ASSIGNMENT - 2 ON LAPLACE TRANSFORM

1.

(a) $y''(t) + 5y'(t) + 4y(t) = e^{3t}$ $\quad \begin{cases} y(0) = 0 \\ y'(0) = 3 \end{cases}$

$\mathcal{L}[y''(t)] + 5\mathcal{L}[y'(t)] + 4\mathcal{L}[y(t)] = \mathcal{L}[e^{3t}]$

Let $\mathcal{L}[y(t)] = F(s)$.

$\therefore \left[s^2F(s) - s\overbrace{y(0)}^{=0} - \overbrace{y'(0)}^{=3}\right] + 5\left[sF(s) - \overbrace{y(0)}^{=0}\right] + 4F(s) = \dfrac{1}{s-3}$

$F(s)\left[s^2 + 5s + 4\right] = \dfrac{1}{s-3} + 3$

$$\boxed{F(s) = \frac{(3s-8)}{(s-3)(s+1)(s+4)}}$$

Let $\dfrac{(3s-8)}{(s-3)(s+1)(s+4)} = \dfrac{A}{s-3} + \dfrac{B}{s+1} + \dfrac{C}{s+4}$

$= \dfrac{A(s+1)(s+4) + B(s-3)(s+4) + C(s-3)(s+1)}{(s-3)(s+1)(s+4)}$

$\Rightarrow$ From this, we get

$$\boxed{A = \frac{1}{28}, \quad B = \frac{11}{12}, \quad C = -\frac{20}{21}}$$

$\therefore \mathcal{L}[y(t)] = \dfrac{1}{28}\cdot\dfrac{1}{s-3} + \dfrac{11}{12}\cdot\dfrac{1}{s+1} - \dfrac{20}{21}\cdot\dfrac{1}{s+4}$

$$\boxed{y(t) = \frac{1}{28}e^{3t} + \frac{11}{12}e^{-t} - \frac{20}{21}e^{-4t}}$$

(c) $2y''(t) - y'(t) - y(t) = \cos(t)$ $\quad y(0) = 1,\ y'(0) = 0$

$$2\mathcal{L}[y''(t)] - \mathcal{L}[y'(t)] - \mathcal{L}[y(t)] = \mathcal{L}[\cos(t)]$$

$\cancel{\mathcal{L}[f(s)]}$

$$2[s^2F(s) - s\,y(0) - y'(0)] - [sF(s) - y(0)] - F(s) = \frac{s}{s^2+1}$$

$$F(s)[2s^2 - s - 1] - 2s + 1 = \frac{s}{s^2+1}$$

$$F(s) = \left(\frac{1}{2s^2 - s - 1}\right)\cdot\left(\frac{2s^3 - s^2 + 3s - 1}{s^2+1}\right)$$

$$= \frac{1}{2}\ \frac{(2s^3 - s^2 + 3s - 1)}{(s^2+1)(s-1)(s+1/2)}$$

Let $(2s^3 - s^2 + 3s - 1) = (As + B)(s-1)(s+1/2) + C(s^2+1)(s+1/2) + D(s^2+1)(s-1)$

Substituting $s = 1$: $\quad 3 = C(2)\left(\frac{3}{2}\right) \Rightarrow C = 1$

$s = -1/2$: $\quad -\cancel{3} = D\left(\frac{5}{4}\right)\left(\frac{-\cancel{3}}{2}\right) \Rightarrow D = 8/5$

$\cancel{s = i}$: $\cancel{1} = \cancel{(Ai+B)(i-1)(i+1/2)}$

$2\cdot\frac{-1}{8}$

$-\frac{1}{4} - \frac{1}{4} - \frac{3}{2} - 1$

$-\frac{1}{2} - \frac{3}{2}$

$\cancel{A} = \cancel{-\frac{3}{5}},\ \cancel{B} =$

$2i^3 -$

$-2i + 1 + 3i - 1$

$s = 0$: $\quad -1 = B(-1)(1/2) + 1(1)(1/2) + \frac{8}{5}(1)(-1)$

$$-1 = -\frac{B}{2} + \left(\frac{-11}{10}\right)$$

$$-\frac{B}{2} = \frac{1}{10} \Rightarrow \boxed{\begin{array}{l} B = -1/5 \\ A = -3/5 \end{array}}$$

$$\therefore\ y(t) = \frac{1}{2}\mathcal{L}^{-1}\left[-\frac{1}{5}\ \frac{(3s+1)}{s^2+1} + \frac{1}{s-1} + \frac{8}{5}\cdot\frac{1}{s+1/2}\right]$$

$$= \boxed{-\frac{3}{10}\cos(t) - \frac{1}{10}\sin(t) + \frac{4}{5}\left(e^{-t/2}\right) + \frac{1}{2}e^{t}}$$

(n) $f(t) = 1 + t + 2\int_0^t \sin\tau f(t-\tau)\,d\tau$

$\underbrace{}$ $\hookrightarrow = \int_0^t g(\tau) f(t-\tau)\,d\tau$, where $g(\tau) = \sin\tau$

$$\therefore \mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[t] + 2\mathcal{L}[g(t) * f(t)]$$

$$= \mathcal{L}[1] + \mathcal{L}[t] + 2\mathcal{L}[g(t)]\mathcal{L}[f(t)]$$

$$= \frac{1}{s} + \frac{1}{s^2} + 2 \cdot \frac{1}{1+s^2}\mathcal{L}[f(t)]$$

$$\mathcal{L}[f(t)]\left[\frac{s^2-1}{s^2+1}\right] = \frac{s+1}{s^2}$$

$$\hookrightarrow \mathcal{L}[f(t)] = \frac{s^2+1}{s^2(s-1)}$$

$$= \frac{1-s^2}{s^2(s-1)} + \frac{2s^2}{s^2(s-1)}$$

$$= \boxed{\frac{-1}{s^2} - \frac{1}{s} + \frac{2}{s-1}}$$

$$\boxed{\therefore f(t) = -t - 1 + 2e^t}$$

(v) $f(t) = a\sin t + 2\int_0^t f'(z)\sin(t-z)\,dz$.

$$\underbrace{\int_0^t f'(z)\sin(t-z)\,dz}_{= f'(t) * g(t)}$$

$\mathcal{L}[f(t)] = \mathcal{L}[a\sin t] + 2\mathcal{L}[f'(t)]\mathcal{L}[g(t)]$.

$$F(s) = \frac{a}{s^2+1} + \frac{2}{s^2+1}[sF(s) - f(0)]$$

$$\Rightarrow \boxed{f(0) = a\sin(0) + 2\int_0^0 f'(z)\sin(t-z)\,dz = 0}$$

$$\therefore F(s) = \frac{1}{s^2+1}[a + 2sF(s)]$$

$$(s-1)^2 F(s) = a.$$

$$F(s) = \frac{a}{(s-1)^2}$$

$$\mathcal{L}^{-1}[F(s)] = a\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = ae^t\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$= \boxed{ate^t}$$