RBE 549: Homework Assignment 11

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Problem 1

a $\overrightarrow{f} = [0, 0, f]$, $\overrightarrow{X^W} = [x^W, y^W, z^W]$, $\overrightarrow{B} = [b, 0, 0]$
$\overrightarrow{X_L^C} = \overrightarrow{X^W} + \overrightarrow{b}/2$
$\overrightarrow{X_R^C} = \overrightarrow{X^W} - \overrightarrow{b}/2$
$\overrightarrow{X_L'} = \frac{|\overrightarrow{f}|^2}{\overrightarrow{f}.\overrightarrow{X^W}}[X_L^C]$
$\begin{bmatrix} \overrightarrow{X_L'} \\ \overrightarrow{Y_L'} \end{bmatrix} = \overrightarrow{X_L'} = \frac{|\overrightarrow{f}|^2}{\overrightarrow{f}.\overrightarrow{X^W}} \begin{bmatrix} \overrightarrow{X^W} + \overrightarrow{b}/2 \\ \overrightarrow{Y^W} \end{bmatrix} = \frac{\overrightarrow{f}}{\overrightarrow{Z^W}} \begin{bmatrix} \overrightarrow{X^W} + \overrightarrow{b}/2 \\ \overrightarrow{Y^W} \end{bmatrix} \qquad (\because \overrightarrow{f}.\overrightarrow{X^W} = \overrightarrow{f}\,\overrightarrow{Z^W})$
Similarly, $\begin{bmatrix} \overrightarrow{X_R'} \\ \overrightarrow{Y_R'} \end{bmatrix} = \overrightarrow{X_R'} = \frac{\overrightarrow{f}}{\overrightarrow{Z^W}} \begin{bmatrix} \overrightarrow{X^W} - \overrightarrow{b}/2 \\ \overrightarrow{Y^W} \end{bmatrix}$
Error $E = |\overrightarrow{X_L'} - \overrightarrow{X_L}|^2 + |\overrightarrow{X_R'} - \overrightarrow{X_R}|^2$
$\Rightarrow (X_L' - X_L)^2 + (Y_L' - Y_L)^2 + (X_R' - X_R)^2 + (Y_R' - Y_R)^2$
$\Rightarrow (\frac{\overrightarrow{f}}{\overrightarrow{Z^W}}(\overrightarrow{X^W} + b/2) - X_L)^2 + (\frac{\overrightarrow{f}}{\overrightarrow{Z^W}}\overrightarrow{Y^W} - Y_L)^2 + (\frac{\overrightarrow{f}}{\overrightarrow{Z^W}}(\overrightarrow{X^W} - b/2) - X_R)^2 + (\frac{\overrightarrow{f}}{\overrightarrow{Z^W}}\overrightarrow{Y^W} - Y_R)^2$

b $\frac{dE}{d\overrightarrow{x^W}} = 2(\frac{\overrightarrow{f}}{\overrightarrow{z^W}}(x^w + b/2) - X_L)\frac{\overrightarrow{f}}{\overrightarrow{z^W}} + 2(\frac{\overrightarrow{f}}{\overrightarrow{z^W}}(x^w - b/2) - X_R)\frac{\overrightarrow{f}}{\overrightarrow{z^W}}$
$\frac{dE}{d\overrightarrow{x^W}} = 0 = \frac{f}{z^W}(2x^W) = X_L + X_R$
$\Rightarrow x^W = \frac{X_L + X_R}{2}\frac{z^W}{f}$
$\frac{dE}{d\overrightarrow{y^W}} = 2(\frac{f}{z^W}y^w - Y_L)\frac{f}{z^W} + 2(\frac{f}{z^W}y^w - Y_R)\frac{f}{z^W}$
$\frac{dE}{d\overrightarrow{y^W}} = 0 = \frac{f}{z^W}(2y_W) = Y_L + Y_R$
$\Rightarrow y^W = \frac{Y_L + Y_R}{2}\frac{z^W}{f}$

c $\frac{dE}{d\overrightarrow{z^W}} = -2\frac{f}{z^{W2}}[\frac{f}{z^W}(\overrightarrow{x^W} + b/2) - X_L)(\overrightarrow{x^W} + b/2) + \frac{f}{z^W}(\overrightarrow{y^W}) - Y_L)(\overrightarrow{y^W}) +$

$$\frac{f}{z^W}(\overrightarrow{x^W} - b/2) - X_R)(\overrightarrow{x^W} - b/2) + \frac{f}{z^W}(\overrightarrow{y^W}) - Y_R)(\overrightarrow{y^W})]$$

$\frac{dE}{d\overrightarrow{z^W}} = 0$

$\Rightarrow X^W(\frac{f}{z^W}(\overrightarrow{x^W}+b/2+\overrightarrow{x^W}-b/2)-X_L-X_R)+\frac{b}{2}(\frac{f}{z^W}(\overrightarrow{x^W}+b/2-(\overrightarrow{x^W}-b/2)-(X_L-X_R))+$
$y^W(\frac{f}{z^W}2y^W-Y_L-Y_R)=0$

$$\Rightarrow x^W\left(\frac{f}{z^W}(2\frac{X_L+X_R}{2}\frac{z^W}{f})-X_L-X_R\right)+y^W\left(\frac{f}{z^W}(2\frac{Y_L+Y_R}{2}\frac{z^W}{f})-Y_L-Y_R\right)$$

$$+\frac{b}{2}\left(\frac{f}{z^W}(2\frac{b}{2}\frac{z^W}{f})-(X_L-X_R)\right)=0$$

$\Rightarrow \frac{b}{2}(\frac{f}{z^W}b-\Delta_X)=0$
where, $\Delta_X=X_L-X_R$
$\Rightarrow Z_W=\frac{fb}{\Delta_X}$
$\Rightarrow \frac{Z_W}{f}=\frac{b}{\Delta_X}$
$\Delta_X=\frac{\overrightarrow{\Delta_X}.\overrightarrow{B}}{|\overrightarrow{B}|}$ $\quad(\because \overrightarrow{B}=[b,0,0]),|\overrightarrow{B}|=b)$
$\frac{Z_W}{f}=\frac{b^2}{\overrightarrow{\Delta_X}.\overrightarrow{B}}$
From **b** we have:
$\Rightarrow x^W=\begin{bmatrix}x^W\\ y^W\end{bmatrix}=\frac{z^W}{f}\begin{bmatrix}(X_L+X_R)/2\\ (Y_L+Y_R)/2\end{bmatrix}$
$=\frac{|\overrightarrow{B}|^2}{\overrightarrow{\Delta}.\overrightarrow{B}}\overrightarrow{X_{AVG}}$

Problem 2

We have $E=min|\overrightarrow{X_L^W}-\overrightarrow{X_R^W}|^2=min|l\overrightarrow{X_L}-r\overrightarrow{X_R}-\overrightarrow{B}|^2$
$\frac{dE}{dl}=2\overrightarrow{X_L}(l\overrightarrow{X_L}-r\overrightarrow{X_R}-\overrightarrow{B})=0$
$\frac{dE}{dr}=2\overrightarrow{X_R}(l\overrightarrow{X_L}-r\overrightarrow{X_R}-\overrightarrow{B})=0$
$\Rightarrow\begin{cases}\overrightarrow{X_L}^2-r\overrightarrow{X_L}\overrightarrow{X_R}-\overrightarrow{X_L}\overrightarrow{B}\\ l\overrightarrow{X_R}\overrightarrow{X_L}-r\overrightarrow{X_R}^2-\overrightarrow{X_R}\overrightarrow{B}\end{cases}$
$\Rightarrow l\overrightarrow{X_L}^2-\frac{l\overrightarrow{X_R}\overrightarrow{X_L}-\overrightarrow{X_R}\overrightarrow{B}}{\overrightarrow{X_R}^2}=\overrightarrow{X_R}\overrightarrow{X_L}$
$\Rightarrow l(|\overrightarrow{X_L}|^2|\overrightarrow{X_R}|^2-(\overrightarrow{X_L}\overrightarrow{X_R})^2)=|\overrightarrow{X_R}^2|(\overrightarrow{X_L}\overrightarrow{B})-(\overrightarrow{X_L}\overrightarrow{X_R}\overrightarrow{B}\overrightarrow{X_R})$
$\Rightarrow l=\frac{|\overrightarrow{X_R}|^2(\overrightarrow{X_R}\overrightarrow{B})-(\overrightarrow{X_L}\overrightarrow{X_R}\overrightarrow{X_R}\overrightarrow{B})}{|\overrightarrow{X_L}|^2|\overrightarrow{X_R}|^2-(\overrightarrow{X_L}\overrightarrow{X_R})}$
Similarly,
$\Rightarrow r=\frac{(\overrightarrow{X_L}\overrightarrow{X_R}\overrightarrow{X_L}\overrightarrow{B})-|\overrightarrow{X_L}|^2(\overrightarrow{X_R}\overrightarrow{B})}{|\overrightarrow{X_L}|^2|\overrightarrow{X_R}|^2-(\overrightarrow{X_L}\overrightarrow{X_R})}$
Therefore, $\overrightarrow{X^W}=\frac{1}{2}(\overrightarrow{X_L^W}+\overrightarrow{X_R^W})=\frac{1}{2}(l\overrightarrow{X_L}+r\overrightarrow{X_R})$
$\overrightarrow{X^W}=\frac{1}{2}\frac{(|\overrightarrow{X_R}|^2(\overrightarrow{X_L}\overrightarrow{B})-(\overrightarrow{X_L}\overrightarrow{X_R}\overrightarrow{X_R}\overrightarrow{B}))\overrightarrow{X_L}+((\overrightarrow{X_L}\overrightarrow{X_R}\overrightarrow{X_L}\overrightarrow{B})-|\overrightarrow{X_L}|^2(\overrightarrow{X_R}\overrightarrow{B}))\overrightarrow{X_R}}{|\overrightarrow{X_L}|^2|\overrightarrow{X_R}|^2-(\overrightarrow{X_L}\overrightarrow{X_R})}$

Problem 3

a $z_1=\frac{1}{2}ln(x^2+y^2)$, $z_2=tan^{-1}(\frac{x}{y})$
Let $p_1=\frac{\delta z_1}{\delta x}=\frac{x}{x^2+y^2}$
$q_1=\frac{\delta z_1}{\delta y}=\frac{y}{x^2+y^2}$

$\hat{n_1} = \frac{[\frac{x}{x^2+y^2} \frac{y}{x^2+y^2} 1]}{\sqrt{(\frac{x}{x^2+y^2})^2+(\frac{y}{x^2+y^2})^2+1^2}}$
Similarly,
$p_2 = \frac{\delta z_2}{\delta x} = \frac{y}{x^2+y^2}$
$q_2 = \frac{\delta z_2}{\delta y} = \frac{x}{x^2+y^2}$
$\hat{n_2} = \frac{[\frac{y}{x^2+y^2} \frac{x}{x^2+y^2} 1]}{\sqrt{(\frac{y}{x^2+y^2})^2+(\frac{x}{x^2+y^2})^2+1^2}}$

b $\hat{n_1}$ and $\hat{n_2}$ are the same vectors rotated about Z-axis. $R(p_1^2+q_1^2) = R(p_2^2+q_2^2)$. $\therefore$ if R is rotationally symmetric, z_1 & z_2 result in the same shading.

Problem 4

a To calculate the equation of line that starts from 0, 0, -1, we need to figure out the direction of line. The direction of line will be the sum of vectors $\hat{n}$ and vector from 0, 0, -1 to origin. The vector is given by:

$$\begin{bmatrix} 0-0 \\ 0-0 \\ 0+1 \end{bmatrix} + \begin{bmatrix} \hat{n_x} \\ \hat{n_y} \\ \hat{n_z} \end{bmatrix} = \begin{bmatrix} \hat{n_x} \\ \hat{n_y} \\ \hat{n_z}+1 \end{bmatrix}$$

The unit vector is given by:

Direction: $\frac{1}{\hat{n_x}^2\hat{n_y})^2(\hat{n_z}+1)^2} \begin{bmatrix} \hat{n_x} \\ \hat{n_y} \\ \hat{n_z}+1 \end{bmatrix}$

$\Rightarrow \frac{1}{\sqrt{2n_z+2}} \begin{bmatrix} \hat{n_x} \\ \hat{n_y} \\ \hat{n_z}+1 \end{bmatrix}$

The equation of line is:

L = [0, 0, -1] + $k\frac{1}{\sqrt{2n_z+2}} \begin{bmatrix} \hat{n_x} \\ \hat{n_y} \\ \hat{n_z}+1 \end{bmatrix}$

Let $k = \frac{t}{\hat{n_z}+1}$
L = $[00-1]^T - k[00-1]^T + k[\hat{n_x}\hat{n_y}\hat{n_z}]$
L = (1 - k) $[00-1]^T + k\hat{n_z}$
To find f & g, we must project the line to z = 1 plane. The z coordinate of our line is given by:
z = (1 - k)(-1) + $k\hat{n}$
$\Rightarrow k(1+\hat{n_z}) = 2$
$\Rightarrow k = \frac{2}{(1+\hat{n_z})}$
Substituting k in the equation of line, we get f & g:

$$[\text{f, g, 1}] = (1 - \frac{2}{(1+\hat{n_z})}) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \frac{2}{(1+\hat{n_z})} \begin{bmatrix} \hat{n_x} \\ \hat{n_y} \\ \hat{n_z} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} f \\ g \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2n_x}{1+n_z} \\ \frac{2n_y}{1+n_z} \\ 1 \end{bmatrix}$$

b When $n_z = 0$

$$\Rightarrow \begin{bmatrix} f \\ g \\ 1 \end{bmatrix} = \begin{bmatrix} 2n_x \\ 2n_y \\ 1 \end{bmatrix}$$

We know that,

$n_x^2 + n_y^2 + n_z^2 = 1$

Since $n_z = 0$

$\therefore n_x^2 + n_y^2 = 1$

From the above equation, it is clear that n_x and n_y form a circle of radius 1.

$2n_x$ and $2n_y$ also lie on a circle. The radius of the circle is given by:

$r = \sqrt{(2n_x)^2 + (2n_y)^2}$

$r = \sqrt{4(n_x + n_y)}$

$r = \sqrt{4}$

$r = 2$