bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = 0$$

reason → we used the divergence formula for spherical coordinates
↳ not cartesian.

sphere of radius : R ↖ S

$$\oint_S v\cdot da = \int \frac{1}{R^2}\hat{r} \cdot R^2 \sin\theta \, d\theta d\phi \, \hat{r}$$

$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$= [-\cos\theta]_0^{\pi} \cdot [\phi]_0^{2\pi}$$

$$= (1+1)(2\pi) = \underline{4\pi}$$

now, $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)$

quiz 2 - questions →

Q2) (a) $\int_{-1}^{+1} (r^2+2)\, \bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) dv$

$$= \int_{-1}^{+1} (r^2+2)\, 4\pi\delta^3(r)\, dv$$

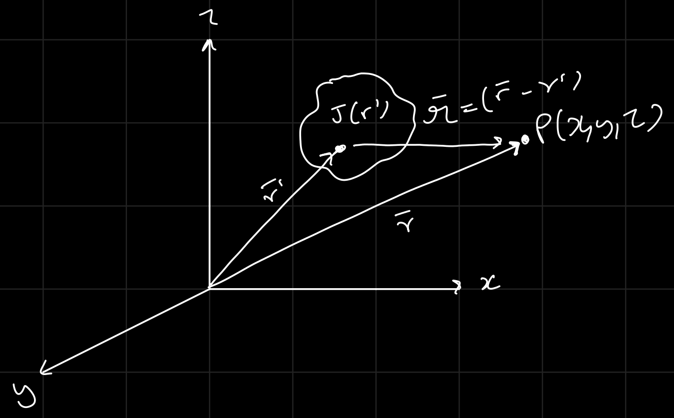
$$= (0+2)(4\pi) = \underline{8\pi}$$

(b) $\int_1^{\infty} (r^2+2)\, \bar{\nabla}\left(\frac{\hat{r}}{r^2}\right) dv$

$$= \int_1^{\infty} (r^2+2)\, \underbrace{4\pi\delta^3(r)}_{\text{always } = 0}\, dv = \underline{0}$$

always = 0
∵ $r=0$ not included in the integral

Lecture after midsem



$$J(\bar{r}') \rightarrow A/m^2 = \frac{dI}{da_\perp}$$

→ dependent on x', y', z'

$$\bar{B}(r) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\bar{\nabla} f = -\bar{\nabla}' f$$

note: assuming steady current

$$\bar{\nabla} \cdot \bar{B} = \frac{\mu_0}{4\pi} \int \nabla \left(\bar{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\mu_0}{4\pi} \int \underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} (\bar{\nabla} \times \bar{J}(r')}_{\text{zero because dir wrt r but J(r')}} - \bar{J} \left(\bar{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

$$\bar{\nabla} \cdot (\bar{P} \times \bar{Q}) = \bar{Q} (\bar{\nabla} \times \bar{P}) - \bar{P} (\bar{\nabla} \times \bar{Q})$$

and since $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla\left(\frac{1}{r}\right) \rightarrow$ curl $= 0$

So, $\boxed{\nabla \cdot B = 0}$

Curl of magnetic field

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

↳ magnetic vector potential
(need not be unique)

$\bar{A}_{new} = \bar{A} + \bar{\nabla}\theta$ ← gauge freedom

$\bar{\nabla} \times \bar{A}_{new} = \bar{\nabla} \times A + 0 = \bar{B}$ always

if $\bar{B}$ given, find the magnetic vector potential
or given, $\bar{J}$, find the magnetic field

since $B \propto \frac{1}{r^2} \rightarrow A \propto \frac{1}{\mathscr{r}}$ because $\bar{B} = \bar{\nabla} \times \bar{A}$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \left(\bar{J}(r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) dV'$$

we know $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla\left(\frac{1}{\mathscr{r}}\right)$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \underbrace{\left(\bar{J}(r') \times \bar{\nabla}\left(\frac{1}{\mathscr{r}}\right) \right)}_{(\bar{P} \times \bar{\nabla} f)} dv'$$

$$\bar{\nabla}\times(f\bar{P}) = f(\bar{\nabla}\times\bar{P}) - \bar{P}\times(\bar{\nabla}f)$$

$$\rightarrow \quad P\times(\bar{\nabla}f) = f(\bar{\nabla}\times\bar{P}) - \bar{\nabla}\times(f\bar{P})$$

here $\bar{\nabla}$ wrt $\mathscr{r}$ but $J(r')$

so, $f(\bar{\nabla}\times\bar{P}) = 0$

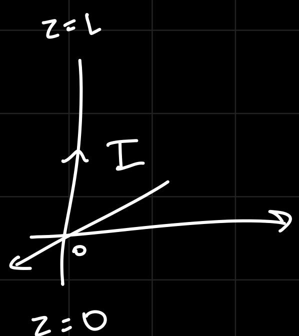
$$\bar{\nabla}\times\bar{A} = \frac{\mu_0}{4\pi}\int\left(\bar{J}(r')\times\bar{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right)dv'$$

$$\bar{\nabla}\times\bar{A} = \frac{\mu_0}{4\pi}\int\bar{\nabla}\times\left(\frac{\bar{J}(r')}{\mathscr{r}}\right)dv'$$

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi}\int\frac{\bar{J}(r')\,dv'}{\mathscr{r}}}$$

current carrying filamentary conductor

$\hat{k}$ $\uparrow$



$\bar{J}(r') = ?$

$$\bar{J} = \frac{dI}{da_\perp}$$

$$J = \bar{I}\,\delta(x)\,\delta(y)$$

$$\bar{A} = \frac{\mu_0}{4\pi} \iiint_{V'} \frac{\bar{J}(r')}{\mathscr{r}} \, dx' dy' dz'$$

$$\bar{J}(r') \times dx'dy' = I$$

for $x, y \to$ infinitesimally small

$$\bar{A} = \frac{\mu_0}{4\pi} \underbrace{\int_0^\infty \int_0^\infty \delta(x)\delta(y)\,dx\,dy}_{1} \cdot I\hat{z} \int_0^L \frac{dz}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{1/2}}$$

$\hookrightarrow \mathscr{r}$

$=$ function of x, y, z

curl$(E) = 0$ but

div$(E) \Rightarrow$ Gauss's Law

div$(B) = 0$ but

curl of $B \Rightarrow$ Ampere's Law

$$\bar{B} = \frac{\mu_0}{4\pi} \int \bar{J}(r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, dV'$$

$$\bar{\nabla} \times \bar{B} = ?$$

problem $\to$ cannot apply chain rule to $\nabla \times (\bar{P} \times \bar{\varphi})$

$$\bar{\nabla}\times(\bar{P}\times\bar{\varphi}) = \underbrace{(\bar{\varphi}\cdot\bar{\nabla})\bar{P}}_{\left(\varphi_x \frac{\partial P}{\partial y} + \varphi_y \frac{\partial P}{\partial y} + \varphi_z \frac{\partial P}{\partial z}\right)} - (\bar{P}\cdot\bar{\nabla})\varphi + \bar{P}(\bar{\nabla}\cdot\bar{\varphi}) - \bar{\varphi}(\bar{\nabla}\cdot\bar{P})$$

$\bar{P} = \bar{J}$ but since $\bar{J}$ depends on r', not r

so, 1st and last term $= 0$

non zero term:

$$\bar{\nabla}\times(\bar{P}\times\bar{\varphi}) = \bar{P}(\bar{\nabla}\cdot\bar{\varphi}) - (\bar{P}\cdot\bar{\nabla})\bar{\varphi}$$

$$\bar{\nabla}\times\bar{B} = \frac{\mu_0}{4\pi}\left[\int \bar{J}\left(\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\right) dv' + \int (\bar{J}\cdot\bar{\nabla}')\frac{\hat{r}}{r^2}\, dv'\right]$$

not minus
because $\bar{\nabla} f = -\bar{\nabla}' f$

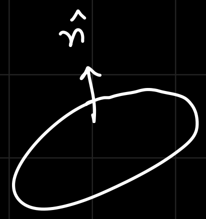
$$= \frac{\mu_0}{4\pi}\left[\int \bar{J}(\bar{r}')\cdot 4\pi\delta^3(\bar{r}-\bar{r}')\, dv + \int (\bar{J}\cdot\bar{\nabla})\frac{\hat{r}}{r^2}\, dv'\right]$$

$$= \frac{\mu_0}{4\pi}\times 4\pi\times J(\bar{r}) + \frac{\mu_0}{4\pi}\int (\bar{J}\cdot\bar{\nabla})\frac{\hat{r}}{r^2}\, dv'$$

$$= \mu_0 J(\bar{r}) + \frac{\mu_0}{4\pi}\int (\bar{J}\cdot\bar{\nabla})\frac{\hat{r}}{r^2}\, dv'$$

assuming that 2nd term is zero

$\bar{\nabla} \times \bar{B} = \mu_0 J(\bar{r})$ → differential form of Ampere's Law



$$\int (\bar{\nabla} \times \bar{B}) \cdot ds = \mu_0 \int \bar{J} \cdot d\bar{s}$$

$$\boxed{\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed}}$$

Integral form of Ampere's Law

Proving the previous assumption

steady current → $\bar{\nabla} \bar{J}(r') = 0$

so, $\bar{\nabla}' \bar{J}(r') = 0$ also

We need to prove = $\int (\bar{J}(r') \cdot \bar{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} dV$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mathscr{r}\hat{\mathscr{r}}}{\mathscr{r}^3} = \frac{\hat{x}(x-x') + \hat{y}(y-y') + \hat{z}(z-z')}{\mathscr{r}^3}$$

we need contribution of entire system but we will be checking for individual components

$$= \int_{V'} \left(\bar{J}(r') \cdot \bar{\nabla} \right) \frac{(x - x')}{r^3} \, dv'$$

$$\bar{A} \cdot (\bar{\nabla} f) = \bar{\nabla}' \cdot (f \bar{A}) - \underbrace{\bar{f} (\bar{\nabla}' \cdot \bar{A})}_{0 \text{ because } \bar{\nabla}' \cdot J = 0}$$

$$= \int_{V'} \bar{\nabla}' \cdot \left(\frac{J(r')(x - x')}{r^3} \right) dv'$$

$$= \oint_{S'} \frac{\bar{J}(r')(x - x')}{r^3} \, ds$$

since $J(r) = 0$ outside V'

this $\int$ holds for $V' + \Delta$ as well

$$\oint_{S' + \Delta} \underbrace{\bar{J}(r')\Big|_{surface}}_{\text{zero}} \frac{(x - x')}{r^3} \, ds'$$

$\rightarrow$ Assumption correct

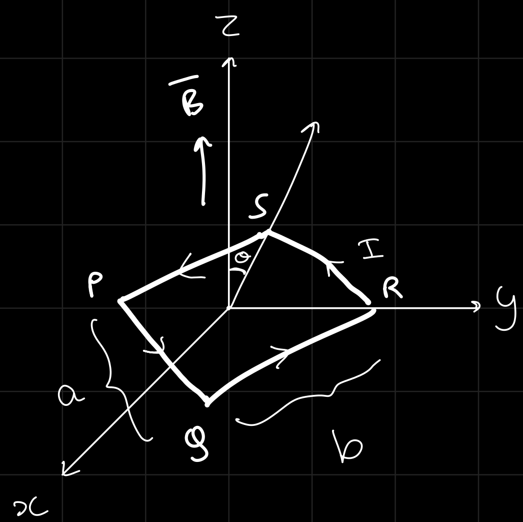
Magnetic field in a matter

note: $\bar{\nabla}\cdot\bar{B} = 0 \rightarrow \int(\nabla\cdot B)\,dV = \oint B\,ds = 0$

Gauss Law

Diamagnets
Paramagnets
Ferromagnets

assuming a rectangular closed loop



θ: angle of plane's normal with z-axis

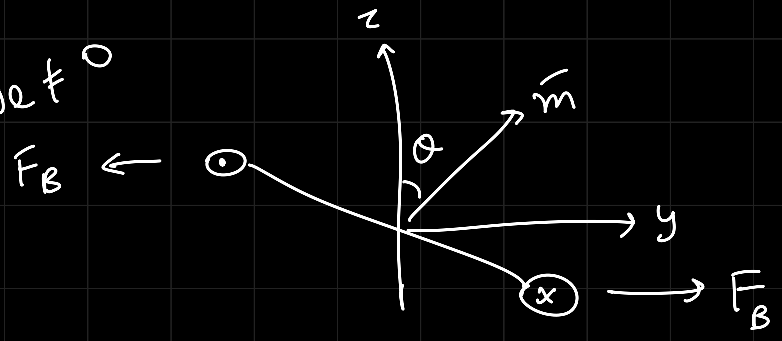
$\bar{F}_B = I(\bar{l} \times \bar{B})$

from lorentz Law

$\bar{F}_B = q(\bar{v} \times \bar{B})$

The PQ and RS sides' force will cancel out but the force due to the other two sides will make the rectangle rotate

net Force = 0
but net torque ≠ 0



at equilibrium
$\tau = 0 \Rightarrow \bar{m}$ aligns paralles with $\bar{B}$

$$\bar{\tau} = \bar{F}\cdot a\sin\theta\ \hat{x}$$
$$= IbBa\sin\theta\ \hat{x}$$
$$= IAB\sin\theta\ \hat{x}$$

area

$IA = \bar{m}$: magnetic moment

$$= |\bar{m}||\bar{B}|\sin\theta\ \hat{x} \Rightarrow \boxed{\tau = \bar{m}\times\bar{B}}$$

assuming the loop is kept in a uniform field

In case of dipoles →

{ constant thermal fluctuation throws dipole out of equilibrium }

Similar reason for "out of equilibrium" state in this case as well requirements:

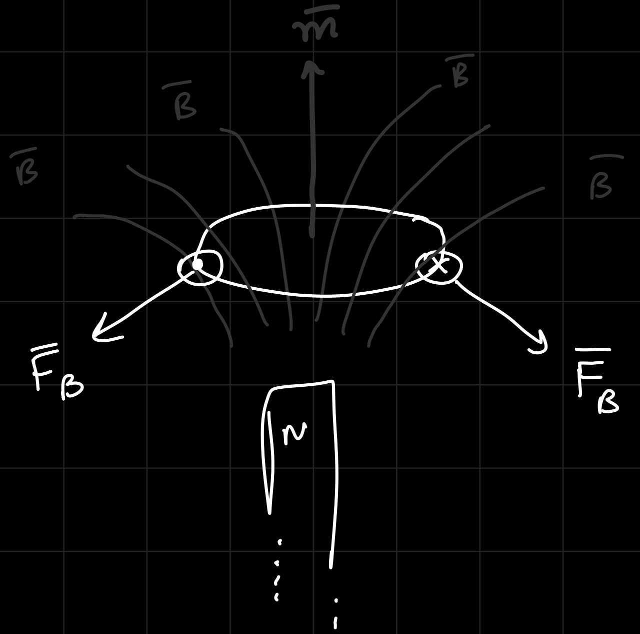
liquid (not gas)

low temperature

or else these small loops wont respond to the magnetic field

non-uniform magnetic field $\equiv$ $\bar{B}$ varying in space

eg:



Diamagnetic material

↳ tries to align $\bar{m}$ in the opposite direction to $\bar{B}$

Paramagnetism

↳ $\bar{m}$ and $\bar{B}$ align in the same direction

$ cannot use gas because of not enough density for it to exhibit enough magnetization

hence we go for the next lightest type of materials → liquid

magnetic susceptibility
↳ determines para/ferro/diamagnetism

liquid nitrogen → diamagnetic
doesn't hold

liquid oxygen → paramagnetic
sticks b/w poles because of very high magnetic susceptibility

aluminium → paramagnetic
but doesn't hold because gravitational pull is greater because of high mass (solid)

$\bar{E}$ arises due to q

$\bar{B}$ arises due to I

$\bar{B} = \bar{\nabla} \times \bar{A}$

↳ magnetic vector potential

$\bar{A}$ and $\bar{V}$ vary with $\frac{1}{r}$ but for dipole → $\frac{1}{r^2}$

↓ magnetic vec potential

↓ electrostatic potential

magnetic dipole moment

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times \hat{r}}{r^2}$$

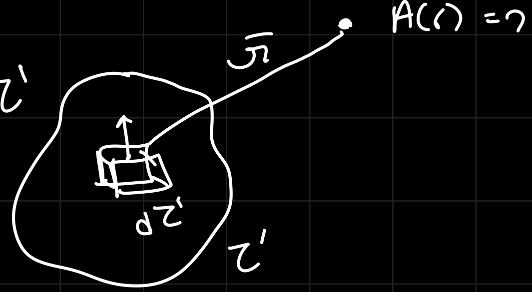
just like $\bar{P}$: polarization,

we have $\bar{M}$: magnetization

($\bar{m}$ / volume)

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int_V \frac{\bar{m} \times \hat{r}\, d\tau'}{r^2}$$ due to continuous material

$$= \frac{\mu_0}{4\pi} \int_V \left[\bar{m} \times \bar{\nabla}' \left(\frac{1}{r} \right) \right] d\tau'$$



$$(\bar{\nabla}' \times f\bar{p}) = f(\bar{\nabla} \times \bar{P}) - \bar{P} \times (\bar{\nabla}' f)$$

$$\Rightarrow \quad \bar{P} \times \bar{\nabla}' f = f(\bar{\nabla}' \times \bar{P}) - \bar{\nabla}(f\bar{P})$$

$\bar{P} = \bar{m}$, $f = \frac{1}{r}$

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathfrak{r}} (\bar{\nabla}' \times \bar{m}) - \bar{\nabla} \times \left(\frac{\bar{m}}{\mathfrak{r}} \right) \right] d\tau'$$

remember →

$$\bar{J}_b(\mathfrak{r}') = \bar{\nabla} \times \bar{m}(r')$$

↳ bound volume current density

Gauss's divergence theorem: $\int \bar{\nabla} \bar{P} \, dV = \oint P \cdot ds$

if $\bar{P} = \bar{V} \times \bar{C}$

where $\bar{V}$ varies with space and

$\bar{C}$ doesnt varies with space

then $\therefore$ $\bar{P}$ varies with space

$$\int_V \bar{\nabla} \cdot (\bar{V} \times \bar{C}) \, dV = \oint_S (\bar{V} \times \bar{C}) \, ds$$

$$\int \bar{C} \cdot (\bar{\nabla} \times \bar{V}) - \bar{V} \cdot \underbrace{(\bar{\nabla} \times \bar{C})}_{\text{zero because C doesnt vary with space}} dv = \oint \bar{C} \cdot (d\bar{S} \times \bar{V})$$

$$\int C \cdot (\nabla \times V) \, dv = - \oint \bar{C} \cdot (\bar{V} \times d\bar{S})$$

$$\int C \cdot (\bar{\nabla} \times V) \, dV = -C \oint_S (\bar{V} \times d\bar{S})$$

$$\int (\bar{\nabla} \times \bar{V})\, dV = -\oint (\bar{V} \times d\bar{s})$$

$$\frac{\mu_0}{4\pi} \int \frac{\bar{J}(r')\, d\tau'}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi} \int \left(\frac{\bar{\nabla} \times \bar{m}}{\mathscr{r}}\right) d\tau'$$

$$= -\frac{\mu_0}{4\pi} \oint \left(\frac{\bar{m}}{\mathscr{r}}\right) ds$$

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} (\bar{\nabla}' \times \bar{m}) - \bar{\nabla} \times \left(\frac{\bar{m}}{\mathscr{r}}\right)\right] d\tau'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\overbrace{J_b(\mathscr{r}')}^{\bar{\nabla}' \times \bar{m}(r')}\, d\tau'}{\mathscr{r}} + \frac{\mu_0}{4\pi} \oint \frac{\overbrace{K_b(r')}^{\bar{m} \times \hat{n}}}{\mathscr{r}} ds$$

volume current density

surface current density

$$\bar{A}(\mathscr{r}) = \frac{\mu_0}{4\pi} \int \frac{J_b(\mathscr{r}')\, d\tau'}{\mathscr{r}} + \frac{\mu_0}{4\pi} \oint \frac{K_b(\mathscr{r}')}{\mathscr{r}} ds$$

Lec (Revision) 17/03/25

Ampere's Law

$\bar{\nabla}\times\bar{B} = \mu_0 \bar{J}$ $\oint B \cdot dl = \mu_0 I_{enc}$

$\bar{m} = I\bar{A}$: magnetic moment

magnetic vector potential : $\bar{A}(r)$

from multiple expansion

due to a single magnetic dipole : $\bar{A}(r) = \frac{\mu_0}{4\pi} \frac{\bar{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$

$\bar{M}$: magnetic dipole moment per unit volume

for a continuous distribution :

$$\bar{A}(r) = \int \frac{\mu_0}{4\pi} \frac{\bar{m}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} dV'$$

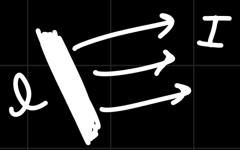
$$= \frac{\mu_0}{4\pi} \int \frac{J_b(\mathscr{r}')}{\mathscr{r}} dV' + \frac{\mu_0}{4\pi} \oint \frac{K_b(\mathscr{r}')}{\mathscr{r}} ds'$$

summation of and bound volume current density and bound surface current density where, $J_b(r) = \bar{\nabla} \times \bar{M}$, $K_b(r) = \bar{M} \times \hat{n}$

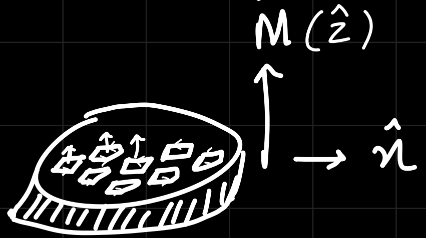
J_b: Am^{-2} ; K_b: Am^{-1}

$$K = \frac{I}{l_\perp}$$

$l_\perp$: length perpendicular to flow of current



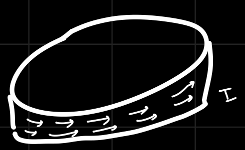
Uniform magnetized material



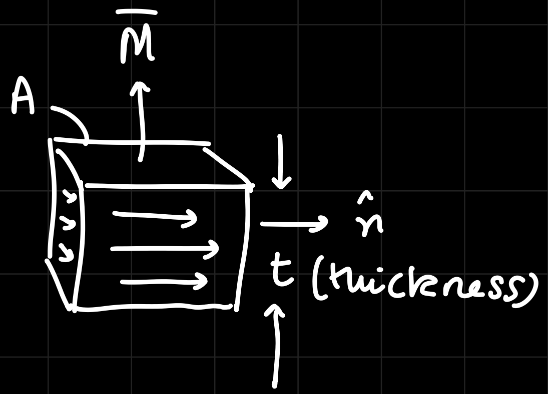
internal currents are cancelled out

$\bar{m} = I\bar{A}$

↓



called bound surface current like a ribbon wrapped around the material carrying the current



$\bar{m} = \bar{M}\bar{A}t$

$I\bar{A} = \bar{M}\bar{A}t$

$\bar{M} = \dfrac{\bar{I}}{t} = K_b$

So, $\boxed{\bar{K}_b = \bar{M} \times \hat{n}}$

because $\bar{m}, \hat{n}, \bar{I}$ are mutually $\perp$ and dir of K_b is = dir of current

<u>NOTE:</u> volume current density is zero for uniform magnetized material

NON-UNIFORM MAGNETIZATION

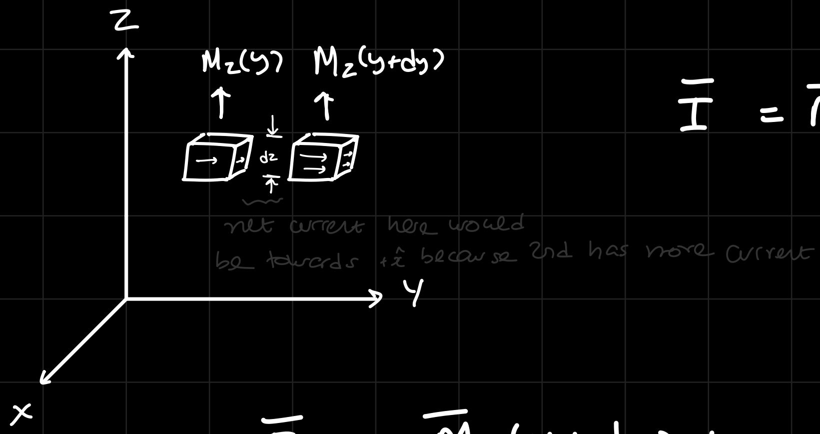
$\bar{M}$ will not be same in 2 different dipoles and hence, same goes for current

$\bar{M}$ varies with pos^n

from above $\rightarrow \bar{I} = \bar{M}t$

$\bar{J}_b = \bar{J}_{b-x}\hat{x} + \bar{J}_{b-y}\hat{y} + \bar{J}_{b-z}\hat{z}$

consider slabs



$$\bar{I} = \bar{M} t$$

$$\bar{I}_x = \bar{M}_z(y+dy)\,dz - \bar{M}_z(y)\,dz$$
$$= [\bar{M}_z(y+dy) - \bar{M}_z(y)]\,dz$$

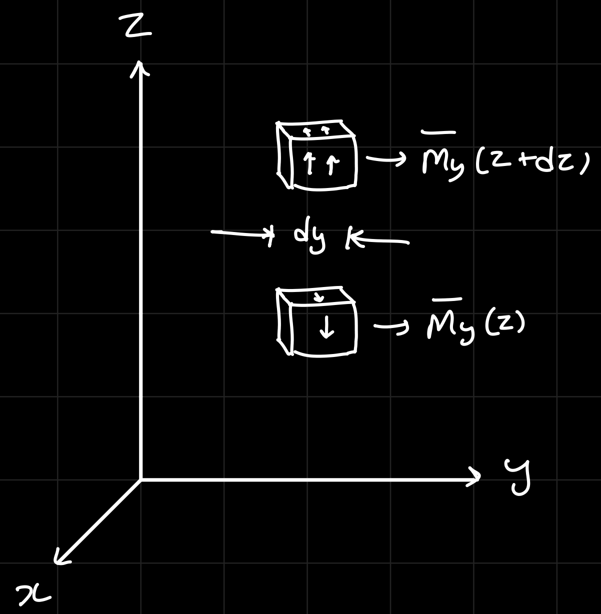
taylor series expansion

$$\bar{I}_x = \left[M_z(y) + \frac{\partial M_z}{\partial y} dy\right] dz - M_z(y)\,dz$$

$$= \frac{\partial M_z}{\partial y}\,dy\,dz$$

So, $\boxed{J_{b\_x} = \frac{\partial M_z}{\partial y}}$

but magnetization can have a x and y component as well



$$I_x = -\bar{M}_y(z+dz)\,dy + \bar{M}_y(z)\,dy$$

again taylor series expansion

$$= -\frac{\partial M_y}{\partial z}\,dy\,dz$$

$$\boxed{J_{b\_x} = -\frac{\partial M_y}{\partial z}}$$

Total volume current density : $\boxed{J_{b_x} = \left(\frac{\partial \bar{M}_z}{dy} - \frac{\partial \bar{M}_y}{dz}\right)}$

which is equal to $\underline{\underline{\bar{\nabla} \times \bar{M}}}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{M}_x & \bar{M}_y & \bar{M}_z \end{vmatrix}$$

So, $J_{b_x} = \hat{x}\left(\frac{\partial}{\partial y} M_z - \frac{\partial}{\partial z} M_y\right)$

again Ampere's Law $\rightarrow$ $\quad \bar{\nabla}\times\bar{B} = \mu_0 J_{total}$

$$\frac{\bar{\nabla}\times\bar{B}}{\mu_0} = \bar{J}_b + \bar{J}_{free}$$

$$\bar{J}_{free} = \nabla\times\left(\frac{\bar{B}}{\mu_0} - \bar{M}\right)$$

let $\vec{H} = \dfrac{\bar{B}}{\mu_0} - \bar{M}$

So, $\quad \bar{J}_f = \bar{\nabla}\times\bar{H}$

and $\oint H\cdot dl = I_{free}$

$\bar{B} = \mu_0\bar{H} + \bar{m}$

$\bar{m} = \chi_m\bar{H}$

$\hookrightarrow$ magnetic susceptibility

$$\bar{B} = \mu_0(\bar{H} + \chi_m\bar{H}) = \underbrace{\mu_0(\chi_m + 1)}_{\mu}\bar{H}$$

$\mu = \mu_0(1 + \chi_m) \rightarrow \quad \bar{B} = \mu\bar{H}$

magnetic materials

linear
$(\bar{M} = \chi_m \bar{H})$

Para-
-magnetic
$(\chi_m > 0)$

Dia-
-magnetic
$(\chi_m < 0)$

$$\mu = \mu_0(1 + \underbrace{\chi_m}_{<0})$$

$\mu \simeq \mu_0$

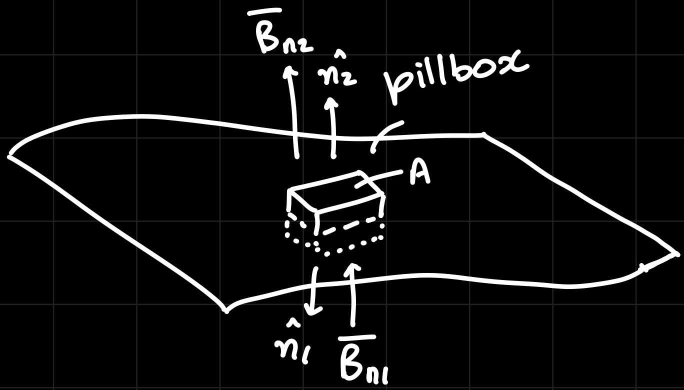
very low magnetization (at room temp)
in general for both
linear elements

non-linear

(ferromagnetic)

$\chi_m \sim 10^5$

Boundary Conditions (magnetostatics)



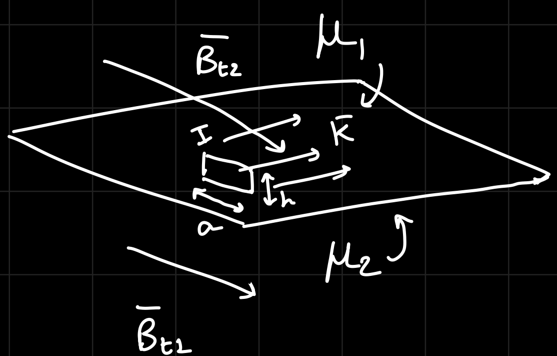
flux through sides = 0

because area of sides → 0

Gauss's Law: $\oint \bar{B} \cdot d\bar{s} = 0$

$$B_{n2}A - B_{n1}A = 0$$

$$B_{n1} = B_{n2}$$ normal component : equal



$\bar{I}_{enc} = \bar{K}a$

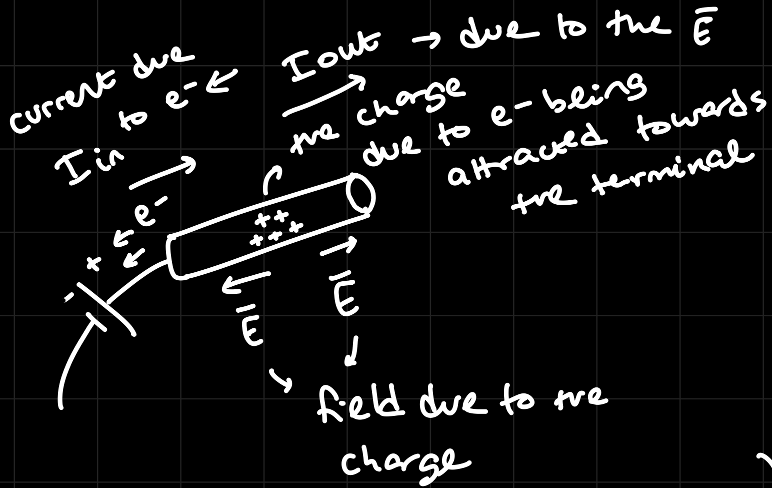
line current density

Ampere's Law ⇒ $\oint \bar{H} \cdot d\bar{l} = I_{enc} = \bar{K}a \Rightarrow H_{t1}a - H_{t2}a = K\bar{a}$

$$H_{t1} - H_{t2} = K$$ tangential component

* Electrodynamics

EMF : Voltage across a cell without the internal resistance.



This process is repeated throughout

I_{in} is stopped and

I_{out} is allowed due to $\bar{E}$

$$\bar{f} = \bar{f_s} + \bar{E} \rightarrow \text{drives the current}$$

total force

effect of this force is confined to the close vicinity of the source

taking closed loop integral

$$\oint \bar{f} \cdot d\bar{l} = \oint \bar{f_s} \cdot d\bar{l} + \underbrace{\oint \bar{E} \cdot d\bar{l}}_{0}$$

assumption:

because DC source { electrostatic field

for a circuit with no resistance, we dont need any force to push the current

So, for zero resistance,

$$\bar{f}_s + \bar{E} = 0$$

$$\Rightarrow \oint f_s \cdot dl = -\oint E \cdot dl = 0$$

but for a non closed loop

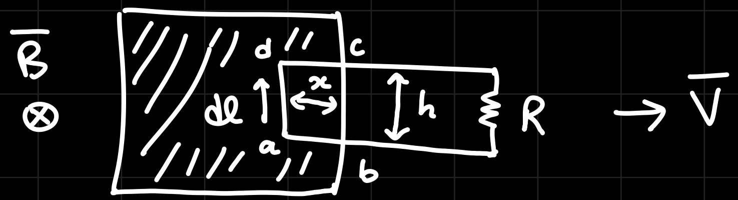
$$\underbrace{\int_A^B \bar{f}_s \cdot dl}_{\text{EMF } (\mathcal{E})} = -\int_A^B E \cdot dl = V_{BA}$$

So, EMF is the voltage when there is no internal resistance

now, we can also say

$$\oint \bar{F}_s \cdot d\bar{l} = \underset{\text{closed loop EMF}}{\text{EMF}} = 0$$

→ Proving Faraday's Law using an example



$$\bar{f}_{mag} = q\bar{v}\bar{B} = vB$$

$$\oint \bar{f}_{mag} \cdot \bar{dl} = vBh = EMF$$

assuming $q=1$

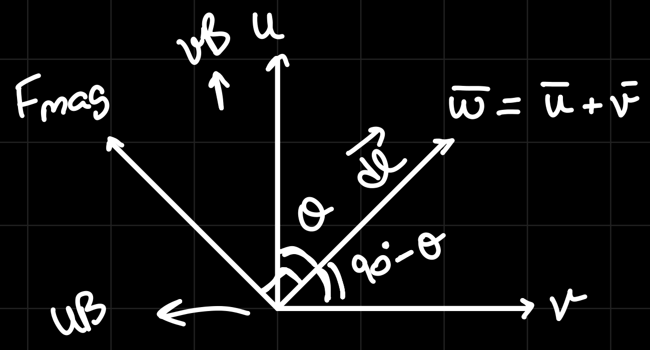
note: only ad contribute to the closed loop integral because ab and cd have $\bar{B} \perp dl$ and so dot product $= 0$ and rest of the ckt is considered outside the magnetic field $\bar{B}$

magnetic force doesn't do any work
current will flow through the resistance and there is EMF induced

but f_{mag} cant do any work → f_{mag} is not the source of the EMF then

the current flowing through R is not due to f_{mag} but due to the ckt being pulled to right with velocity $\bar{v}$

let the force due to the motion be equal to $\vec{f}_{pull}$



$\otimes \bar{B}$

fmag would now be ⊥ to $\bar{w}$ (motion)

$$\bar{f}_{mag} = (\bar{w} \times \bar{B}) \quad \text{assuming } q=1$$

$$= (\bar{u}+\bar{v}) \times \bar{B}$$

$$\text{work done} = \int \bar{f}_{pull} \cdot d\bar{l}$$

$$= \int uB\sin\theta \, dl$$

$$= uB\sin\theta \frac{h}{\cos\theta}$$

$$= u\tan\theta \, Bh = vBh = \underline{\mathcal{E}MF}$$

$$\phi = B\overset{\text{area}}{\widetilde{xh}} \leftarrow \int \bar{B} \cdot d\bar{s}$$

rate of decrease of flux → $\frac{d\phi}{dt} = B\frac{dx}{dt}h = Bvh = \mathcal{E}MF$

(x is decreasing) → rate of change = $-\mathcal{E}MF$

$$\oint \bar{E}\cdot d\bar{l} = -\frac{d\phi}{dt} = -\frac{d\left(\int B\cdot ds\right)}{dt} = -\int \frac{\partial B}{\partial t}\cdot ds$$

if B is not varying with time $\rightarrow \oint \bar{E}\cdot d\bar{l} = 0$

electrostatics

$$\oint E\cdot dl = \int (\nabla \times E)\, ds = -\int \frac{\partial B}{\partial t}\, ds$$

$$\boxed{\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

So, change in magnetic field induces and electric field

$$\bar{\nabla}\times\bar{B} = \mu_0 J \; + (\ldots) \text{ for electrodynamics}$$

$$\text{so, } \bar{\nabla}\cdot\bar{J} = 0 \quad \text{(div of curl)}$$

only valid in case of _statics ↳ electro/magneto

but $\bar{\nabla}\cdot\bar{J} = -\frac{\partial \rho}{\partial t} \rightarrow$ conservation of charge

now we need a modified ampere's law for electrodynamics

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\oint E \cdot dl = -\frac{\partial}{\partial t} \int B \cdot ds$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{J}) = \mu_0 \bar{\nabla} \cdot \bar{J} = 0$$

so, $\bar{\nabla} \cdot \bar{J} = 0$

conservation of charge →

$$\bar{\nabla} \cdot \bar{J} = -\frac{\partial \rho}{\partial t}$$

$$\bar{\nabla} \cdot \bar{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\bar{\nabla} \cdot \bar{J} + \bar{\nabla} \cdot \frac{\partial \bar{D}}{\partial t} = 0$$ where $\bar{D} = \varepsilon \bar{E}$

$$\nabla \cdot \left(\bar{J} + \frac{\partial \bar{D}}{\partial t} \right) = 0$$

so, $\bar{\nabla} \times \bar{B} = \mu_0 \left(J + \frac{\partial \bar{D}}{\partial t} \right)$

so, we just need a time varying electric field for magnetic field. No need for even a conduction current

Maxwell's modification:

time varying $\bar{E}$ gives rise to $\bar{B}$

Faraday's Law of Induction:

time varying $\bar{B}$ gives rise to $\bar{E}$

remember: Faraday's Law

$$\bar{\nabla} \times \bar{E} = -\frac{d\bar{B}}{dt} \qquad \oint E \cdot dl = -\frac{\partial}{\partial t} \iint_S B \cdot ds$$

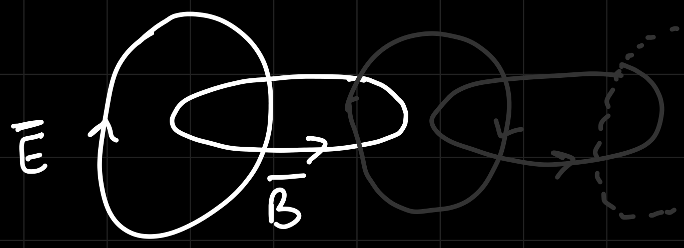
So, yes, $\bar{E}$ can be generated with just a time varying $\bar{B}$ even without charge / conductor.

How will the $\bar{E}$ field lines look like without a conductor / charges

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon \frac{\partial \bar{E}}{\partial t}$$

$$= \mu_0 (\underbrace{\bar{J}_D}_{\text{displacement current}} + \underbrace{\bar{J}_c}_{\text{conduction current}}) + \mu_0 \varepsilon \frac{\partial \bar{E}}{\partial t}$$

the $\bar{B}$ field lines will always loop and be $\perp$ to $\bar{E}$ field lines



self reproducing $\bar{E}$ and $\bar{B}$ fields from just a single initial $\bar{E}$ or $\bar{B}$ on all sides

These imaginary chains are Electromagnetic waves

Static case (no time variation)

$$\oint E \cdot ds = \frac{q_{enc}}{\varepsilon_0}$$

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint E \cdot dl = 0$$

$$\bar{\nabla} \times \bar{E} = 0 \Leftarrow \bar{E} = -\bar{\nabla}\phi$$

$$\oint \bar{B} \cdot d\bar{s} = 0$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 \bar{I}_{enclosed}$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

Dynamic case

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho(t)}{\varepsilon_0}$$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{B} = \mu_0 J + \mu_0 \left(\frac{\partial \bar{E}}{\partial t}\right) \varepsilon \quad \text{— maxwell}$$

EM waves

Potential in Electrodynamics

we can still write $\bar{B} = \bar{\nabla} \times \bar{A}$

but not $\bar{E} = -\bar{\nabla}\phi$

from (2) $\rightarrow$ $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\frac{\partial (\bar{\nabla} \times \bar{A})}{\partial t}$

$$\Rightarrow \nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

from 2 statics $\rightarrow$ $\nabla \times E = 0$

where $E = -\nabla\phi$

so, $\bar{E} + \frac{d\bar{A}}{dt} = -\nabla\phi$

so, $\bar{E} = -\left(\bar{\nabla}\phi + \frac{d\bar{A}}{dt}\right)$

everything defined till now in dynamics is in time domain. We will be moving to the frequency domain now.

square $\rightleftharpoons$ sinc

if $x(t)$ given as square wave and
system's freq response given,
we find $y(t)$ by $\to$

$x(t) \to X(\omega)$ {sinc}
multiply with $H(\omega)$
then IFT to get $y(t)$

This works for a linear system

Time → Freq (maxwell's equations)

$$\bar{\nabla}\bar{E}(t) = \frac{\rho(t)}{\varepsilon_0}$$

$$\bar{\nabla}\times\bar{E}(t) = -\frac{\partial B(t)}{\partial t}$$

$$\bar{\nabla}\cdot\bar{B}(t) = 0$$

$$\bar{\nabla}\times\bar{B}(t) = \mu_0\left(J(t) + \frac{\partial D(t)}{\partial t}\right)$$

Just take Fourier Transform

notation : $\bar{E}(t) \rightleftharpoons \bar{E}(\omega)$

$$\bar{\nabla}\cdot\bar{E}(\omega) = \frac{\rho(\omega)}{\varepsilon}$$

$$\bar{\nabla}\times\bar{E}(\omega) = -i\omega\bar{B}(\omega)$$

$$\bar{\nabla}\bar{B}(\omega) = 0$$

$$\bar{\nabla}\times\bar{B}(\omega) = \mu\left(\bar{J}(\omega) + i\omega\varepsilon\bar{E}(\omega)\right)$$

we know $\bar{J} = \sigma\bar{E}$

$$\boxed{\bar{\nabla}\times\bar{B} = \mu\bar{E}(\sigma + i\omega\varepsilon)}$$

In a source free region →

Time
$$\left\{\begin{array}{ll} \bar{\nabla}\cdot\bar{E} = 0 & \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{E} = -\dfrac{\partial\bar{B}}{\partial t} & \bar{\nabla}\times\bar{B} = \mu\varepsilon\dfrac{\partial\bar{E}}{\partial t} \end{array}\right.$$

Freq
$$\left\{\begin{array}{ll} \bar{\nabla}\cdot\bar{E} = 0 & \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{E} = -i\omega\bar{B} & \bar{\nabla}\times\bar{B} = -i\omega\mu\varepsilon\bar{E} \end{array}\right.$$

wave eqn in 1d

↳ $f(x-vt)$

$$\frac{\partial^2 f}{\partial x^2} = f''(x-vt) = \frac{\partial^2 f(x-vt)}{\partial (x-vt)^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial t}$$

$$= f'(x-vt) \times (-v)$$

$$\frac{\partial^2 f}{\partial t^2} = v^2 \; f''(x-vt)$$

so, $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{v^2} \dfrac{\partial^2 f}{\partial t^2}$

for 3d →

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 \bar{f} = \frac{1}{v^2} \frac{\partial^2 \bar{f}}{\partial t^2}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t}(\nabla \times B)$$

$$= \frac{-\partial}{\partial t}\left(\mu \varepsilon \frac{\partial \bar{E}}{\partial t}\right)$$

considering source free region

$$\bar{\nabla} \cdot \cancel{(\bar{\nabla} \bar{E})}^{0} - \bar{\nabla}^2 \bar{E} = -\mu \varepsilon \left(\frac{\partial^2 \bar{E}}{\partial t^2}\right)$$

$$\Rightarrow \quad \nabla^2 E = \mu \varepsilon \left(\frac{\partial^2 \bar{E}}{\partial t^2}\right)$$

So, the velocity $\Rightarrow \frac{1}{v^2} = \mu \varepsilon$

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

$$\bar{\nabla} \times \bar{B} = \mu\varepsilon \frac{\partial \bar{E}}{\partial t}$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{B}) = \mu\varepsilon \frac{\partial}{\partial t} (\nabla \times \bar{E})$$

$$\bar{\nabla} (\cancelto{0}{\bar{\nabla} \bar{B}}) - \nabla^2 B = -\mu\varepsilon \frac{\partial^2 \bar{B}}{\partial t^2}$$

$$\nabla^2 B = \mu\varepsilon \frac{\partial^2 B}{\partial t^2}$$

velocity: $\dfrac{1}{\sqrt{\mu\varepsilon}}$

no material involved
this wave still propagates
even through vacuum
we just need time varying
magnetic field and electric field
for EM waves

$$\nabla^2 E = \mu\varepsilon\left(\frac{\partial^2 \bar{E}}{\partial t^2}\right)$$

$$\nabla^2 B = \mu\varepsilon \frac{\partial^2 B}{\partial t^2}$$

in frequency domain :

$$\bar{\nabla}^2\bar{E} = \mu\varepsilon(-\omega^2\bar{E})$$

$$\nabla^2\bar{E} + (\mu\varepsilon\omega^2)\bar{E} = 0$$

let $K = \omega\sqrt{\mu\varepsilon} = \dfrac{\omega}{v}$ or $\dfrac{\omega}{c}$ velocity

$$\nabla^2\bar{E} + K^2\bar{E} = 0 \longrightarrow \text{Helmholtz eq}^n$$

$$\bar{\nabla}^2 \underset{(r,t)}{\bar{E}} = \frac{1}{c^2} \frac{\partial^2 \bar{E}(r,t)}{\partial t^2}$$

let $K = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v}$ or $\frac{\omega}{c}$ velocity

$$\nabla^2 \underset{(r,\omega)}{\bar{E}} + K^2 \underset{(r,\omega)}{\bar{E}} = 0 \longrightarrow \text{Helmholtz eq}^n$$

freq domain $\rightarrow$ time ?

IFT

$$\bar{E}(r,t) = \int_{-\infty}^{\infty} \bar{E}(r,\omega)\, e^{i\omega t}\, d\omega$$

(for multiple freq)

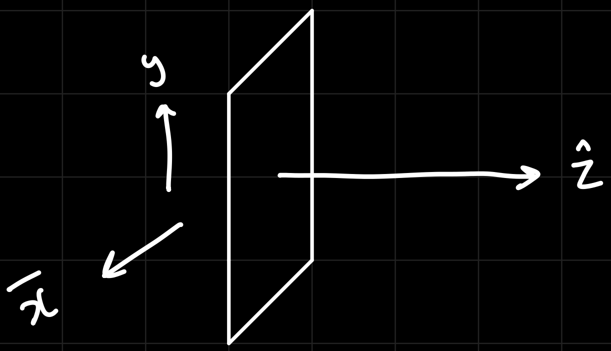
$$\delta(x) \overset{IFT}{\rightleftharpoons} 1$$

for single frequency

$$E_0(r)\,\delta(\omega-\omega_0) \rightleftharpoons E_0(r)\, e^{i\omega t}$$

↳ only a func of space

1D → Uniform Plane wave



$\bar{E}$ and $\bar{B}/\bar{H}$ only vary with z
they are uniform in the xy plane

considering a monochromatic wave
i.e. only one frequency

So, $\bar{E}(z,t) \rightleftharpoons \bar{E}(z)e^{i\omega t}$

Helmholtz equation →

$$\frac{\partial^2 \bar{E}(z)}{\partial z^2} + k^2 \bar{E}(z) = 0$$

assume $\bar{E}(z) \simeq e^{mz}$

$$m^2 e^{mz} + k^2 e^{mz} = 0$$
$$m^2 + k^2 = 0$$
$$m = \pm ik$$

$$E(z) = e^{\pm ikz}$$

$$\bar{E}(z,\omega) = \underset{(-\hat{z})}{\bar{A}e^{ikz}} + \underset{(\hat{z})}{\bar{B}e^{-ikz}}$$

$$\bar{E}(z,t) = \bar{A}e^{i(\omega t+kz)} + \bar{B}e^{i(\omega t-kz)}$$

lets take the case of forward travelling wave

$$\bar{E}(z,t) = \underset{\text{(Amplitude)}}{\bar{E}_0}\, e^{i(\omega t-kz)} \rightarrow \text{(phase)}$$

direction of EM wave $\equiv$ polarization of the wave
$(\bar{E})$

$\underline{\underline{t=t_1}}$ Locus of the points at the same phase = ?
we will look at it now.

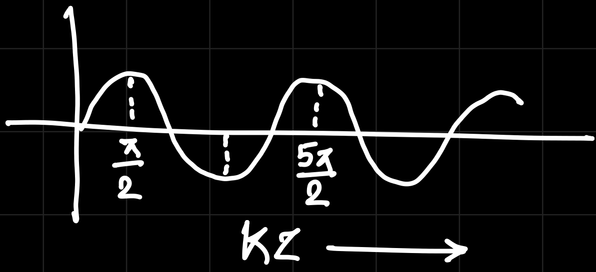
$$\omega t_1 - kz = \underline{M}$$

↳ same phase M at $t=t_1$.
Locus = ?

$$z = \frac{\omega t_1 - M}{K} = 0 \rightarrow \text{totally a constant}$$

Locus $\Rightarrow$ Z = constant
Plane parallel to xy-plane

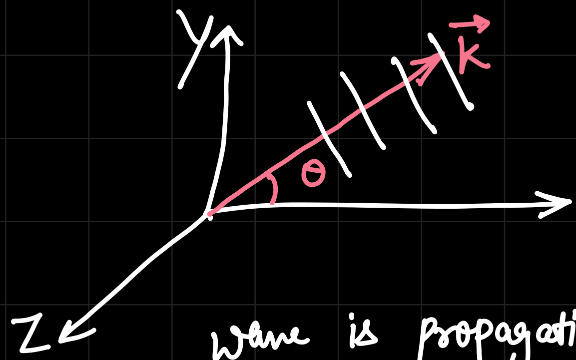
$$\bar{E}(z,t) = \bar{E}_0\, e^{i(\omega t-kz)}\, e^{i2n\pi}$$



$2n\pi$ gap in Kz axis.

* wave = planar but not uniform. Possible?

YES



Still a plane wave but not uniform.

wave is propagating but tilted (at an angle θ) w.r.t. x-axis.

There can be other examples but this is simplest one.

Basic Helmh. eqn we studied till now, we assumed $\sigma = 0$ (source-free)

↳ No conduction current.

$$\nabla \cdot \vec{E} = 0$$
$$\vec{\nabla} \times \vec{E} = -i\omega \vec{B}$$
$$\vec{\nabla} \times \vec{B} = \mu\sigma \vec{E} + i\omega\mu\epsilon E$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

↳ freq. domain

$\sigma = 0$

If not this, it will be very tough to analyze.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = i\omega\mu\epsilon(\vec{\nabla} \times \vec{E})$$
$$= i\omega\mu\epsilon(-i\omega \vec{B})$$
$$= -i^2\omega^2 \mu\epsilon \vec{B}$$

$$\bar{\nabla}\times\bar{B} = i\omega\mu\epsilon_c\vec{E} = i\omega\mu\epsilon\vec{E}$$ (written in simpler)

$$i\omega\mu\epsilon_c = \mu\sigma + i\omega\mu\epsilon$$ (equating both $\bar{\nabla}\times\bar{B}$ eqn.)

$$\epsilon_c = \epsilon + \frac{\sigma}{i\omega}$$

$$\boxed{\epsilon_c = \epsilon - i\frac{\sigma}{\omega}}$$

Permittivity of a medium becomes complex no.

$$\epsilon_c = \epsilon\left(1 - i\frac{\sigma}{\omega\epsilon}\right)$$

* $K = \omega\sqrt{\mu\epsilon_c} = K' - i\underbrace{K''}_{>0}$

Initially, K was simpler. $(K = \omega\sqrt{\mu\epsilon})$
Now, it is complex. $(\epsilon \rightarrow \epsilon_c)$

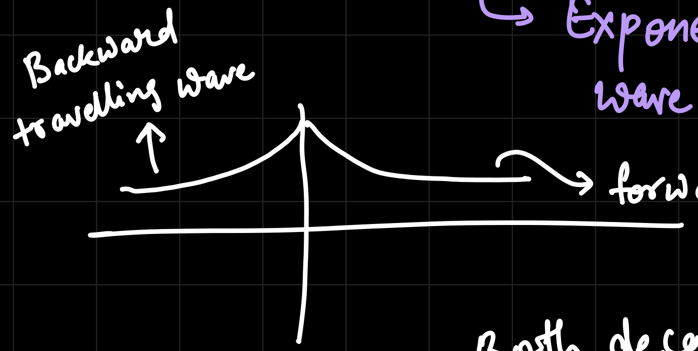
Just ϵ changed to a complex no. (ϵ_c).

* $$\vec{E}_0 e^{i(\omega t - Kz)} = \vec{E}_0 e^{i\omega t} \cdot e^{-i(K' - iK'')z}$$

$$= E_0 e^{i\omega t} e^{-iK'z} e^{-K''z}$$

Nature of this wave = ?

↳ Exponentially decaying wave moving forward



Both decaying in diff. dirxns.

If we had taken +ve from $k' + ik''$, we would have an exp. increasing wave and at an ∞ amplitude, Energy will become ∞ (which is not possible obv.)

* When to assume ϵ_c = real and ϵ_c = complex

$\sigma \neq 0$, current $\neq 0$

Lost energy will be released in form of heat because of conduction current.

What if $\sigma \ll \omega\epsilon$

↳ Perfect dielectric
$\epsilon_c = \epsilon$ (real ϵ)

$\sigma \sim \omega\epsilon$

↳ Lot of heat will be generated.
medium with decreasing constant

$\sigma \gg \omega\epsilon$

EM wave will die inside a P.E.C.

It will decay very fast.

* Static $\vec{E}$
  Time-varying $\vec{E}$ } inside P.E.C. $= 0$

* Static $\vec{B}$ can exist inside P.E.C.

Q: Can Time-varying $\vec{B}$ exist inside P.E.C ?

NO, as T-V $\vec{B}$ will generate $\vec{E}$ but $\vec{E} = 0$ inside P.E.C.

Ag, Cu → good conductors at microwave frequencies
→ not good conductors at optical frequencies

Lecture before Quiz 3

for the 1d uniform plane wave:

$$\bar{E}(z,t) = \left(\bar{A} e^{i(\omega t + kz)} + \bar{B} e^{i(\omega t - kz)} \right) \hat{x}$$

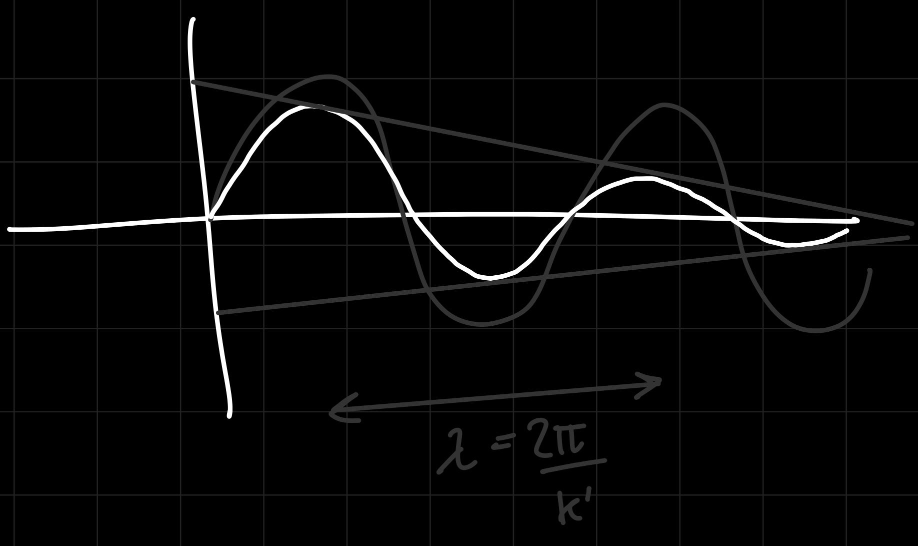
polarization of the EM wave

$$K = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \longrightarrow \text{wave number}$$

$$\lambda = \frac{2\pi}{K}$$

K'' defines the decay

K' defines the wavelength $\lambda = \frac{2\pi}{K'}$



Just like time and frequency form a fourier pair,
k and λ also form a fourier pair

also → k domain ≡ momentum space

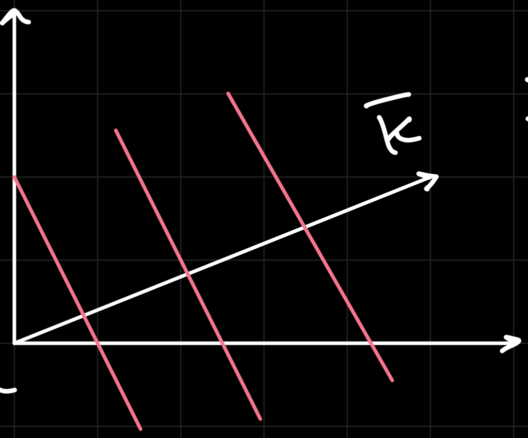
$$\omega = \frac{2\pi}{T} \quad , \quad k = \frac{2\pi}{\lambda}$$

What is the physical meaning of k?
↳ represents the number of cycles in a unit distance

$$k\lambda = 2\pi$$

in a given phase of 2π, what fraction of a complete wavelength can fit?
eg → $1\lambda = 2\pi$ or 1.5λ or 2.5λ

Can we have a non uniform plane wave ?

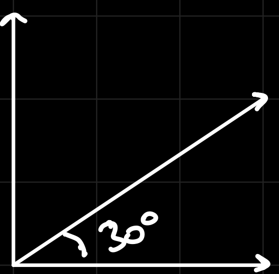


Still planar equinormal surfaces

$\bar{E}$ and $\bar{B}$ are non uniform because wave varies not in a single axis

$$\bar{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$

currently only a 2d problem so $k_z = 0$

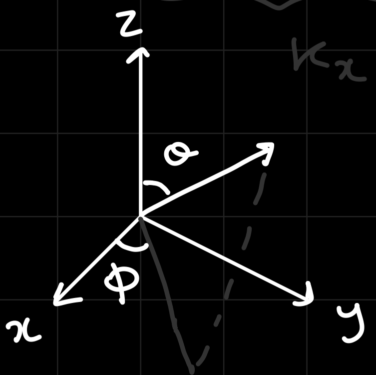


$$\bar{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$

$$= \hat{x} |k| \cos\theta + \hat{y} |k| \sin\theta$$

for a vector in sperical coordinates

$$\bar{k} = \underbrace{\hat{x} |k| \sin\theta \cos\phi}_{k_x} + \underbrace{\hat{y} |k| \sin\theta \sin\phi}_{k_y} + \underbrace{\hat{z} |k| \cos\theta}_{k_z}$$



$$\frac{k_x}{k_y} = \tan\phi$$

$$\phi = \tan^{-1}\left(\frac{k_x}{k_y}\right)$$

from $\bar{k}$, we ca find

frequency
direction of propagation

$$f = \frac{2\pi c}{\lambda} = \frac{2\pi c}{2\pi/|k|} = |k|c$$

going back:

$$\nabla^2 \bar{E} + k^2 E = 0$$

↳ # of eq^{n}s = ?

↳ 3 because of the 3 components

$$\left.\begin{aligned} \nabla^2 \bar{E}_x + k^2 E_x &= 0 \\ \nabla^2 \bar{E}_y + k^2 E_y &= 0 \\ \nabla^2 \bar{E}_z + k^2 E_z &= 0 \end{aligned}\right\}$$

also note:

$$k^2 = \left(\frac{\omega}{c}\right)^2 = \omega^2 \mu \varepsilon$$

↓

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

Variable separation :

$$E_x(x,y,z) = X(x)\,Y(y)\,Z(z)$$

we proved earlier :

solution of 1d wave

↳ plane wave

now we prove,

soln of 3d is also

plane wave

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}} = -k^2$$

let → $-k_x^2$ $\quad -k_y^2 \quad -k_z^2$

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

(sphere eqn

radius $= k^2 = \left(\frac{\omega}{c}\right)^2$)

We can only choose 2 variables arbitrarily (independently).
The 3rd must be a fixed value in order to satisfy

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

because $k^2 = \left(\frac{\omega}{c}\right)^2$ is const

since ω (frequency) is given

$$k_z = \pm\sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

solⁿ comes out
in the form →
$$X \simeq e^{\pm i k_x x}$$
$$Y \simeq e^{\pm i k_y y}$$
$$Z \simeq e^{\pm i k_z z}$$

with some constants but we will just assume that in final solⁿ

$$E_x(x, y, z) = E_{x_0} e^{\pm i k_x x} e^{\pm i k_y y} e^{\pm i k_z z}$$

↳ wave of 3 planes

$$E_x(x,y,z) = E_{x_0} e^{\pm i(k_x x + k_y y + k_z z)}$$

as a dot product of 2 vectors

$$= E_{x_0} e^{\pm i(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)(\hat{x}x + \hat{y}y + \hat{z}z)}$$

$$E_x(x,y,z) = E_{x_0} e^{\pm i\vec{k}\vec{r}}$$ → radius vector

plane wave ✓
but not uniform

now back the $\bar{E}$ equation
(general)

$$\bar{E} = e^{\pm i\bar{k}\cdot\bar{r}} (\hat{x}E_{x_0} + \hat{y}E_{y_0} + \hat{z}E_z)$$

so, $\bar{k}$ determines the propagation of the wave and the general eqn of the wave is $e^{\pm i(\bar{k}\cdot\bar{r})}$

Remember Maxwells Eqn in freq domain

$$\bar{\nabla}\cdot\bar{E} = 0 \qquad \bar{\nabla}\cdot\bar{B} = 0$$
$$\bar{\nabla}\times\bar{E} = -i\omega B \qquad \bar{\nabla}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

we can achieve the time domain equivalent of a wave as the superposition of different wavelengths in the k domain.

$$g(t) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega t}\, d\omega$$

Fourier optics

↳ used for wide angle clicks in portable computing devices like VR headsets, Mobile phones etc.

$$E_x = E_{x0}\, e^{i(k_x x + k_y y + k_z z)}$$

$$\frac{\partial E_x}{\partial x} = -ik_x\left(E_{x0}\, e^{i(k_x x + k_y y + k_z z)}\right)$$

$$= -ik_x E_x$$

like fourier formulas

$$\bar{K}\cdot\bar{E} = 0 \quad , \quad \bar{K}\cdot\bar{B} = 0$$

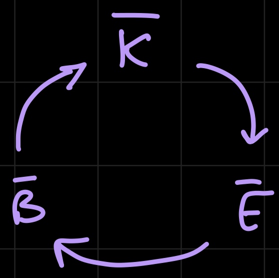
so, $\bar{K}$, $\bar{E}$ and $\bar{B}$ fields are perpendicular to each other

$$\bar{\nabla}\times\bar{E} = -i\omega\bar{B} \qquad\qquad \bar{\nabla}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

$$-i\bar{K}\times\bar{E} = -i\omega\bar{B} \qquad\qquad -i\bar{K}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

$$\underbrace{\bar{K}\times\bar{E} = \omega\bar{B} \qquad\qquad \bar{K}\times\bar{B} = -\omega\mu\varepsilon\bar{E}}$$

not just direction,
we have this relation



we know, $\dfrac{|E|}{|B|} = c$ but how?

$$\bar{K}\times\bar{E} = \omega\bar{B}$$

$$|K||E| = \omega|B|$$

$$\frac{|E|}{|B|} = \frac{\omega}{|K|} = \underline{\underline{c}}$$

We knew $\bar{B} = \mu \bar{H}$

and

Voltage / Current

$$\frac{\vec{E}}{\vec{H}} = \mu\frac{\bar{E}}{\bar{B}} = \mu c = \frac{\mu}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

Impedance of a medium

Why not called Resistance? $\longrightarrow$ because it can be complex

lecture

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = \eta : \Omega$$

$$w_e = \frac{1}{2}\varepsilon |E|^2$$

$$w_m = \frac{1}{2}\mu |E|^2$$

$$w_e + w_m = \frac{1}{2}\left(\varepsilon|E|^2 + \mu|H|^2\right)$$

$$W_e = \frac{1}{2}\varepsilon \int |E|^2 dz$$

$$W_m = \frac{1}{2}\mu \int |H|^2 dz$$

$$\bar{\nabla}\cdot(\bar{E}\times\bar{H}) = \bar{H}\cdot(\bar{\nabla}\times\bar{E}) - \bar{E}(\bar{\nabla}\times\bar{H})$$

$$= \bar{H}\left(-\mu\frac{\partial \bar{H}}{\partial t}\right) - \bar{E}(\dots)$$

$$= -\mu\, \bar{H}\cdot\frac{\partial \bar{H}}{\partial t} - \bar{E}\left(\bar{J} + \varepsilon\frac{\partial \bar{E}}{\partial t}\right)$$

$$= -\frac{\mu}{2}\frac{\partial |H|^2}{\partial t} - \bar{E}\cdot\bar{J} - \frac{\varepsilon}{2}\frac{\partial |E|^2}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\mu|\bar{H}|^2 + \varepsilon|\bar{E}|^2\right) - \bar{E}\cdot\bar{J}\right\}$$

(heat) (outgoing)

$$-\frac{\partial}{\partial t}(w_e + w_m) = \bar{E}\cdot\bar{J} + \bar{\nabla}(\bar{E}\times\bar{H})$$

decay rate of stored energy in an electromagnetic field

we said in the last class

$\bar{E}$ = Voltage

$\bar{B}$ = current

So, $\bar{E}\cdot\bar{J} \simeq$ Power

$$dw\ \bar{F}\cdot d\bar{l} = q(\bar{E} + \underbrace{(\bar{v}\times\bar{B})) \cdot \bar{v}}_{0}\, dt$$

$$dw\ \bar{F}\cdot d\bar{l} = q\bar{E}\bar{v}\, dt$$

$$\frac{dw}{dt} = \rho\, dz\, \bar{E}\cdot\bar{v}$$

$$\frac{dw}{dt} = \bar{E}\cdot\bar{J}\, dz$$

The Energy is either used by doing some work $(\bar{E}\cdot\bar{J})$ or comes out of the surface $(\bar{\nabla}\cdot(\bar{E}\times\bar{H}))$

$$-\frac{\partial}{\partial t}\int (w_e + w_m)\,d\tau = \int \bar{E}\cdot\bar{J}\,d\tau + \int \bar{\nabla}\cdot(\bar{E}\times\bar{H})\,d\tau$$

$$= \int \bar{E}\cdot\bar{J}\,d\tau + \oiint (\bar{E}\times\bar{H})\,d\bar{s}$$

So, $P_{outgoing} = \oiint \underbrace{(\bar{E}\times\bar{H})}_{\vec{S}}\,d\bar{s}$

(Poynting vector)

$$\bar{S} = \bar{E}\times\bar{H}$$

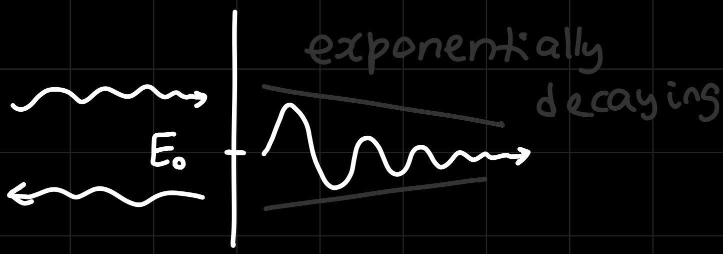
$|\bar{S}| = |E||H| = |E|\cdot\frac{|E|}{\eta} = \frac{|E|^2}{\eta}$: ≃ power per unit area (Watt/m²)

last topic for EM waves →

Next : transmission lines

SOMETHINGIDK123

$$\varepsilon_c = \varepsilon\left(1 - \frac{i\sigma}{\omega\varepsilon}\right)$$



$$k = \omega\sqrt{\mu\varepsilon_c}$$

$$k = k' - ik''$$

$$e^{-ikz} = e^{-i(k'-ik'')z} = e^{-ik'z}e^{-k''z}$$

Skin depth of a medium:

$$\delta = \frac{1}{k''}$$

$\hookrightarrow$ 0 for a PEC ($\infty\times$)

distance for the signal to decay

for $z = \delta$,

$$\bar{E} = E_0 e^{-ik'z} e^{-k''/k''} = \frac{E_0}{e} e^{-ik'z}$$

$$k^2 = \omega^2\mu\varepsilon\left(1 - \frac{i\sigma}{\omega\varepsilon}\right)$$

$$(k' - ik'')^2 = k'^2 - k''^2 - 2ik'k'' = \omega^2\mu\varepsilon - i\omega\sigma\mu$$

$$2k'k'' = \omega\sigma\mu$$

$$k'^2 - k''^2 = \omega^2\mu\varepsilon \quad \text{— ①}$$

for simplicity, we find

$$(k'^2 + k''^2)^2 = (k'^2 - k''^2)^2 + 4(k'k'')^2$$

$$= \omega^4 \mu^2 \varepsilon^2 + \omega^2 \mu^2 \sigma^2$$

$$(k'^2 + k''^2) = \omega^2 \mu\varepsilon \left[1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right]^{1/2} \quad \text{— } 2$$

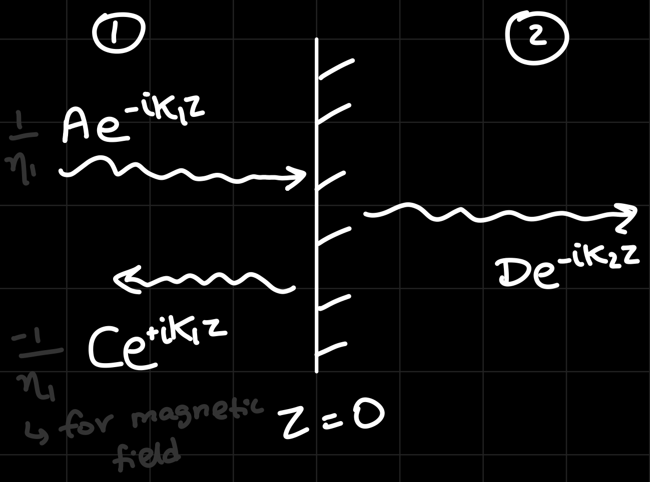
using 1 and 2 →

$$k' = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[1 + \left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} \right]^{1/2}$$

$$k'' = \omega \sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} - 1 \right]^{1/2}$$

Note: Lossless medium → no decay

We figure out reflection and trasmission of the em waves



(μ_1, ε_1) (μ_2, ε_2)

$k_1 = \omega\sqrt{\mu_1\varepsilon_1}$ $k_2 = \omega\sqrt{\mu_2\varepsilon_2}$

assuming SRC ≡ medium 1

trasmitted ≡ medium 2

$$\Gamma(z) = \frac{Ce^{-ik_1z}}{Ae^{-ik_2z}}$$

Reflection coefficient:

(gamma)

$$\Gamma(z) = \frac{C}{A} e^{i2k_1z}$$

at the interface → $z=0$

$$\Gamma_0 = \frac{C}{A}$$

$$\boxed{\Gamma(z) = \Gamma_0 e^{i2k_1z}}$$

lossless medium ≡ no conduction current (pure dielectric)

for a lossless medium → Γ_0 decreases as we move away from the interface and $|\Gamma(z)| = |\Gamma_0|$

periodicity:

$$e^{i2\pi} = 1 \rightarrow 2k_1 z = 2\pi$$

$$2 \cdot \frac{2\pi}{\lambda} \cdot z = 2\pi \rightarrow \boxed{z = \frac{\lambda}{2}}$$

Transmission Coefficient : $T = \frac{D}{A}$

defined at the interface only, not all points because there exists only the transmitted wave in medium 2.

at $Z=0$: $\frac{E_{t1}}{H_{t1}} = \frac{E_{t2}}{H_{t2}}$

$$\frac{A+C}{A/\eta_1 + C/(-\eta_1)} = \eta_2$$

$$\frac{1+(\Gamma_0)}{\frac{1}{\eta_1}(1-\Gamma_0)} = \eta_2$$

$$\frac{\eta_2}{\eta_1} = \frac{1+\Gamma_0}{1-\Gamma_0}$$

$$A + C = D$$

$$1 + C/A = D/A$$

$$\boxed{1 + \Gamma_0 = T}$$

$$T = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \rightarrow \boxed{T = \frac{2\eta_2}{\eta_1 + \eta_2}}$$

$$E_{incident} = E_0 e^{-ikz}$$
$$E_{reflected} = \Gamma_0 E_0 e^{ikz}$$

if Γ_0 is negative then $\bar{E}$ has changed the direction

$$\Gamma_0 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \eta_1 = \left(\frac{\mu_1}{\varepsilon_1}\right)^{1/2}, \ \eta_2 = \left(\frac{\mu_2}{\varepsilon_2}\right)^{1/2}$$

for non magnetic media

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}}, \quad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

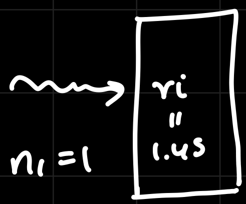
$$\Gamma_0 = \frac{\sqrt{\mu_0/\varepsilon_1} - \sqrt{\mu_0/\varepsilon_2}}{\sqrt{\mu_0/\varepsilon_1} + \sqrt{\mu_0/\varepsilon_2}} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

$$\frac{\sqrt{\varepsilon_{r_1}} - \sqrt{\varepsilon_{r_2}}}{\sqrt{\varepsilon_{r_1}} + \sqrt{\varepsilon_{r_2}}} \longleftarrow \quad = \frac{\sqrt{\varepsilon_0 \varepsilon_{r_1}} - \sqrt{\varepsilon_0 \varepsilon_{r_2}}}{\sqrt{\varepsilon_0 \varepsilon_{r_1}} + \sqrt{\varepsilon_0 \varepsilon_{r_2}}}$$

$$\downarrow$$

$$\boxed{\frac{n_1 - n_2}{n_1 + n_2}}$$

these n are not η (eeta)
this is refractive index



for non magnetic material

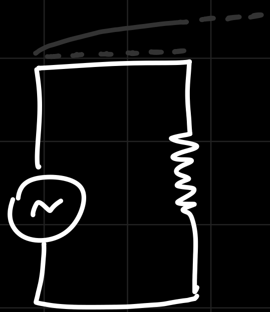
$$\Gamma_0 = \frac{n_1 - n_2}{n_1 + n_2} = -ve$$

and so, $\bar{E}_r$ = opposite to $\bar{E}_i$

Initial KVL/KCL problem

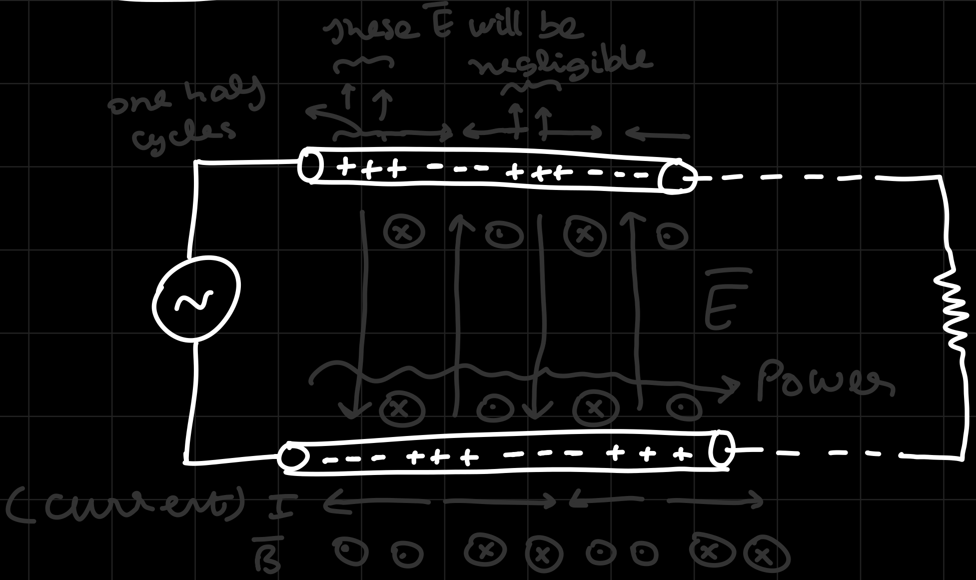
wavelength for freq = 50 Hz → $\frac{3\times10^8}{50} = 6\times10^8$ cm

Since wavelength is very big, we can assume for a very small circuit that voltage remains constant along a R=0 line



but for very long ckts, we cannot assume:

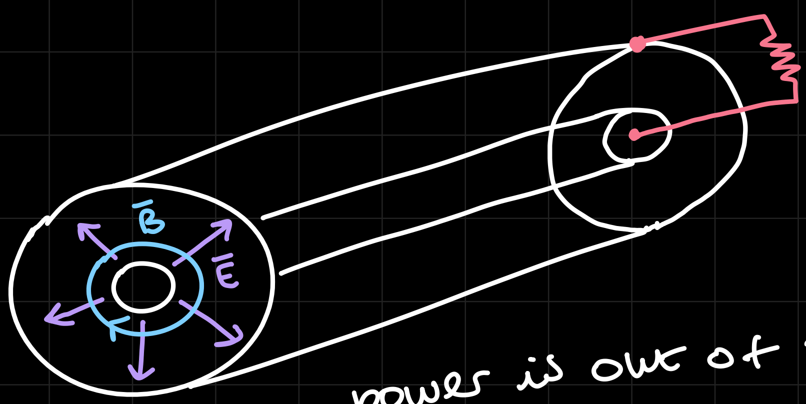
$$\left.\begin{aligned} V_{SRC} &= V_{Load} \\ I_{SRC} &= I_{load} \end{aligned}\right\} \times$$



$\bar{E} \times \bar{H}$ is almost zero outside the circuit because of $\bar{E}$

($\bar{E} \times \bar{H}$) Power is held / concentrated within the 2 conductors and being guided from SRC → Load at all points because both $\bar{E}$ and $\bar{H}$ switch directions together.

Guided wave propagation



power is out of the screen ⊙

ERROR POSSIBLE: These conductors might not remain as PEC at higher frequency

Distributed Resistance (not lump resistance ×) throughout the conductors because not PEC

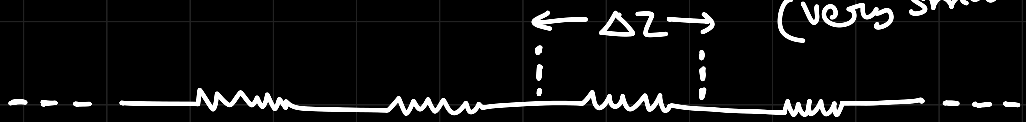
Assume that distributed resistance per unit length as R_s (Ω/m)

↳ not just Ω because distributed

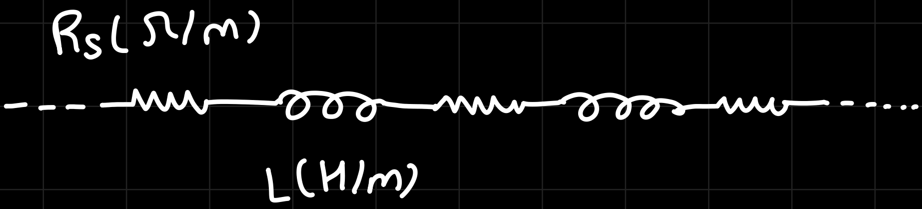
↓

series resistances

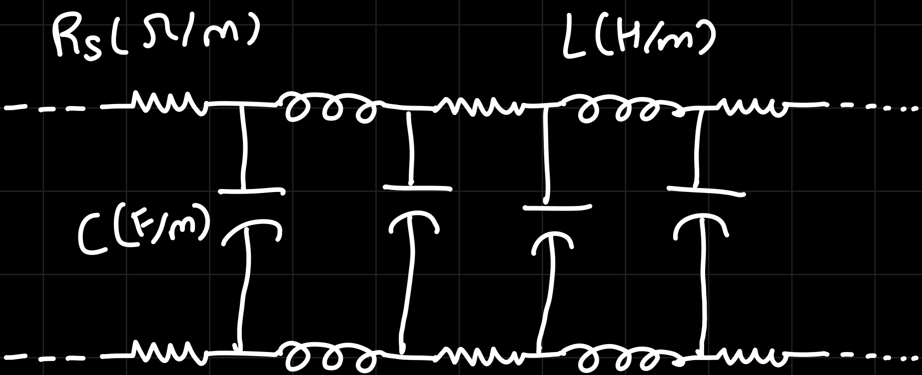
→ tends to zero (very small)



along with this we have inductance as well because of the $\bar{B}$

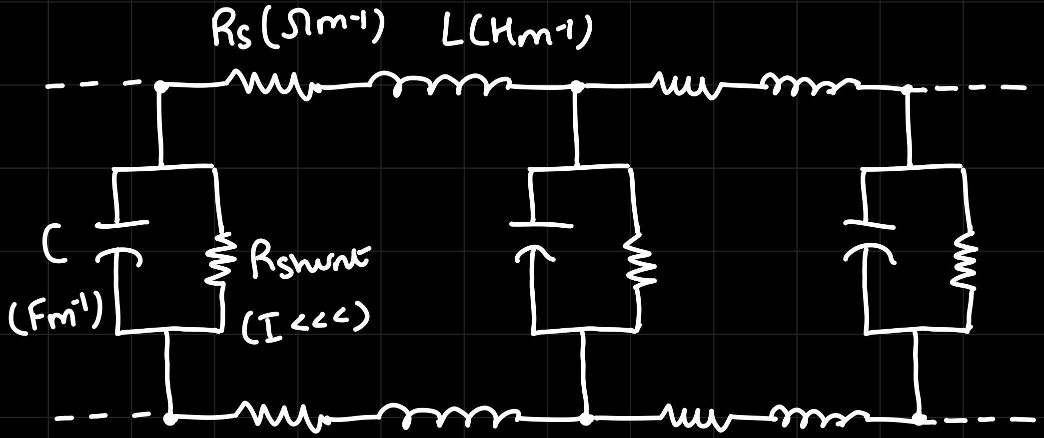


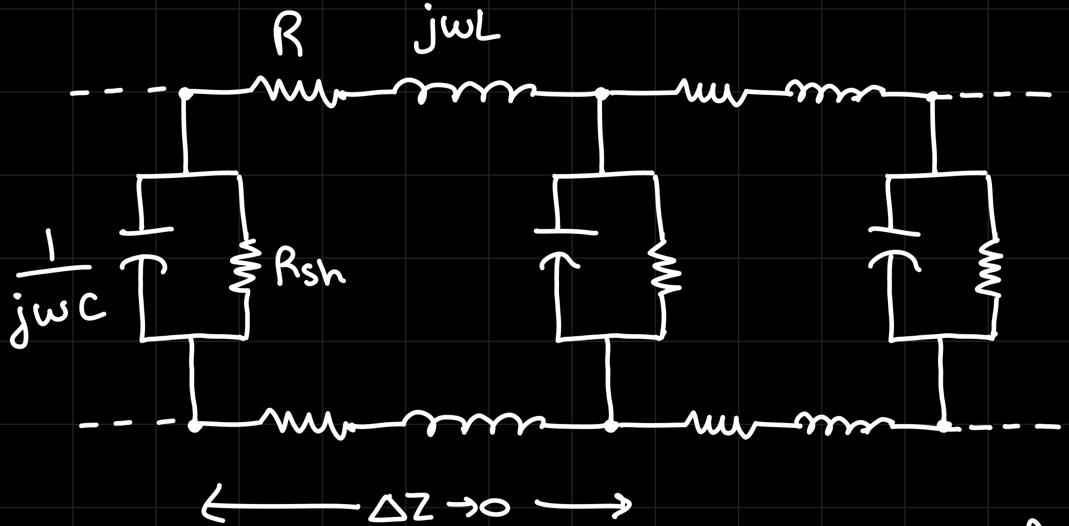
additionally, we also have distributed capacitance due to the $\bar{E}$



lastly, we have σ very small between the two conductors so that means very high resistance between them (dielectric medium)

(leakage current)





<fig L>

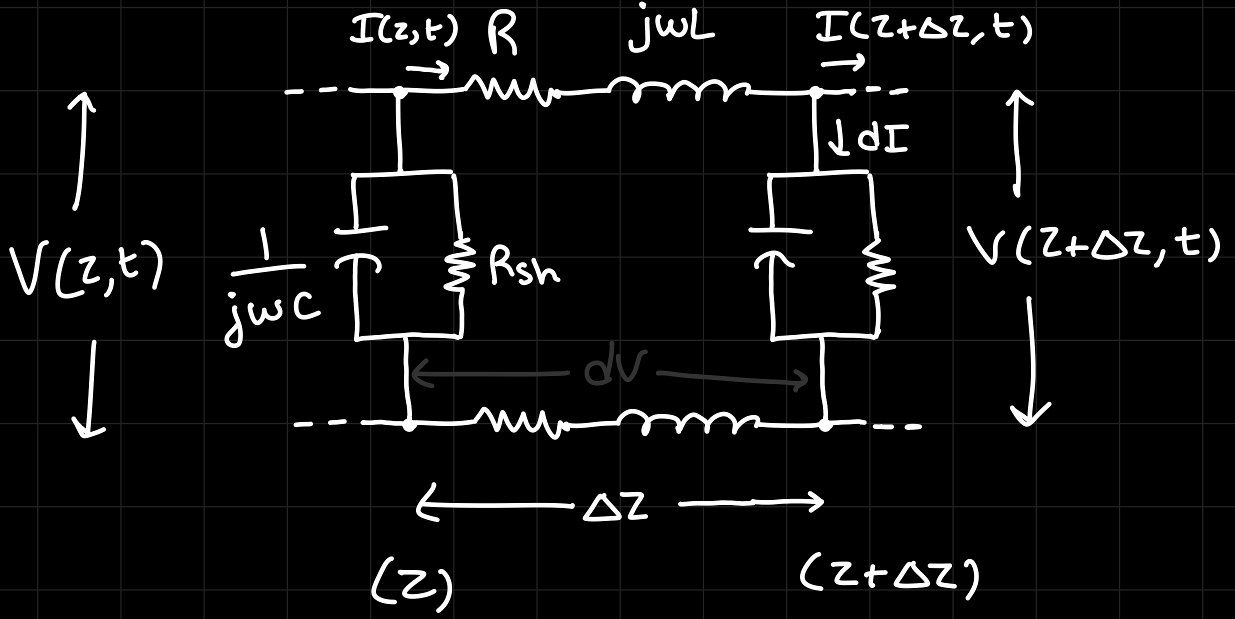
let $X = R + j\omega L$

(admittance) $Y = j\omega C + \frac{1}{R_{sh}} = (G + j\omega C)$

↳ very less because $R_{sh} >>>$

Since we have all these non-idealities, there must be some voltage drop and hence KVL / KCL would fail without considering these factors.

KCL / KVL fails for macroscopic circuitry but still work for <fig1> because $\Delta z \to 0$ (because distance very small)



$$V(z,t) - I(z,t)\underbrace{(R+j\omega L)}_{X}\Delta z - V(z+\Delta z,t) = 0$$

because R and L are per unit values

$$-I(z,t) X \Delta z = V(z+\Delta z,t) - V(z,t)$$

$$\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -X I(z,t)$$

$\lim \Delta z \to 0$

$$\boxed{\frac{\partial V}{\partial z} = -X\, I(z,t)}$$

what about the current ?

$$I(z,t) = I(z+\Delta z,t) + dI$$

$$I(z+\Delta z,t) - I(z,t) = -dI$$

$$I(z+\Delta z,t) - I(z,t) = -VY\Delta z$$

$$\boxed{\frac{\partial I}{\partial z} = -VY}$$

now we find the wave equation
(requires 2nd derivation)

$$\frac{\partial^2 V}{\partial z^2} = -X\frac{\partial I}{\partial z} = -X(-VY) = V\cdot XY$$

note: $\left.\begin{array}{l} X = R+j\omega L \\ Y = G+j\omega C \end{array}\right\}$ complex nums

$$\frac{\partial^2 V}{\partial z^2} - \gamma^2 V = 0$$

where $\gamma = \sqrt{XY}$

$\gamma = \left((R+j\omega L)(G+j\omega C)\right)^{1/2}$

(helmholtz eqn)

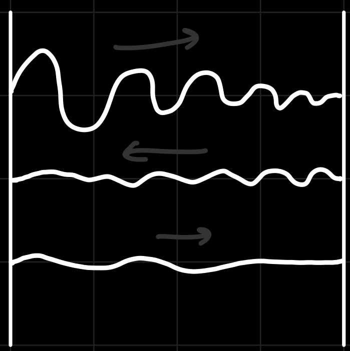
$\gamma = \alpha + j\beta$

general soln:

$$V = V_+ e^{-\gamma z} + V_- e^{\gamma z}$$

$$V = V_+ e^{-\alpha z} e^{-j\beta z} + V_- e^{\alpha z} e^{j\beta z}$$

$$V(z,t) = V_+ e^{-\alpha z} e^{j(\omega t - \beta) z} + V_- e^{\alpha z} e^{j(\omega t + \beta) z}$$



continuous transmission
and reflection

Now, when will have lossless transmission
i.e. no attenuation in signal?

when $e^{-\alpha z}$ and $e^{\alpha z} = 0$

↳ $\alpha = 0$ i.e. γ = purely complex

so, $R = 0$ and $G = 0$

↳ series resistance $= 0$

↳ leakage current $= 0$ (R_{shunt} ↑↑↑)

so, $\gamma = j\beta = j\omega\sqrt{LC} \rightarrow \quad \underline{\beta = \omega\sqrt{LC}}$

wavelength : $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{1}{f\sqrt{LC}}$

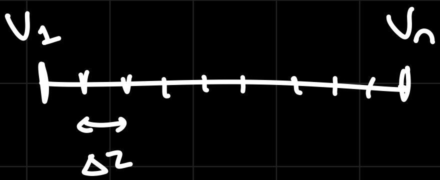
and velocity → $\boxed{v = \frac{1}{\sqrt{LC}}}$

this is of the guided wave

quiz 3 and quiz 4 syllabus same
(open book) (closed book)

quiz 5 and quiz 6 syllabus : till today
(open book) (closed book)
30 mins 30 mins

PROJECT
{finite difference method}



$\frac{V_{i+1} - V_i}{\Delta z}$ forward difference

$\frac{V_i - V_{i-1}}{\Delta z}$ backward difference

central difference (best)

Lecture: TRANSMISSION LINES

$$\frac{\partial I}{\partial z} = -VY \qquad \frac{\partial V}{\partial z} = -IZ$$

admittance (Y) · impedance (Z) · same as X

$$\frac{\partial^2 I}{\partial z^2} - \gamma^2 I = 0 \qquad \frac{\partial^2 V}{\partial z^2} - \gamma^2 V = 0$$

$$\gamma = \sqrt{XY}$$

$$X = R + j\omega L$$

$$Y = G + j\omega C$$

$$\frac{1}{R_s}$$

forward travelling · backward travelling

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$= V_0^+ e^{-i\beta z} e^{-\alpha z} + V_0^- e^{i\beta z} e^{\alpha z}$$

$$v = \frac{1}{\sqrt{LC}} \quad : \text{Velocity}$$

Tx line with no reflected wave

$$V(z) = V_0^+ e^{-\gamma z}$$
$$I(z) = I_0^+ e^{-\gamma z}$$

$$\frac{V(z)}{I(z)} = Z_0 = \frac{V_0^+}{I_0^+}$$

Characteristic Impedance
$\hookrightarrow Z_0$

$$\frac{\partial V}{\partial z} = \frac{\partial (V_0 e^{-\gamma z})}{\partial z} = -XI = -X I_0 e^{-\gamma z}$$

$$-\gamma V_0^+ e^{-\gamma z} = -X I_0^+ e^{-\gamma z}$$

$$\frac{V_0^+}{I_0^+} = \frac{X}{\gamma} = \frac{X}{\sqrt{XY}} = \sqrt{\frac{X}{Y}}$$

So, $$Z_0 = \sqrt{\frac{X}{Y}} = \frac{\sqrt{R+j\omega L}}{\sqrt{G+j\omega C}}$$

In lossless transmission:

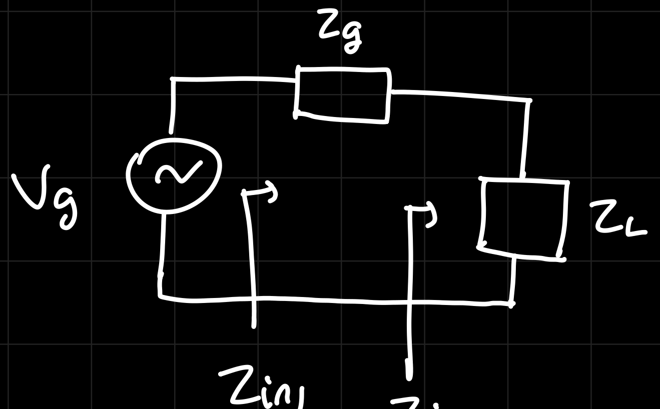
$$Z_0 = \sqrt{\frac{X}{Y}} = \sqrt{\frac{0+j\omega L}{0+j\omega C}} = \sqrt{\frac{L}{C}}$$

$$\boxed{Z_0 = \sqrt{\frac{L}{C}}}$$

for backward travelling wave:

$$Z_0 \to -Z_0$$

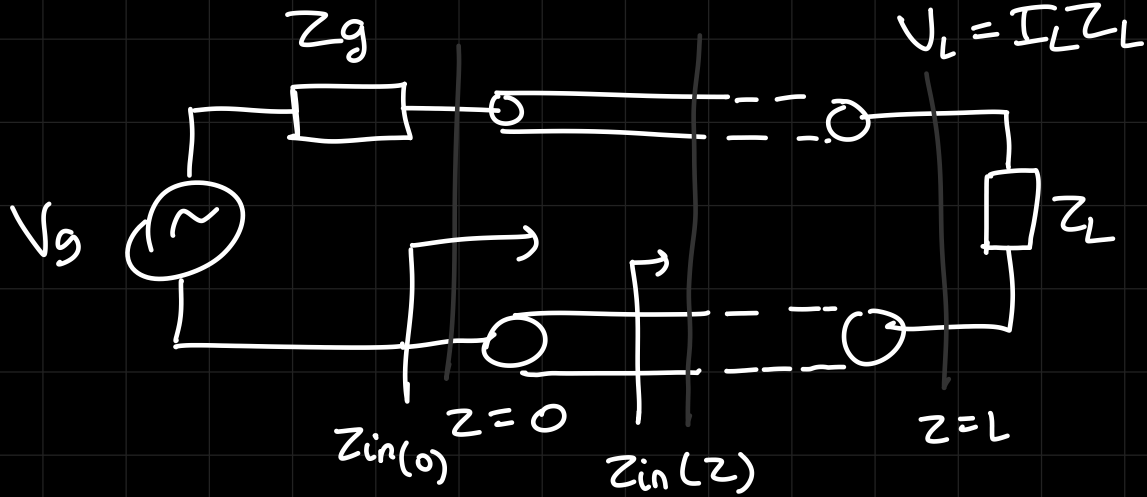
① * $\begin{cases} V_0^- = -Z_0 I_0^- \\ V_0^+ = Z_0 I_0^+ \end{cases}$ $\quad \dfrac{V_0^-}{I_0^-} = -Z_0$



$Z_{in1} = Z_g + Z_L \qquad Z_{in2} = Z_L$

for a small circuit like this ← it ez to determine input impedances

for transmission lines we have to consider the π ckts of R, L, C, Rs as well



Q) given L; what is Z_{in}?

$$Z_{in}(z) = \frac{V(z)}{I(z)}$$

$$Z_{in} = Z_{in}(0) = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{I_0^+ + I_0^-}$$

$$\{\text{from } 1\} = \frac{V_0^+ + V_0^-}{\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}} = Z_0\left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right)$$

2

$$I_L = I_0^+ e^{-\gamma L} + I_0^- e^{\gamma L}$$

$-/+$

$$\begin{cases} V_L = V_0^+ e^{-\gamma L} + V_0^- e^{\gamma L} \\ Z_0 I_L = V_0^+ e^{-\gamma L} - V_0^- e^{\gamma L} \end{cases}$$

$\downarrow$

$$V_L + Z_0 I_L = 2V_0^+ e^{-\gamma L}$$

$$V_L - Z_0 I_L = 2V_0^- e^{\gamma L}$$

$$V_0^+ = \frac{1}{2}(Z_L + Z_0) I_L e^{\gamma L}$$

$$V_0^- = \frac{1}{2}(Z_L - Z_0) I_L e^{-\gamma L}$$

back to ②

$$Z_{in} = Z_0 \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)$$

$$= Z_0 \left(\frac{(Z_L+Z_0)e^{\gamma L} + (Z_L-Z_0)e^{-\gamma L}}{(Z_L+Z_0)e^{\gamma L} - (Z_L-Z_0)e^{-\gamma L}} \right)$$

$$= Z_0 \left(\frac{Z_L(e^{\gamma L}+e^{-\gamma L}) + Z_0(e^{\gamma L}-e^{-\gamma L})}{Z_L(e^{\gamma L}-e^{-\gamma L}) + Z_0(e^{\gamma L}+e^{-\gamma L})} \right)$$

$$= Z_0 \left(\frac{Z_L \cdot \cosh(\gamma L) + Z_0 \cdot \sinh(\gamma L)}{Z_L \cdot \sinh(\gamma L) + Z_0 \cdot \cosh(\gamma L)} \right)$$

$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh(\gamma L)}{Z_L \tanh(\gamma L) + Z_0} \right]$

for a lossless transmission: $\gamma = j\beta$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tanh(\beta L)}{Z_0 + jZ_L \tanh(\beta L)} \right]$$

remember : reflection coefficient

$$\Gamma(z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \frac{V_0^+}{V_0^-} e^{2\gamma z}$$

$$\Gamma_L(z) = \frac{V_0^- e^{\gamma L}}{V_0^+ e^{-\gamma L}}$$

$$\left.\begin{aligned} \tfrac{1}{2}(V_L + Z_0 I_L) &= V_0^+ e^{-\gamma L} \\ \tfrac{1}{2}(V_L - Z_0 I_L) &= V_0^- e^{\gamma L} \end{aligned}\right\} \text{from before}$$

$$\Gamma_L(z) = \frac{V_L - Z_0 I_L}{V_L + Z_0 I_L} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$\Gamma_L(z) = \frac{Z_L - Z_0}{Z_L + Z_0}$

reflection coeff right at the load

We won't have any reflected wave

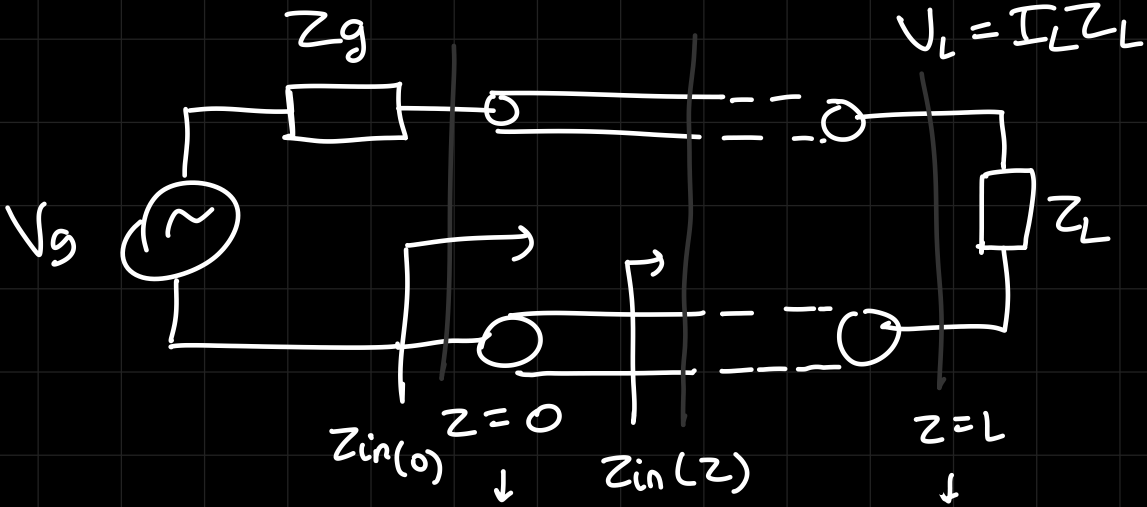
① when: $\Gamma_L(z) = 0$ i.e. $Z_L = Z_0$

(matched impedance/Load)

RFIC

② or when the 1st medium is infinitely large i.e. the interface is at ∞

$(L \to \infty)$



let : $l' = l$ $\qquad$ $l' = l - z$

$$\Gamma(z) = \frac{V_0^- e^{\gamma L}}{V_0^+ e^{-\gamma L}} \times \frac{e^{-\gamma L}}{e^{-\gamma L}} \times e^{2\gamma z}$$

$$= \Gamma_L e^{-2\gamma(l-z)} = \Gamma_L e^{-2\gamma l'}$$

$\Gamma(z) = \Gamma_o e^{-2\gamma l'}$

$$\rightarrow Z_{in}(z) = \frac{V(z)}{I(z)} = \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}} \qquad \times \frac{e^{\gamma z}}{e^{\gamma z}}$$

$$= \frac{V_0^+ + V_0^- e^{2\gamma z}}{\frac{V_0^+}{Z_0} - \frac{V_0^- e^{2\gamma z}}{Z_0}} \qquad \div \frac{V_0^+}{V_0^+}$$

$$= \left(\frac{1 + V_0^-/V_0^+ \, e^{2\gamma z}}{1 - V_0^-/V_0^+ \, e^{2\gamma L}} \right) Z_0$$

$Z_{in} = Z_0 \left(\frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)$

reflection coeff (current)

$$\Gamma_{current_L} = \frac{I_0^- e^{\gamma L}}{I_0^+ e^{-\gamma L}} = -\frac{V_0^-}{V_0^+} e^{2\gamma L} = -\Gamma_{voltage_L}$$

$\Gamma_{I_L} = -\Gamma_{V_L}$

Considering lossless transmission

$$V(z) = V_0^+ e^{-i\beta z} + V_0^- e^{i\beta z}$$

$$= V_0^+ \cos\beta z - j V_0^+ \sin\beta z + V_0^- \cos\beta z + j V_0^- \sin\beta z$$

$$= \cos\beta z (V_0^+ + V_0^-) - j \sin\beta z (V_0^+ - V_0^-)$$

$$\text{amplitude} = |V(z)| = \sqrt{\cos^2\beta z (V_0^+ + V_0^-)^2 + \sin^2\beta z (V_0^+ - V_0^-)^2}$$

$$= V_0^{+2} + V_0^{-2} + 2V_0^+ V_0^- \cos(2\beta z)$$

maxima : $\cos(2\beta z) = \cos(2n\pi) = 1$

$\downarrow$

$$z = \frac{n\pi}{\beta}$$

$$\boxed{|V(z)|_{max} = V_0^+ + V_0^-}$$

note: $\beta = \omega\sqrt{LC}$

minima : $\cos(2\beta z) = \cos((2n+1)\pi) = -1$

$$z = \frac{(2n+1)\pi}{\beta} \longrightarrow \boxed{|V(z)|_{min} = V_0^+ - V_0^-}$$

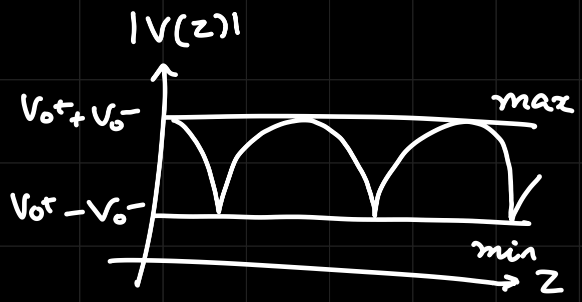
$\beta \rightarrow \frac{2\pi}{\lambda}$

max : at $z = \frac{n\pi}{\beta} = \frac{n\lambda}{2}$

min : at $z = (n + \frac{1}{2})\lambda$

$\beta \equiv$ wave numbe

and hence $\beta = \frac{2\pi}{\lambda}$



Standing wave pattern

if $V_0^+ = V_0^-$ → pure standing wave

if no reflected back i.e $V_0^- = 0$,

we get a straight line standing wave pattern

VSWR (voltage standing wave ratio)

$$s = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

if $\Gamma_L = 1/-1$ i.e. everything is getting reflected back

↓

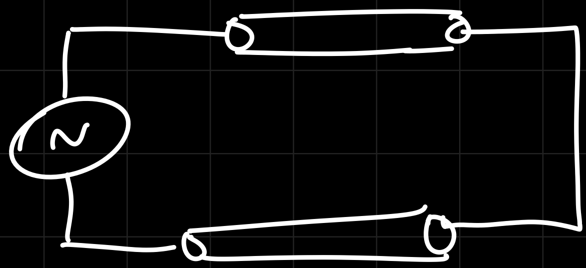
$s = \infty$ (standing wave)

if $|\Gamma_L| = 0 \rightarrow s = 1$

$$\boxed{1 \leq s \leq \infty}$$

Lecture

transmission line ended with short circuit



for low freq ckts, KVL and KCL applies and Zin = 0

for high freq → $Z_{in} = Z_0 \left[\dfrac{Z_L + Z_0 \tan(\gamma L)}{Z_0 + Z_L \tan(\gamma L)} \right]$

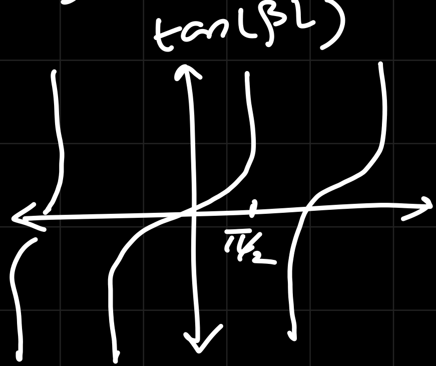
but $Z_L = 0$

$$Z_{in} = Z_0 \tan(\gamma L)$$

for no transmission loss:

$$Z_{in} = j Z_0 \tan(\beta L)$$

inductive or capacitive? depends on L



reflection coefficient?

we know $\Gamma_L(z) = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

and $Z_L = 0$

so $\Gamma_V(z) = -1$

and $\Gamma_I(z) = -\Gamma_V(z) = 1$

if $\beta L = \dfrac{\pi}{2}$

$\beta = \dfrac{2\pi}{\lambda} \rightarrow \dfrac{2\cancel{\pi}}{\lambda} L = \dfrac{\cancel{\pi}}{2}$

$\boxed{L = \dfrac{\lambda}{4}}$ and $Z_{in} = \infty$ because $\tan(\pi/2) = \infty$

for $L = \dfrac{\lambda}{2} \rightarrow \beta L = \pi \rightarrow Z_{in} = 0$

what if transmission line ended by open circuit ?

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tan(\gamma L)}{Z_0 + Z_L \tan(\gamma L)} \right]$$

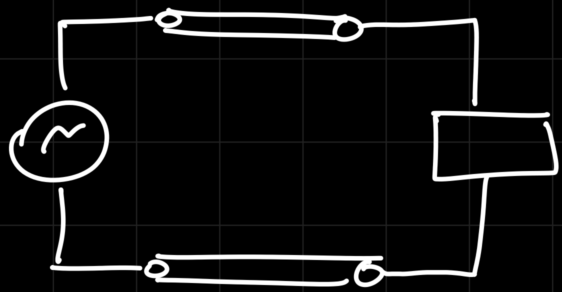
but $Z_L = \infty$

$$Z_{in} = Z_0 \left[\frac{1 + \frac{Z_0}{Z_L} \tan(\gamma L)}{\frac{Z_0}{Z_L} + \tan(\gamma L)} \right]$$

$$= Z_0 \left[\frac{1}{j\tan(\beta L)} \right] = -jZ_0 \cot(\beta L)$$

$$Z_{in} = -jZ_0 \cot(\beta L)$$

So if we really want ∞ input impedance we don't just do oc we short output and put $L = \frac{\lambda}{2}$



$$Z_L = j\beta$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{j\beta - Z_0}{j\beta + Z_0}$$

$|\Gamma_L|$ = reflection coefficient = 1

perfect / pure reflection

no absorption of energy

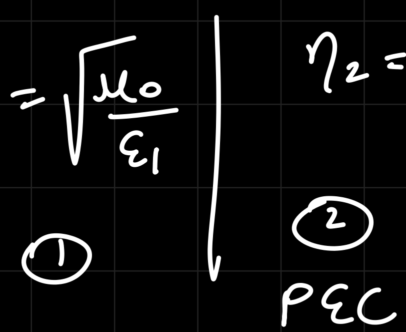
goes back to SRC

Now prove,

A Perfect Electric conductor is a perfect reflector

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

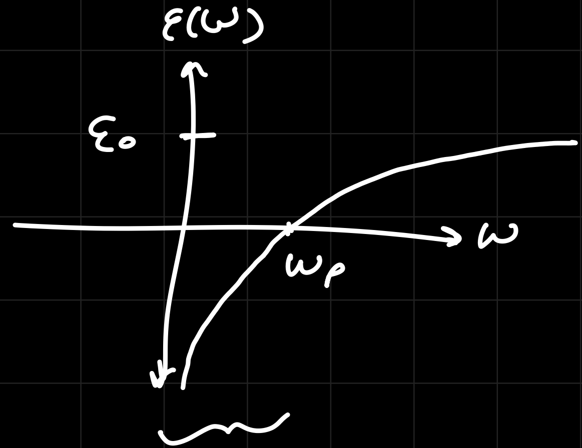
$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$



plasma frequency: $\omega_p \sim$ UV Range

metal $\rightarrow \varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$

for $\omega < \omega_p$ and lossless transmission,



$$\varepsilon(\omega) = -ve$$

So, $\eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{-a}}$

$$= jm$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{jm - \eta_1}{jm + \eta_1}$$

$$|\Gamma| = \frac{m^2 + n^2}{m^2 + n^2} = \underline{\underline{1}}$$

hence for $\omega < \omega_p$ and lossloss transmission, a metal acts as a perfect reflector