ASSIGNMENT - 6

1. $\omega = \dfrac{1}{z-i}$. Let $z = u + iv$

$\therefore \omega = \dfrac{1}{u+(v-1)i}$. ω is undefined ONLY

WHEN $u = 0$ & $v - 1 = 0 \Rightarrow v = 1$.

$\therefore \boxed{z = i}$

When $z = 1 - 2i$,

$$\omega = \frac{1}{1-2i-i} = \frac{1}{1-3i} = \frac{(1+3i)}{(1-3i)(1+3i)}$$

$$= \boxed{\frac{1+3i}{10}}$$

2. $$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$= 1 + \lim_{x\to 0} \frac{2\sin^2(x/2)}{(x/2)^2 \times 4}$$

$$= 1 + \frac{1}{2} \lim_{x\to 0} \left(\frac{\sin(x/2)}{(x/2)}\right)^2$$

$$= \boxed{3/2}$$

$\frac{y}{x+iy} = \frac{1}{\left(\frac{x}{y}\right)+i} - v^2 + 2vu\, i\,][u+iv]$

$u^3 - uv^2 - 2v^2u$

~~3. (a) Let $z = u + iv$.~~

~~$\therefore \frac{iz^3+1}{z^2+1} = \frac{i[(u^3-3uv^2) + i[3vu^2 - v^3]] + 1}{[u^2-v^2+1] + [2vu]i}$~~

3. (a) $\frac{iz^3+1}{z^2+1} = \frac{iz^3+1}{(z+i)(z-i)}$

$$= \frac{i(z^3 - i)}{(z+i)(z-i)}$$

$$= \frac{i(z^3 + i^3)}{(z+i)(z-i)}$$

$$= \frac{i\cancel{(z+i)}(z^2 - iz + i^2)}{\cancel{(z+i)}(z-i)}$$

$$= \boxed{\frac{iz^2 + z - i}{z-i}}$$

$$\therefore \lim_{z \to -i} \frac{iz^3+1}{z^2+1} = \lim_{z \to -i} \frac{iz^2+z-i}{z-i}$$

$$= \frac{i(-1) - i - i}{-i-i}$$

$$= -\frac{3i}{-2i} = \boxed{3/2}$$

$\therefore$ LIMIT EXISTS.

(b) $\lim_{z \to 0} \frac{Im(z)}{z}$

Let $z = u + iv$.

$$\frac{Im(z)}{z} = \frac{v}{u+iv} = \frac{1}{\left(\frac{u}{v}\right)+i}$$

$\left(\lim_{\substack{u \to 0, \\ v \to 0}} \frac{u}{v}\right)$ is UNDEFINED.

$\therefore$ Limit does NOT exist

4. $a_n = \frac{n^3 x^{3n}}{n^4+1}$

$a_{n+1} = \frac{(n+1)^3 x^{3n+3}}{(n+1)^4+1}$

$$\therefore \frac{a_{n+1}}{a_n} = x^3 \left[\frac{n^4+1}{(n+1)^4+1}\right]\left[\left(\frac{n+1}{n}\right)^3\right]$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = x^3$$

$\therefore$ Radius of convergence is 1

[since for sequence to be convergent, $\frac{a_{n+1}}{a_n} < 1 \Rightarrow$
$x^3 < 1 \Rightarrow x < 1$]

5. $$\sum_{1}^{\infty} \frac{1+2^n}{3^{n-1}} = \sum_{1}^{\infty} \frac{1}{3^{n-1}} + 2\sum_{1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$$

$$= \left(1+\frac{1}{3}+\frac{1}{9}+\cdots\right) + 2\left[1+\frac{2}{3}+\frac{4}{9}+\cdots\right]$$

$$= \frac{1}{1-(1/3)} + 2 \cdot \frac{1}{(1-2/3)}$$

$$= \frac{3}{2} + 2\cdot(3)$$

$$= \boxed{7.5}$$

6. Theorem - A series $\sum (-1)^n x_n$ is convergent if $\lim_{n\to\infty} x_n = 0$ and $\{x_n\}$ is monotonically decreasing.

~~since~~ $\cancel{\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \Rightarrow 0}$, $\boxed{\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0}$ $\to$ ①

~~$\sum (-1)^n x_n$ is CONDITIONALLY CONVERGENT~~

$\boxed{\frac{1}{\sqrt{n^2+1}} \text{ is MONOTONICALLY DECREASING}}$ $\to$ ②

However, $\sum_{n=0}^{\infty} \left|(-1)^n \frac{1}{\sqrt{n^2+1}}\right|$ is not convergent.

Since $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}} \geq \sum \frac{1}{\sqrt{(n+1)^2}} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

and $\sum_{n=0}^{\infty} \frac{1}{n+1}$ is DIVERGENT.

$\therefore$ $\sum (-1)^n \frac{1}{\sqrt{n^2+1}}$ is CONDITIONALLY CONVERGENT

7. $f(x) = \begin{cases} \ln x & 0 < x < 1 \\ ax^2+b & 1 \le x < \infty \end{cases}$

f is continuous at $x=1$.

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\lim_{x \to 1^-} \ln(x) = \lim_{x \to 1^+} (ax^2+b)$

$0 = a+b$

$\boxed{a+b=0} \to (1)$

$f(2) = 3 \Rightarrow a(2^2)+b = 3$

$\boxed{4a+b=3} \to (2)$

$(2)-(1)$:

$3a = 3 \Rightarrow \boxed{\begin{matrix} a=1 \\ b=-1 \end{matrix}}$

8. $f(z) = 3z^2 + 2z.$

$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z \to z_0} \frac{(3z^2+2z)-(3z_0^2-2z_0)}{z-z_0}$

$= \lim_{z \to z_0} \frac{3(z^2-z_0)+2(z-z_0)}{z-z_0}$

$= \lim_{z \to z_0} \frac{3(z+z_0)\cancel{(z-z_0)}+2\cancel{(z-z_0)}}{\cancel{(z-z_0)}}$

$= \lim_{z \to z_0} 3(z+z_0)+2$

$= 3(z_0+z_0)+2$

$= \boxed{6z_0+2}$

9. $$\frac{z^2-z+1-i}{z^2-2z+2} = \frac{(z^2-2z+2)+(z-i-1)}{z^2-2z+2}$$

$$= 1 + \frac{(z-i-1)}{z^2-2z+2}$$

$$= 1 + \frac{\cancel{(z-(1+i))}}{\cancel{(z-(1+i))}(z-(1-i))}$$

$$= 1 + \frac{1}{(z-1+i)}$$

$$\therefore \lim_{z\to 1+i} \frac{z^2-z+1-i}{z^2-2z+2} = \lim_{z\to i+1} 1 + \frac{1}{(z-1+i)}$$

$$= 1 + \frac{1}{(1+i-1+i)}$$

$$= 1 + \frac{1}{2i} = \boxed{1 - \frac{i}{2}}$$

10. $$(1+x)(1+x^2)(1+x^4)(1+x^8) + \cdots\cdots (1+x^{2^n})$$

$$= (1+x+x^2+x^3+x^4+\cdots\cdots\cdots x^{(1+2+4+\cdots 2^n)})$$

$$= \boxed{1+x+x^2+x^3+\cdots\cdots x^{2^{n+1}-1}}$$

$(1+x+x^2+x^3)(1+x^4)$

$\frac{a(r^n-1)}{r-1}$. $1(2^{n+1}-1)$

11. $f(z)$ is cont. at z_0.

$\therefore \lim_{z \to z_0^-} f(z) = f(z_0) = \lim_{z \to z_0^+} f(z)$

$\therefore \left|\lim_{z \to z_0^-} f(z)\right| = |f(z_0)| = \left|\lim_{z \to z_0^+} f(z)\right|$

(or) $\lim_{z \to z_0^-} |f(z)| = |f(z_0)| = \lim_{z \to z_0^+} |f(z)|$

If $g(z) = |f(z)|$

$$\boxed{\text{then } \lim_{z \to z_0^-} g(z) = g(z_0) = \lim_{z \to z_0^+} g(z)}$$

$\therefore g(z)$ is continuous $\Rightarrow$
$|f(z)|$ is continuous.

12. Let $f(z) = az+b$.

$\therefore f(1) = a+b = 3+i \rightarrow ①$

$f(3i) = a(3i)+b = -2+6i \rightarrow ②$

② − ①:

$(3i-1)a = 5i-5$

$a = \frac{5(i-1)(-3i-1)}{10}$

$= \frac{(1-i)(1+3i)}{2} = \frac{4+2i}{2} = \boxed{2+i}$

subs $a = 2+i$ in (1),

$$(2+i) + b = 3+i$$

$$\boxed{\cancel{b = 1+2i}} \quad \boxed{b = 1}$$

$$\therefore f(z) = (2+i)z + \cancel{(1+2i)}\ 1$$

$$= \boxed{\cancel{(2z+1) + i(2-z)}} = \cancel{2}$$

13. z^3+1 can be written as

$$z^3+1 = (z-e^{i\pi/3})(z-e^{-i\pi/3})(z+1)$$

$$\therefore \frac{(z-e^{i\pi/3})(z)}{z^3+1} = \frac{\cancel{(z-e^{i\pi/3})}(z)}{(z+1)\cancel{(z-e^{i\pi/3})}(z-e^{-i\pi/3})}$$

$$= \frac{z}{(z+1)(z-e^{-i\pi/3})}$$

$$\therefore \lim_{z\to e^{i\pi/3}} \frac{(z-e^{i\pi/3})(z)}{z^3+1} = \lim_{z\to e^{i\pi/3}} \frac{z}{(z+1)(z-e^{-i\pi/3})}$$

$$= \frac{e^{i\pi/3}}{(1+e^{i\pi/3})(e^{i\pi/3}-e^{-i\pi/3})}$$

$$= \frac{(1/2+\sqrt{3}/2\,i)}{(3/2+\sqrt{3}/2\,i)(\sqrt{3}\,i)}$$

$$= -\cancel{(1/2+\sqrt{3}/2\,i)}$$

$$= -\frac{(1+\sqrt{3}\,i)\,i}{(\sqrt{3})(\sqrt{3})(\sqrt{3}+i)}$$

$$= -\frac{i}{3}\,\frac{(1+\sqrt{3}\,i)(\sqrt{3}-i)}{4}$$

$$= -\frac{i}{12}(2\sqrt{3}+2i)$$

$$= -\frac{i}{6}(\sqrt{3}+i)$$

$$= \boxed{\frac{1-\sqrt{3}\,i}{6}}$$

14. $x = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \cdots$

$y = -\frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 3\theta + \cdots$

$$x + iy = 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} + \cdots$$

$$= 1 - \frac{(1/2)e^{i\theta}}{1!} + \frac{(1/2)(3/2)(e^{i\theta})^2}{2!} - \frac{(1/2)(3/2)(5/2)(e^{i\theta})^3}{3!} + \cdots$$

$$= (1 + e^{i\theta})^{-1/2} \quad \text{[Taylor Series Expansion]}$$

$$= (1 + \cos\theta + i\sin\theta)^{-1/2}$$

$$= \frac{1}{\sqrt{2\cos^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)}}$$

$$= \frac{1}{\sqrt{2\cos(\theta/2)} \times \sqrt{e^{i\theta/2}}}$$

$$= \frac{e^{-i\theta/4}}{\sqrt{2\cos(\theta/2)}} = \frac{\cos(\theta/4) - i\sin(\theta/4)}{\sqrt{2\cos(\theta/2)}}$$

$$\boxed{\therefore x = \frac{\cos(\theta/4)}{\sqrt{2\cos(\theta/2)}}}$$

$$(n^2-n-1)+i(2n-1)$$

15. $\sum_{n=2}^{\infty} T_n \Rightarrow T_n = \dfrac{1}{(n+i)(n+i-1)}$

$$= \frac{1}{(n+i-1)} - \frac{1}{(n+i)}$$

$$\sum_{n=2}^{N} T_n = \frac{1}{(1+i)} - \frac{1}{(2+i)}$$

$$+ \frac{1}{(2+i)} - \frac{1}{(3+i)}$$

$$+ \quad \vdots$$

$$= \frac{1}{(1+i)} - \frac{1}{(N+i)}$$

$$\lim_{N\to\infty} \sum_{n=2}^{N} T_n = \frac{1}{(1+i)}$$

$$= \boxed{\frac{1-i}{2}}$$

It is convergent and the sum is $\dfrac{1-i}{2}$