MA1140 ELEMENTARY LINEAR ALGEBRA

ASSIGNMENT - 2

1. For any matrix to form a vector space over $\mathbb{R}$, it should satisfy additive and multiplicative closure, associativity, commutativity and multiplicative distributivity, and additive identity & inverse.

(a) $\begin{bmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{bmatrix}$ does not satisfy additive closure
V

(eg) $$\boxed{\underset{\in V}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 1 & 5 & 6 \end{bmatrix}} + \underset{\in V}{\begin{bmatrix} 7 & 8 & 1 \\ 9 & 1 & 10 \\ 1 & 11 & 12 \end{bmatrix}} = \underset{\notin V}{\begin{bmatrix} 8 & 10 & 2 \\ 12 & 2 & 14 \\ 2 & 16 & 18 \end{bmatrix}}}$$

Hence matrices of the above form don't form a vector space over $\mathbb{R}$.

(b) $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ forms a vector space if

① $v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$

Let $v_1 = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$, $v_2 = \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix}$

Where $a, b, c, d, e, f \in \mathbb{R}$ [Blank spaces denote 0]

$\therefore v_1 + v_2 = \begin{bmatrix} (a+d) & & \\ & (b+e) & \\ & & (c+f) \end{bmatrix} \in V$

(since $a+d \in \mathbb{R}$, $b+e \in \mathbb{R}$, $c+f \in \mathbb{R}$)

② $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

Let v_1 & v_2 be same as in ①.

$v_3 = \begin{bmatrix} g & & \\ & h & \\ & & i \end{bmatrix}$, $g, h, i \in \mathbb{R}$.

$(v_1 + v_2) + v_3 = \begin{bmatrix} (a+d) & & \\ & (b+e) & \\ & & (c+f) \end{bmatrix} + \begin{bmatrix} g & & \\ & h & \\ & & i \end{bmatrix}$

$= \begin{bmatrix} (a+d+g) & & \\ & (b+e+h) & \\ & & (c+f+i) \end{bmatrix}$

$v_1 + (v_2 + v_3) = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} + \begin{bmatrix} (d+g) & & \\ & (e+h) & \\ & & (f+i) \end{bmatrix}$

$= \begin{bmatrix} (a+d+g) & & \\ & (b+e+h) & \\ & & (c+f+i) \end{bmatrix}$

$\therefore (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

$$\begin{bmatrix} a-c & b-d \\ c & d \end{bmatrix} = \begin{bmatrix} -b & -c \\ c & d \end{bmatrix} = \begin{bmatrix} -d & -a \\ c & d \end{bmatrix}$$

③ $\exists\ 0_V$ s.t $V + 0_V = V$

$$\underset{V}{\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}} + \underset{0_V}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = \underset{V}{\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}}$$

④ $V_1 \in V$ s.t $V + V_1 = 0_V$

$$\underset{V}{\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}} + \underset{V_1}{\begin{bmatrix} -a & & \\ & -b & \\ & & -c \end{bmatrix}} \in \underset{0_V}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$V_1 \in V$ since if $a \in \mathbb{R}$, $-a \in \mathbb{R}$ as well.

⑤ $V_1 + V_2 = V_2 + V_1$

$$V_1 + V_2 = \begin{bmatrix} a+d & & \\ & b+e & \\ & & c+f \end{bmatrix}$$

$$V_2 + V_1 = \begin{bmatrix} d+a & & \\ & e+b & \\ & & f+c \end{bmatrix}$$

since $\mathbb{R}$ is commutative additively,

$V_1 + V_2 = V_2 + V_1$

Since $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ satisfies all ⑤ conditions of a vector space, all matrices of the above form form a vector space.

2. $u, v, w \in V$ [V is a vector space over field F]

$$w + u = w + v$$

We know $\exists\ a \in V$ s.t $a + w = 0_v$

$\therefore$ Adding a on both sides

$$a + w + u = a + w + v$$

$$(a + w) + u = (a + w) + v \quad \text{[Associativity]}$$

$$0_v + u = 0_v + v \quad [0_v \in V + a \in V = a]$$

$$\boxed{u = v}$$

3. $S = \left\{ \underset{A}{\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}}, \underset{C}{\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}} \right\}$ is linearly

independent if $a, b, c \in \mathbb{R}$ s.t. $aA + bB + cC = 0$, then $a = 0, b = 0, c = 0$ is the only solution.

$$2a + 4b - 3c = 0 \rightarrow (1)$$

$$a + c = 0 \rightarrow (2) \Rightarrow a = -c$$

$$3a + 2b + 2c = 0 \rightarrow (3)$$

$$-a + 3b + c = 0 \rightarrow (4)$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -3 \\ 1 & 0 & 1 \\ 3 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix}_{4\times 3} \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

① $R_1 \to R_1/2$
② $R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 2 & -1.5 \\ 0 & -2 & 2.5 \\ 0 & -4 & 6.5 \\ 0 & 5 & -0.5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

① $R_2 \to -R_2/2$
② $R_1 \to R_1 - 2R_2$
$R_3 \to R_3 + 4R_2$
$R_4 \to R_4 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1.25 \\ 0 & 0 & 1.5 \\ 0 & 0 & 5.75 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

① $R_3 \to \frac{2}{3} R_3$
② $R_1 \to R_1 - R_3$
$R_2 \to R_2 + 1.25R_3$
$R_4 \to R_4 - 5.75R_3$

$-0.5 + 6.25$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $a = 0$
$b = 0$ is the ONLY Solution
$c = 0$

$\Rightarrow$ S is linearly independent.

4.

$$S = \left\langle \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \right\rangle$$

$$\quad\quad\quad v_1 \quad\quad\quad v_2 \quad\quad\quad v_3$$

$$S = \langle \{v_1, v_2, v_3\} \rangle = \{ av_1 + bv_2 + cv_3 ; a, b, c \in \mathbb{R} \}$$

$$\therefore \boxed{A \in S \text{ iff } \exists\, a, b, c \in \mathbb{R} \text{ s.t. } A = av_1 + bv_2 + cv_3}$$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = a \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} + c \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -3 \\ 1 & 0 & 1 \\ 3 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 6 \\ -4 \end{bmatrix}$$

① $R_1 \to R_1/2$

② $R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 2 & -1.5 \\ 0 & -2 & 2.5 \\ 0 & -4 & 6.5 \\ 0 & 5 & -0.5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1.5 \\ 4.5 \\ 10.5 \\ -5.5 \end{bmatrix}$$

① $R_2 \to -R_2/2$

② $R_1 \to R_1 - 2R_2$

$R_3 \to R_3 + 4R_2$

$R_4 \to R_4 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1.25 \\ 0 & 0 & 1.5 \\ 0 & 0 & 5.75 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -2.25 \\ 1.5 \\ 5.75 \end{bmatrix}$$

① $R_3 \to \frac{2}{3} R_3$

② $R_1 \to R_1 - R_3$

$R_2 \to R_2 + 1.25 R_3$

$R_4 \to R_4 - 5.75 R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} a &= 2 \\ b &= -1 \\ c &= 1 \end{aligned}}$$

$$\boxed{\therefore \quad A = 2V_1 - V_2 + V_3 \Leftrightarrow A \in S}$$

5. $a + b = c \rightarrow (1)$

$b + c = d \rightarrow (2)$

$c + d = a \rightarrow (3)$

$(1) + (3)$:

$b + d = 0 \Rightarrow \boxed{b = -d}$

$b + c = d \ [(2)] \Rightarrow \boxed{c = d - b = 2d}$

$a = c + d \ [(3)] \Rightarrow \boxed{a = 3d}$

$$\therefore \ S = d\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}.$$

$\therefore$ Dimensions of $S = 1$.

6. $$A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix} \quad R_1 \rightarrow R_1/2$$

$$\begin{bmatrix} 1 & -0.5 & -1.5 & 5.5 & 4.5 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -0.5 & -1.5 & 5.5 & 4.5 \\ 0 & 2.5 & 2.5 & -12.5 & -7.5 \\ 0 & 2.5 & 1.5 & -10.5 & -5.5 \\ 0 & 2 & 5 & -16 & -12 \end{bmatrix} \quad R_2 \rightarrow 0.4R_2$$

$$\begin{bmatrix} 1 & -0.5 & -1.5 & 5.5 & 4.5 \\ 0 & 1 & 1 & -5 & -3 \\ 0 & 2.5 & 1.5 & -10.5 & -5.5 \\ 0 & 2 & 5 & -16 & -12 \end{bmatrix} \quad \begin{matrix} R_1 \rightarrow R_1 + 0.5R_2 \\ R_3 \rightarrow R_3 - 2.5R_2 \\ R_4 \rightarrow R_4 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 & 3 \\ 0 & 1 & 1 & -5 & -3 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 3 & -6 & -6 \end{bmatrix} \quad R_3 \to -R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 & 3 \\ 0 & 1 & 1 & -5 & -3 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 3 & -6 & -6 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - R_3 \\ R_4 \to R_4 - 3R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduced Echleon form

$\therefore$ Rank(A) = 3

We know by Rank-Nullity Theorem,

$$\text{Rank}(A) + \dim(N(A)) = \text{No. of columns of } A.$$

$$\therefore \quad 3 + \dim(N(A)) = 5$$

$$\boxed{\dim(N(A)) = 2}$$

$\Rightarrow$ Dimensions of null space of A = 2.