Lecture before Quiz 3

for the 1d uniform plane wave:

$$\bar{E}(z,t) = \left(\bar{A}e^{i(\omega t+kz)} + \bar{B}e^{i(\omega t-kz)}\right)\hat{x}$$

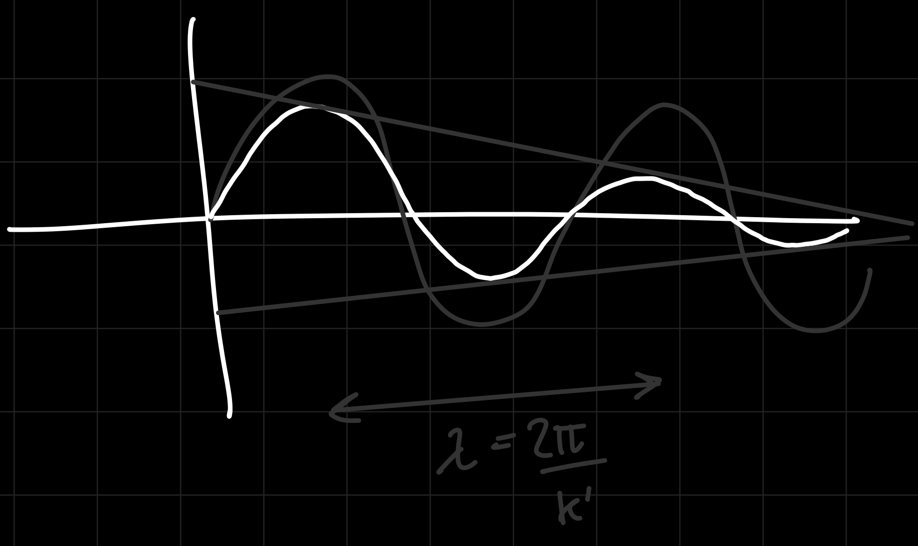
polarization of the EM wave

$$K = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$ → wave number

$$\lambda = \frac{2\pi}{K}$$

k'' defines the decay

k' defines the wavelength $\lambda = \frac{2\pi}{k'}$



Just like time and frequency form a fourier pair,
k and λ also form a fourier pair

also → k domain ≡ momentum space

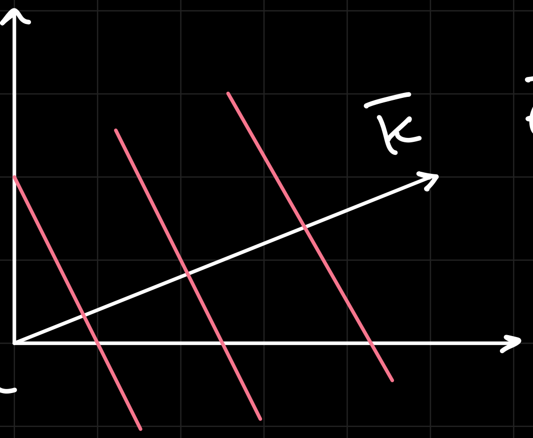
$$\omega = \frac{2\pi}{T} \quad , \quad k = \frac{2\pi}{\lambda}$$

What is the physical meaning of k?
↳ represents the number of cycles in a unit distance

$$k\lambda = 2\pi$$

in a given phase of 2π, what fraction of a complete wavelength can fit?
eg → $1\lambda = 2\pi$ or 1.5λ or 2.5λ

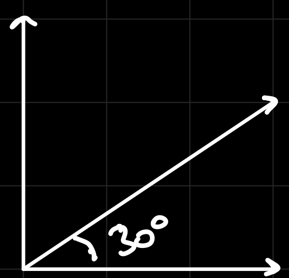
Can we have a non uniform plane wave?



$\bar{E}$ and $\bar{B}$ are non uniform because wave varies not in a single axis

$$\bar{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$

currently only a 2d problem
so $k_z = 0$

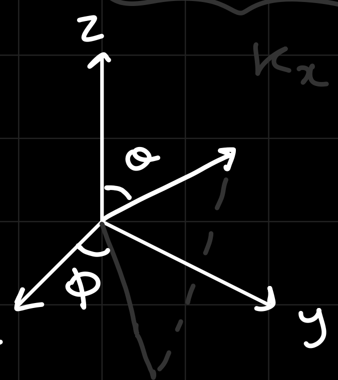


$$\bar{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$

$$= \hat{x} |k| \cos\theta + \hat{y} |k| \sin\theta$$

for a vector in sperical coordinates

$$\bar{k} = \underbrace{\hat{x} |k| \sin\theta \cos\phi}_{k_x} + \underbrace{\hat{y} |k| \sin\theta \sin\phi}_{k_y} + \underbrace{\hat{z} |k| \cos\theta}_{k_z}$$



$$\frac{k_x}{k_y} = \tan\phi$$

$$\phi = \tan^{-1}\left(\frac{k_x}{k_y}\right)$$

from $\bar{k}$, we ca find

frequency
direction of propagation

$$f = \frac{2\pi c}{\lambda} = \frac{2\pi c}{2\pi/|k|} = |k|c$$

going back:

$$\nabla^2 \bar{E} + k^2 E = 0$$

↳ # of eq^{n}s = ?

↳ 3 because of the 3 components

$$\left.\begin{aligned} \nabla^2 \bar{E}_x + k^2 E_x = 0 \\ \nabla^2 \bar{E}_y + k^2 E_y = 0 \\ \nabla^2 \bar{E}_z + k^2 E_z = 0 \end{aligned}\right\}$$

also note:

$$k^2 = \left(\frac{\omega}{c}\right)^2 = \omega^2 \mu \varepsilon$$

↓

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

Variable separation :

$$E_x(x,y,z) = X(x)\,Y(y)\,Z(z)$$

we proved earlier :

solution of 1d wave

↳ plane wave

now we prove,

soln of 3d is also

plane wave

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{} = -k^2$$

let → $-k_x^2$ $\quad -k_y^2 \quad -k_z^2$

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

(sphere eqn

radius $= k^2 = \left(\frac{\omega}{c}\right)^2$)

we can only choose 2 variables arbitrarily (independently).
The 3rd must be a fixed value in order to satisfy

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

because $k^2 = \left(\frac{\omega}{c}\right)^2$ is const

since ω (frequency) is given

$$k_z = \pm\sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

soln comes out
in the form →

$$X \simeq e^{\pm i k_x x}$$
$$Y \simeq e^{\pm i k_y y}$$
$$Z \simeq e^{\pm i k_z z}$$

with some constants but we will just assume that in final soln

$$E_x(x,y,z) = E_{x_0} e^{\pm i k_x x} e^{\pm i k_y y} e^{\pm i k_z z}$$

↳ wave of 3 planes

$$E_x(x,y,z) = E_{x_0} e^{\pm i(k_x x + k_y y + k_z z)}$$

as a dot product of 2 vectors

$$= E_{x_0} e^{\pm i(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)(\hat{x}x + \hat{y}y + \hat{z}z)}$$

$$E_x(x,y,z) = E_{x_0} e^{\pm i\vec{k}\vec{r}}$$ → radius vector

plane wave ✓
but not uniform

now back the $\bar{E}$ equation
(general)

$$\bar{E} = e^{\pm i\bar{k}\cdot\bar{r}}(\hat{x}E_{x_0} + \hat{y}E_{y_0} + \hat{z}E_z)$$

so, $\bar{k}$ determines the propagation of the wave and the general eq^n of the wave is $e^{\pm i(\bar{k}\cdot\bar{r})}$

Remember Maxwells Eq^n in freq domain

$$\bar{\nabla}\cdot\bar{E} = 0 \qquad \bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla}\times\bar{E} = -i\omega B \qquad \bar{\nabla}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

we can achieve the time domain equivalent of a wave as the superposition of different wavelengths in the K domain.

$$g(t) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega t}\, d\omega$$

Fourier optics

↳ used for wide angle clicks in portable computing devices like VR headsets, Mobile phones etc.

$$E_x = E_{x0}\, e^{i(k_x x + k_y y + k_z z)}$$

$$\frac{\partial E_x}{\partial x} = -ik_x\left(E_{x0}\, e^{i(k_x x + k_y y + k_z z)}\right)$$

$$= -ik_x E_x$$

like fourier formulas

$$\bar{K}\cdot\bar{E} = 0 \quad , \quad \bar{K}\cdot\bar{B} = 0$$

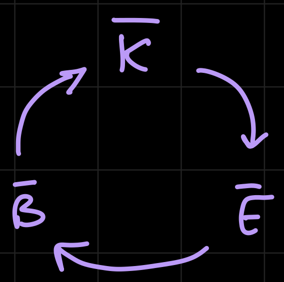
So, $\bar{K}$, $\bar{E}$ and $\bar{B}$ fields are perpendicular to each other

$$\bar{\nabla}\times\bar{E} = -i\omega\bar{B} \qquad\qquad \bar{\nabla}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

$$-i\bar{K}\times\bar{E} = -i\omega\bar{B} \qquad\qquad -i\bar{K}\times\bar{B} = i\omega\mu\varepsilon\bar{E}$$

$$\underbrace{\bar{K}\times\bar{E} = \omega\bar{B} \qquad\qquad \bar{K}\times\bar{B} = -\omega\mu\varepsilon\bar{E}}$$

not just direction,
we have this relation



we know, $\dfrac{|E|}{|B|} = c$ but how?

$$\bar{K}\times\bar{E} = \omega\bar{B}$$

$$|K|\,|E| = \omega|B|$$

$$\frac{|E|}{|B|} = \frac{\omega}{|K|} = \underline{\underline{c}}$$

We knew $\bar{B} = \mu \bar{H}$

and

Voltage / Current $\quad \dfrac{\vec{E}}{\vec{H}} = \mu \dfrac{\bar{E}}{\bar{B}} = \mu c = \dfrac{\mu}{\sqrt{\mu \varepsilon}} = \sqrt{\dfrac{\mu}{\varepsilon}}$

Impedance of a medium

Why not called Resistance? $\longrightarrow$ because it can be complex