MA1140 Elementary Linear Algebra

Assignment - 4

1. $A = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix}$

$$char_A(x) = (-1-x)(6-x) + 12$$
$$= x^2 - 5x + 6$$
$$= (x-3)(x-2)$$

Roots of $char_A(x) = 2, 3$

$\boxed{\lambda = 2, 3 \text{ are the eigenvalues of } A}$

$\Rightarrow \lambda = 2$:

$\boxed{alg_A(2) = 1}$ $[char_A(x) = (x-2)^1 (x-3)^1]$

$$(A - 2I) = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

$N(A-2I)$:

$$\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 = 2x_2 \Rightarrow \boxed{x_1 = 2/3 \, x_2}$$

$\boxed{\therefore N(A-2I) = \left\langle \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\rangle \text{ is the eigenspace of } A \text{ for } \lambda = 2}$

$\boxed{geo_A(2) = \dim(N(A-2I)) = 1}$

$\rightarrow$ $\lambda = 3$:

$$\boxed{alg_A(3) = 1}$$

$$A - 3I = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$$

$N(A-3I)$:

$$\begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 = 2x_2 \Rightarrow \boxed{x_1 = x_2/2}$$

$$\boxed{\therefore N(A-3I) = \left\langle \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \right\rangle \text{ is the eigenspace of } A \text{ for } \lambda = 3}$$

$$\boxed{geo_A(3) = \dim(N(A-3I)) = 1}$$

2. For $A \in M_{n \times n}(\mathbb{C})$, we know

$$char_A(x) = (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2} \ldots (x-\lambda_r)^{k_r}(-1)^n \quad [n = \sum k_i]$$

~~substituting x~~

$$\boxed{\text{The constant term in } char_A(x) = \lambda_1^{k_1}\lambda_2^{k_2}\ldots\lambda_r^{k_r}(-1)^n}$$

Also, $char_A(0) = \lambda_1^{k_1}\lambda_2^{k_2} \ldots \lambda_r^{k_r}(-1)^n$

$= \det(A - 0I)$

$= \det(A)$

$$\boxed{\therefore \text{ We get that the constant term in } char_A(x) = \det(A)}$$

3. A & B are similar $\Rightarrow \exists$ invertible $S \in M_{n\times n}(\mathbb{C})$ such that

$$A = SBS^{-1}$$

$$\therefore A^m = (SBS^{-1})(SBS^{-1}) \ldots \text{ m times}$$

$$= SB(S^{-1}S)B(S^{-1}S) \ldots \text{ m times}$$

$$\boxed{A^m = SB^m S^{-1}} \quad (S^{-1}S = I_{n\times n})$$

$\therefore A^m$ & B^m are similar (By definition)

4. $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$

$char_A(x) = -(x-3)^2(x-4)$

Eigenvalues are 3, 3, 4

$\lambda = 3$

$alg_A(3) = 2$

$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

$(3x_1 = x_2 + 9x_3)$

Rank $(A - 3I) = 1$

$\therefore$ Nullity $(A-3I) = geo_A(3)$

$= 3 - 1 = 2$

$\therefore \boxed{alg_A(3) = geo_A(3) = 2}$

$N(A-3I) = \left\langle \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

$\lambda = 4$

$alg_A(4) = 1$

$A - 4I = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 0 & 9 \\ 0 & 0 & -1 \end{bmatrix}$

$N(A-4I) = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$

$\therefore \dim(N(A-4I)) = 1$

$\boxed{alg_A(4) = geo_A(4) = 1}$

since $alg_A(\lambda) = geo_A(\lambda)$ $\forall$ λ are eigenvalues,

A is diagonalizable.

$$\therefore \; S = \begin{bmatrix} 1/3 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 3 & 0 & -9 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix}$$

$$\boxed{A = SDS^{-1}}$$

5. $A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$$char_A(x) = (1-x)^2 - 1$$
$$= x(x-2) \quad \text{Eigenvalues are } \lambda = 0, 2$$

$\lambda = 0$	$\lambda = 2$
$N(A-0I): \left[\begin{array}{cc\|c}1 & -1 & 0\\ -1 & 1 & 0\end{array}\right] R_2 \to R_2 + R_1$	$N(A-2I): \left[\begin{array}{cc\|c}-1 & -1 & 0\\ -1 & -1 & 0\end{array}\right] R_2 \to R_2 - R_1$
$\left[\begin{array}{cc\|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$	$\left[\begin{array}{cc\|c}-1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$
$\boxed{x_1 = x_2}$	$\boxed{x_1 = -x_2}$
$\therefore N(A) = \left\langle \begin{bmatrix}1\\1\end{bmatrix} \right\rangle$	$\therefore N(A-2I) = \left\langle \begin{bmatrix}-1\\1\end{bmatrix} \right\rangle$
$alg_A(0) = 1$	$alg_A(2) = 1$
$geo_A(0) = 1$	$geo_A(2) = 1$

Since $alg_A(\lambda) = geo_A(\lambda) \ \forall \lambda =$ eigenvalues of A,
A is diagonalizable.

$\therefore A = SDS^{-1}$, where

$$S = \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}, \quad D = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}, \quad S^{-1} = \begin{bmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}$$

$$\therefore e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}$$

$$\cancel{= \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix} = \begin{bmatrix}-e^{2t} & 1\\ e^{2t}+1 & 1/2\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}}$$

$$\cancel{= \begin{bmatrix}-e^{2t} & 0\\ e^{2t} & 0\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix} = \begin{bmatrix}1/2\, e^{2t} & -1/2\, e^{2t}\\ -1/2\, e^{2t} & 1/2\, e^{2t}\end{bmatrix}}$$

$$\cancel{= \frac{1}{2} e^{2t}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}$$

$$= \begin{bmatrix}1/2(e^{2t}+1) & 1/2(1-e^{2t})\\ 1/2(1-e^{2t}) & 1/2(e^{2t}+1)\end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = -A$$

$$\therefore A^3 = A.A^2 = -A^2 = A$$

$$\Leftrightarrow A^n = (-1)^{n+1} A$$

$$\therefore e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = A.\sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^n}{n!} + I$$

$$= A(-1) \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} + I$$

$$= -A(e^{-t} - 1) + I$$

$$= I + A(1 - e^{-t})$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^{-t} - 1 & 1 - e^{-t} \\ 0 & 0 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^{-t} & 1 - e^{-t} \\ 0 & 1 \end{bmatrix}}$$

6. $u, v \in \mathbb{R}^3$ are linearly independent; $T = \langle u, v \rangle$

(a) $v^\perp = v - \dfrac{(u,v)}{(u,u)} u$

$v^\perp$ can be expressed as $av + bu$, $a, b \in \mathbb{R}$.
By definition, $v^\perp \in \langle u, v \rangle$

(b) Angle btw $v^\perp$ & u is $\cos^{-1}\left(\dfrac{(v^\perp, u)}{\|v^\perp\|\|u\|}\right)$

$$(v^\perp, u) = \left(v - \frac{(u,v)}{(u,u)} u, u\right)$$

$$= (v, u) - \frac{(u,v)\cdot(u,u)}{(u,u)} = \boxed{0}$$

$\therefore$ Angle btw $v^\perp$ & u $= \cos^{-1}(0)$
$= 90^\circ$ ($\pi/2$ rad)

(c) Let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

$$v^\perp = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{u_1^2 + u_2^2 + u_3^2}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Let $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ be $\perp$r to u & $v^\perp$

$(w, u) = 0$:
$w_1 u_1 + w_2 u_2 + w_3 u_3 = 0$.

$(w, v^\perp) = 0$:

$$(w_1 v_1 + w_2 v_2 + w_3 v_3) - \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{u_1^2 + u_2^2 + u_3^2}\right)\underbrace{(u_1 w_1 + u_2 w_2 + u_3 w_3)}_{=0} = 0$$

$\therefore$ $w_1 v_1 + w_2 v_2 + w_3 v_3 = 0$

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right] \quad R_2 \to R_2 - \frac{v_1 R_1}{u_1}$$

$$\cancel{\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ 0 & \frac{v_2u_1 - u_2v_1}{u_1} & & 0 \end{array}\right]}$$

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ 0 & \left(\frac{v_2u_1 - u_2v_1}{u_1}\right) & \left(\frac{v_3u_1 - u_3v_1}{u_1}\right) & 0 \end{array}\right]$$

w_1 & w_2 are dependant:

$$\therefore w_2 = \frac{(u_3v_1 - u_1v_3)}{(v_2u_1 - u_2v_1)} w_3$$

$$w_1 = \frac{(u_2v_3 - u_3v_2)}{(u_1v_2 - u_2v_1)} w_3$$

$$\boxed{w = \left\langle \begin{bmatrix} \left(\frac{u_2v_3 - u_3v_2}{u_1v_2 - u_2v_1}\right) \\ \left(\frac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1}\right) \\ 1 \end{bmatrix} \right\rangle}$$

(d) To construct an orthogonal basis in $\mathbb{R}^3$, we need a third vector $\in \mathbb{R}^3$, $\notin \langle u, v \rangle$

We can use "w" from part (c).

Let r_1, r_2, r_3 be the orthogonal basis

By Gram - Schmidt process,

$r_1 = u$.

$r_2 = v_2 + \alpha r_1$

$= v + \alpha u. \Rightarrow (r_2, r_1) = 0.$

$\therefore \alpha = -\dfrac{(v \cdot u)}{(u \cdot u)}$

$$\boxed{\therefore r_2 = v - \frac{(v \cdot u)}{(u \cdot u)} u = v^{\perp}}$$

$r_3 = v_3 + \alpha r_1 + \beta r_2. \Rightarrow (r_3, r_1) = 0, \ (r_3, r_2) = 0$

$$\alpha = -\frac{(v_3 \cdot r_1)}{(r_1 \cdot r_1)} = -\frac{\overbrace{(w, u)}^{=0}}{(u, u)} = \boxed{0}$$

$$\beta = -\frac{(v_3 \cdot r_2)}{(r_2 \cdot r_2)} = -\frac{\left(w, v - \frac{(v \cdot u)}{(u \cdot u)} u\right)}{\left\| v - \frac{(v \cdot u)}{(u \cdot u)} u \right\|^2}$$

$$= \frac{-\overbrace{(w, v)}^{=0} + \frac{(v, u)}{(u, u)} \overbrace{(w, u)}^{=0}}{\left\| v - \frac{(v, u)}{(u, u)} u \right\|^2}$$

$$= \boxed{0}$$

$$\boxed{\therefore r_3 = v_3 = w}$$

$\therefore$ The orthonormal basis is

$$B = \left\{ \frac{u}{\|u\|}, \frac{v^{\perp}}{\|v^{\perp}\|}, \frac{w}{\|w\|} \right\}$$

$\frac{4}{25} + \frac{1}{25} + 1$ $\qquad \frac{4+1}{25} = \frac{1}{5}$ $\qquad \frac{1}{5} + 1 = \frac{6}{5}$ $\qquad \sqrt{\frac{6}{5}}$ $\qquad -\frac{2}{5} \times \frac{\sqrt{5}}{\sqrt{6}}$

(e) $u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{pmatrix} u_1 = 1 & v_1 = 0 \\ u_2 = 2 & v_2 = 1 \\ u_3 = 0 & v_3 = 1 \end{pmatrix}$

$(u, v) = 2 \quad ; \quad (u, u) = 5$

$\rightarrow \therefore v^{\perp} = v - \frac{(u,v)}{(u,u)} u$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 1/5 \\ 1 \end{bmatrix}$$

$\rightarrow (v^{\perp}, u) = u^T v^{\perp} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -2/5 \\ 1/5 \\ 0 \end{bmatrix}$

$$= \boxed{0}$$

$\rightarrow w = \left\langle \begin{bmatrix} \left(\frac{u_2 v_3 - u_3 v_2}{u_1 v_2 - u_2 v_1}\right) \\ \left(\frac{u_3 v_1 - u_1 v_3}{u_1 v_2 - u_2 v_1}\right) \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \left(\frac{2}{1}\right) \\ \left(\frac{-1}{1}\right) \\ 1 \end{bmatrix} \right\rangle$

$$= \left\langle \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\rangle$$

$\rightarrow$ orthogonal basis $B = \left\{ \frac{u}{\|u\|}, \frac{v^{\perp}}{\|v^{\perp}\|}, \frac{w}{\|w\|} \right\}$

$$= \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -2/\sqrt{30} \\ 1/\sqrt{30} \\ \sqrt{5/6} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

6

$2/\sqrt{6}$

$\frac{4}{30} + \frac{1}{30} + \frac{5}{6}$