~~2B-2C~~ ASSIGNMENT - 8

1. $\oint \frac{(z^2-2z)\,dz}{(z+1)^2(z^2+4)}$ $\neq$ $\cancel{\oint \frac{\left(\frac{z^2-2z}{z^2+4}\right)}{(z+1)^2}\,dz}$

~~Let $f(z) = \frac{z^2-2z}{z^2+4}$~~

~~By Cauchy's theorem,~~

~~$\oint \frac{f(z)}{(z+1)^2} = 2\pi i \times \frac{1}{1!} f'(z)$ at $z = -1$~~

~~$= 2\pi i \times$~~

Let $(Az+B)(z^2+4) + (Cz+D)(z+1)^2 = z^2-2z$

$A + C = 0 \qquad B - 2C + D = 1$

$4B + D = 0 \qquad 4A + C - 2D = -2$

$-3B - 2C = 1 \qquad 3A + 8B = -2$

$2A - 3B = 1$

$6A - 9B = 3$

$6A + 16B = -4$

$-25B = 7$

$\cancel{B = -7/25}$

$\cancel{A = 46/25}$

$$\boxed{\begin{aligned} B &= -7/25 \\ A &= 2/25 \\ C &= -2/25 \\ D &= 28/25 \end{aligned}}$$

$1 + \frac{21}{25}$

$1 - \frac{21}{25}$

$\therefore \frac{z^2-2z}{(z+1)^2(z^2+4)} = \frac{1}{25}\left(\frac{2z-7}{(z+1)^2} + \frac{(-2z+28)}{(z-2i)(z+2i)}\right)$

$\therefore \oint \frac{z^2-2z\,dz}{(z+1)^2(z^2+4)} = \frac{1}{25}\left[\oint \frac{(2z-7)dz}{(z+1)^2} + \frac{1}{4i}\oint \frac{(-2z+28)}{z-2i} - \frac{1}{4i}\oint \frac{(-2z}{z+2}\right.$

$$= \frac{2\pi i}{25}\left[\frac{1}{1!}\frac{d}{dz}(2z-7)\Big|_{z=-1} + \frac{1}{4i}(-2z+28)\Big|_{z=2i} - \frac{1}{4i}(-2z+28)\Big|_{z=-2i}\right]$$

$$= \frac{2\pi i}{25}\left[2 + \frac{1}{4i}\left[28 - 4i - 4i - 28\right]\right]$$

$$= \frac{2\pi i}{25}\left[0\right] = \boxed{0}$$

(a) Area $= \iint_R dx\,dy = \frac{1}{2i}\oint_C \bar{z}\,dz$ over the hypocycloid.

$$\underbrace{\oint_C \bar{z}\,dz}_{I}$$

$$I = \oint (a\cos^3\theta - ia\sin^3\theta)(-3a\cos^2\theta\sin\theta + 3ia\sin^2\theta\cos\theta)\,d\theta$$

$$= 3a^2 \oint_0^{2\pi} (\cos^3\theta - i\sin^3\theta)(-\cos^2\theta\sin\theta + i\sin^2\theta\cos\theta)\,d\theta$$

$$= 3a^2 \int_0^{2\pi} [\cos^5\theta(-\sin\theta) + \sin^5\theta\cos\theta]\,d\theta$$

$$+ 3ia^2 \int_0^{2\pi} [\cos^4\theta\sin^2\theta + \cos^2\theta\sin^4\theta]\,d\theta$$

$$= 3a^2\left[\frac{\cos^6\theta}{6} + \frac{\sin^6\theta}{6}\Big|_0^{2\pi}\right] + \frac{3ia^2}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{2\pi}$$

$$= 3a^2[(0)] + \frac{3ia^2}{8}(2\pi) = \boxed{\frac{3}{4}i\pi a^2}$$

$$\boxed{\text{Area} = \frac{1}{2i} \times \frac{3}{4} i\pi a^2 = \boxed{\frac{3}{8}\pi a^2}}$$

(b) $\oint (8\bar{z}+3z)\,dz = 2i \iint_R \frac{d(8\bar{z}+3z)}{d\bar{z}}\,dx\,dy$

$= 2i \iint_R (8+0)\,dx\,dy$

$= 8 \times \text{Area} \times 2i$

$= 16i \times \frac{3}{8}\pi a^2$

$= \boxed{6\pi a^2 i}$

3. $$I = \int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$$

(a) $|z| = 2$: ($z = 1, -3/2$ lie in the circle)

$$\therefore\ I = 2\pi i \left[\frac{d}{dz}\left(\frac{12z-7}{2z+3}\right)(z=1) + \left(\frac{12z-7}{2(z-1)^2}\right)(z=-3/2) \right]$$

$$= 2\pi i \left[\frac{50}{(2z+3)^2}(z=1) + \frac{1}{2}\cdot\frac{(-18-7)}{(25/4)} \right]$$

$$= 2\pi i \left[\frac{50}{25} - \frac{1}{2}\cdot\frac{25(4)}{25} \right]$$

$$= \boxed{0}$$

(b) $|z+i| = \sqrt{3}$ (only $z = 1$ lies in circle)

$$\therefore\ I = 2\pi i \left[\frac{50}{25} \right]$$

$$= \boxed{4\pi i}$$

4. $$\int_C \frac{z}{\bar{z}}\,dz = \int_{-2}^{-1} 1\,dz + \int_{\pi}^{0} \frac{e^{i\theta}}{e^{-i\theta}} \times ie^{i\theta}\,d\theta + \int_{1}^{2} 1\,dz$$

$$+ \int_{0}^{\pi} \frac{2e^{i\theta}}{2e^{-i\theta}} \times 2ie^{i\theta}\,d\theta$$

$$= 2 + i\left[-\int_0^{\cancel{2}\pi} e^{3i\theta}\,d\theta + 2\int_0^{\cancel{2}\pi} e^{3i\theta}\,d\theta\right]$$

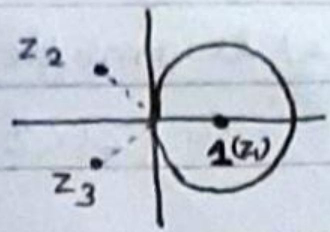
$$= 2 + i\left[\int_0^{\cancel{2}\pi} e^{3i\theta}\,d\theta\right]$$

$\cos(3\pi)$ $\quad e^{2i\pi}$ $\quad e^{i\pi}$

$$= 2 + \frac{i}{3i}\left[e^{3i\theta}\Big|_0^{\cancel{2}\pi}\right] = 2 + \frac{1}{3}(-1-1)$$

$$= 2 - 2/3 = \boxed{4/3}$$

5. $\oint \frac{dz}{(z^3-1)^2}$ Poles are $z_1 = 1$, $z_2 = e^{i(2\pi/3)}$, $z_3 = e^{i(4\pi/3)}$ all of order 2.



since z_2 & $z_3 \notin C$ [circle], they can be ignored

$$\therefore\ I = 2\pi i \times \frac{1}{1!} \frac{d}{dz}\left(\frac{1}{(z^2+z+1)^2}\right) \text{ at } z=1$$

$$= -2\pi i \times \frac{2(2z+1)}{(z^2+z+1)^3} \text{ at } z=1$$

$$= -2\pi i \times \frac{2(3)}{3^3}$$

$$= \boxed{-\frac{4\pi i}{9}}$$

Since counterclockwise,

6. $I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$

Let $z = e^{i\theta}$

$\therefore \sin\theta = \frac{z-\bar{z}}{2i} = \frac{z - 1/z}{2i} = \frac{z^2-1}{2i}$

$d\theta = \frac{dz}{iz}$

$\therefore I = -i \oint_C \frac{dz/z}{a + \frac{b(z^2-1)}{2i}} = \oint_C \frac{2dz}{bz^2 + 2aiz - b}$

[c is circle of radius = 1 & centre at (0,0)]

$\therefore I = 2 \oint \frac{dz}{\left(z-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)i\right)\left(z-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)i\right)}$

Out of these 2 poles,

only $\frac{-a+\sqrt{a^2-b^2}}{b}$ lies in $|z|=1$

$\left(\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} > 1\right)$

$\therefore I = \frac{2\pi i}{b}\left(\frac{1}{\left[\frac{-a+\sqrt{a^2-b^2}}{b} + \frac{a+\sqrt{a^2-b^2}}{b}\right]i}\right)$

$\frac{-a+\sqrt{a^2-b^2}+a+}{b}$

$= \boxed{\frac{2\pi}{\sqrt{a^2-b^2}}}$

7. $I = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta$

$= Re\left\{ \int_0^{2\pi} e^{\cos\theta} \left[\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)\right] d\theta \right.$

$= Re\left\{ \int_0^{2\pi} e^{\cos\theta + i\sin\theta - in\theta}\, d\theta \right\} = Re\{I_1\}$

$I_1 = \int_0^{2\pi} e^{\cos\theta + i\sin\theta} \cdot e^{-in\theta}\, d\theta$

Let $z = \cos\theta + i\sin\theta$

$z^{-n} = e^{-in\theta}$

$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$= i(\cos\theta + i\sin\theta)$

$= \boxed{iz}$

$\boxed{d\theta = \frac{dz}{iz}}$

$\therefore I_1 = \oint e^z \cdot z^{-n} \cdot \frac{dz}{iz}$

$= \cancel{\frac{1}{i} \oint e^z z^{-n-1} dz} = \boxed{-i \oint \frac{e^z}{z^{n+1}} dz}$

By residue theorem,

$I_1 = -i \times 2\pi i \cdot \frac{1}{n!} \frac{d^n}{dz^n}(e^z)$ at $z = 0$

$= \boxed{\frac{2\pi}{n!} e^z} \Rightarrow \therefore \boxed{I = \frac{2\pi}{n!}}$

8. $\int e^x \cos 2x \, dx = \text{Re}\left\{\int e^x [\cos 2x + i\sin 2x]\, dx\right\}$

$= \text{Re}\left\{\int e^x \cdot e^{2ix}\, dx\right\}$

$= \text{Re}\left\{\int e^{x(1+2i)}\, dx\right\}$

$= \text{Re}\left\{\frac{e^{x(1+2i)}}{1+2i}\right\} = \text{Re}\left\{\frac{e^x [\cos 2x + i\sin 2x][1-2i]}{5}\right\}$

$= \boxed{\frac{e^x}{5}[\cos 2x + 2\sin 2x] + C}$

$a\sqrt{1-e^2}$

9. $$\oint \bar{z}\,dz = \oint (x - iy)(dx + i\,dy)$$

$$= \oint (\overset{P}{x}\,dx + \overset{Q}{y}\,dy) + i \oint \overset{P_1}{(-y\,dx)} + \overset{Q_1}{(x\,dy)}$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R (0 - 0)\,dx\,dy + i \iint_R (1 - (-1))\,dx\,dy$$

$$= 2i \iint_R dx\,dy$$

$$= 2i \times (\text{Area of ellipse}) \quad \begin{bmatrix} a = 10/2 = 5 \\ e = 3/a = 3/5 \Rightarrow b = 4 \end{bmatrix}$$

$$\left(\frac{ae}{a}\right) \qquad \boxed{b = a\sqrt{1-e^2}}$$

$$= 2i \times (\pi \times 5 \times 4)$$

$$= \boxed{40 i \pi}$$

$$2\pi i\left[\frac{1}{1!}\frac{d}{dz}\left(\frac{e^z}{z^2+4}\right) + \left(\frac{e^z}{(z-1)^2(z-2i)}\right)_{z=-2i}\right]$$

10. $$\oint \frac{e^z}{(z-1)^2(z^2+4)}dz$$

$$= \oint \frac{\left(e^z/z^2+4\right)}{(z-1)^2}dz$$

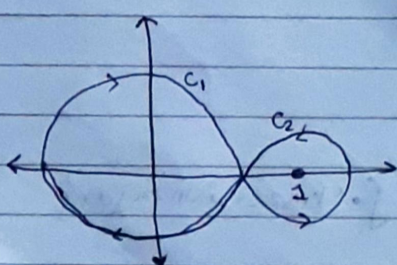
Let $f(z) = \dfrac{e^z}{z^2+4}$ [$f(z)$ is analytic in $|z|<2$]

$$\therefore \oint \frac{f(z)}{(z-1)^2}dz = \frac{2\pi i}{1!} f'(1)$$

$$= 2\pi i \left[\frac{e^z(z^2+4-2z)}{(z^2+4)^2} \text{ at } z=1\right]$$

$$= 2\pi i\left[\frac{e(3)}{25}\right]$$

$$= \boxed{\frac{6}{25}e\pi i}$$

11. 

$$\int_c \frac{z-2}{z(z-1)}dz$$

Poles are $z=0$ [$\in C_1$]
$z=1$ [$\in C_2$]

counter-clockwise

$$\therefore I = \int_{C_1} \frac{(z-2)dz}{z} - \left[-\int_{C_2}\frac{(z-2)}{(z-1)}dz\right]$$

clockwise

$$= 2\pi i\left[(z-2) \text{ at } z=0\right] + 2\pi i\left[(z-2) \text{ at } z=1\right]$$

$$= -4\pi i - 2\pi i$$

$$= \boxed{-6\pi i}$$