ASSIGNMENT - 5

1. $x = \cos\theta + i\sin\theta$

(a) $$\frac{2}{1+x} = \frac{2}{(1+\cos\theta) + i\sin\theta}$$

$$= \frac{1}{\cos(\theta/2)\left[\cos(\theta/2) + i\sin(\theta/2)\right]}$$

$$= \frac{\cos(\theta/2) - i\sin(\theta/2)}{\cos^2(\theta/2)}$$

$$= \boxed{1 - i\tan(\theta/2)}$$

(b) $$\frac{1+x}{1-x} = \frac{(1+\cos\theta) + i\sin\theta}{(1-\cos\theta) - i\sin\theta}$$

$$= \frac{\cos(\theta/2)\left[\cos(\theta/2) + i\sin(\theta/2)\right]}{\sin(\theta/2)\left[\sin(\theta/2) - i\cos(\theta/2)\right]}$$

$$= \cot(\theta/2) \cdot \frac{\left[e^{i\theta/2}\right]}{\left[e^{i(\theta/2 - \pi/2)}\right]}$$

$$= e^{i\pi/2}\cot(\theta/2)$$

$$= \boxed{i\cot(\theta/2)}$$

2. $$\frac{x + (x-2)i}{3+i} + \frac{2y - (3y)i}{3-i} = i$$

$$\frac{[3x + x - 2] + i[3x - 6 - x] + [6y + 3y] + i[2y - 9y]}{10} = i$$

$$[4x + 9y - 2] + i[2x - 7y - 6] = 10i$$

$2x - 7y = 16$

$2 + \frac{14 \times 9}{23}$

$\therefore \quad 4x + 9y = 2$

$\ominus \; 4x - 14y = \cancel{16}\, 32$

$\cancel{23y = -14}$

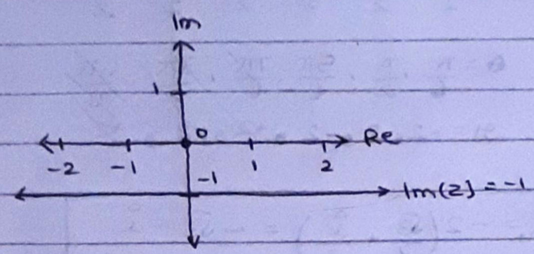
$\cancel{y = -14/23}$

$\cancel{x = 43/23}$

$23y = -30 \Rightarrow$

$\boxed{y = -30/23 \quad x = 79/23}$

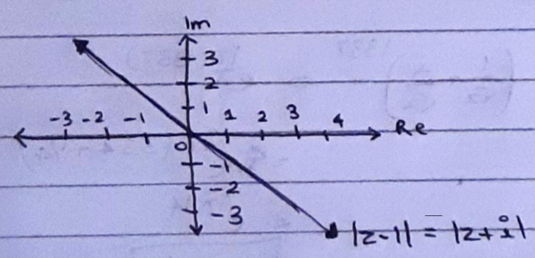
3. (a) $\mathrm{Im}(z) = -1$



(b) $|z-1|^2 = |z+i|^2$

$(x-1)^2 + y^2 = x^2 + (y+1)^2$

$-2x + 1 = 2y + 1 \Rightarrow \boxed{x + y = 0}$



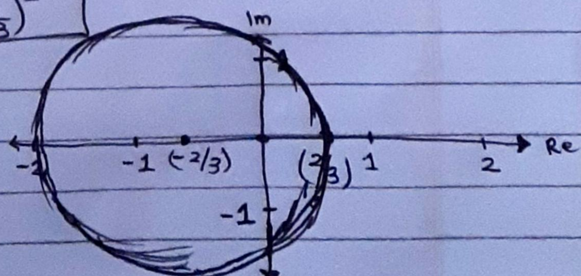
(c) $4|z|^2 = |z-2|^2$

$4x^2 + 4y^2 = x^2 - 4x + 4 + y^2$

$3x^2 + 3y^2 + 4x = 4$

$x^2 + y^2 + \frac{4}{3}x = \frac{4}{3}$

$\boxed{\left(x + \frac{2}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2}$



4. (a) Let $z = (-8i)^{1/3}$

$\therefore z^3 = -8i$

$z = re^{i\theta}$

$\therefore r^3 e^{i3\theta} = -8i$

$r^3\cos(3\theta) = 0$ $\qquad$ $r^3\sin(3\theta) = -8$

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \cancel{\frac{7\pi}{6}}, \cancel{\frac{3\pi}{2}}, \cancel{\frac{11\pi}{6}}$

$r = -2, 2, -2, \cancel{+2}, \cancel{-2}, \cancel{+2}$

$$\boxed{\begin{aligned} z_1 &= -2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = -\sqrt{3} - i \\ z_2 &= 2(i) = 2i \\ z_3 &= -2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\right) = +\sqrt{3} - i \end{aligned}}$$

(b) $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{1337} = e^{\frac{i\pi}{4}(1337)}$

$= e^{i\pi(334+1/4)}$

$= e^{i\pi/4}$

$= \boxed{\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}$

5. $i^{\alpha+i\beta} = \alpha + i\beta \quad \left[i^{i} = e^{-\pi/2} \right]$

$\therefore\ i^{\alpha} \cdot e^{-\frac{\pi\beta}{2}} = \alpha + i\beta$

(OR)

$\alpha + i\beta = i^{\alpha} \cdot e^{-\frac{(4n+1)}{2}\pi\beta}$

$|\alpha + i\beta|^2 = |i^{\alpha}|^2 \left| e^{-\frac{(4n+1)}{2}\pi\beta} \right|^2$

since $i^{i} = \cancel{e^{i\pi/2 \times i}}$
$= e^{(4n+1)i\pi/2 \times i}$
$= e^{-(4n+1)\pi/2}$

$$\boxed{\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}}$$

6. $x^4 - x^3 + x^2 - x + 1 = 0$

$(x+1)(x^4 - x^3 + x^2 - x + 1) = 0$

$\therefore\ x^5 + 1 = 0$

$$\boxed{x^5 = -1}$$

$x = re^{i\theta}$

$r^5 \cos(5\theta) = -1 \quad | \quad r^5 \sin(5\theta) = 0$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$r = -1, 1, -1, 1, -1$

$$\therefore\ \boxed{\begin{array}{ll} x_1 = e^{\pi/5} & x_4 = -e^{4\pi/5} \\ x_2 = -e^{\frac{2\pi}{5}} & \\ x_3 = e^{3\pi/5} & \end{array}}$$

7. $z^6 = -9$

$z^3 = 3i$

$z = r_1 e^{i\alpha}$

$r_1^3 \cos(3\alpha) = 0 \quad r_1^3 \sin(3\alpha) = 3$

$\alpha = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

$r_1 = 3^{1/3}, -3^{1/3}, 3^{1/3}$

$\frac{3\pi}{2}$

$z^3 = -3i$

$z = r_2 e^{i\beta}$

$\cancel{\alpha = }$

$$\boxed{\text{Roots are } \pm 3^{1/3} e^{i\pi/6}, \pm 3^{1/3} e^{i\pi/2}, \pm 3^{1/3} e^{5i\pi/6}}$$

8. $z^4 = -16$

$z^2 = 4i, -4i$

$z = re^{i\theta}$

$r^2 \cos(2\theta) = 0$

$r^2 \sin(2\theta) = 4$

$\theta = \frac{\pi}{4}, \cancel{\frac{3\pi}{4}}, \frac{5\pi}{4}, \cancel{\frac{7\pi}{4}}$

$r = \pm 2 \quad , \pm 2$

$r^2 \cos(2\theta) = 0$

$r^2 \sin(2\theta) = -4$

$\theta = \frac{3\pi}{4}$

$r = \pm 2$

$\therefore$

$z_1 = 2e^{\pi/4}$

$z_2 = -2e^{\pi/4}$

$z_3 = 2e^{3\pi/4}$

$z_4 = -2e^{3\pi/4}$

$z_1 + z_4 = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= 2\sqrt{2} \Rightarrow z_1 z_4 = 4$

& $z_2 + z_3 = -2\sqrt{2} \Rightarrow z_2 z_3 = 4$

$\therefore z^4 + 16 = [z^2 - 2\sqrt{2}z + 4][z^2 + 2\sqrt{2}z + 4]$

$\frac{2010}{3} = 670$

9. (a) $1+\sqrt{3}i = 2e^{i\pi/3} = z$

$$\therefore z^{2011} = (2)^{2011} \cdot e^{i\pi(670+1/3)}$$

$$= (2)^{2011}\, e^{i\pi/3}$$

$$= (2)^{2011}\left(\frac{z}{2}\right) = \boxed{2^{2010} z}$$

(b) $1+\sqrt{3}i = 2e^{i\pi/3} = z.$

$$\therefore z^{-2011} = \cancel{\frac{1}{2^{2010} z}} \Rightarrow \frac{1}{2^{2011}\, e^{i\pi/3}}$$

$$= \boxed{\frac{e^{-i\pi/3}}{(2^{2011})}}$$

10. $z = x + iy$

$$\therefore \sin z = \sin x \cos(iy) + \cos x \sin(iy) \quad \left[\begin{array}{l} \sin(iy) = i\sinh y \\ \cos(iy) = \cosh y \end{array}\right]$$

$$= \sin x \cosh y + i\cos x \sinh y$$

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x\,[\cosh^2 y - \sinh^2 y] + \sinh^2 y$$

$$\boxed{|\sin z|^2 = \sin^2 x + \sinh^2 y}$$

11. $\cos\left(\frac{\pi}{3}+i\right) = \cos\left(\frac{\pi}{3}\right)\cos(i) - \sin\left(\frac{\pi}{3}\right)\sin(i)$

$$= \boxed{\frac{\cosh(1)}{2} - \frac{\sqrt{3}\,i\sinh(1)}{2}}$$

12. Let $x^2 = -3+4i \quad [z = x+iy]$

$$x^2 - y^2 = -3$$

$$xy = 2$$

$$\therefore x^2 - \frac{4}{x^2} = -3$$

$$x^4 + 3x^2 - 4 = 0 \Rightarrow (x^2-1)(x^2+4) = 0$$

$$x^2 = 1 \Rightarrow \boxed{x = \pm 1}$$

$$\boxed{y = \pm 2}$$

$\therefore$ Roots of $(-3+4i)$ are $\boxed{\pm(1+2i)}$

13. $\sinh(z_1 + z_2)$

$$= \cancel{e^{z_1}} - \cancel{e^{z_2}} \frac{e^{z_1+z_2} - e^{-(z_1+z_2)}}{2}$$

$$= \frac{e^{z_1}e^{z_2} - e^{-z_1}e^{-z_2}}{2}$$

$$= \cancel{e^{z_1}e^{z_2}}$$

$$= \frac{2e^{z_1}e^{z_2} - 2e^{-z_1}e^{-z_2}}{4}$$

$e^{z_1}e^{-z_2}$

$$= \frac{[e^{z_1}e^{z_2} + e^{z_1}e^{-z_2} - e^{-z_1}e^{z_2} - e^{-z_1}e^{-z_2}] + [e^{z_1}e^{z_2} + e^{-z_1}e^{z_2} - e^{z_1}e^{-z_2} - e^{-z_1}e^{-z_2}]}{4}$$

$$= \frac{[e^{z_1} - e^{-z_1}][e^{z_2} + e^{-z_2}] + [e^{z_1} + e^{-z_1}][e^{z_2} - e^{-z_2}]}{4}$$

$$= \left[\frac{e^{z_1} - e^{-z_1}}{2}\right]\left[\frac{e^{z_2} + e^{-z_2}}{2}\right] + \left[\frac{e^{z_1} + e^{-z_1}}{2}\right]\left[\frac{e^{z_2} - e^{-z_2}}{2}\right]$$

$$= \boxed{\sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2)}$$

14. (i) $f(z) = \dfrac{1}{i - z} = \dfrac{1}{(-x) + i(1-y)}$

$$= \frac{-x + i(y-1)}{x^2 + (y-1)^2}$$

$$\boxed{f(z) = \left(\frac{-x}{x^2+(y-1)^2}\right) + i\left[\frac{(y-1)}{x^2+(y-1)^2}\right]}$$

(ii) $f(z) = \overline{e^{z^2}}$

$= \cancel{e^{\bar{z}^2}}$

$\cancel{f(z)}$ $e^{z^2} = e^{(x^2-y^2) + (2xyi)}$

$= e^{(x^2-y^2)} \cdot e^{i(2xy)}$

$= \left[e^{(x^2-y^2)}\cos(2xy)\right] + i\left[e^{(x^2-y^2)}\sin(2xy)\right]$

$$\boxed{\therefore f(z) = \overline{e^{z^2}} = \left[e^{(x^2-y^2)}\cos(2xy)\right] - i\left[e^{(x^2-y^2)}\sin(2xy)\right]}$$

15. ① $(1-i)z_1 + z_2 = 3+2i$

② $(1-i)z_1 + (1-i)(2-i)z_2 = (2+i)(1-i)$

② $(1-i)z_1 + (1-3i)z_2 = (3-i)$

$\cancel{①} \cancel{-②/x}$ ② − ①:

$(-3\cancel{i})z_2 = (-3\cancel{i})$

$\boxed{z_2 = 1}$

$(1-i)z_1 = 2+2i$

$z_1 = \dfrac{\cancel{2}(1+i)(1+i)}{\cancel{2}}$

$\boxed{z_1 = 2i}$

$\boxed{z_1 = 2i, \ z_2 = 1}$