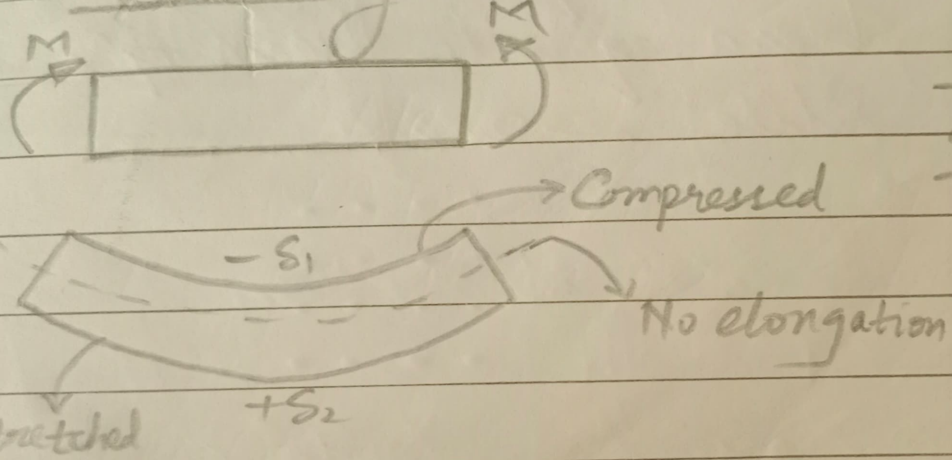
$\boxed{\text{ME 2110}}$

Bending

Linear Elastic

$\sigma \quad \epsilon$

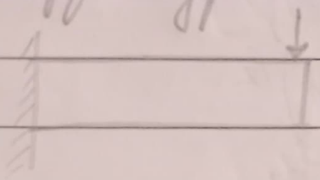
$\tau \quad \gamma$



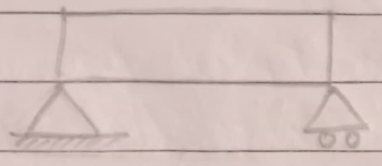
In Bending, we have normal stresses on

Due to variations in deformation at each layer of the beam, there would also be shear forces acting b/w differ layers of the beam

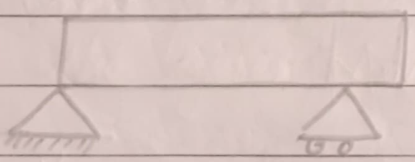
Diff types of beams



Cantilever



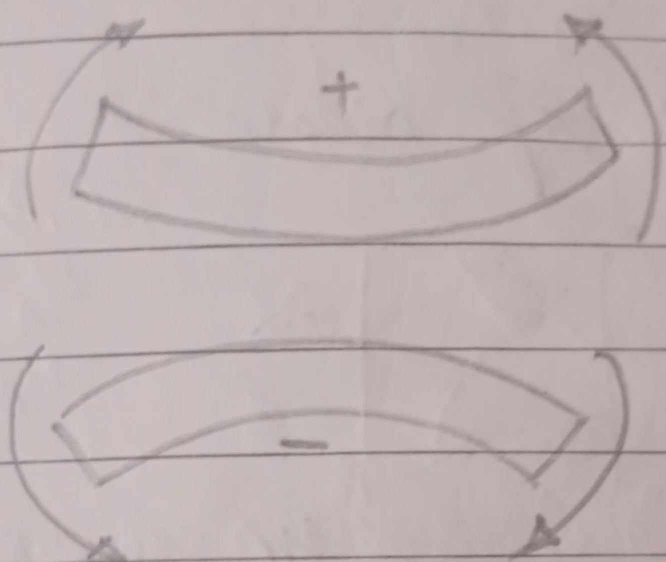
Simply supported

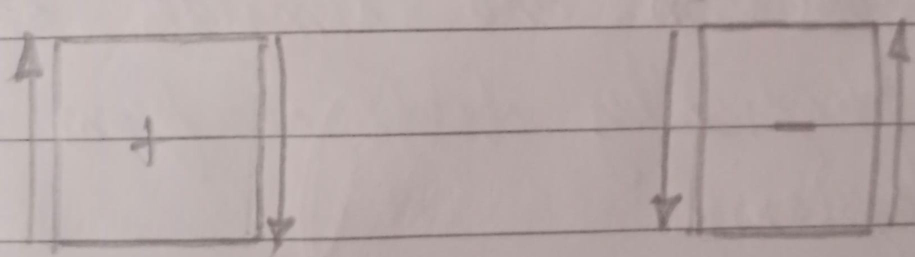


Overhanging

1. Find reac^{n}s
2. Resultant forces & moments, SFD & BMD
3. Strain-displacement relations, Euler Bernoulli beam eq^{n}s
4. Relate σ & ε

Sign convention (for SFD - BMD)





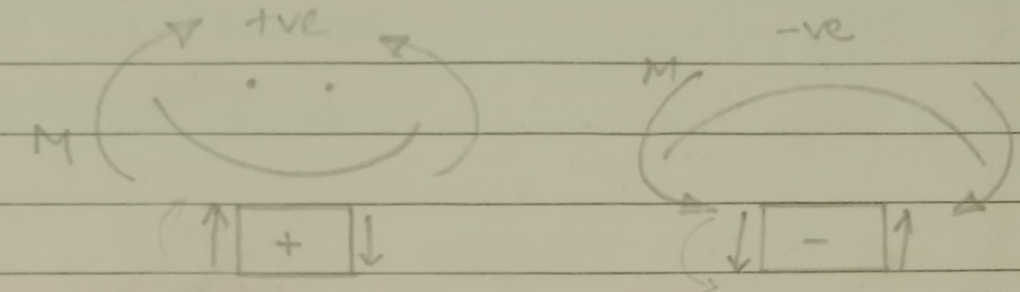
ME211a

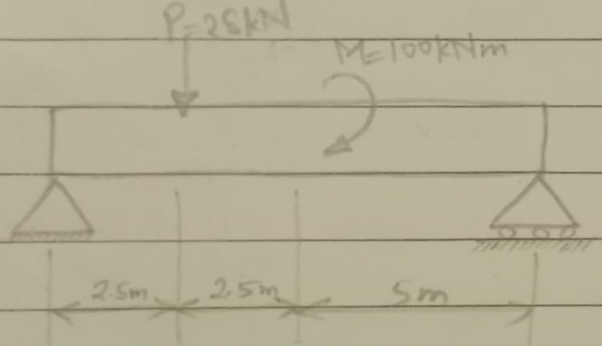
Summary

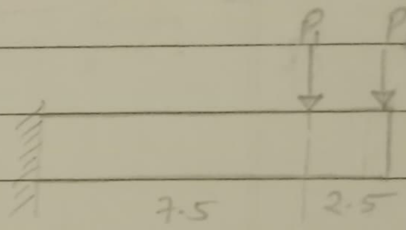
1) Introduction to bending (Boundary condⁿ)

2) Stresses / Strain in bending (Simply supported, cantilever, overhanging, concentrated loads, distributed loads)
3) Strain - displacement, stress - internal resultants
Hooke's law
4) V, M, q

$$\frac{dV}{dx} = -q \qquad \frac{dM}{dx} = V$$



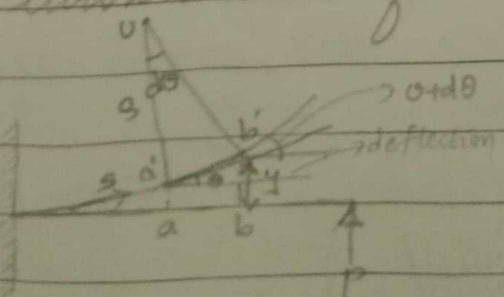




$P_1 = 10kN$
$P_2 = 20kN$

1. Find reaction forces
2. Draw shear force diagram
3. Draw bending-moment diagram

strain - displacement
1) strain - curvature
2) curvature - deflection



$$k = \frac{1}{\rho}$$

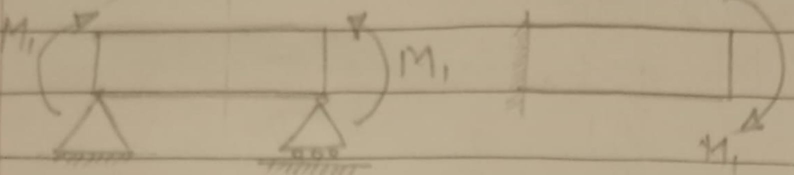
$$\rho\, d\theta = ds$$

$$\Rightarrow k = \frac{d\theta}{ds}$$

For Euler Bernoulli beams, $ds \approx dx$ (small deflections)
$\therefore k = \frac{d\theta}{dx}$

Pure bending

Bending moment is const. along length
($V=0$)

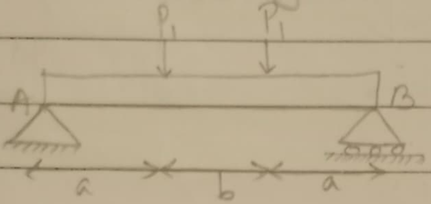


$M - M_1 = 0$ $\qquad$ $M + M_1 = 0$

$M = M_1 = \text{const.}$ $\qquad$ $M = -M_1 = \text{const.}$

$\Downarrow$ $\qquad$ $\Downarrow$

Pure bending $\qquad$ pure bending



$R_{Ax} = 0$

$R_{Ay} + R_{By} = 2P_1$

$R_{By}(2a+b) - P_1(a+b) - P_1 a = 0$

$R_{By} = P_1 \qquad R_{Ay} = P_1$

$0 < x < a$ $\quad V - R_{Ay} = 0 \Rightarrow V = P_1$

$M - R_{Ay}x = 0 \Rightarrow M = P_1 x$

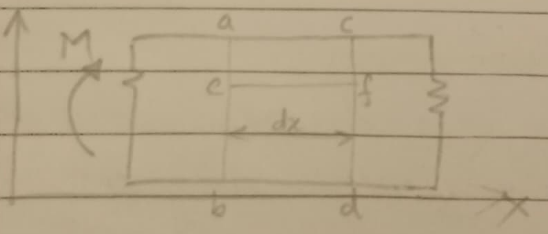
$a < x < a+b$ $\quad V + P_1 - R_{Ay} = 0 \Rightarrow V = 0$

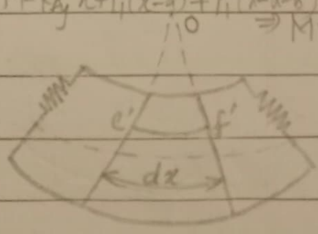
$M - R_{Ay}x + P_1(x-a) = 0 \Rightarrow M = P_1 a$

$a+b < x < b$ $\quad V + 2P_1 - R_{Ay} = 0 \Rightarrow V = -P_1$

$M - R_{Ay}x + P_1(x-a) + P_1(x-a-b) = 0$

$\Rightarrow M = P_1(2a+$





Neutral Axis (doesn't change its length)

$dx = \rho\, d\theta$

$dx' = e'f' = (\rho - y)\,d\theta = dx - y\dfrac{dx}{\rho}$

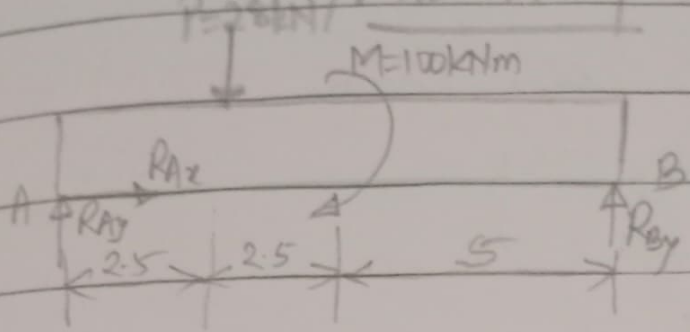
Strain $= \dfrac{dx' - dx}{dx} = \dfrac{-y\frac{dx}{\rho}}{dx} = -yk$

$\boxed{\varepsilon_x = -yk}$

$\boxed{\sigma_x = -yEk}$

Curvature $\Rightarrow$ rate of change of slope

ME2110



$R_{Ax} = 0$

$R_{Ay} + R_{By} - 28 = 0$

$28(2.5) + 100 - R_{By}(10) = 0$

$R_{By} = 17\ kN$

$R_{Ay} = 11\ kN$

$0 < x < 2.5$ $\quad V - R_{Ay} = 0 \Rightarrow V = 11\ kN$

$M - R_{Ay}x = 0 \Rightarrow M = 11x$

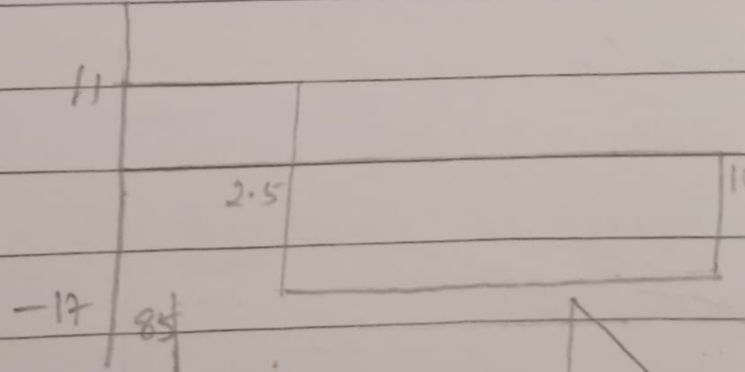
$2.5 < x < 5$ $\quad V - R_{Ay} + 28 = 0 \Rightarrow V = -17\ kN$

$M - R_{Ay}x + 28(x - 2.5) = 0 \Rightarrow M = 70 - 17x$

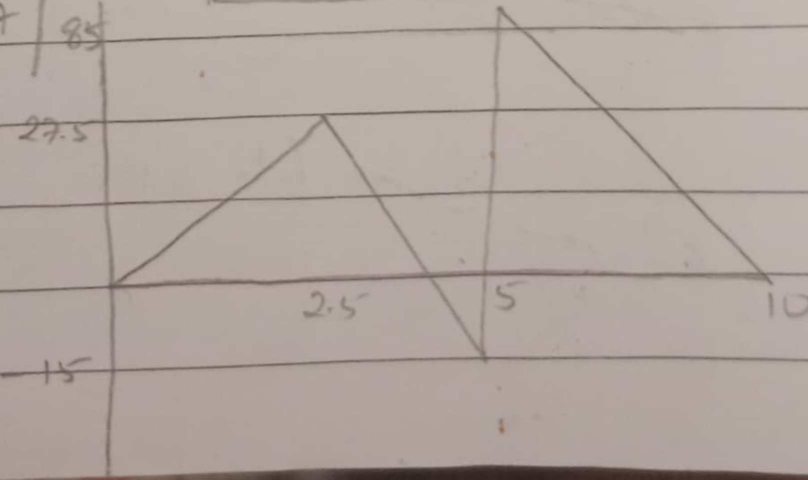
$5 < x < 10$ $\quad V - R_{Ay} + 28 = 0 \Rightarrow V = -17\ kN$

$M - R_{Ay}x + 28(x-2.5) - 100 = 0$

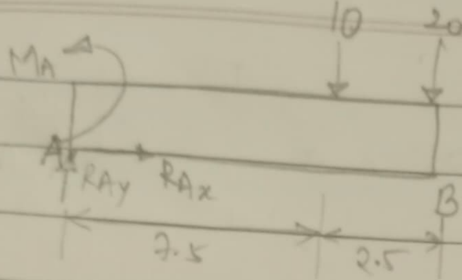
$\Rightarrow M = 170 - 17x$



SFD



BMD



$R_{Ax} = 0$

$R_{Ay} - 10 - 20 = 0$

$\Rightarrow R_{Ay} = 30\,kN$

$M_A - 10(7.5) - 20(10) = 0$

$M_A = 275\,kNm$

$0 < x < 7.5$

$V - R_{Ay} = 0 \Rightarrow V = 30\,kN$

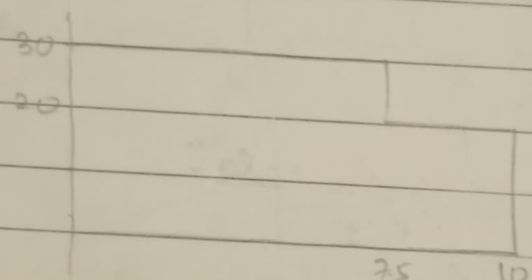
$M + M_A - R_{Ay}\,x \Rightarrow M = 30x - 275$

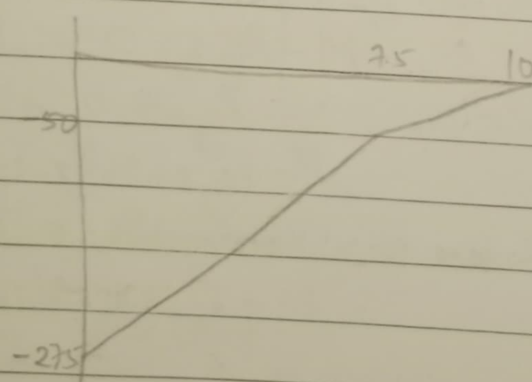
$7.5 < x < 10$

$V + 10 - R_{Ay} = 0 \Rightarrow V = 20\,kN$

$M + M_A - R_{Ay}\,x + 10(x - 7.5) = 0$

$\Rightarrow M = 20x - 200$

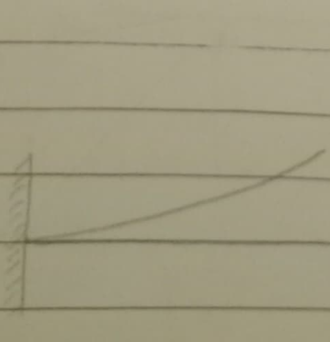




Summary

$k = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx}$

(small deflection)

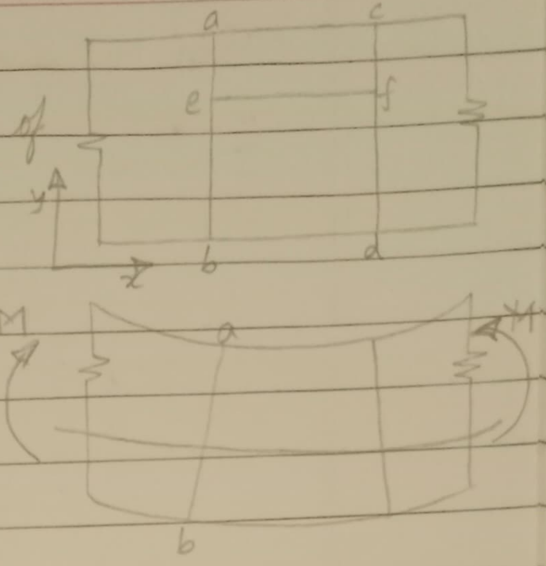


Pure bending

1) Loading is in the plane of bending (xy)

2) CS is symmetric w.r.t y axis

3) CS remains in plane



Bending strain depends
- Material prop.
- Geometric prop.

$$\varepsilon_x = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta}$$

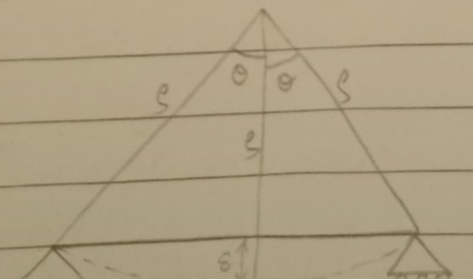
$$= \frac{-y}{\rho} = -yK$$

$$\boxed{\varepsilon_x = -yK}$$

Q) Simply supported beam

$L = 4.9m$ $\varepsilon_x|_{C_2} = 0.00125$ $C_2 = 150mm$

$\rho = ?$, $K = ?$ deflection of the beam?



$\varepsilon_x = -yK = \frac{-y}{\rho}$

$\because$ we are given $\varepsilon_x|_{C_2}$, put $y = C_2$

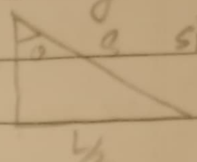
i.e. $\varepsilon_x|_{C_2} = \frac{-C_2}{\rho}$

$\Rightarrow \rho = \frac{-C_2}{\varepsilon_x|_{C_2}} = 120m$

And $k = \frac{1}{S} = 0.0083/m$

Deflection of the beam is given by

$$\delta = S - S\cos\theta$$



$\sin\theta = \frac{L}{2S}$

$$= S\left(1 - \sqrt{\frac{S^2 - \left(\frac{L}{2}\right)^2}{S^2}}\right)$$

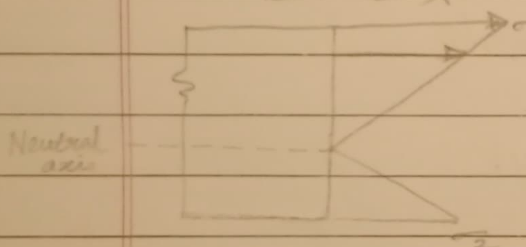
$$= \frac{2S^2 - \sqrt{4S^2 - L^2}}{2S}$$

$$= 119\ m$$

Stress in bending

Hooke's law

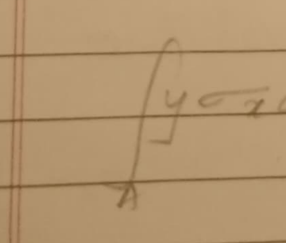
$$\sigma_x = E\epsilon_x = -yEk$$



$$\int_A \sigma\, dA = 0$$

$$\int_A -yEk\, dA = 0$$

$$-Ek\int y\, dA = 0$$



$$\int y\, dA = 0$$ → neutral axis/surface coincides with z-axis

$$dM = y\sigma_x\, dA$$

$$\int_A y\sigma_x\, dA + M = 0$$

$$M = -\int y\sigma_x\, dA$$

$$= +\int y\, y\, Ek\, dA$$

$$= Ek\underbrace{\int y^2\, dA}_{I}$$

I → M.I w.r.t neutral axis

$$M = EkI$$

$$K = \frac{M}{EI}$$ $\rightarrow$ Flexural rigidity

ME2110

Summary (Bending)

Neutral surface/axis

Strain - curvature relⁿ

$$\varepsilon_x = -\kappa y \qquad \left(\kappa = \frac{1}{\rho}\right)$$

Linearly elastic $\sigma_x = -E\kappa y$ (Hooke's law)

Location of neutral axis : $\int E\kappa y\, dA = 0$

$$\Rightarrow \int y\, dA = 0$$

Moment - curvature relⁿ

$$M = \int y E\kappa y\, dA$$

$$M = E\kappa I \quad \ldots \left(\int y^2 dA = I\right)$$

$$\kappa = \frac{M}{EI}$$

flexural rigidity

Flexure formulas

$$\sigma_x = -E\kappa y = -E\frac{M}{EI}y = -\frac{My}{I}$$

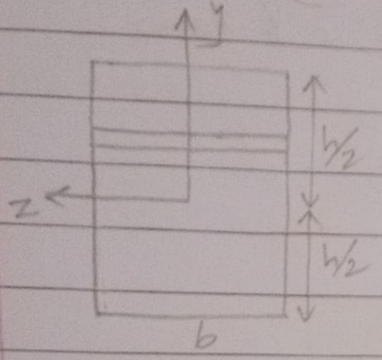
$$\boxed{\sigma_x = -\frac{My}{I}}$$

$$\sigma_1 = -\frac{M}{S_1}$$

$$\sigma_2 = -\frac{M}{S_2}$$

S ⇒ section moduli

$S_1 = S_2$ (unit-m^3)



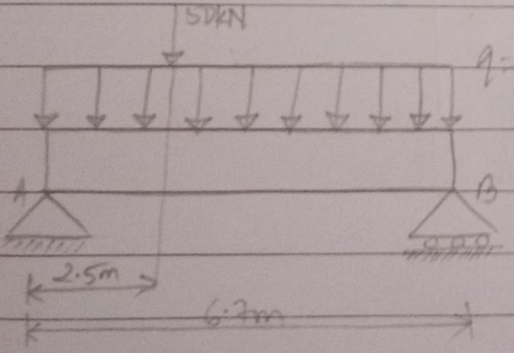
$$I = \int y^2 dA = b\int_{-h/2}^{h/2} y^2 dy = \frac{bh^3}{12}$$

from $y = c_1$ to $y = -c_2$

$c_1 = h/2$, $c_2 = h/2$

$$S = \frac{I}{c} = \frac{\frac{bh^3}{12}}{h/2} = \frac{bh^2}{6}$$

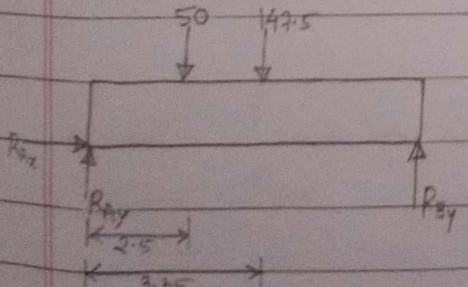
Q



q = 22 kN/m

h = 700mm b = 220mm

$\sigma_1 = ?$ $\sigma_2 = ?$



$R_{Ax} = 0$

$R_{Ay} + R_{By} = 22(6.7) + 50 = 197.4$

$50(2.5) + 147.5(3.35) = R_{By}(6.7)$

$R_{By} = 92.36$ kN

$R_{Ay} = 105.04$ kN

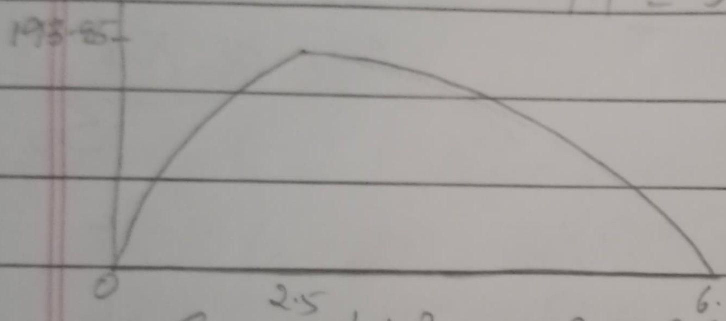
$0 < x < 2.5$ $V - R_{Ay} + 22x = 0 \Rightarrow V = 105.04 - 22x$

$M - R_{Ay}x + 22x\left(\frac{x}{2}\right) = 0 \Rightarrow M = 105.04x - 11x^2$

$2.5 < x < 6.7$ $V - R_{Ay} + 22x + 50 = 0 \Rightarrow V = 55.04 - 22x$

$$M - R_{Ay}x + 50(x-2.5) + 22x\left(\frac{x}{2}\right) = 0$$

$$M = 55.04x + 125 - 11x^2$$



BMD → from this we get $BM_{max} = 193.85$ kN

$$S = \frac{bh^2}{6} = 0.017977\ m^3$$

$$\sigma_1 = \frac{-M}{S} = -10.79\ MPa$$

$$\sigma_2 = \frac{-M}{-S} = 10.79\ MPa$$

put $M = BM_{max} = 193.85$ kN

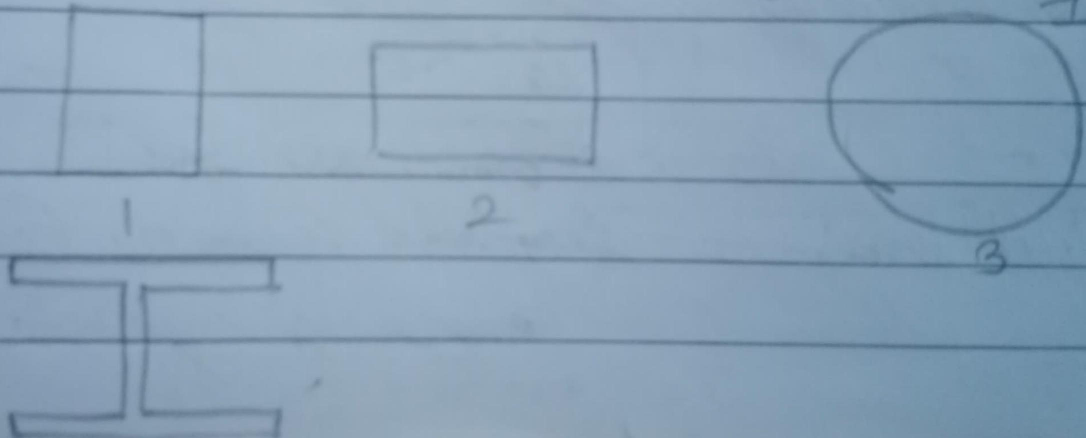
ME2110

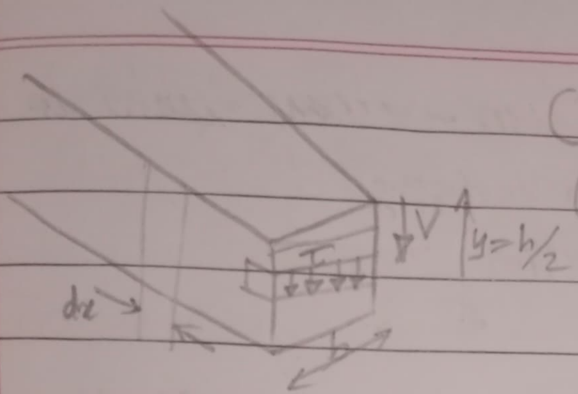
Summary (Bending)

$\varepsilon_x = -\frac{y}{\rho} = -y\kappa$

$\sigma = -yE\kappa$

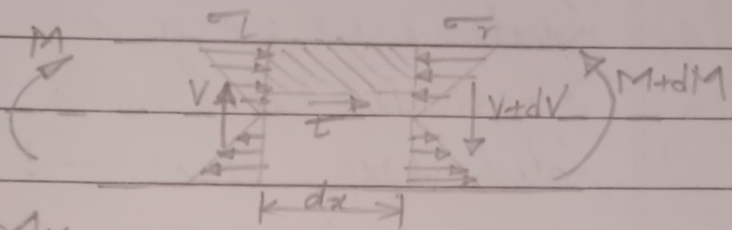
$M = E\kappa I \quad \kappa = \frac{M}{EI}$ } Flexure formulae $\sigma = -\frac{M}{I}y \quad \frac{M}{S}$



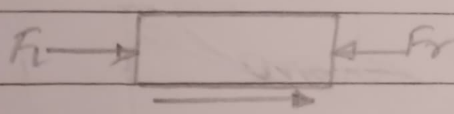


① τ is acting ∥ to y-axis

② τ is uniform along b



$$|\sigma_l| = \frac{My}{I}$$



$$F_l + F_2 = F_r$$

$$F_l = \int_{y_1}^{h/2} \frac{My}{I}\, dA$$

$$F_r = \int_{y_1}^{h/2} \left(\frac{M+dM}{I}\right) y$$

$$F_2 = \tau\, dx\, b$$

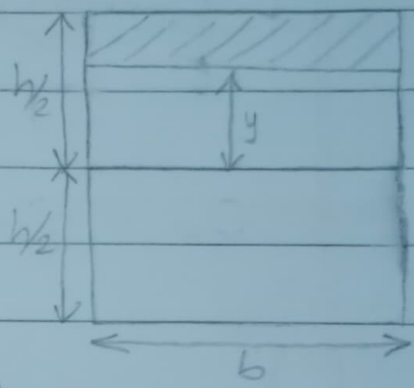
$$F_r - F_l = \frac{b\, dM}{I} \int_{y_1}^{h/2} y\, dy$$

$$\tau b\, dx = \frac{dM}{I} \int_{y_1}^{h/2} y b\, dy$$

$$\tau = \frac{dM}{dx} \frac{Q}{Ib} \ldots \left[\text{where } Q = \int_{y_1}^{h/2} y \underbrace{b\, dy}_{dA} = \int y\, dA\right]$$

$$\boxed{\tau = \frac{VQ}{Ib}}$$

$\tau_{avg} = \frac{V}{A}$ for a C.S.



$\tau = \frac{VQ}{Ib}$ where $I = \frac{1}{12} b h^3$

And $Q = \int y\, dA = A\bar{y} = b\left(\frac{h}{2} - y\right)\left(y + \frac{\left(\frac{h}{2} - y\right)}{2}\right)$

$\bar{y} \quad y + \frac{\left(\frac{h}{2} - y\right)}{2}$

$A = b\left(\frac{h}{2} - y\right)$

$$= b\left(\frac{h}{2} - y\right)\left(\frac{y}{2} + \frac{h}{4}\right)$$

$$\therefore Q = \frac{b}{2}\left(\frac{h}{2} - y\right)\left(\frac{h}{2} + y\right)$$

$$Q = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\tau = \frac{V}{Ib}\,\frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\boxed{\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)} \longrightarrow \text{quadratic in } y$$

Shear strain will also be quadratic in y

$$\tau\Big|_{y=0} = \frac{Vh^2}{8I} \leftarrow \text{Max} \qquad \tau\Big|_{y=\pm h/2} = 0 \leftarrow \text{Min}$$

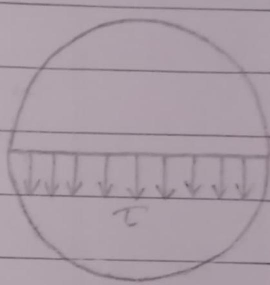
$$\tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2 \times 12}{8bh^3} = \frac{3}{2}\frac{V}{A}$$

$$\boxed{\tau_{max} = \frac{3}{2}\tau_{avg}}$$

Normal stress / strain obtained for the case of pure bending holds true (to a very good extent) for beams with non-uniform shear force variation

For Circular CS

$$\tau = \frac{VQ}{Ib}$$



$$Q = \frac{\pi r^2}{2} \times \frac{4r}{3\pi}, \quad I = \frac{\pi r^4}{4}, \quad b = 2r$$

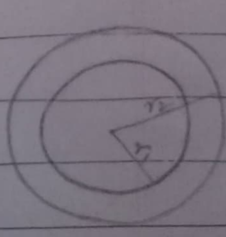
$$= \frac{2}{3} r^3$$

$$\therefore \tau = \frac{V \frac{2r^3}{3}}{\frac{\pi r^4}{4}(2r)}$$

$$\tau = \frac{4V}{3\pi r^2}$$

$$\tau = \frac{4}{3}\frac{V}{A}$$

Circular Ring



$$Q = \frac{2}{3}(r_2^3 - r_1^3)$$

$$I = \frac{\pi}{4}(r_2^4 - r_1^4)$$

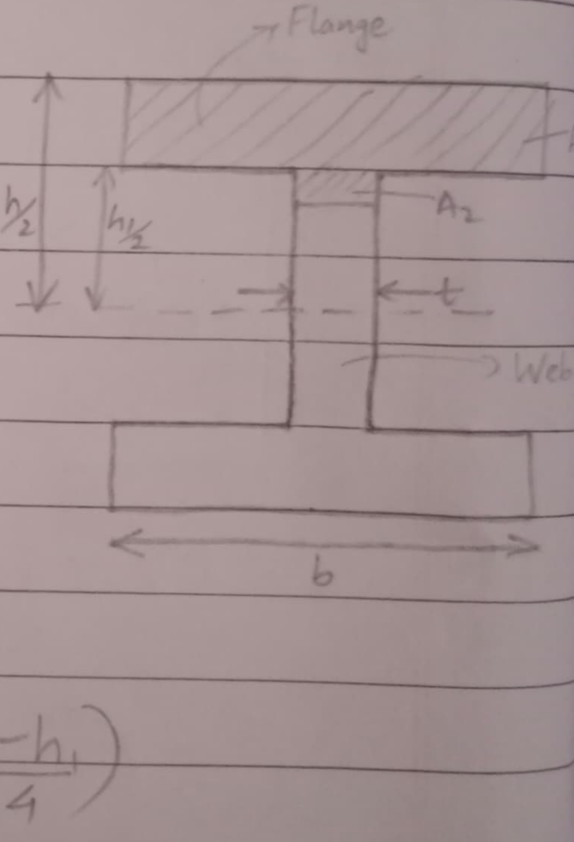
$$b = 2(r_2 - r_1)$$

$$\tau = \frac{V \times \frac{2}{3}\cancel{(r_2 - r_1)}(r_1^2 + r_1 r_2 + r_2^2)}{\frac{\pi}{2}\cancel{(r_2 - r_1)}(r_1 + r_2)(r_1^2 + r_2^2)(r_2 - r_1)}$$

$$= \frac{4}{3}\frac{V(r_1^2 + r_1 r_2 + r_2^2)}{\pi(r_2^4 - r_1^4)}$$

$$\boxed{\tau = \frac{4}{3}\frac{V}{A}\frac{(r_1^2 + r_1 r_2 + r_2^2)}{r_1^2 + r_2^2}}$$

I- Section



$$\tau = \frac{VQ}{It}$$

$$I = \frac{1}{12}bh^3 - \frac{(b-t)h_1^3}{12}$$

$$A_1 = b\left(\frac{h}{2} - \frac{h_1}{2}\right)$$

$$\bar{y}_1 = \frac{h}{2} + \frac{h - h_1}{4}$$

$$Q_1 = b\left(\frac{h - h_1}{2}\right) \times \left(\frac{h_1}{2} + \frac{h - h_1}{4}\right)$$

$$Q_2 = t\left(\frac{h_1}{2} - y_1\right)\left(y_1 + \frac{h}{4} - \frac{y_1}{2}\right)$$

$$Q = Q_1 + Q_2$$

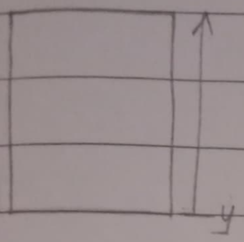
classmate

Date ____
Page ____

ME2110

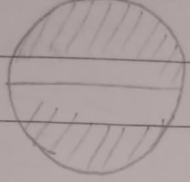
Summary

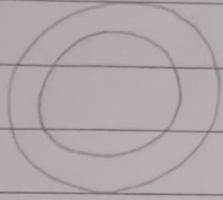
Shear formula, $\tau = \frac{VQ}{It}$



$$\tau = \frac{VQ}{It} \qquad \tau = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)$$

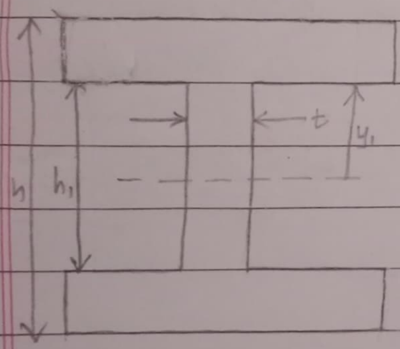
$$\tau_{max} = \frac{3}{2}\left(\frac{V}{A}\right)$$





$$\tau = \frac{VQ}{It}$$

$$\tau_{max} = \frac{4}{3}\frac{V}{A}$$



$$\tau = \frac{VQ}{It} \quad Q \rightarrow Q_1 + Q_2$$

$$I = \frac{1}{2}bh^3 - \frac{(b-t)h_1^3}{12}$$

$$Q_1 = b\left(\frac{h}{2} - \frac{h_1}{2}\right) \times \left(\frac{h_1}{2} + \frac{h-h_1}{4}\right)$$

$$Q_2 = t\left(\frac{h_1}{2} - y_1\right)\left(y_1 + \frac{h_1}{4} - \frac{y_1}{2}\right)$$

$$\tau = \frac{V}{8It}\left[b(h^2 - h_1^2) + t(h_1^2 - 4y_1^2)\right]$$

at $y = \frac{h_1}{2}$, τ = minimum for web

$y = 0$, τ = maximum " "

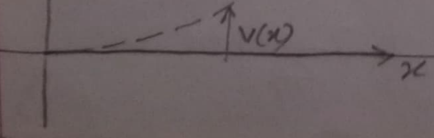
$$\tau_{max} = \frac{V}{8It}\left[b(h^2 - h_1^2) + t(h_1^2)\right]$$

$$\tau_{min} = \frac{V}{8It}(h^2 - h_1^2)$$

$V = h_1\tau_{min} + \frac{2}{3}(\tau_{max} - \tau_{min})th$

$\frac{V}{h_1 t} = 10\%\ \tau_{max}$

Deflection



$$k = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{dv'}{dx} = v''$$

$$\tan\theta = v' = \frac{dy}{dx}$$

Initial & Final condⁿs - w.r.t time
Boundary condⁿs - w.r.t space

for small deflection $\tan\theta \approx \theta$

$$\therefore \theta \approx v'$$

$$\frac{d\theta}{dx} = \frac{dv'}{dx} = v''$$

$$k = \frac{M}{EI} = v''$$

For beams with varying CS $I = I(x)$

$$\therefore \boxed{M = EI_x v''} \rightarrow \text{Bending moment eq}^n$$

$$\boxed{V = (EI_x v'')'} \rightarrow \text{Shear force eq}^n$$

$$\boxed{-q = (EI_x v'')''} \rightarrow \text{Loading eq}^n$$

For prismatic beams

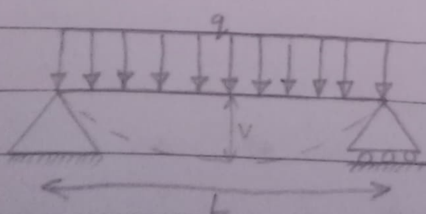
$$M = EIv'', \quad V = EIv''', \quad q + EIv^{IV} = 0$$

When not assuming small deformation

$$\frac{d\theta}{ds} = \frac{d}{ds}\tan^{-1}(v') = \frac{d\tan^{-1}(v')}{dx}\frac{dx}{ds} = \frac{v''}{1+(v')^2} \times \frac{1}{(1+(v')^2)^{1/2}}$$

$$\Rightarrow \frac{d\theta}{ds} = \frac{v''}{(1+(v')^2)^{3/2}} \quad \left[\because (ds)^2 = (dx)^2 + (dv)^2 \quad \therefore \frac{ds}{dx} = (1+(v')^2)^{1/2}\right]$$

Integration constants
1> Boundary condⁿs
2> Continuity "
3> Symmetric "



$$M = \frac{qLx}{2} - \frac{qx^2}{2} = EIv''$$

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1$$

Using symmetry, at $x = \frac{L}{2}$ $v' = 0$ (i.e. slope = 0 ∴ horizontal tangent can be drawn)

$$0 = qL^3\left(\frac{1}{16} - \frac{1}{48}\right) + C_1$$

$$C_1 = -\frac{qL^3}{24}$$

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24}$$

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24} + C_2$$

Now at $x = 0$ & $x = L$, $v = 0$ — Boundary condⁿs

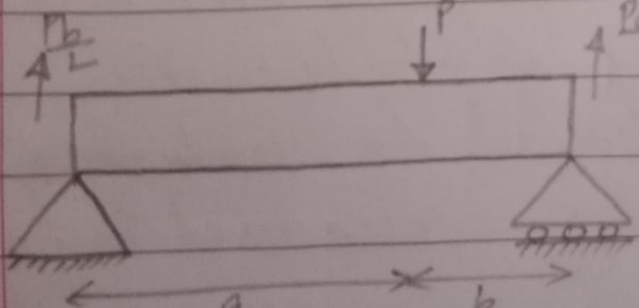
$\therefore\ 0 = C_2$

$$EIv = \frac{qx}{12}\left(Lx^2 - \frac{x^3}{2} - \frac{L^3}{2}\right)$$

$$v = \frac{qx}{24EI}\left(2Lx^2 - x^3 - L^3\right)$$

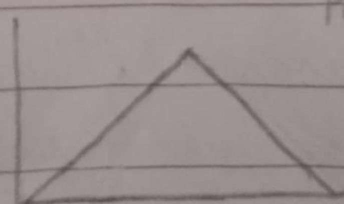
For max deflection put $x = L/2$

$$v(L/2) = \frac{qL}{48EI}\left(\frac{L^3}{2} - \frac{L^3}{8} - L^3\right) = \frac{-5qL^4}{384EI}$$

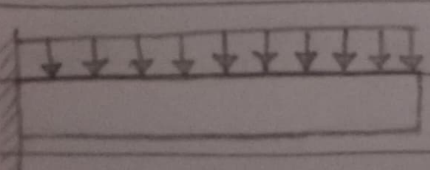


$$M = \begin{cases} \dfrac{Pbx}{L} & 0<x<a \\ \dfrac{Pbx}{L} - P(x-a) = Pa\left(1-\dfrac{x}{L}\right) & a<x<b \end{cases}$$

BMD



$$\underbrace{\begin{matrix} v_1(0)=0 \\ v_2(L)=0 \end{matrix}}_{\text{Boundary cond}^n\text{s}} \qquad \underbrace{\begin{matrix} v_1(a)=v_2(a) \\ v_1'(a)=v_2'(a) \end{matrix}}_{\text{Continuity cond}^n\text{s}}$$



$$\left.\begin{matrix} v(0)=0 \\ v'(0)=0 \end{matrix}\right\} \text{Boundary cond}^n\text{s}$$

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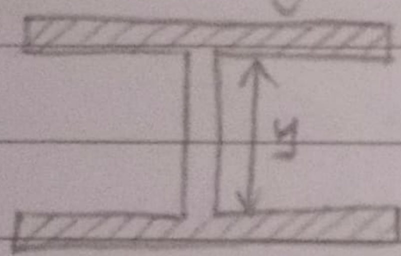
Summary



$\tau(y)$, τ_{max}, τ_{min}

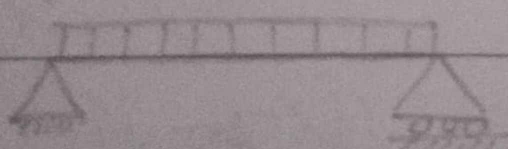
V $\qquad$ $\tau_{avg} \sim \tau_{max} \pm 10\%$

$\tan\theta$, $\sin\theta \sim \theta$ $\qquad$ $y \ll 1$

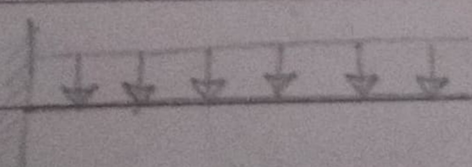
$EIv'' = M$, $(EIv'')' = V$, $(EIv'')'' + q = 0$

Boundary cond^{n}s
Continuity " "
Symmetric " "

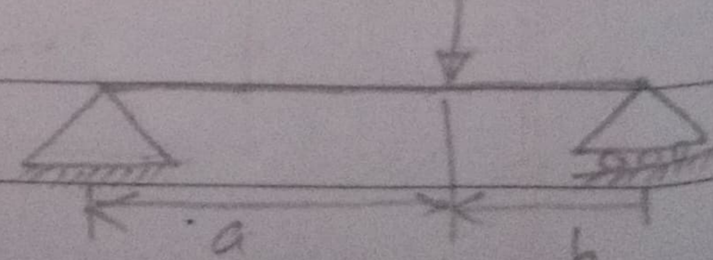
} for finding integation const... to aid "successive integration method" for obtaining deflection

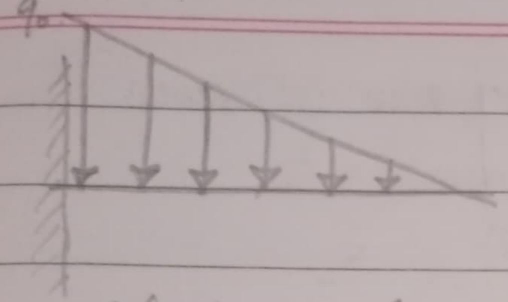


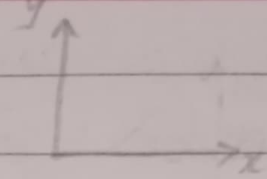
$v(0) = 0$
$v'(L/2) = 0$



$v(0) = 0$
$v'(0) = 0$







$$q(x) = q_0\left(1 - \frac{x}{L}\right)$$

We know that, $(EIv'')'' = -q$

$$(EIv'')'' = -q_0\left(1 - \frac{x}{L}\right)$$

$$EIv^{IV} = -q_0\left(1 - \frac{x}{L}\right)$$

$$EIv''' = \frac{q_0 L}{2}\left(1 - \frac{x}{L}\right)^2 + C_1$$

At $x = L$, $V = EIv''' = 0 \Rightarrow C_1 = 0$

$$\therefore EIv''' = \frac{q_0 L}{2}\left(1 - \frac{x}{L}\right)^2$$

$$EIv'' = -\frac{q_0 L^2}{6}\left(1 - \frac{x}{L}\right)^3 + C_2$$

At $x = L$, $M = EIv'' = 0 \Rightarrow C_2 = 0$

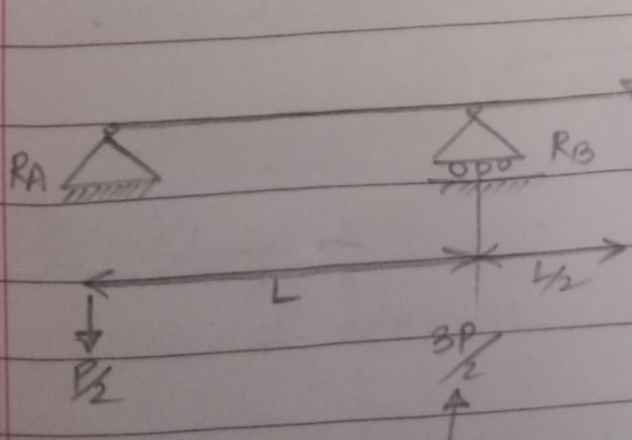
$$\therefore EIv'' = -\frac{q_0 L^2}{6}\left(1 - \frac{x}{L}\right)^3$$

$$EIv = -\frac{q_0 L^4}{120}\left(1 - \frac{x}{L}\right)^5 + C_3 x + C_4$$ Double integration

Imposing boundary cond^{n}s $v(0) = 0$, $v'(0) = 0$

we get $C_3 = -\frac{q_0 L^3}{24EI}$ $\quad C_4 = \frac{q_0 L^4}{120EI}$

$$v = -\frac{q_0 L^3}{24EI}\left(\frac{L}{5}\left(\left(1 - \frac{x}{L}\right)^5 - 1\right) + 1\right)$$



$V = -P/2 \qquad 0 < x < L$

$V = P \qquad L < x < \frac{3L}{2}$

6 Integration const

$v_1(0) = 0$	$v_1'(L) = v_2'(L)$
$v_1(L) = 0$	$EIv_2''\left(\frac{3L}{2}\right) = 0$
$v_2(L) = 0$	$EIv_1''(0) = 0$

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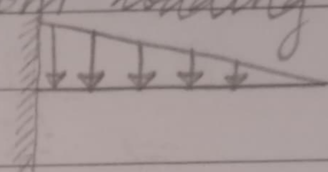
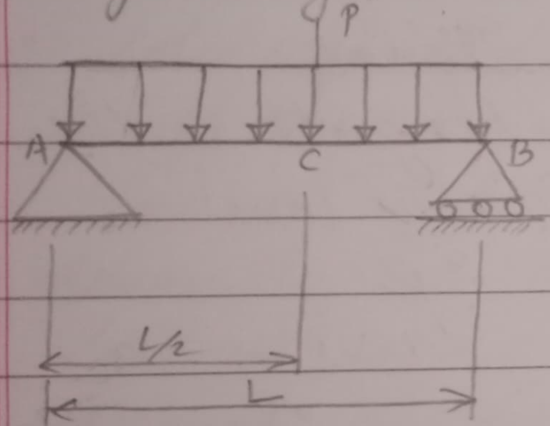
Summary

Successive integration from loading eq^n

(4)  (6)

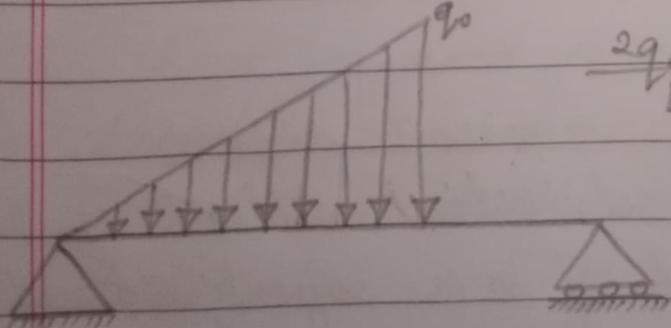
Symmetry, boundary, continuity cond^ns



$$\delta_{c_1} = \frac{5qL^4}{384EI} \qquad \delta_{c_2} = \frac{PL^3}{48EI}$$

$$\delta_c = \frac{5qL^4}{384EI} + \frac{PL^3}{48EI}$$

$$\theta_A = \theta_{A_1} + \theta_{A_2}$$
$$= \frac{qL^3}{24EI} + \frac{PL^2}{16EI}$$



$$\frac{2q_0 x}{L} \qquad P = \frac{2q_0 x}{L} dx$$

$$d\delta_c = \left(\frac{2q_0 x\, dx}{L}\right) \frac{x(3L^2 - 4x^2)}{48EI}$$

$$\delta_c = \int_0^{L/2} \frac{2q_0}{48EIL} x^2 (3L^2 - 4x^2)\, dx$$

When does superposition principle hold?

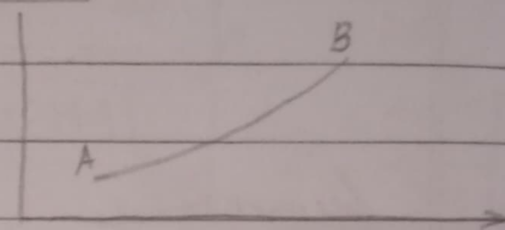
Hooke's law (stress & strain are linearly dependent)

deflections/rotations are very small
deflection due to load are not affecting other loads present

Note: All our governing eqⁿs are linear in loading

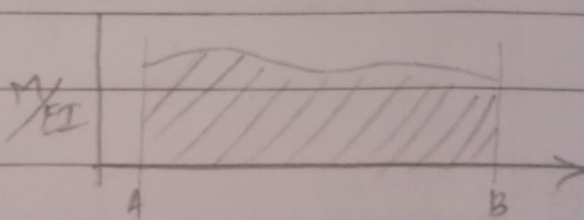
Moment-area methods

$K = \frac{M}{EI}$



$\frac{d\theta}{dx} = \frac{dv'}{dx} = \frac{M}{EI}$

$d\theta = \frac{M}{EI} dx$



First moment-area theorem

The angle, $\theta_{B/A}$, b/w tangents to the deflection curve at 2 pts. A & B is equal to area of M/EI diagram b/w those 2 pts.

Second theorem of moment-area

The tangential deviation at B is equal the first moment of area of M/EI diagram b/w A & B evaluated w.r.t B

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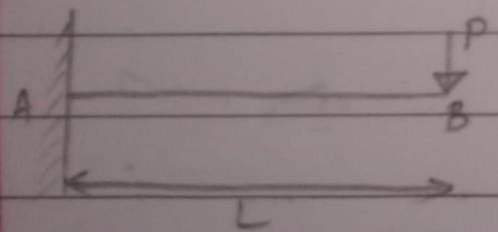
Summary

Superposition principle & its requirements

Deflection of beams (graphical)

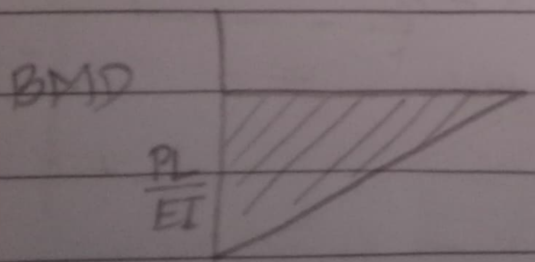
1) $\displaystyle\int_{x_A}^{x_B} \frac{M}{EI}\,dx = \theta_{B/A}$

2) $\displaystyle\int_{x_A}^{x_B} \frac{M}{EI}\,x\,dx = t_{B/A}$



$$|\theta_{B/A}| = |\theta_B - \underset{=0}{\theta_A}| = |\theta_B| = \frac{1}{2} L \frac{PL}{EI}$$

$$= \frac{PL^2}{2EI}$$



$$t_{B/A} = \delta_B = \frac{2L}{3}\frac{PL^2}{2EI} = \frac{PL^3}{3EI}$$

Strain energy

Bending

$$U = \int \frac{\sigma \varepsilon}{2} dV$$

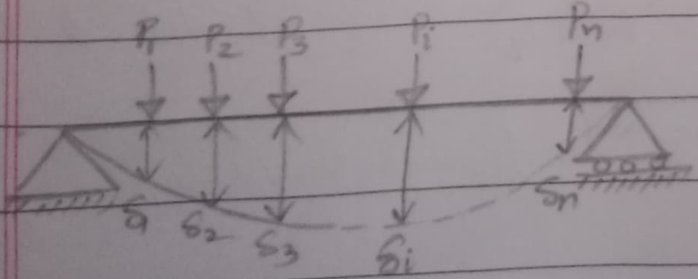
$$= \int \frac{\sigma^2}{2E} dV$$

$$= \int \frac{M^2 y^2}{2EI^2} dV$$

$$= \int_x \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx \quad = I$$

$$= \int \frac{M^2}{2EI} dx \qquad = \int_x \frac{E^2 k^2 I^2}{2EI} dx$$

$$= \frac{M^2 L}{2EI} \qquad = \int \frac{EI}{2} (v'')^2 dx$$



If $P_i \to P_i + dP_i$

$U \to U + \frac{\partial U}{\partial P_i} dP_i$

$U(P_1, P_2, P_3, P_i, P_n)$ Work done by $dP_i = \frac{dP_i \, d\delta_i}{2}$

Now if dP_i is applied first & after that all big loads are applied, $W = \frac{dP_i \, d\delta_i}{2} + U + dP_i \delta_i$

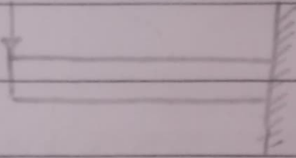
$$\cancel{\frac{dP_i \, d\delta_i}{2}}^{=0} + \cancel{U} + dP_i \delta_i = \cancel{U} + \frac{\partial U}{\partial P_i} dP_i$$

Castligano's theorem $\boxed{\frac{\partial U}{\partial P_i} = \delta_i}$

$$\delta_A = \frac{\partial U}{\partial P_0} = \frac{2P_0L^3}{6EI} + \frac{M_0L^2}{2EI}$$

$$\theta_A = \frac{\partial U}{\partial M_0} = \frac{M_0L}{EI} + \frac{P_0L^2}{2EI}$$

Dummy load Method



for calculating δ at centre, assu
loading Q to be acting on entire beam
& solve. Then put $Q=0$

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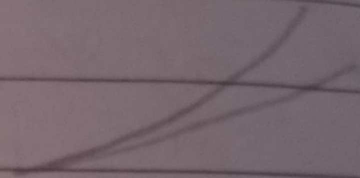
Summary

* $t_{B/A} = \int_A^B x' \frac{M}{EI} dx$



* Strain energy in Bending, $U = \int \frac{M^2 dx}{2EI}$
* Castigliano's theorem

$U(P_1, P_2, \ldots P_n)$ $\boxed{\frac{\partial U}{\partial P_i} = \delta_i}$

* Dummy-Load method

Modified Castigliano's theorem

$$\frac{\partial U}{\partial P_i} = \delta_i \qquad U = \int_0^L \frac{M^2\,dx}{2EI}$$

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(x,t)\,dt\right) = f(x, v(x))\,v'(x) - f(x, u(x))\,u'(x)$$

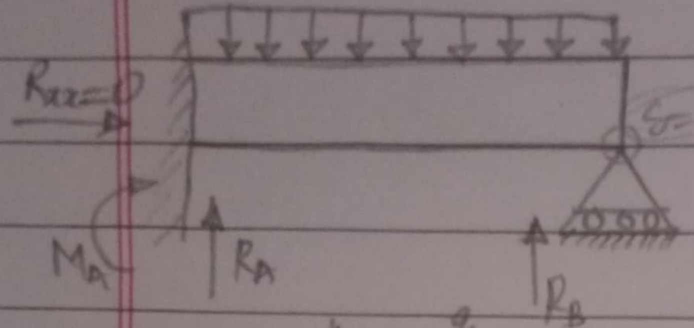
$$\therefore \delta_i = \frac{\partial U}{\partial P_i} = \int_0^L \frac{\cancel{2}M}{\cancel{2}EI}\,\frac{\partial M}{\partial P_i}\,dx$$

Steps: M piecewise

Write $\frac{\partial M}{\partial P_i}$ piecewise

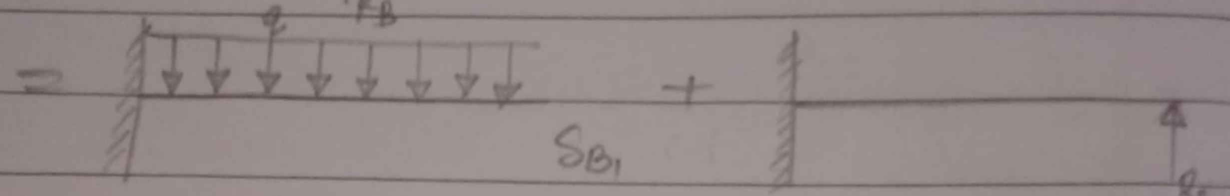
SIP

If there are more restrictions than d.o.f problem becomes statically indeterminate



$\therefore$ 3 unknowns & 2 eq^n's

$\sum F_y = 0$, $\sum M = 0$



Compatibility eq^n $\delta = \delta_{B_1} - \delta_{B_2} = 0$

<u>Eq^{n}s</u>

$$\left.\begin{aligned} R_A + R_B &= qL \\ M_A + R_B L &= \frac{qL^2}{2} \end{aligned}\right\} \quad \frac{\text{unknowns}}{R_A, R_B, M_A}$$

$$\delta_{B_1} = \delta_{B_2}$$

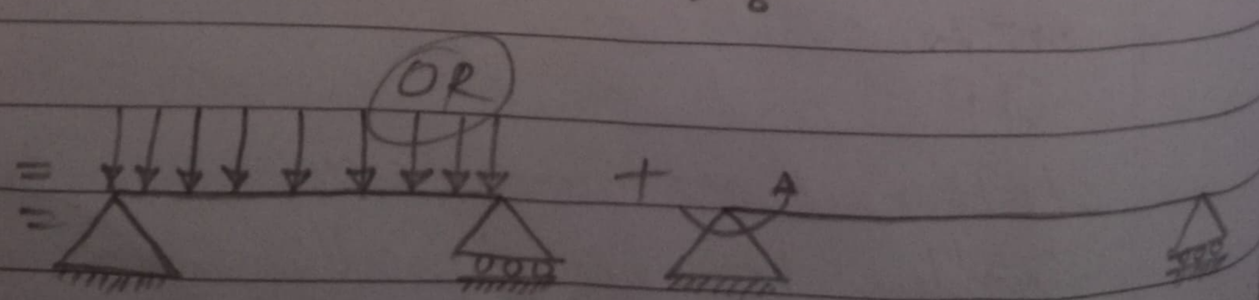
$$\delta_{B_1} = \frac{qL^4}{8EI} = \delta_{B_2} = \frac{R_B L^3}{3EI}$$

$$R_B = \frac{qL^4}{8EI} \frac{3EI}{L^3} = \frac{3qL}{8}$$

$$R_A = qL - \frac{3qL}{8} = \frac{5qL}{8}$$

$$M_A = \frac{qL^2}{2} - \frac{3qL^2}{8} = \frac{qL^2}{8}$$

OR



$$\theta_{A_1} = \theta_{A_2}$$

$$\frac{qL^{\cancel{3}\,2}}{\cancel{24}\,8\,\cancel{EI}} = \frac{M_0 \cancel{L}}{\cancel{3}\,\cancel{EI}} \Rightarrow M_0 = \frac{qL^2}{8}$$