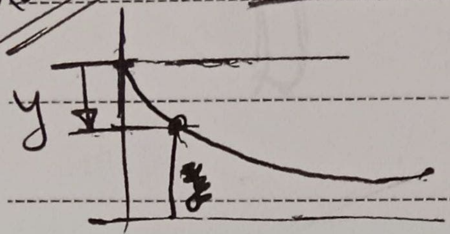
Example 3: Brachistochrone



$$\Delta KE = \Delta PE$$

$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gy$$

$$v = \sqrt{2gy}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$\text{time} = \int \frac{dz}{\sqrt{2gy}}$$

$$\text{time} = \int \underbrace{\frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}} dx$$

↳ Minimize time taken

$\Rightarrow$ Euler's eq^n on $f(y, y')$

$$\Rightarrow y' \frac{\partial f}{\partial y'} - f = \text{const.}$$

$$\Rightarrow \frac{(y')^2}{\sqrt{2gy}\,[1+(y')^2]} - \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} = \text{const.}$$

$$\Rightarrow \frac{1}{\sqrt{y}\sqrt{1+(y')^2}} = \text{const.}$$

$$\Rightarrow y(1+(y')^2) = C$$

$$y\left(1+(y')^2\right) = a$$

$$(y')^2 = \frac{a}{y} - 1$$

$$\int \sqrt{\frac{y}{a-y}}\, dy = \int dx$$

put $y = a\sin^2\theta \Rightarrow dy = 2a\sin\theta\cos\theta\, d\theta$

$$\int \sqrt{\frac{\cancel{a}\sin^2\theta}{\cancel{a}\cancel{\cos^2\theta}}}\; 2a\sin\theta\, \cancel{\cos\theta}\, d\theta$$

$$= 2a\int \sin^2\theta\, d\theta$$

$$= \frac{a}{2}\left[2\theta - \sin 2\theta\right] = \int dx = x + c_2$$

Since curve passes through origin,

$x = y = \theta = 0 \Rightarrow \boxed{c_2 = 0}$

$\therefore\ x = \frac{a}{2}\left(2\theta - \sin 2\theta\right) = a(\phi - \sin\phi)$ — ①

$y = a\sin^2\theta = a(1 - \cos\phi)$ — ②

\# Equations ① & ② form a cycloid!