ASSIGNMENT-1 ON LAPLACE TRANSFORM

1.

(a) $g(t) = te^{t^2}\sin(e^{t^2})$

(i) Each one of $t, e^{t^2}, \sin(e^{t^2})$ are continuous in $[0,\infty)$ hence the function is continuous on $[0,\infty)$.

$\Rightarrow$ If $\lim_{t\to\infty} e^{-\alpha t} g(t)$ exists for any α, it is of exponential order 'α'

$\lim_{t\to\infty} e^{-\alpha t} te^{t^2}\sin(e^{t^2}) = \lim_{t\to\infty} te^{t(t-\alpha)} \sin(e^{t^2}) \Rightarrow$ Limit doesnt exist

$\therefore$ $g(t)$ is NOT of exponential order.

(ii)

$g(t) = f'(t)$ where $f(t) = -\frac{1}{2}\cos(e^{t^2})$

we know, if $f(t)$ is Laplace Transformable, so is $f'(t)$

$\{L[f'(t)] = sL[f(t)] - f(0)\}$

$f(t)$ is of exponential order & piecewise continous $\Rightarrow L[f(t)]$ exists.

$\therefore$ $\mathcal{L}[g(t)]$ exist ~~for~~ for $Re(s) > 0$.

(iii) g is the derivative of $f(t) = -\frac{1}{2}\cos e^{t^2}$, which is of exponential order

$$\boxed{-\frac{1}{2}\lim_{t\to\infty} e^{-\alpha t}\cos e^{t^2} = 0}$$

2.

(a) $f(t) = \cos(at+b)$

$= \cos at \cos b - \sin at \sin b$

$\therefore \mathcal{L}[f(t)] = \mathcal{L}[\cos at \cos b - \sin at \sin b]$

$= \cos b\,\mathcal{L}[\cos at] - \sin b\,\mathcal{L}[\sin at]$

$= \frac{\cos b \cdot s}{s^2+a^2} - \sin b \frac{\cdot a}{s^2+a^2}$

$= \boxed{\frac{s\cos b - a\sin b}{s^2+a^2}}$

(c) $f(t) = \begin{cases} 0 & 0 \le t \le \pi \\ \sin t & t \ge \pi \end{cases}$

$f(t) = \underbrace{\sin t}_{g(t)} . H(t-\pi)$

we know $\mathcal{L}[g(t)H(t-\pi)] = e^{-\pi s} \mathcal{L}[g(t+\pi)]$

$= e^{-\pi s} \mathcal{L}[\sin(t+\pi)]$

$= e^{-\pi s} \mathcal{L}[-\sin t]$

$= \boxed{\dfrac{-e^{-\pi s}}{s^2+1}}$

(e) $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$

$\mathcal{L}[f(t)] = \cancel{\int_0^1 0.\,dt} \int_0^1 0.e^{-st}\,dt + \int_1^2 (t-1)e^{-st}\,dt + \int_2^\infty 0.e^{-st}\,dt$

$= \int_1^2 [te^{-st} - e^{-st}]\,dt$

$= \left[-\frac{1}{s}e^{-st}\left(t+\frac{1}{s}\right) + \frac{1}{s}e^{-st}\right]_1^2$

$= \left[\frac{e^{-st}}{s}\left[1-t-\frac{1}{s}\right]_1^2\right]$

$= \frac{e^{-2s}}{s}\left[-\frac{1}{s}-1\right] - \frac{e^{-s}}{s}\left[-\frac{1}{s}\right]$

$= \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} = \boxed{\dfrac{e^{-s} - e^{-2s}(1+s)}{s^2}}$

(m) $f(t) = (1+te^{-t})^3$

$$= \cancel{1 + e^{-3t}t^3}$$

$$= 1 + 3e^{-t}\cdot t^2 + 3e^{-t}\cdot t + t^3e^{-3t}$$

$$\therefore \mathcal{L}[f(t)] = \mathcal{L}[1] + 3\mathcal{L}[te^{-t}] + 3\mathcal{L}[t^2e^{-2t}] + \mathcal{L}[t^3e^{-3t}]$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{3.2!}{(s+2)^3} + \frac{3!}{(s+3)^4}$$

$$= \boxed{\frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}}$$

3.

(b) $\cancel{\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt =}$

(a) $F(2) = \int_0^\infty te^{-2t}\cos t\,dt \to I$, where $F(s) = \int_0^\infty e^{-st}\,t\cos t\,dt$

$$= \mathcal{L}[t\cos t]$$

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}\mathcal{L}[f(t)]$$

$$\therefore \mathcal{L}[t\cos t] = -\frac{d}{ds}\cdot\frac{s}{s^2+1}$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

$$\therefore I = F(2) = \frac{4-1}{(4+1)^2} = \boxed{\frac{3}{25}}$$

4.

(a) $F(s) = \dfrac{s^2+2s+5}{(s-1)(s-2)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

Comparing,

$A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) = s^2+2s+5$

$\Rightarrow$ $\underline{s=2}$: $0 - B + 0 = 13$, $\boxed{B=-13}$ | $\underline{s=1}$: $A(2)+0+0 = 8$, $\boxed{A=4}$ | $\underline{s=3}$: $0+0+2C = 20$, $\boxed{C=10}$

$\therefore \mathcal{L}^{-1}[F(s)] = 4\mathcal{L}^{-1}\left[\dfrac{1}{s-1}\right] - 13\mathcal{L}^{-1}\left[\dfrac{1}{s-2}\right] + 10\mathcal{L}^{-1}\left[\dfrac{1}{s-3}\right]$

$= \boxed{4e^{t} - 13e^{2t} + 10e^{3t}}$

(g) $F(s) = \dfrac{6s^2+22s+18}{s^3+6s^2+11s+6}$

$= \dfrac{6s^2+22s+18}{(s+1)(s+2)(s+3)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+2)} + \dfrac{C}{(s+3)}$

Comparing.

$A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) = 6s^2+22s+18$

$\underline{s=-1}$: $A(2) = 2$, $\boxed{A=1}$ | $\underline{s=-2}$: $-B = -2$, $\boxed{B=2}$ | $\underline{s=-3}$: $C(2) = 6$, $\boxed{C=3}$

$\therefore \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right] + 2\mathcal{L}^{-1}\left[\dfrac{1}{s+2}\right] + 3\mathcal{L}^{-1}\left[\dfrac{1}{s+3}\right]$

$= \boxed{e^{-t} + 2e^{-2t} + 3e^{-3t}}$

(h) $F(s) = \dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$$= \frac{1}{(b^2-a^2)}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{(b^2-a^2)}\left\{\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right]\right\}$$

$$= \frac{1}{(b^2-a^2)}\left[\cos(at) - \cos(bt)\right]$$

$$= \boxed{\frac{\cos(at) - \cos(bt)}{b^2-a^2}}$$

4) $\cancel{F(s) = \dfrac{4s+5}{(s-1)^2(s+2)}} = \cancel{\dfrac{As+B}{(s-1)^2} + \dfrac{C}{(s+2)}}$

(b) $F(s) = \dfrac{3s+5}{s^2+6s+12} = \dfrac{3(s+3) - 4}{(s+3)^2 + (\sqrt{3})^2}$

$$= \frac{3(s+3)}{(s+3)^2+(\sqrt{3})^2} - \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{(s+3)^2+(\sqrt{3})^2}$$

$$\therefore \mathcal{L}^{-1}[F(s)] = 3\mathcal{L}^{-1}\left[\frac{(s+3)}{(s+3)^2+\sqrt{3}^2}\right] - \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{(s+3)^2+\sqrt{3}^2}\right]$$

$$= 3e^{-3t}\mathcal{L}^{-1}\left[\frac{s}{s^2+\sqrt{3}^2}\right] - \frac{4e^{-3t}}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{s^2+\sqrt{3}^2}\right]$$

$$= \boxed{\frac{e^{-3t}}{\sqrt{3}}\left[3\sqrt{3}\cos(\sqrt{3}t) - 4\sin(\sqrt{3}t)\right]}$$