CS/RBE 549 Computer Vision Name:______________________________

Fall 2021 Due: Thursday 28 Oct 2021

HW #7

1. **Singular Value Decomposition (5 pts):** This is an important tool that allows us to solve systems of equations $A\vec{x} \approx \vec{0}, |\vec{x}| = 1$. This is useful in camera calibration, where we get such a system of equations and want a non-zero parameter vector $\vec{x}$. See Szeliski for an introduction to SVD.

   You can think of the SVD as follows:

   1. It takes the component of input $\vec{x}$ in the v_1 direction, scales it by σ_1 and outputs it in the u_1 direction.
   2. Repeat for all v_i, σ_i, and u_i.
   3. The output vector is the sum of all the contributions.
   4. The singular values are sorted from largest to smallest, $|\sigma_1| \geq |\sigma_2| \geq \cdots |\sigma_N|$.
   5. Input direction v_1 is scaled by the largest factor, σ_1.
   6. Input direction v_N is scaled by the smallest factor, σ_N, which might be at or near 0.
   7. The v_is are mutually orthogonal. So are the u_is.

   a. **(3 pts)** Suppose that 2×2 matrix M has singular value decomposition

$$U = [u_1|u_2] = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right],$$

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$V^T = \begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix} = \left[\begin{array}{cc} 3/5 & 4/5 \\ \hline -4/5 & 3/5 \end{array}\right]$$

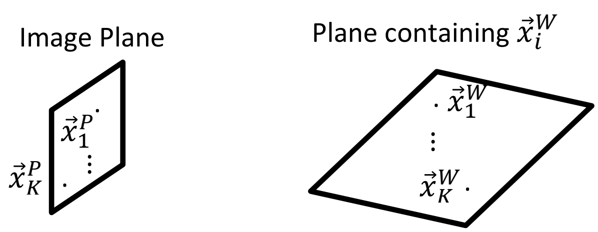
   Without computing M, predict the value of $M\vec{x}$ for $\vec{x} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$ based on the SVD properties.

   Note: The purpose of this problem is to reinforce the main ideas behind SVD. Understanding the 7 points above is all you need to reason through this problem without computing anything.

   Hint: Note that in both cases $V^T\vec{x}$ has a simple form, so does $DV^T\vec{x}$, and finally, so does $UDV^T\vec{x}$. Now compute M and use it to verify $M\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ and $M\begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$.

b. **(2 pts)** Let matrix N have singular value decomposition $N = UDV^T$. Show that matrix $N^T N$ has SVD $N^T N = VD^2V^T$.

2. **Camera Calibration (10 pts):** One restriction in camera calibration is that the world points chosen must not lie in a single plane, that is, they cannot be co-planar, otherwise calibration will fail. To see this, suppose that there are K world points $\vec{x}_i^W, 1 \leq i \leq K$. We know that we need at least 6 points for calibration, $K \geq 6$. Consider what happens if all world points lie in a single plane represented by $\tilde{p} = (a, b, c, d)$, defined by $\tilde{x}^W \cdot \tilde{p} = 0$ (*Szeliski*, eqn. 2.7).



The calibration equation is given by $A\vec{m} = \vec{0}$, written out as

$$\begin{bmatrix} x_1^W & y_1^W & z_1^W & 1 & 0 & 0 & 0 & 0 & x_1^P x_1^W & x_1^P y_1^W & x_1^P z_1^W & x_1^P \\ 0 & 0 & 0 & 0 & x_1^W & y_1^W & z_1^W & 1 & y_1^P x_1^W & y_1^P y_1^W & y_1^P z_1^W & y_1^P \\ & \vdots & & & & & & & & \vdots & & \\ 0 & 0 & 0 & & & \cdots & z_K^W & 1 & y_K^P x_K^W & y_K^P y_K^W & y_K^P z_K^W & y_K^P \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{12} \end{bmatrix}$$

Ideally, there should be a single vector $\vec{m}$ such that $A\vec{m} = \vec{0}$ (or $\approx \vec{0}$). This is the $\vec{m}$ that we hope to find through the singular value decomposition of A. The key concept here is the *rank* of a matrix, which is the number of independent rows, also the number of independent columns. For this $\vec{m}$ to exist and be unique, A must have rank = 11 (with $\sigma_{12} = 0$) or rank = 12 (with $\sigma_{12} \approx 0$). Show that if all world points are co-planar, then A cannot have rank greater than 9 by finding 3 independent non-zero vectors $\vec{m}$ such that $A\vec{m} = \vec{0}$. (These independent vectors establish that the nullspace of A has rank at least 3; hence A's rank cannot exceed 12-3=9.) In this case, it is not possible to find a unique $\vec{m}$ such that $A\vec{m} = \vec{0}$ and the calibration procedure fails.

Hint: The non-zero vectors $\vec{m}$ are mostly 0s. Use the fact that if all world points are co-planar, then $\tilde{x}_i^W \cdot \tilde{p} = 0$. This problem is not as hard as it appears once you realize that you don't have to understand anything about calibration and simply follow this hint. But having followed the hint and solved the problem, if you go back and read through it all again, you might just understand something about calibration!

3. **Camera Parameters (9 pts)** From Lecture 13, using the SVD and our calibration procedure, we can find a unique value of the camera parameter vector $\vec{m}$ with components

$$k\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{bmatrix} = \begin{bmatrix} f_x r_{11} + c_x r_{31} \\ f_x r_{12} + c_x r_{32} \\ f_x r_{13} + c_x r_{33} \\ f_x T_x + c_x T_z \\ f_y r_{21} + c_y r_{31} \\ f_y r_{22} + c_y r_{32} \\ f_y r_{23} + c_y r_{33} \\ f_y T_y + c_y T_z \\ -r_{31} \\ -r_{32} \\ -r_{33} \\ -T_z \end{bmatrix}$$

In this problem, we find values for $R, \vec{T}, \vec{C}, f_x, f_y$. First, we must deal with the scaling factor. The SVD gives us $\vec{m}$ up to some scaling factor *k*. Because *R* is orthonormal, we must have $r_{31}^2 + r_{32}^2 + r_{33}^2 = 1$. Therefore $k = 1 \big/ \sqrt{r_{31}^2 + r_{32}^2 + r_{33}^2}$. Define $\vec{m}' = k\vec{m}$.

a. **(2 pts)** What are r_{31}, r_{32}, r_{33}, and T_z in terms of m'_i?
b. **(2 pts)** Show how you can use part a. and m'_1, m'_2, m'_3 to get c_x. Hint: Use the orthonormality of *R*.
c. **(2 pts)** Show how you can use parts a. and b., and m'_1, m'_2, m'_3 to get f_x. Hint: Use the orthonormality of *R* again.
d. **(3 pts)** Show how you can use parts a., b., and c. and m'_1through m'_4 to get r_{11}, r_{12}, r_{13}, and T_x.

We could get $c_y, f_y, r_{21}, r_{22}, r_{23},$ and T_y the same way, although we won't do so here. This gives us all 16 camera parameters from only 12 equations! (However, we took advantage of the orthonormality of *R* which actually supplies another 6 equations.)

4. **Object Representation (<u>6</u> pts):** We can represent an object by its boundary $(x(s), y(s)), 0 \leq s \leq S$ where *S* is the length of the object's boundary and *s* is distance along that boundary from some arbitrary starting point. Combine *x* and *y* into a single complex function $z(s) = x(s) + jy(s)$. The Discrete Fourier Transform (DFT) of *z* is

$$Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s), 0 \le k \le S-1$$

We can use the coefficients $Z(k)$ to represent the object boundary. The limit on s is S-1 because for a closed contour $z(S) = z(0)$. The Inverse Discrete Fourier Transform is

$$z(s) = \frac{1}{S} \sum_{k=0}^{S-1} e^{+2\pi j \frac{ks}{S}} Z(k), 0 \le s \le S-1$$

a. **(2 pts)** Suppose that the object is translated by $(\Delta x, \Delta y)$, that is, $z'(s) = z(s) + \Delta x + j\Delta y$. How is z''s DFT $Z'(k)$ related to $Z(k)$?

b. **(2 pts)** What object has $z(s) = R \cos \frac{2\pi s}{S} + jR \sin \frac{4\pi s}{S}$? Sketch it. Hint: This is infinitely easy!

c. **(2 pts)** What is $Z(k)$ corresponding to $z(s)$ from Part b? Hint: Most coefficients are 0.