MA1140 - Elementary Linear Algebra

Assignment 3

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

$$\boxed{\text{Let } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \begin{array}{l} 2a + b = 3 \to (1) \\ 2c + d = 4 \to (2) \end{array}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{array}{l} a + b = -1 \to (3) \\ c + d = 2 \to (4) \end{array}$$

(A) (1) - (3) :

$$a = 3 - (-1) = 4$$

$$\therefore b = -1 - a = -5$$

(B) (2) - (4) :

$$c = 4 - 2 = 2$$

$$\therefore d = 2 - c = 0$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & 0 \end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 & -5 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

2. $T: \mathbb{P}_2 \to \mathbb{R}^2$ given by

$T(a + bx + cx^2) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$ is a linear transform

if and only if $T(kv_1 + v_2) = kT(v_1) + T(v_2)$ $\forall$ $k \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{P}_2$

Let $v_1 = a_1 + b_1 x + c_1 x^2$, $(a_1, b_1, c_1 \in \mathbb{R})$

$v_2 = a_2 + b_2 x + c_2 x^2$ $(a_2, b_2, c_2 \in \mathbb{R})$

$\therefore T(v_1) = \begin{bmatrix} 2a_1 - b_1 \\ b_1 + c_1 \end{bmatrix}$, $T(v_2) = \begin{bmatrix} 2a_2 - b_2 \\ b_2 + c_2 \end{bmatrix} \to (1)$

$\Rightarrow T(kv_1 + v_2) = T\left(a_1 k + b_1 k x + c_1 k x^2 + a_2 + b_2 x + c_2 x^2\right)$

$= T\left([ka_1 + a_2] + [kb_1 + b_2]x + [kc_1 + c_2]x^2\right)$

$= \begin{bmatrix} 2(ka_1 + a_2) - (kb_1 + b_2) \\ (kb_1 + b_2) + (kc_1 + c_2) \end{bmatrix}$

$= \begin{bmatrix} k(2a_1 - b_1) + (2a_2 - b_2) \\ k(b_1 + c_1) + (b_2 + c_2) \end{bmatrix}$

$= \begin{bmatrix} k(2a_1 - b_1) \\ k(b_1 + c_1) \end{bmatrix} + \begin{bmatrix} 2a_2 - b_2 \\ b_2 + c_2 \end{bmatrix}$

$= k\begin{bmatrix} 2a_1 - b_1 \\ b_1 + c_1 \end{bmatrix} + \begin{bmatrix} 2a_2 - b_2 \\ b_2 + c_2 \end{bmatrix}$

$= \boxed{kT(v_1) + T(v_2)}$ [from (1)]

$\therefore$ Since $T(kv_1 + v_2) = kT(v_1) + T(v_2)$,

$T : \mathbb{P}_2 \to \mathbb{R}^2$ is a linear transformation.

3. $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 2b - c \\ 3a - b + 2c \\ a + b + 2c \end{bmatrix}$$

We know $T_A(x)$ can be expressed as $A.x$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

(a) preimages of $\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ is the solution of

$$A.x = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 3 & -1 & 2 & 5 \\ 1 & 1 & 2 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 5 & 5 & 11 \\ 0 & 3 & 3 & 5 \end{array}\right] \quad \begin{array}{l} \underline{R_2 \to R_2/5} \\ R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 12/5 \\ 0 & 1 & 1 & 11/5 \\ 0 & 0 & 0 & -8/5 \end{array}\right]$$

$5 - 33/5$

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the solution.

$\frac{22}{5} - 2$

$\therefore a + c = 12/5$

$b + c = 11/5$

$\boxed{0 = -8/5} \to$ contradiction.

$\therefore$ No solution exists $\Rightarrow$ There are NO preimages of $\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

(b) $A \cdot x = \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 3 & -1 & 2 & 5 \\ 1 & 1 & 2 & 7 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 0 & 5 & 5 & 20 \\ 0 & 3 & 3 & 12 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/5 \\ R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the solution. (a & b are dependent variables)

$\therefore \; a + c = 3 \Rightarrow a = 3 - c$

$b + c = 4 \Rightarrow b = 4 - c$

$\therefore$ The preimages of $\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ are

$$\boxed{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}} = \boxed{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle}$$

4. $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined as

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2a + b + c \\ -a + 3b + c - d \\ 3a + b + 2c - 2d \end{bmatrix}.$$

x

~~is~~/ ~~injective~~

We know $T_A(x)$ can be represented as $A.x$

$$\boxed{\begin{array}{l} \text{For } T_A(x) \text{ to be injective,} \\ T_A(x_1) = T_A(x_2) \Rightarrow x_1 = x_2 \quad [x_1, x_2 \in \mathbb{R}^4] \end{array}}$$

Let $T_A(x_1) = T_A(x_2) \Rightarrow A.x_1 = Ax_2 \Rightarrow A.(x_1 - x_2) = 0$

Let $x_1 - x_2 = x_3$.

For $T_A(x)$ to be injective,

$\therefore A.x_3 = 0, \; x_3 = 0.$

i.e. $N(A) = \ker(T) = \{0\}$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 1 & -1 \\ 3 & 1 & 2 & -2 \end{bmatrix}$$

$A.x = \{0\}$

Aug. Matrix :

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 3 & 1 & 2 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/2 \\ \hline R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0.5 & 0.5 & 0 & 0 \\ 0 & 3.5 & 1.5 & -1 & 0 \\ 0 & -0.5 & 0.5 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/3.5 \\ \hline R_1 \to R_1 - R_2/2 \\ R_3 \to R_3 + R_2/2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2/7 & 1/7 & 0 \\ 0 & 1 & 3/7 & -2/7 & 0 \\ 0 & 0 & 5/7 & -15/7 & 0 \end{array}\right] \begin{array}{l} R_3 \to R_3/5 \\ R_1 \to R_1 - 2R_3/7 \\ R_2 \to R_2 - 3R_3/7 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right] \quad \left(\begin{array}{c}\text{Reduced Row} \\ \text{Ech. form}\end{array}\right)$$

$$\therefore \quad a + d = 0 \Rightarrow a = -d$$

$$b + d = 0 \Rightarrow b = -d$$

$$c - 3d = 0 \Rightarrow c = 3d$$

$$\therefore \text{Ker}(T) = d \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix} \neq 0$$

$\therefore$ $\boxed{T(x) \text{ is NOT INJECTIVE}}$

5.

(a) $\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ $\quad \text{char}_A x = \left|\begin{bmatrix} 1-x & 0 & 0 \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{bmatrix}\right|$

$$= (1-x)^3$$

$\boxed{\therefore \text{ roots of } \text{char}_A x = 1. \Rightarrow 1 \text{ is the ONLY eigenvalue of } A}$

eigenvectors associated with $\lambda = 1$:

$$A . x = x \quad (A = I_3)$$

$\boxed{\therefore \quad x = \text{any vector} \in \mathbb{C}^3}$

(b) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$\text{char}_A x = \begin{vmatrix} (1-x) & -1 & 1 \\ -1 & (1-x) & -1 \\ 1 & -1 & (1-x) \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{vmatrix} (1-x) & -1 & 1 \\ -x & -x & 0 \\ x & 0 & -x \end{vmatrix}$$

$$= x^2 \begin{vmatrix} (1-x) & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= x^2 \left[(1-x) + 1 + 1 \right] = \boxed{x^2(3-x)}$$

Roots of $char_n\ x = 0, 3$

$\therefore \lambda = 0, 3$ are the eigenvalues of A

Eigen functions corresponding to

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\therefore x = x_2 \begin{bmatrix} +1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\boxed{x_2, x_3 \in \mathbb{C}}$$

$\lambda = 3$

$(A - 3I)$

$$\left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to -R_1/2 \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0.5 & -0.5 & 0 \\ 0 & -1.5 & -1.5 & 0 \\ 0 & -1.5 & -1.5 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 / -1.5 \\ R_1 \to R_1 - R_2/2 \\ R_3 \to R_3 + 1.5R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\therefore x = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle \qquad x_3 \in \mathbb{C}}$$

$x_1 - x_2 + x_3 = 0$

$x_3 - x_2$

$$\begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = -x_3$

$-2(3) + 1(3) + 1(3)$

6. Eigenvalue & Eigenvectors of Idempotent matrix

$$\boxed{A^2 = A} \quad (A \in M_{n\times n})$$

we know $|A| = 0$ or $A = I$ for an idempotent matrix.

$$A.x = \lambda x$$

$$A(Ax) = \lambda Ax \quad [A.A = A^2 = A] \quad \text{(Pre-Multiplying by A on both sides)}$$

$$A.x = \lambda Ax \quad [A.x = \lambda x]$$

$$\lambda x = \lambda(\lambda x)$$

$$\text{(or)} \quad \boxed{\lambda x = \lambda^2 x}$$

Since $x \neq 0$, we get

$$\lambda = \lambda^2 \Rightarrow \boxed{\lambda = 0, 1 \text{ ONLY}} \rightarrow (1)$$

Eigenvectors ~~Eigenvalues~~ corresponding to

$\lambda = 0$: $x = N(A)$

$\lambda = 1$: $x = N(A - I)$

For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$

$\therefore$ A is idempotent.

$$\text{char}_A(x) = \left|\begin{bmatrix} 1-x & 0 \\ 0 & -x \end{bmatrix}\right| = x(x-1)$$

eigenvalues are 0, 1 (satisfies eqn (1))

eigenvectors corresponding to

$\lambda = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{x_1 = 0}$$

$$\therefore \boxed{x = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle}$$

$\lambda = 1$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0$$

$$\boxed{x = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle}$$