RBE 549: Homework Assignment 7

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Problem 1

a $U = \left[\begin{array}{c|c} u_1 & u_2 \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right]$

$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$V^T = \left[\begin{array}{c} v_1 \\ \hline v_2 \end{array}\right] = \left[\begin{array}{cc} 3/5 & 4/5 \\ \hline -4/5 & 3/5 \end{array}\right]$

$\overrightarrow{x} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

Now,

$V^T\overrightarrow{x} = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$D\ V^T\overrightarrow{x} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

$M\ \overrightarrow{x} = UDV^T\overrightarrow{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\overrightarrow{x} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$

$M\ \overrightarrow{x} = UDV^T\overrightarrow{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$

$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

Now, M = U D V^T = $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \frac{1}{5} & \frac{11}{10} \\ 1 & \frac{1}{2} \end{bmatrix}$

For $\overrightarrow{x} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, M\overrightarrow{x} = \begin{bmatrix} 2 & 1.1 \\ 1 & 0.5 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\overrightarrow{x} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}, M\overrightarrow{x} = \begin{bmatrix} 2 & 1.1 \\ 1 & 0.5 \end{bmatrix}\begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

b $N = U.D.V^T$

N$^T = (U.D.V^T)^T$

$\Rightarrow = (V^T)^T D^T U^T$

Now,

N$^T N = V D^T U^T.U.D.V^T$

$\Rightarrow = V D^T D V^T \qquad (\because U^T U = I)$

Since D is a diagonal matrix,

D$^T D = D^2$

$\therefore N^T N = V D^2 V^T$

Problem 2

One can find three vectors $\overrightarrow{m}$ as

$\overrightarrow{m_1} = \begin{bmatrix} a & b & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\overrightarrow{m_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & a & b & c & d & 0 & 0 & 0 & 0 \end{bmatrix}$

$\overrightarrow{m_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & c & d \end{bmatrix}$

According to definition,

$$\begin{bmatrix} {x_1}^w & {y_1}^w & {z_1}^w & 1 \\ \vdots & \vdots & \vdots & \vdots \\ {x_k}^w & {y_k}^w & {z_k}^w & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

A $\overrightarrow{m_1} = a\overrightarrow{{X_k}^w} + b\overrightarrow{{Y_k}^w} + c\overrightarrow{{Z_k}^w} + 1 = 0$

$$\text{A } \overrightarrow{m_2} = A(1:12, 4:8)\overrightarrow{m_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ {x_1}^w & {y_1}^w & {z_1}^w & 1 \\ \vdots & \vdots & \vdots & \vdots \\ {x_k}^w & {y_k}^w & {z_k}^w & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$$A\ \overrightarrow{m_3} = A(1:12, 8:12)\overrightarrow{m_3} = \begin{bmatrix} {x_1}^p {x_1}^w & {x_1}^p {y_1}^w & {x_1}^p {z_1}^w & 1 \\ {y_1}^p {x_1}^w & {y_1}^p {y_1}^w & {y_1}^p {z_1}^w & 1 \\ \vdots & \vdots & \vdots & \vdots \\ {y_1}^p {x_k}^w & {y_1}^p {y_k}^w & {y_1}^p {z_k}^w & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

So $\overrightarrow{m_1}, \overrightarrow{m_2}, \overrightarrow{m_1}$ satisfy $A\vec{m} = 0$

Alternatively, if all points are co-planar, $a\overrightarrow{X_i^w} + b\overrightarrow{Y_i^w} + c\overrightarrow{Z_i^w} + 1 = 0$ which means $\overrightarrow{Z_i^w}$ can be represented by $\overrightarrow{X_i^w}, \overrightarrow{Y_i^w}$. So the $3^{rd}, 7^{th} \& 11^{th}$ columns can be represented by other columns, i.e., these columns are linearly dependent on other columns.

Rank $\leq 12 - 3 = 9$

Problem 3

a $\overrightarrow{m'} = km$

$r_{31} = -m'_9$

$r_{32} = -m'_{10}$

$r_{33} = -m'_{11}$

$T_z = -m'_{12}$

b $\because$ R is orthonormal

$r_{31}^2 + r_{32}^2 + r_{33}^2 = 1$

$\therefore r_{11}.r_{31} + r_{12}.r_{32} + r_{13}.r_{33} = 0$

$m'_1 = f_x r_{11} + c_x r_{31} \Rightarrow m'_1 m'_9 = -f_x r_{11} r_3 1 - c_x r_{31} r_{31}$

$m'_2 = f_x r_{12} + c_x r_{32} \Rightarrow m'_2 m'_{10} = -f_x r_{12} r_3 2 - c_x r_{32} r_{32}$

$m'_3 = f_x r_{13} + c_x r_{33} \Rightarrow m'_3 m'_{11} = -f_x r_{13} r_3 3 - c_x r_{33} r_{33}$

$m'_1 m'_9 + m'_2 m'_{10} + m'_3 m'_{11} = -f_x(r_{11} r_3 1 + r_{12} r_3 2 + r_{13} r_3 3) - c_x(r_{31} + r_{32} + r_{32})^2$

$\Rightarrow c_x = -(m'_1 m'_9 + m'_2 m'_{10} + m'_3 m'_{11})$

c $m_1'^2 = (f_x r_{11} + c_x r_{31})^2 = f_x^2 r_{11}^2 + 2 f_x r_{11} c_x r_{31} + c_x^2 r_{31}^2$

$m_2'^2 = (f_x r_{12} + c_x r_{32})^2 = f_x^2 r_{12}^2 + 2 f_x r_{12} c_x r_{32} + c_x^2 r_{32}^2$

$m_3'^2 = (f_x r_{13} + c_x r_{33})^2 = f_x^2 r_{13}^2 + 2 f_x r_{13} c_x r_{33} + c_x^2 r_{33}^2$

$m_1'^2 + m_2'^2 + m_3'^2 = f_x^2(r_{11}^2 + r_{12}^2 + r_{13}^2) + 2 f_x c_x (r_{11} r_{31} + r_{12} r_{32} + r_{13} r_{33}) + c_x^2(r_{31}^2 + r_{32}^2 + r_{33}^2)$

$\because$ R is orthonormal

$m_1'^2 + m_2'^2 + m_3'^2 = f_x^2 + c_x^2$

$f_x^2 = m_1'^2 + m_2'^2 + m_3'^2 + (m'_1 m'_9 + m'_2 m'_{10} + m'_3 m'_{11})^2$

$f_x = \sqrt{m_1'^2 + m_2'^2 + m_3'^2 + (m'_1 m'_9 + m'_2 m'_{10} + m'_3 m'_{11})^2}$

d $m'_1 = f_x r_{11} + c_x r_{31}$

$(m'_1 - c_x r_{31})/f_x = r_{11}$

$\Rightarrow r_{11} = \frac{m'_1+m'_9(m'_1m'_9+m'_2m'_{10}+m'_3m'_{11})}{\sqrt{m'^2_1+m'^2_2+m'^2_3+(m'_1m'_9+m'_2m'_{10}+m'_3m'_{11})^2}}$
Similarly,
r$_{12} = (m'_2 - c_x m'_{10})/f_x$
r$_{13} = (m'_3 - c_x m'_{11})/f_x$
T$_x = (m'_4 - c_x m'_{12})/f_x$
Where, c$_x = -(m'_1m'_9+m'_2m'_{10}+m'_3m'_{11}), and f_x = \sqrt{m'^2_1 + m'^2_2 + m'^2_3 + (m'_1m'_9 + m'_2m'_{10} + m'_3m'_{11})^2}$

Problem 4

a $Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s)$
For z'(s) = z(s) + $\Delta x + j\Delta y$
Z'(K) = $\sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z'(s)$
$\Rightarrow \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} (x + jy + \Delta x + j\Delta y)$
$\Rightarrow Z(k) + \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} (\Delta x + j\Delta y)$
$\Rightarrow Z(k) + (\Delta x + j\Delta y) \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}}$
$\Rightarrow Z(k) + (\Delta x + j\Delta y)\delta(k)$
$\therefore Z'(K) = Z(K), K \neq 0;$
Z'(K) = Z(b) + $(\Delta x + j\Delta y), K = 0$

b For $z(s) = Rcos(\frac{2\pi ks}{S}) + jRsin(\frac{4\pi ks}{S})$, we get the plot as shown in Fig. 1. Y axis represents R while the X axis represents the function z(s).

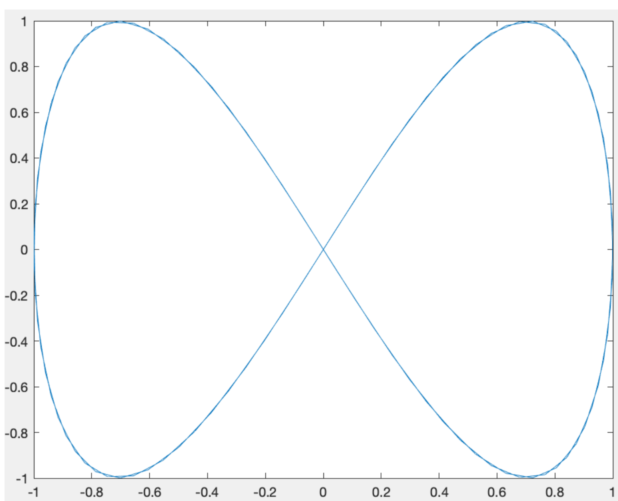

Figure 1: Curve

c We know that, $cosx = (e^{jx} + e^{-jx})/2 \& sinx = (e^{jx} - e^{-jx})/2j$
So,

Z(k) $= \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s)$
$\Rightarrow \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} (Rcos(\frac{2\pi ks}{S}) + jRsin(\frac{4\pi ks}{S}))$
$\Rightarrow \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} \left(R(e^{j\frac{2\pi ks}{S}} + e^{-j\frac{2\pi ks}{S}})/2 + jR(e^{j\frac{4\pi ks}{S}} - e^{-j\frac{4\pi ks}{S}})/2j \right)$
$\Rightarrow R/2(\delta(k-1) + \delta(k+1) + \delta(k+2) - \delta(k-2))$