ASSIGNMENT - 3

1. out of the six faces of the cube, let

$\hat{n}_1 = -\hat{i}$, $\hat{n}_2 = +\hat{i}$

$\hat{n}_3 = -\hat{j}$, $\hat{n}_4 = +\hat{j}$

$\hat{n}_5 = -\hat{k}$, $\hat{n}_6 = +\hat{k}$

For the first surface $[\hat{n}_1 = -\hat{i}]$; $x = 0$:

$$\therefore \iint_S u\,dS = \int_0^1\int_0^1 (y^2+z^2)(-\hat{i})\,dy\,dz$$

$$= -\int_0^1 \left(\frac{1}{3} + z^2\right) dz\ \hat{i}$$

$$\cancel{= -\left[\frac{z^2}{6} + \frac{z^3}{3}\right]_0^1 \hat{i}}$$

$$= -\left[\frac{z}{3} + \frac{z^3}{3}\right]_0^1 \hat{i} = \boxed{-\frac{2}{3}\hat{i}}$$

For second surface $[\hat{n}_2 = \hat{i}]$, $x = 1$:

$$\therefore \iint_S u\,dS = \int_0^1\int_0^1 (y^2 + z^2 + 1)\,dy\,dz\ \hat{i}$$

$$= \int_0^1 \left[\frac{y^3}{3} + (z^2+1)y\right]_0^1 dz\ \hat{i}$$

$$= \int_0^1 \left(\frac{4}{3} + z^2\right) dz\ \hat{i}$$

$$= \frac{4}{3}z + \frac{z^3}{3}\Big|_0^1 = \boxed{\frac{5}{3}\hat{i}}$$

$\therefore$ On the 2 opp. surfaces,

total $= \left(\frac{5}{3} - \frac{2}{3}\right)\hat{i} = \boxed{\hat{i}}$

2.

3.

By symmetry, the final value of $\iint_S u \cdot ds$

$$= \boxed{\hat{i} + \hat{j} + \hat{k}}$$

2. $d\bar{S} = dx\,dy\,\hat{k}$

$\therefore \bar{F} \times d\bar{S} = y^2 dx\,dy\,\hat{i} - x^2 dx\,dy\,\hat{j}$

$$\therefore \iint \bar{F} \times d\bar{s} = \int_{-1}^{1}\int_{-1}^{1} y^2 dx\,dy\,\hat{i} - \int_{-1}^{1}\int_{-1}^{1} x^2 dx\,dy\,\hat{j}$$

$$= \int_{-1}^{1} 2y^2 dy\,\hat{i} - \int_{-1}^{1} \frac{2}{3}\,dy\,\hat{j}$$

$$= \left.\frac{2y^3}{3}\right|_{-1}^{1} \hat{i} - \left.\frac{2y}{3}\right|_{-1}^{1} \hat{j}$$

$$= \boxed{\frac{4}{3}(\hat{i} - \hat{j})}$$

3. $\bar{F} = x^2 y\,\hat{i} + (x - z)\,\hat{j} + 2xz^2\,\hat{k}$

$\bar{\nabla} \cdot \bar{F} = 2xy + 4xz$

$$\therefore \int_0^1\int_0^1\int_0^1 (2xy + 4xz)\,dx\,dy\,dz = \int_0^1\int_0^1 \left.(x^2 y + 2x^2 z)\right|_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 (y + 2z)\,dy\,dz$$

$$= \int_0^1 \left.\left(\frac{y^2}{2} + 2yz\right)\right|_0^1 dz$$

$$= \frac{z}{2} + z^2 \Big|_0^1 = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$

4. $I = \int \bar{a} \cdot d\bar{s}$

Converting to polar coordinates,

$$\bar{a} = x\,\hat{i}$$

$$= \cancel{(a\sin\theta\cos\phi)}$$

$$\therefore d\bar{s} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{e}_r$$

$$\boxed{x}$$

$$\bar{a} \cdot d\bar{s} = (a\sin\theta\cos\phi)(a^2\sin\theta \, d\theta \, d\phi)(\hat{i} \cdot \hat{e}_r)$$

$$= a^3 \sin^2\theta \cos\phi \, d\theta \, d\phi \, (\sin\theta\cos\phi)$$

$$= \boxed{a^3 \sin^3\theta \cos^2\phi \, d\theta \, d\phi}$$

$$\therefore \iint_S \bar{a} \cdot d\bar{s} = a^3 \int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta \cos^2\phi \, d\theta \, d\phi$$

$$= a^3 \int_0^{2\pi} \left(\frac{\cos^3\theta}{3} - \cos\theta \Big|_0^{\pi/2} \right) d\phi \, . \cos^2\phi$$

$$= a^3 \int_0^{2\pi} \left(\frac{2}{3}\right) \cos^2\phi \, d\phi$$

$$= \frac{2}{3} a^3 (\pi) = \boxed{\frac{2\pi}{3} a^3}$$

(a) Flux [cylinders] :

$$\text{Flux} = \iint \rho \times \frac{F_0}{a} \rho \cos \lambda z \, d\phi \, dz \quad \text{between } \rho = a, 2a$$

$$= \frac{F_0 \rho^2}{a} \int_{-\frac{a\pi}{2}}^{\frac{a\pi}{2}} \int_0^{2\pi} \cos \lambda z \, d\phi \, dz$$

$$= \frac{F_0 \rho^2}{a} \int_{-\frac{a\pi}{2}}^{a\pi/2} 2\pi . \cos \lambda z \, dz$$

$$= \frac{2\pi F_0 \rho^2}{a\lambda} \left[\sin \lambda z \right]_{-\frac{a\pi}{2}}^{\frac{a\pi}{2}}$$

$$= \frac{4\pi F_0 \rho^2}{a\lambda} \sin\left(\frac{a\pi\lambda}{2}\right) \quad \text{between } \rho = a, 2a$$

$$\therefore \text{Flux} = \frac{4\pi F_0}{a\lambda} \sin\left(\frac{a\lambda\pi}{2}\right) \left[4a^2 - a^2\right]$$

$$= \boxed{\frac{12\pi F_0 a}{\lambda} \sin\left(\frac{\lambda\pi a}{2}\right)}$$

(b) Flux [Surface]

$$\left[\pi(4a^2) - \pi a^2\right]\left[F_0 \sin\left(\frac{\lambda a\pi}{2}\right) - F_0 \sin\left(-\frac{\lambda a\pi}{2}\right)\right]$$

$$= \boxed{6F_0\pi a^2 \sin\left(\frac{\lambda a\pi}{2}\right)}$$

$$\boxed{\therefore \text{Total Flux} = 6\pi a F_0 \sin\left(\frac{\lambda\pi a}{2}\right)\left[\frac{2}{\lambda} + a\right]}$$

6. $$I = \iint_S e^{-x^2-y^2}\, dS$$

converting to ~~[illegible] polar~~ cylindrical coordinates
$(z=0)$:

$$x^2+y^2 = \rho^2$$
$$dS = \rho\, d\phi\, d\rho$$

$$\therefore I = \int_0^{2\pi}\int_0^{\infty} e^{-\rho^2} \rho\, d\rho\, d\phi$$

$$= -\frac{1}{2}\int_0^{2\pi} \left(e^{-\rho^2}\right)\Big|_0^{\infty} d\phi$$

$$= \frac{1}{2}\int_0^{2\pi} d\phi = \boxed{\pi}$$

7. $$M = \int_0^1\int_0^1\int_0^1 (1+x+y+z)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 (3/2+y+z)\, dy\, dz$$

$$= \int_0^1 \left(\frac{3}{2}+\frac{1}{2}+z\right) dz$$

$$= 2z+\frac{z^2}{2}\Big|_0^1 = \boxed{\frac{5}{2}}$$

8. $\phi = x^2+y^2+z^2$.

$$I = \iiint_V \phi \, dV = \int_0^3\int_1^2\int_0^1 (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^3\int_1^2 \left(\frac{1}{3} + y^2 + z^2\right) dy\,dz$$

$$= \int_0^3 \left(\frac{y}{3} + \frac{y^3}{3} + z^2 y \Big|_1^2\right) dz$$

$$= \int_0^3 \left[\left(\frac{10}{3} + 2z^2\right) - \left(\frac{2}{3} + z^2\right)\right] dz$$

$$= \int_0^3 \left[\frac{8}{3} + z^2\right] dz$$

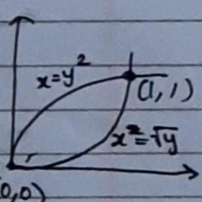
$$= \frac{8z}{3} + \frac{z^3}{3} \Big|_0^3$$

$$= \frac{24}{3} + \frac{27}{3}$$

$$= 8+9 = \boxed{17}$$

9. $\bar{u}\cdot\bar{n}\,dS = \iint (x^2)\,dx\,dy$.

$$\therefore\ I = \int_0^1\int_{y^2}^{\sqrt{y}} x^2\,dx\,dy$$



$$= \int_0^1 \frac{y\sqrt{y} - y^6}{3}\,dy$$

$$= \frac{1}{3}\int_0^1 (y^{3/2} - y^{6})dy$$

$$= \frac{1}{3}\left[\frac{2y^{5/2}}{5} - \frac{y^{7}}{7}\Big|_0^1\right]$$

$$= \frac{1}{3}\left[\frac{2}{5} - \frac{1}{7}\right] = \cancel{\frac{1}{3}\times\frac{3}{}}\ \frac{1}{3}\times\frac{9}{35}$$

$$= \boxed{\frac{3}{35}}$$

10. $\iiint x^2\,dx\,dy\,dz$

converting to spherical coordinates,

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} (r\sin\theta\cos\phi)^2\, r^2 \sin\theta \; dr\, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{[\sin^3\theta\cos^2\phi]}{5}\, d\phi\, d\theta$$

$$= \frac{1}{5}\int_0^{\pi} \cancel{\cos}\sin^3\theta\,[\pi]\, d\theta$$

$$= \frac{\pi}{5}\int_0^{\pi} \sin^3\theta\, d\theta$$

$$= \frac{\pi}{5}\left[-\cos\theta + \frac{\cos^3\theta}{3}\Big|_0^{\pi}\right]$$

$$= \frac{\pi}{5}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

$\sin\theta - \cos^2\theta\sin\theta$

$$= \boxed{\frac{4\pi}{15}}$$

$$J = \int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^3\theta \cos^2\phi \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \cos^2\phi \, d\phi \left[\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} \Big|_0^{\pi} \right]$$

$$= \int_0^{2\pi} \cos^2\phi \, d\phi \left[\left(-\frac{1}{5} + \frac{1}{3} \right) \times 2 \right]$$

$$= \frac{4}{15} \int_0^{2\pi} \cos^2\phi \, d\phi$$

$$= \boxed{\frac{4\pi}{15}}$$