Dirac Delta Function

let $\bar{v} = \frac{1}{r^2}\hat{r}$

$$\bar{\nabla}\cdot\bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\right) = 0$$

reason → we used the divergence formula for spherical coordinates
↪ not cartesian.

sphere of radius : R ↖

$$\oint_S v\cdot da = \int \frac{1}{R^2}\hat{r}\cdot R^2 \sin\theta \, d\theta d\phi \, \hat{r}$$

$$= \left(\int_0^{\pi}\sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$= [-\cos\theta]_0^{\pi}\cdot[\phi]_0^{2\pi}$$

$$= (1+1)(2\pi) = \underline{4\pi}$$

now,
$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)$$

quiz 2 - questions →

Q2) (a) $\int_{-1}^{+1}(r^2+2)\,\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)dv$

$$= \int_{-1}^{+1}(r^2+2)\,4\pi\delta^3(r)\,dv$$

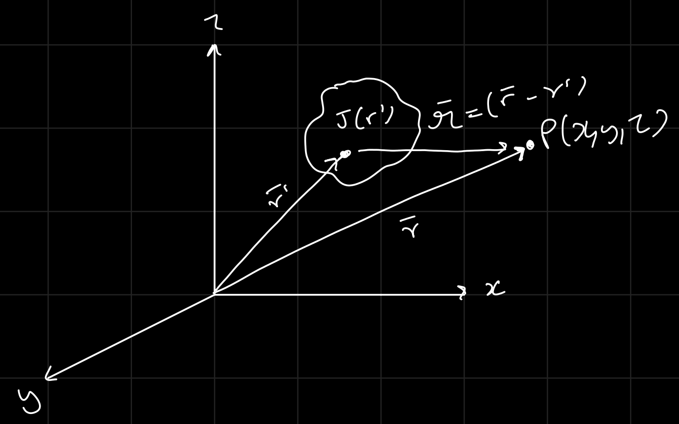
$$= (0+2)(4\pi) = \underline{8\pi}$$

(b) $\int_1^{\infty}(r^2+2)\,\bar{\nabla}\left(\frac{\hat{r}}{r^2}\right)dv$

$$= \int_1^{\infty}(r^2+2)\,\underbrace{4\pi\delta^3(r)}_{\downarrow}\,dv = \underline{0}$$

always $= 0$
$\because r = 0$ not included
in the integral

Lecture after midsem



$$J(\bar{r}') \rightarrow A/m^2 = \frac{dI}{da_\perp}$$

→ dependent on x', y', z'

$$\bar{B}(r) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\bar{\nabla} f = -\bar{\nabla}' f$$

note: assuming steady current

$$\bar{\nabla} \cdot \bar{B} = \frac{\mu_0}{4\pi} \int \nabla \left(\bar{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\mu_0}{4\pi} \int \underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} (\bar{\nabla} \times \bar{J}(r'))}_{\text{zero because div wrt } r \text{ but } J(r')} - \bar{J} \left(\bar{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

$$\bar{\nabla} \cdot (\bar{P} \times \bar{Q}) = \bar{Q} (\bar{\nabla} \times \bar{P}) - \bar{P} (\bar{\nabla} \times \bar{Q})$$

and since $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla \left(\frac{1}{r} \right) \rightarrow$ curl $= 0$

So, $\boxed{\nabla \cdot B = 0}$

Curl of magnetic field

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

↳ magnetic vector potential
(need not be unique)

$\bar{A}_{new} = \bar{A} + \bar{\nabla}\theta$ ← gauge freedom

$\bar{\nabla} \times \bar{A}_{new} = \bar{\nabla} \times A + 0 = \bar{B}$ always

if $\bar{B}$ given, find the magnetic vector potential
or given, $\bar{J}$, find the magnetic field

since $B \propto \frac{1}{r^2} \to A \propto \frac{1}{\mathscr{r}}$ because $\bar{B} = \bar{\nabla} \times \bar{A}$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \left(\bar{J}(r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) dV'$$

we know $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla\left(\frac{1}{\mathscr{r}}\right)$

$$\bar{\nabla} \times \bar{A} = \frac{\mu_0}{4\pi} \int \underbrace{\left(\bar{J}(r') \times \bar{\nabla}\left(\frac{1}{\mathscr{r}}\right) \right)}_{(\bar{P} \times \bar{\nabla} f)} dv'$$

$$\bar{\nabla}\times(f\bar{P}) = f(\bar{\nabla}\times\bar{P}) - \bar{P}\times(\bar{\nabla}f)$$

$$\rightarrow \quad P\times(\bar{\nabla}f) = f(\bar{\nabla}\times\bar{P}) - \bar{\nabla}\times(f\bar{P})$$

here $\bar{\nabla}$ wrt $\mathscr{r}$ but $J(r')$

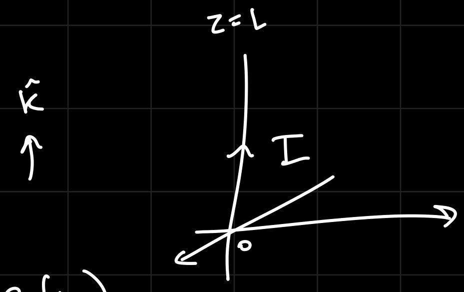
so, $f(\bar{\nabla}\times\bar{P}) = 0$

$$\bar{\nabla}\times\bar{A} = \frac{\mu_0}{4\pi}\int\left(\bar{J}(r')\times\bar{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right)dv'$$

$$\bar{\nabla}\times\bar{A} = \frac{\mu_0}{4\pi}\int\bar{\nabla}\times\left(\frac{\bar{J}(r')}{\mathscr{r}}\right)dv'$$

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi}\int\frac{\bar{J}(r')\,dv'}{\mathscr{r}}}$$

current carrying filamentary conductor



$\bar{J}(r') = ?$

$J = \bar{I}\,\delta(x)\delta(y)$

$$\bar{J} = \frac{dI}{da_\perp}$$

$$\bar{A} = \frac{\mu_0}{4\pi} \iiint_{V'} \frac{\bar{J}(r')}{\mathscr{r}} \, dx' dy' dz'$$

$$\bar{J}(r') \times dx'dy' = I$$

for $x, y \to$ infinitesimally small

$$\bar{A} = \frac{\mu_0}{4\pi} \underbrace{\int_0^{\infty}\int_0^{\infty} \delta(x)\delta(y)\,dx\,dy}_{1} \cdot I\hat{z} \int_0^{L} \frac{dz}{\underbrace{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{1/2}}_{\mathscr{r}}}$$

$=$ function of x, y, z

curl$(E) = 0$ but

div$(E) \Rightarrow$ Gauss's Law

div$(B) = 0$ but

curl of $B \Rightarrow$ Ampere's Law

$$\bar{B} = \frac{\mu_0}{4\pi} \int \bar{J}(r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, dV'$$

$$\bar{\nabla} \times \bar{B} = ?$$

problem $\to$ cannot apply chain rule to $\nabla \times (\bar{P} \times \bar{\varphi})$

$$\bar{\nabla}\times(\bar{P}\times\bar{\varphi}) = \underbrace{(\bar{\varphi}\cdot\bar{\nabla})\bar{P}} - (\bar{P}\cdot\bar{\nabla})\varphi + \bar{P}(\bar{\nabla}\cdot\bar{\varphi}) - \bar{\varphi}(\bar{\nabla}\cdot\bar{P})$$

$$\left(\varphi_x \frac{\partial P}{\partial y} + \varphi_y \frac{\partial P}{\partial y} + \varphi_z \frac{\partial P}{\partial z}\right)$$

$\bar{P} = \bar{J}$ but since $\bar{J}$ depends on r', not r

so, 1st and last term $= 0$

non zero term:

$$\bar{\nabla}\times(\bar{P}\times\bar{\varphi}) = \bar{P}(\bar{\nabla}\cdot\bar{\varphi}) - (\bar{P}\cdot\bar{\nabla})\bar{\varphi}$$

$$\bar{\nabla}\times\bar{B} = \frac{\mu_0}{4\pi}\left[\int \bar{J}\left(\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\right) dv' + \int (\bar{J}\cdot\bar{\nabla}')\frac{\hat{r}}{r^2}\, dv'\right]$$

not minus

because $\bar{\nabla}f = -\bar{\nabla}'f$

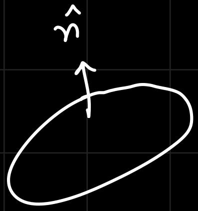
$$= \frac{\mu_0}{4\pi}\left[\int \bar{J}(\bar{r}')\cdot 4\pi\delta^3(\bar{r}-\bar{r}')\,dv + \int (\bar{J}\cdot\bar{\nabla})\frac{\hat{r}}{r^2}\,dv'\right]$$

$$= \frac{\mu_0}{4\pi}\times 4\pi\times J(\bar{r}) + \frac{\mu_0}{4\pi}\int(\bar{J}\cdot\bar{\nabla})\frac{\hat{r}}{r^2}\,dv'$$

$$= \mu_0 J(\bar{r}) + \frac{\mu_0}{4\pi}\int(\bar{J}\cdot\bar{\nabla})\frac{\hat{r}}{r^2}\,dv'$$

assuming that 2nd term is zero

$$\bar{\nabla} \times \bar{B} = \mu_0 J(\bar{r}) \rightarrow$$ differential form of Ampere's Law



$$\int (\bar{\nabla} \times \bar{B}) \cdot ds = \mu_0 \int \bar{J} \cdot d\bar{s}$$

$$\boxed{\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed}}$$

Integral form of Ampere's Law

Proving the previous assumption

Steady current $\rightarrow$ $\bar{\nabla} \bar{J}(r') = 0$

So, $\bar{\nabla}' \bar{J}(r') = 0$ also

We need to prove $\Rightarrow$ $\displaystyle\int (\bar{J}(r') \cdot \bar{\nabla}) \frac{\hat{r}}{r^2} dV$

$$\frac{\hat{r}}{r^2} = \frac{r\hat{r}}{r^3} = \frac{\hat{x}(x-x') + \hat{y}(y-y') + \hat{z}(z-z')}{r^3}$$

we need contribution of entire system but we will be checking for individual components

$$= \int_{V'} \left(\bar{J}(r') \cdot \bar{\nabla} \right) \frac{(x - x')}{r^3} \, dv'$$

$$\bar{A} \cdot (\bar{\nabla} f) = \bar{\nabla}' \cdot (f \bar{A}) - \underbrace{\bar{f} (\bar{\nabla}' \cdot \bar{A})}_{0 \text{ because } \bar{\nabla}' \cdot J = 0}$$

$$= \int_{V'} \bar{\nabla}' \cdot \left(\frac{J(r')(x - x')}{r^3} \right) dv'$$

$$= \oint_{S'} \frac{\bar{J}(r')(x - x')}{r^3} \, ds$$

since $J(r) = 0$ outside V'

this $\int$ holds for $V' + \Delta$ as well

$$\oint_{S' + \Delta} \underbrace{\bar{J}(r') \Big|_{surface}}_{\text{zero}} \frac{(x - x')}{r^3} \, ds'$$

$\rightarrow$ Assumption correct

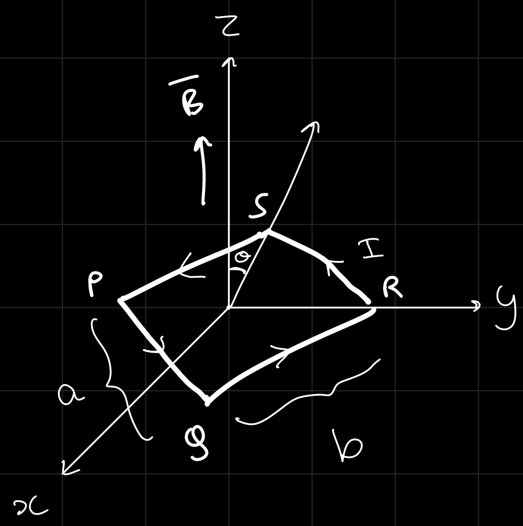
Magnetic field in a matter

note: $\bar{\nabla}.\bar{B} = 0 \rightarrow \int(\nabla.B)dv = \oint B ds = 0$

Gauss Law

Diamagnets
Paramagnets
Ferromagnets

assuming a rectangular closed loop



θ: angle of plane's normal with z-axis

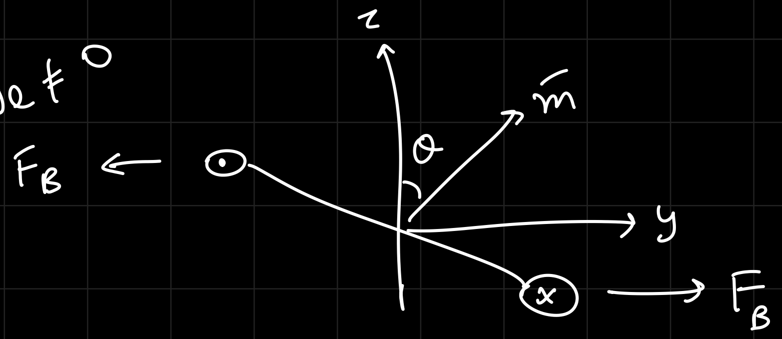
$\bar{F}_B = I(\bar{l} \times \bar{B})$

from lorentz Law

$\bar{F}_B = q(\bar{v} \times \bar{B})$

The PQ and RS sides' force will cancel out but the force due to the other two sides will make the rectangle rotate

net Force = 0
but net torque ≠ 0



at equilibrium
$\tau = 0 \Rightarrow \bar{m}$ aligns paralells with $\bar{B}$

$$\bar{\tau} = \bar{F}.a \sin\theta \, \hat{x}$$
$$= IbBa \sin\theta \, \hat{x}$$
$$= IAB \sin\theta \, \hat{x}$$

area

$IA = \bar{m}$: magnetic moment

$$= |\bar{m}||\bar{B}| \sin\theta \, \hat{x} \Rightarrow \boxed{\tau = \bar{m} \times \bar{B}}$$

assuming the loop is kept in a uniform field

In case of dipoles →

{ constant thermal fluctuation throws dipole out of equilibrium }

Similar reason for "out of equilibrium" state in this case as well requirements:

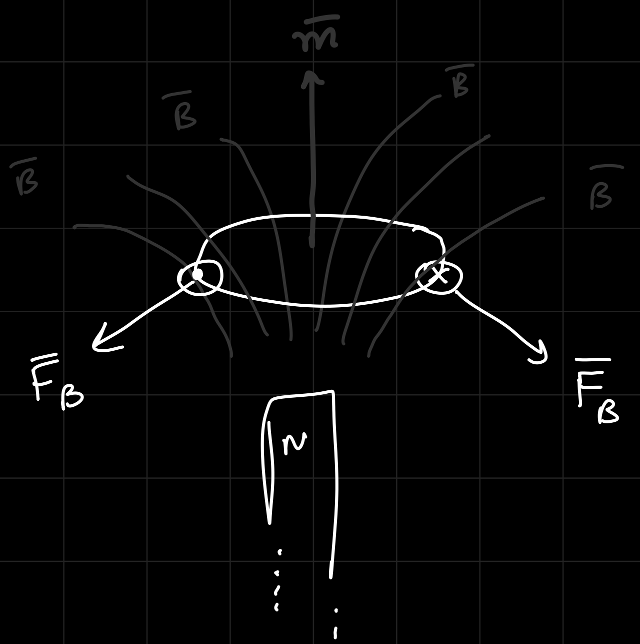
liquid (not gas)

low temperature

or else these small loops wont respond to the magnetic field

non-uniform magnetic field ≡ $\bar{B}$ varying in space

eg:



Diamagnetic material

↳ tries to align $\bar{m}$ in the opposite direction to $\bar{B}$

Paramagnetism

↳ $\bar{m}$ and $\bar{B}$ align in the same direction

$ cannot use gas because of not enough density
for it to exhibit enough magnetization

hence we go for the next lightest type
of materials → liquid

magnetic susceptibility
↳ determines para/ferro/diamagnetism

liquid nitrogen → diamagnetic
doesn't hold

liquid oxygen → paramagnetic
sticks b/w poles because of very
high magnetic susceptibility

aluminium → paramagnetic
but doesn't hold because gravitational
pull is greater because of high mass (solid)

$\bar{E}$ arises due to q

$\bar{B}$ arises due to I

$\bar{B} = \bar{\nabla} \times \bar{A}$

$\hookrightarrow$ magnetic vector potential

$\bar{A}$ and $\bar{V}$ vary with $\frac{1}{\mathscr{r}}$ but for dipole $\rightarrow \frac{1}{\mathscr{r}^2}$

$\bar{A}$ ↓ magnetic vec potential

$\bar{V}$ ↓ electrostatic potential

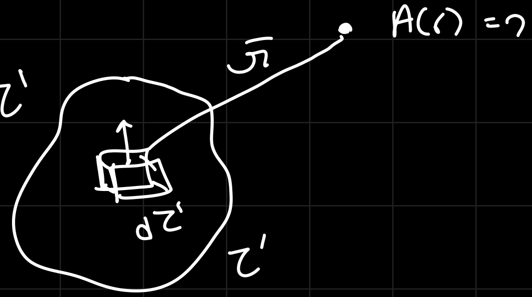
$$\vec{A}(r) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\bar{m}$ ↑ magnetic dipole moment

just like $\bar{P}$: polarization,
we have $\bar{M}$: magnetization
($\bar{m}$ / volume)

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int_V \frac{\bar{m} \times \hat{\mathscr{r}}\, d\tau'}{\mathscr{r}^2}$$ due to continuous material

$$= \frac{\mu_0}{4\pi} \int_V \left[\bar{m} \times \bar{\nabla}' \left(\frac{1}{\mathscr{r}} \right) \right] d\tau'$$



$$(\bar{\nabla}' \times f\bar{p}) = f(\bar{\nabla} \times \bar{P}) - \bar{P} \times (\bar{\nabla}' f)$$

$$\Rightarrow \quad \bar{P} \times \bar{\nabla}' f = f(\bar{\nabla}' \times \bar{P}) - \bar{\nabla} (f\bar{P})$$

$\bar{p} = \bar{m}$, $f = \frac{1}{r}$

$$\bar{A}(r) = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} (\bar{\nabla}' \times \bar{m}) - \bar{\nabla} \times \left(\frac{\bar{m}}{\mathscr{r}} \right) \right] d\tau'$$

remember →

$$\bar{J}_b(\mathscr{r}') = \bar{\nabla} \times \bar{m}(r')$$

↳ bound volume current density

Gauss's divergence theorem: $\int \bar{\nabla} \bar{P} \, dV = \oint P \cdot ds$

if $\bar{P} = \bar{V} \times \bar{C}$

where $\bar{V}$ varies with space and

$\bar{C}$ doesnt varies with space

then o/c $\bar{P}$ varies with space

$$\int_V \bar{\nabla} \cdot (\bar{V} \times \bar{C}) \, dv = \oint_S (\bar{V} \times \bar{C}) \, ds$$

$$\int \bar{C} \cdot (\bar{\nabla} \times \bar{V}) - \bar{V} \cdot \underbrace{(\bar{\nabla} \times \bar{C})}_{\text{zero because c doesnt vary with space}} dv = \oint \bar{C} \cdot (d\bar{s} \times \bar{V})$$

$$\int C \cdot (\nabla \times V) \, dv = -\oint \bar{C} \cdot (\bar{V} \times d\bar{s})$$

$$\int C \cdot (\bar{\nabla} \times V) \, dV = -C \oint_S (\bar{V} \times d\bar{s})$$

$$\int (\bar{\nabla}\times\bar{V})\,dV = -\oint (\bar{V}\times d\bar{s})$$

$$\frac{\mu_0}{4\pi}\int \frac{\bar{J}(r')\,d\tau'}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi}\int \left(\frac{\bar{\nabla}\times\bar{m}}{\mathscr{r}}\right) d\tau'$$

$$= -\frac{\mu_0}{4\pi}\oint \left(\frac{\bar{m}}{\mathscr{r}}\right) ds$$

$$\bar{A}(r) = \frac{\mu_0}{4\pi}\int \left[\frac{1}{\mathscr{r}}(\bar{\nabla}'\times\bar{m}) - \bar{\nabla}\times\left(\frac{\bar{m}}{\mathscr{r}}\right)\right] d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \frac{\overbrace{J_b(\mathscr{r}')}^{\bar{\nabla}'\times\bar{m}(r')}\,d\tau'}{\mathscr{r}} + \frac{\mu_0}{4\pi}\oint \frac{\overbrace{K_b(r')}^{\bar{m}\times\hat{n}}}{\mathscr{r}}\,ds$$

volume current density

surface current density

$$\bar{A}(\mathscr{r}) = \frac{\mu_0}{4\pi}\int \frac{J_b(\mathscr{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint \frac{K_b(\mathscr{r}')}{\mathscr{r}}\,ds$$

Lec (Revision) 17/03/25

Ampere's Law

$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$ $\qquad \oint B \cdot dl = \mu_0 I_{enc}$

$\bar{m} = I\bar{A}$: magnetic moment

magnetic vector potential : $\bar{A}(r)$

from multiple expansion

due to a single magnetic dipole : $\bar{A}(r) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$

$\bar{M}$: magnetic dipole moment per unit volume

for a continuous distribution :

$$\bar{A}(r) = \int \frac{\mu_0}{4\pi} \frac{\bar{m}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} dV'$$

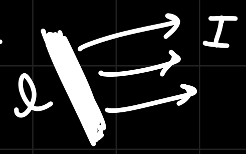
$$= \frac{\mu_0}{4\pi} \int \frac{J_b(\mathscr{r}')}{\mathscr{r}} dV' + \frac{\mu_0}{4\pi} \oint \frac{K_b(\mathscr{r}')}{\mathscr{r}} ds'$$

summation of and bound volume current density and bound surface current density where, $J_b(r) = \bar{\nabla} \times \bar{M}$, $K_b(r) = \bar{M} \times \hat{n}$

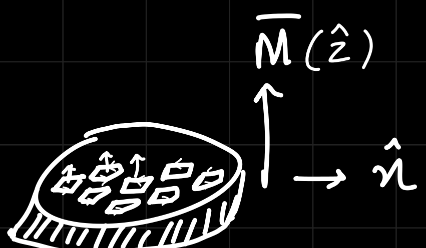
$\uparrow$ Am^{-2} $\uparrow$ Am^{-1}

$$K = \frac{I}{l_\perp}$$

length perpendicular to flow of current



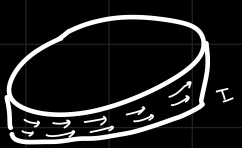
Uniform magnetized material



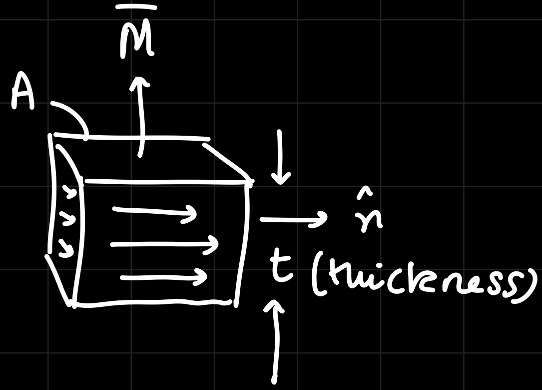
internal currents are concelled out

$\bar{m} = I\bar{A}$

$\downarrow$



called bound surface current like a ribbon wrapped around the material carrying the current



$\bar{m} = \bar{M}\bar{A}t$

$I\bar{A} = \bar{M}\bar{A}t$

$\bar{M} = \dfrac{\bar{I}}{t} = K_b$

So, $\boxed{\bar{K}_b = \bar{M} \times \hat{n}}$

because $\bar{m}, \hat{n}, \bar{I}$ are mutually $\perp$ and dir of K_b is = dir of current

NOTE: volume current density is zero for uniform magnetized material

NON-UNIFORM MAGNETIZATION

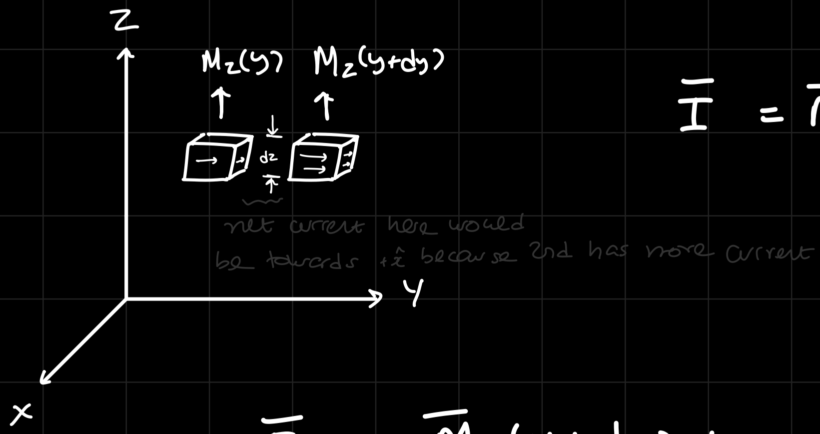
$\bar{M}$ will not be same in 2 different dipoles and hence, same goes for current

$\bar{M}$ varies with pos^n

from above $\rightarrow \bar{I} = \bar{M}t$

$\bar{J}_b = \bar{J}_{b-x}\hat{x} + \bar{J}_{b-y}\hat{y} + \bar{J}_{b-z}\hat{z}$

consider slabs



$$\bar{I} = \bar{M}t$$

$$\bar{I}_x = \bar{M}_z(y+dy)\,dz - \bar{M}_z(y)\,dz$$
$$= [\bar{M}_z(y+dy) - \bar{M}_z(y)]\,dz$$

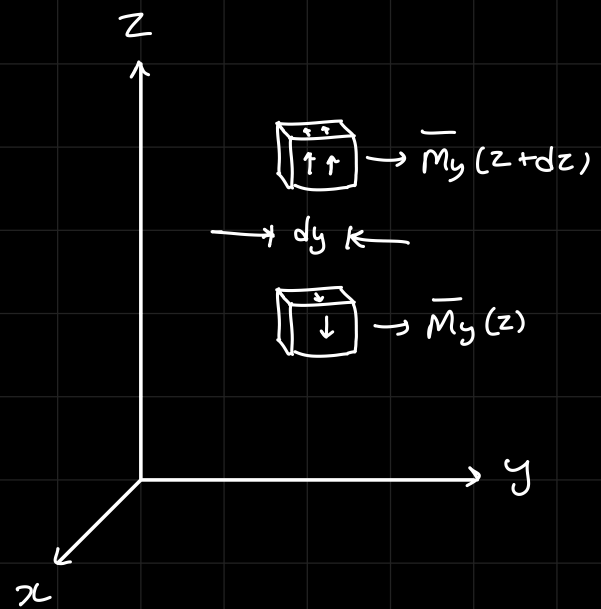
taylor series expansion

$$\bar{I}_x = \left[M_z(y) + \frac{\partial M_z}{\partial y}dy\right]dz - M_z(y)\,dz$$

$$= \frac{\partial M_z}{\partial y}\,dy\,dz$$

So, $\boxed{J_{b\text{-}x} = \frac{\partial M_z}{\partial y}}$

but magnetization can have a x and y component as well



$$I_x = -\bar{M}_y(z+dz)\,dy + \bar{M}_y(z)\,dy$$

again taylor series expansion

$$= -\frac{\partial M_y}{\partial z}\,dy\,dz$$

$$\boxed{J_{b\_x} = -\frac{\partial M_y}{\partial z}}$$

Total volume current density : $\boxed{J_{b_x} = \left(\frac{\partial \bar{M}_z}{dy} - \frac{\partial \bar{M}_y}{dz}\right)}$

which is equal to $\underline{\underline{\bar{\nabla} \times \bar{M}}}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{M}_x & \bar{M}_y & \bar{M}_z \end{vmatrix}$$

So, $J_{b_x} = \hat{x}\left(\frac{\partial}{\partial y} M_z - \frac{\partial}{\partial z} M_y\right)$

again Ampere's Law $\rightarrow$ $\bar{\nabla} \times \bar{B} = \mu_0 J_{total}$

$$\frac{\bar{\nabla} \times \bar{B}}{\mu_0} = \bar{J}_b + \bar{J}_{free}$$

$$\bar{J}_{free} = \nabla \times \left(\frac{\bar{B}}{\mu_0} - \bar{M} \right)$$

let $\vec{H} = \frac{\bar{B}}{\mu_0} - \bar{M}$

So, $\bar{J}_f = \bar{\nabla} \times \bar{H}$

and $\oint H \cdot dl = I_{free}$

$\bar{B} = \mu_0 \bar{H} + \bar{m}$

$\bar{m} = \chi_m \bar{H}$

$\hookrightarrow$ magnetic susceptibility

$$\bar{B} = \mu_0 (\bar{H} + \chi_m \bar{H}) = \underbrace{\mu_0 (\chi_m + 1)}_{\mu} \bar{H}$$

$\mu = \mu_0 (1 + \chi_m)$ $\rightarrow$ $\bar{B} = \mu \bar{H}$

magnetic materials

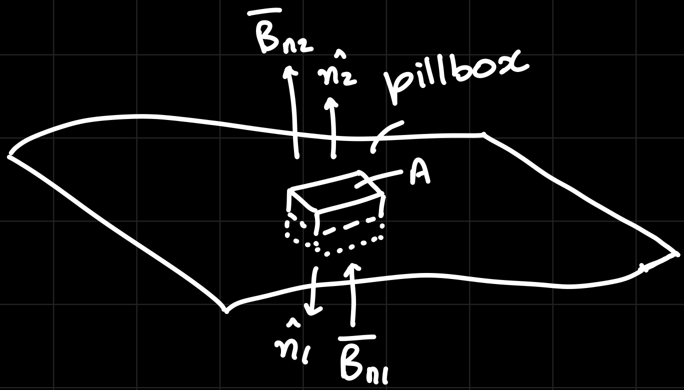
- linear ($\bar{M} = \chi_m \bar{H}$)
  - Para-magnetic ($\chi_m > 0$)
  - Dia-magnetic ($\chi_m < 0$)
- non-linear (ferromagnetic) $\chi_m \sim 10^5$

$$\mu = \mu_0(1 + \underbrace{\chi_m}_{<0})$$

$$\mu \simeq \mu_0$$

very low magnetization (at room temp) in general for both linear elements

Boundary Conditions (magnetostatics)



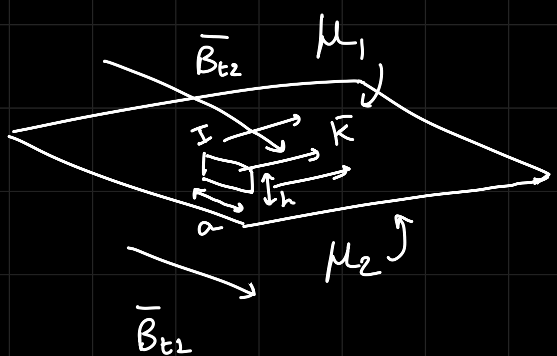
flux through sides = 0
because area of sides → 0

Gauss's Law: $\oint \bar{B} \cdot d\bar{s} = 0$

$$B_{n2}A - B_{n1}A = 0$$

$$B_{n1} = B_{n2}$$

normal component : equal



$\bar{I}_{enc} = \bar{K}a$

line current density

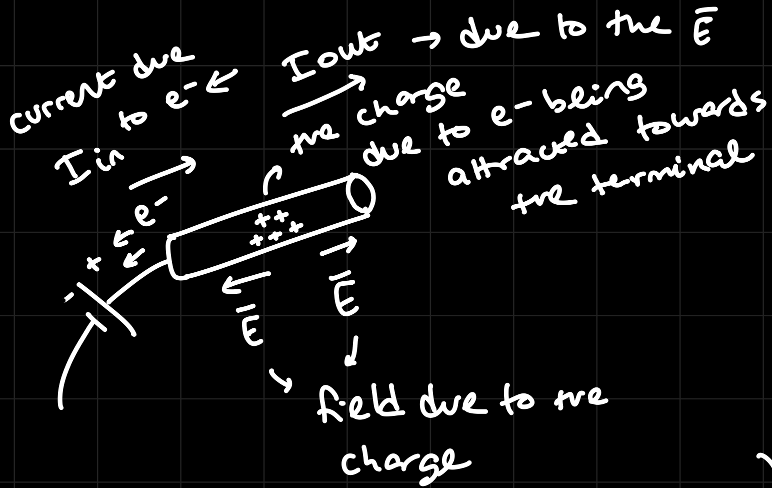
Ampere's Law ⇒ $\oint \bar{H} \cdot d\bar{l} = I_{enc} = \bar{K}a \Rightarrow H_{t1}a - H_{t2}a = K\bar{a}$

$$H_{t1} - H_{t2} = K$$

tangential component

* Electrodynamics

EMF : Voltage across a cell without the internal resistance.



This process is repeated throughout

I_{in} is stopped and
I_{out} is allowed due to $\bar{E}$

$$\bar{f} = \bar{f_s} + \bar{E} \rightarrow \text{drives the current}$$

total force

$\bar{f_s}$: effect of this force is confined to the close vicinity of the source

taking closed loop integral

$$\oint \bar{f}\cdot d\bar{l} = \oint \bar{f_s}\cdot d\bar{l} + \underbrace{\oint \bar{E}\cdot d\bar{l}}_{0}$$

assumption: because DC source { electrostatic field for a circuit with no resistance, we dont need any force to push the current

So, for zero resistance,

$$\bar{f}_s + \bar{E} = 0$$

$$\Rightarrow \oint f_s \cdot dl = -\oint E \cdot dl = 0$$

but for a non closed loop

$$\underbrace{\int_A^B \bar{f}_s \cdot dl}_{\text{EMF}\,(\varepsilon)} = -\int_A^B E \cdot dl = V_{BA}$$

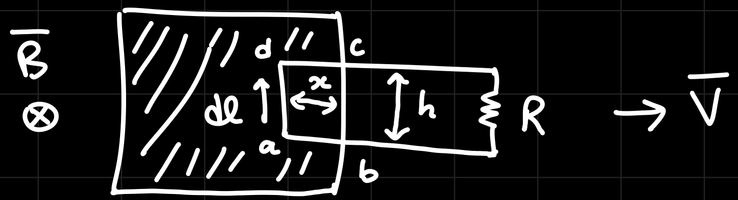
So, EMF is the voltage when there is no internal resistance

now, we can also say

$$\oint \bar{F}_s \cdot d\bar{l} = \text{EMF} = 0$$

closed loop EMF

→ Proving Faraday's Law using an example



$$\bar{f}_{mag} = qvB = vB$$

$$\oint \bar{f}_{mag} \cdot \bar{dl} = vBh = \text{EMF}$$ assuming $q=1$

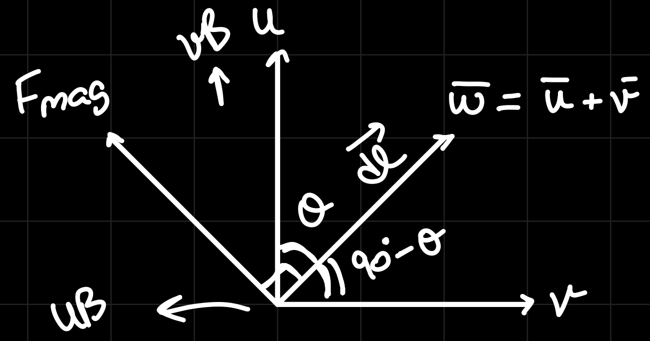
note: only ad contribute to the closed loop integral because ab and cd have $\bar{B} \perp dl$ and so dot product $= 0$ and rest of the ckt is considered outside the magnetic field $\bar{B}$

magnetic force doesn't do any work current will flow through the resistance and there is EMF induced

but f_{mag} can't do any work → f_{mag} is not the source of the EMF then

the current flowing through R is not due to f_{mag} but due to the ckt being pulled to right with velocity $\bar{v}$

let the force due to the motion be equal to $\vec{f}_{pull}$



$\otimes \bar{B}$

fmag would now be $\perp$ to $\bar{w}$ (motion)

$$\bar{f}_{mag} = (\bar{w} \times \bar{B}) \quad \text{assuming } q = 1$$

$$= (\bar{u} + \bar{v}) \times \bar{B}$$

$$\text{work done} = \int \bar{f}_{pull} \cdot d\bar{l}$$

$$= \int uB \sin\theta \, dl$$

$$= uB \sin\theta \frac{h}{\cos\theta}$$

$$= u\tan\theta \, Bh = vBh = \underline{\mathcal{E}MF}$$

$$\phi = B\overset{\text{area}}{\widetilde{xh}} \leftarrow \int \bar{B} \cdot d\bar{s}$$

rate of decrease of flux → $\frac{d\phi}{dt} = B \frac{dx h}{dt} = Bvh = \mathcal{E}MF$

(x is decreasing) → rate of change $= -\mathcal{E}MF$

$$\oint \bar{E}\cdot d\bar{l} = -\frac{d\Phi}{dt} = -\frac{d\left(\int B\cdot ds\right)}{dt} = -\int \frac{\partial B}{\partial t}\cdot ds$$

if B is not varying with time $\rightarrow \oint \bar{E}\cdot d\bar{l} = 0$

electrostatics

$$\oint E\cdot dl = \int (\nabla \times E)\, ds = -\int \frac{\partial B}{\partial t}\, ds$$

$$\boxed{\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

So, change in magnetic field induces and electric field

$+(\ldots)$ for electrodynamics

$\bar{\nabla}\times\bar{B} = \mu_0 J$

So, $\bar{\nabla}\cdot\bar{J} = 0$

(div of curl)

only valid in case of _statics

↳ electro/magneto

but $\bar{\nabla}\cdot\bar{J} = -\frac{\partial \rho}{\partial t} \rightarrow$ conservation of charge

now we need a modified ampere's Law for electrodynamics

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\oint E \cdot dl = -\frac{\partial}{\partial t} \int B \cdot ds$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{J}) = \mu_0 \bar{\nabla} \cdot \bar{J} = 0$$

so, $\bar{\nabla} \cdot \bar{J} = 0$

conservation of charge →

$$\bar{\nabla} \cdot \bar{J} = -\frac{\partial \rho}{\partial t}$$

$$\bar{\nabla} \cdot \bar{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\bar{\nabla} \cdot \bar{J} + \bar{\nabla} \cdot \frac{\partial \bar{D}}{\partial t} = 0$$ where $\bar{D} = \varepsilon \bar{E}$

$$\nabla \cdot \left(\bar{J} + \frac{\partial \bar{D}}{\partial t} \right) = 0$$

so, $\bar{\nabla} \times \bar{B} = \mu_0 \left(J + \frac{\partial \bar{D}}{\partial t} \right)$

so, we just need a time varying electric field for magnetic field. No need for even a conduction current

Maxwell's modification :

time varying $\bar{E}$ gives rise to $\bar{B}$

Faraday's Law of Induction :

time varying $\bar{B}$ gives rise to $\bar{E}$

remember : Faraday's Law

$$\bar{\nabla} \times \bar{E} = -\frac{d\bar{B}}{dt} \qquad \oint E \cdot dl = -\frac{\partial}{\partial t} \iint_S B \cdot ds$$

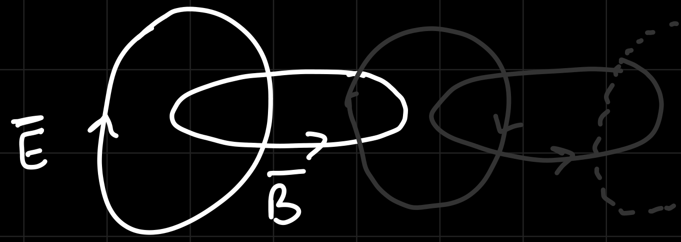
So, yes, $\bar{E}$ can be generated with just a time varying $\bar{B}$ even without charge / conductor.

How will the $\bar{E}$ field lines look like without a conductor / charges

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon \frac{\partial \bar{E}}{\partial t}$$

$$= \mu_0 (\underbrace{\bar{J}_D}_{\text{displacement current}} + \underbrace{\bar{J}_c}_{\text{conduction current}}) + \mu_0 \varepsilon \frac{\partial \bar{E}}{\partial t}$$

the $\bar{B}$ field lines will always loop and be $\perp$ to $\bar{E}$ field lines



self reproducing $\bar{E}$ and $\bar{B}$ fields from just a single initial $\bar{E}$ or $\bar{B}$ on all sides

These imaginary chains are Electromagnetic waves

Static case (no time variation)

$\oint E \cdot ds = \frac{q_{enc}}{\varepsilon_0}$ $\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$

$\oint E \cdot dl = 0$ $\bar{\nabla} \times \bar{E} = 0 \Leftarrow \bar{E} = -\bar{\nabla}\phi$

$\oint \bar{B} \cdot d\bar{s} = 0$ $\bar{\nabla} \cdot \bar{B} = 0$

$\oint \bar{B} \cdot d\bar{l} = \mu_0 \bar{I}_{enclosed}$ $\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$

Dynamic case

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho(t)}{\varepsilon_0}$$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{B} = \mu_0 J + \mu_0 \left(\frac{\partial \bar{E}}{\partial t}\right) \varepsilon$$ — maxwell

EM waves

Potential in Electrodynamics

we can still write $\bar{B} = \bar{\nabla} \times \bar{A}$

but not $\bar{E} = -\bar{\nabla}\phi$

from (2) $\rightarrow$ $\bar{\nabla} \times \bar{E} = -\dfrac{\partial \bar{B}}{\partial t} = -\dfrac{\partial (\bar{\nabla} \times \bar{A})}{\partial t}$

$$\Rightarrow \nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

from ② statics $\rightarrow$ $\nabla \times E = 0$

where $E = -\nabla \phi$

So, $\bar{E} + \dfrac{d\bar{A}}{dt} = -\nabla\phi$

So, $\bar{E} = -\left(\bar{\nabla}\phi + \dfrac{d\bar{A}}{dt} \right)$

everything defined till now in dynamics is in time domain. We will be moving to the frequency domain now.

square $\rightleftharpoons$ sinc

if $x(t)$ given as square wave and
system's freq response given,
we find $y(t)$ by $\rightarrow$

$x(t) \rightarrow X(\omega)$ $\{sinc\}$
multiply with $H(\omega)$
then IFT to get $y(t)$

This works for a linear system

Time → Freq (maxwell's equations)

$$\bar{\nabla}\bar{E}(t) = \frac{\rho(t)}{\varepsilon_0}$$

$$\bar{\nabla}\times\bar{E}(t) = -\frac{\partial B(t)}{\partial t}$$

$$\bar{\nabla}\cdot\bar{B}(t) = 0$$

$$\bar{\nabla}\times\bar{B}(t) = \mu_0\left(J(t) + \frac{\partial D(t)}{\partial t}\right)$$

Just take Fourier Transform

notation : $\bar{E}(t) \rightleftharpoons \bar{E}(\omega)$

$$\bar{\nabla} \cdot \bar{E}(\omega) = \frac{\rho(\omega)}{\varepsilon}$$

$$\bar{\nabla} \times \bar{E}(\omega) = -i\omega \bar{B}(\omega)$$

$$\bar{\nabla} \bar{B}(\omega) = 0$$

$$\bar{\nabla} \times \bar{B}(\omega) = \mu \left(\bar{J}(\omega) + i\omega \varepsilon \bar{E}(\omega) \right)$$

we know $\bar{J} = \sigma \bar{E}$

$$\boxed{\bar{\nabla} \times \bar{B} = \mu \bar{E}(\sigma + i\omega\varepsilon)}$$

In a source free region →

Time
$$\bar{\nabla} \cdot \bar{E} = 0 \qquad \bar{\nabla} \cdot \bar{B} = 0$$
$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \mu \varepsilon \frac{\partial \bar{E}}{\partial t}$$

Freq
$$\bar{\nabla} \cdot \bar{E} = 0 \qquad \bar{\nabla} \cdot \bar{B} = 0$$
$$\bar{\nabla} \times \bar{E} = -i\omega \bar{B} \qquad \bar{\nabla} \times \bar{B} = -i\omega \mu \varepsilon \bar{E}$$

wave eqn in 1d

$\hookrightarrow f(x - vt)$

$$\frac{\partial^2 f}{\partial x^2} = f''(x - vt) = \frac{\partial^2 f(x-vt)}{\partial (x-vt)^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial t}$$

$$= f'(x-vt) \times (-v)$$

$$\frac{\partial^2 f}{\partial t^2} = v^2 \, f''(x-vt)$$

So, $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{v^2} \dfrac{\partial^2 f}{\partial t^2}$

for 3d $\rightarrow$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 \bar{f} = \frac{1}{v^2} \frac{\partial^2 \bar{f}}{\partial t^2}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t}(\nabla \times B)$$

$$= \frac{-\partial}{\partial t}\left(\mu \varepsilon \frac{\partial \bar{E}}{\partial t}\right)$$

considering source free region

$$\bar{\nabla} \cdot (\cancel{\bar{\nabla} \bar{E}}^{\,0}) - \bar{\nabla}^2 \bar{E} = -\mu \varepsilon \left(\frac{\partial^2 \bar{E}}{\partial t^2}\right)$$

$$\Rightarrow \quad \nabla^2 E = \mu \varepsilon \left(\frac{\partial^2 \bar{E}}{\partial t^2}\right)$$

So, the velocity $\Rightarrow \dfrac{1}{v^2} = \mu \varepsilon$

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

$$\bar{\nabla} \times \bar{B} = \mu \varepsilon \frac{\partial \bar{E}}{\partial t}$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{B}) = \mu \varepsilon \frac{\partial}{\partial t} (\nabla \times \bar{E})$$

$$\bar{\nabla} (\cancel{\bar{\nabla} \bar{B}}^{\,0}) - \nabla^2 B = -\mu \varepsilon \frac{\partial^2 \bar{B}}{\partial t^2}$$

$$\nabla^2 B = \mu \varepsilon \frac{\partial^2 B}{\partial t^2}$$

velocity: $\frac{1}{\sqrt{\mu \varepsilon}}$

no material involved
this wave still propagates
even through vacuum
we just need time varying
magnetic field and electric field
for EM waves

$$\nabla^2 E = \mu\varepsilon \left(\frac{\partial^2 \bar{E}}{\partial t^2} \right)$$

$$\nabla^2 B = \mu\varepsilon \frac{\partial^2 B}{\partial t^2}$$

in frequency domain :

$$\bar{\nabla}^2 \bar{E} = \mu\varepsilon (-\omega^2 \bar{E})$$

$$\nabla^2 \bar{E} + (\mu\varepsilon\omega^2) \bar{E} = 0$$

let $K = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v}$ or $\frac{\omega}{c}$ velocity

$$\nabla^2 \bar{E} + K^2 \bar{E} = 0 \longrightarrow \text{Helmholtz eq}^n$$

$$\bar{\nabla}^2 \bar{E}_{(r,t)} = \frac{1}{c^2} \frac{\partial^2 \bar{E}(r,t)}{\partial t^2}$$

let $K = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v}$ or $\frac{\omega}{c}$ velocity

$$\nabla^2 \bar{E}_{(r,\omega)} + K^2 \bar{E}_{(r,\omega)} = 0$$ $\longrightarrow$ Helmholtz eq^n

freq domain $\rightarrow$ time ?

IFT

$$\bar{E}(r,t) = \int_{-\infty}^{\infty} \bar{E}(r,\omega)\, e^{i\omega t}\, d\omega$$

(for multiple freq)

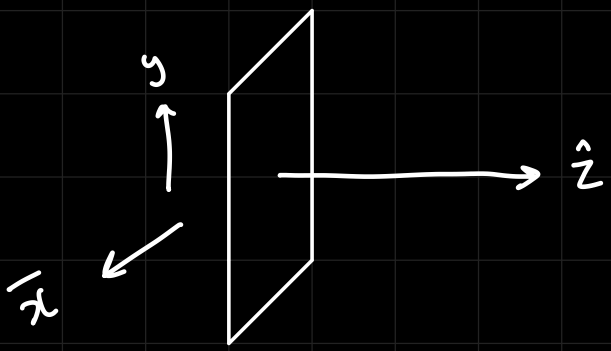
$$\delta(x) \overset{IFT}{\rightleftharpoons} 1$$

for single frequency

$$E_0(r)\, \delta(\omega-\omega_0) \rightleftharpoons E_0(r)\, e^{i\omega t}$$

↳ only a func of space

1D → Uniform Plane wave



$\bar{E}$ and $\bar{B}/\bar{H}$ only vary with z
they are uniform in the xy plane

considering a monochromatic wave
i.e. only one frequency

So, $\bar{E}(z,t) \rightleftharpoons \bar{E}(z)e^{i\omega t}$

helmoltz equation →

$$\frac{\partial^2 \bar{E}(z)}{\partial z^2} + k^2 \bar{E}(z) = 0$$

assume $\bar{E}(z) \simeq e^{mz}$

$$m^2 e^{mz} + k^2 e^{mz} = 0$$
$$m^2 + k^2 = 0$$
$$m = \pm ik$$

$$E(z) = e^{\pm ikz}$$

$$\bar{E}(z,\omega) = \underset{(-\hat{z})}{\bar{A}e^{ikz}} + \underset{(\hat{z})}{\bar{B}e^{-ikz}}$$

$$\bar{E}(z,t) = \bar{A}e^{i(\omega t+kz)} + \bar{B}e^{i(\omega t-kz)}$$

lets take the case of forward travelling wave

$$\bar{E}(z,t) = \underset{\text{(Amplitude)}}{\bar{E}_0}\, e^{i(\omega t - kz)} \rightarrow \text{(phase)}$$

direction of EM wave $(\bar{E})$ = polarization of the wave

$\underline{\underline{t=t_1}}$ Locus of the points at the same phase = ?
We will look at it now.

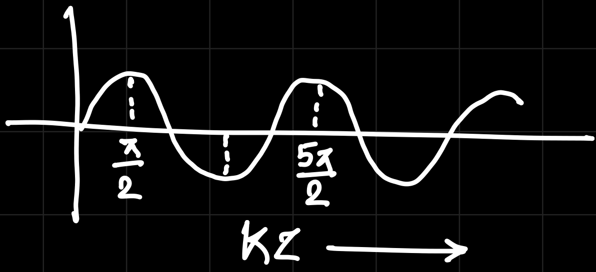
$$\omega t_1 - kz = \underline{M}$$

↳ same phase M at $t=t_1$.
Locus = ?

$$z = \frac{\omega t_1 - M}{K} = 0 \rightarrow \text{totally a constant}$$

Locus ⇒ Z = constant
Plane parallel to xy - plane

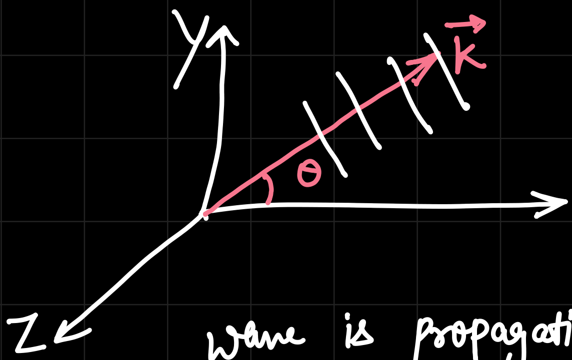
$$\bar{E}(z,t) = \bar{E}_0\, e^{i(\omega t - kz)}\, e^{i 2n\pi}$$



$2n\pi$ gap in kz axis.

* wave = planar but not uniform. Possible?

YES



Still a plane wave but not uniform.

wave is propagating but tilted (at an angle θ) w.r.t. x-axis.

There can be other examples but this is simplest one.

Basic Helmh. eqn we studied till now,

we assumed $\sigma = 0$ (source-free)

↳ No conduction current.

$$\nabla \cdot \vec{E} = 0$$
$$\vec{\nabla} \times \vec{E} = -i\omega\vec{B}$$
$$\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + i\omega\mu\epsilon E$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

→ freq. domain

$\sigma = 0$

If not this, it will be very tough to analyze.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = i\omega\mu\epsilon(\vec{\nabla} \times \vec{E})$$
$$= i\omega\mu\epsilon(-i\omega\vec{B})$$
$$= -i^2\omega^2\mu\epsilon\vec{B}$$

$$\vec{\nabla}\times\vec{B} = i\omega\mu\epsilon_c\vec{E} = i\omega\mu\epsilon\vec{E} \quad \left(\text{written in simpler}\right)$$

$$i\omega\mu\epsilon_c = \mu\sigma + i\omega\mu\epsilon \quad \left(\text{equating both } \vec{\nabla}\times\vec{B} \text{ eqn.}\right)$$

$$\epsilon_c = \epsilon + \frac{\sigma}{i\omega}$$

$$\boxed{\epsilon_c = \epsilon - i\frac{\sigma}{\omega}}$$

→ Permittivity of a medium becomes complex no.

$$\epsilon_c = \epsilon\left(1 - i\frac{\sigma}{\omega\epsilon}\right)$$

* $K = \omega\sqrt{\mu\epsilon_c} = K' - i\underbrace{K''}_{>0}$

Initially, K was simpler. $(K = \omega\sqrt{\mu\epsilon})$
Now, it is complex. $(\epsilon \to \epsilon_c)$

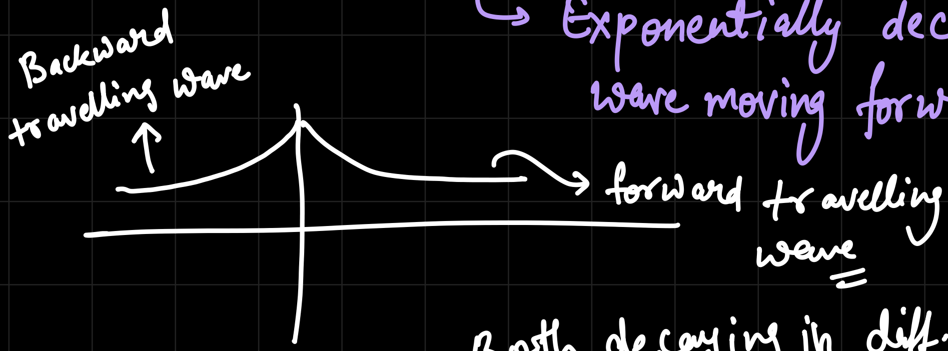
Just ϵ changed to a complex no. (ϵ_c).

* $$\vec{E}_0 e^{i(\omega t - Kz)} = \vec{E}_0 e^{i\omega t} \cdot e^{-i(K' - iK'')z}$$

$$= E_0 e^{i\omega t} e^{-iK'z} e^{-K''z}$$

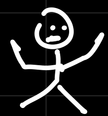
Nature of this wave = ?

↳ Exponentially decaying wave moving forward



Both decaying in diff. dirxns.

If we had taken +ve from $k' + ik''$, we would have an exp. increasing wave and at an ∞ amplitude, Energy will become ∞ (which is not possible obv.)



* When to assume ϵ_c = real and ϵ_c = complex

$\sigma \neq 0$, current $\neq 0$

Lost energy will be released in form of heat because of conduction current.

what if $\sigma \ll \omega\epsilon$

↳ Perfect dielectric

$\epsilon_c = \epsilon$ (real ϵ)

$\sigma \sim \omega\epsilon$

↳ Lot of heat will be generated.

medium with decreasing constant

$\sigma \gg \omega\epsilon$

EM wave will die inside a P.E.C.

It will decay very fast.

* Static $\vec{E}$ / Time-varying $\vec{E}$ } inside P.E.C. $= 0$

* Static $\vec{B}$ can exist inside P.E.C.

Q: Can Time-varying $\vec{B}$ exist inside P.E.C ?

NO, as T-V $\vec{B}$ will generate $\vec{E}$ but $\vec{E} = 0$ inside P.E.C.

Ag, Cu → good conductors at microwave frequencies

→ not good conductors at optical frequencies