ASSIGNMENT - 4

1. $$\iint_S (x+z)\,dy\,dz + (y+z)\,dx\,dz + (x+y)\,dx\,dy$$

$$= \iint_S \left((x+z)\hat{i} + (y+z)\hat{j} + (x+y)\hat{k}\right) \cdot \left(dy\,dz\,\hat{i} + dx\,dz\,\hat{j} + dx\,dy\,\hat{k}\right)$$

$$= \boxed{\iint_S \vec{A} \cdot d\vec{s}}$$

By GDT,

$$\iint_S \vec{A} \cdot d\vec{s} = \iiint_V \vec{\nabla} \cdot \vec{A}\, dV$$

$$= \iiint_V \vec{\nabla} \cdot \left((x+z)\hat{i} + (y+z)\hat{j} + (x+y)\hat{k}\right) dx\,dy\,dz$$

$$= \iiint_V (2)\, dx\,dy\,dz$$

$$= \boxed{2V}$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 2^3 = \frac{32\pi}{3}$$

$$\therefore \boxed{2V = \frac{64\pi}{3}}$$

2. $\oint_C \bar{F} \cdot d\vec{a} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$

$d\vec{S} = dx\,dy\,\hat{k}$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^3 & -(2x+z) \end{vmatrix}$$

$\therefore (\bar{\nabla} \times \bar{F}) \cdot d\vec{S} = \left(\frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial y}(2y^2) \right) dx\,dy$

$= (6x - 4y)\,dx\,dy$

$\therefore$ $\cancel{\iint_S (6x-4y)\,dx\,dy = \iint}$

$\therefore \int_0^2 \int_y^2 (6x-4y)\,dx\,dy = \int_0^2 [(12-8y)+(y^2)]\,dy$

$= \left. \frac{y^3}{3} - 4y^2 + 12y \right|_0^2$

$= \frac{8}{3} - 16 + 24$

$= \frac{8}{3} + 8 = \boxed{\frac{32}{3}}$

$3x^2 - 4yx$

$\cancel{= \int_0^2 (3x^2 - 4yx)\Big|_0^y dy}$

$\cancel{= \int_0^2 (3y^2 - 4y^2)\,dy}$

$\cancel{= \int_0^2 -y^2\,dy}$

$\cancel{= \left. -\frac{y^3}{3} \right|_0^2 = \boxed{-\frac{8}{3}}}$

3. $\bar{F} = xy\,\hat{i} + (4x - yz)\,\hat{j} + (xy - \sqrt{z})\,\hat{k}$

$$\oint \bar{F}\cdot d\bar{a} = \iint_S (\bar{\nabla}\times\bar{F})\cdot d\bar{s}$$

$$\hat{n}\ \cancel{d\bar{s}} = \frac{\bar{\nabla}(x+y+z)}{|\bar{\nabla}(x+y+z)|}$$

$$= \boxed{\frac{i+j+k}{\sqrt{3}}}$$

$$d\bar{s} = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}\times\hat{n}$$

$$= \frac{dx\,dy}{1/\sqrt{3}}\times\frac{(i+j+k)}{\sqrt{3}}$$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 4x-yz & xy-\sqrt{z} \end{vmatrix}$$

$$= \cancel{\hat{i}(x-y) - j(y) + k(x)}$$

$$\cancel{\therefore (\bar{\nabla}\times\bar{F})\cdot d\bar{s} = (x-y) - (y) + x}$$

$$= \hat{i}(x-y) - j(y) + k(4-x)$$

$$\therefore (\bar{\nabla}\times\bar{F})\cdot d\bar{s} = (x-y-y+4-x)\,dx\,dy$$

$$= 4\,dx\,dy = \frac{4}{\sqrt{3}}\times \text{Area}$$

$$\therefore \iint_S (\bar{\nabla}\times\bar{F})\cdot d\bar{s} = 4\times A = \frac{4\pi a^2}{\sqrt{3}} = \pi\sqrt{3}$$

$\boxed{\cancel{a = \frac{3/4}{2}}}$ $\boxed{a = \frac{\sqrt{3}}{2}}$

4. $\bar{F} = x\hat{i} + y\hat{j} + z^2\hat{k}$

$$\iint_S \bar{F}\cdot d\bar{s} = \iiint_V (\bar{\nabla}\cdot\bar{F})\,dV$$

$$\bar{\nabla}\cdot\bar{F} = 1+1+2z$$
$$= 2(1+z)$$

$$\therefore \iiint_V 2(1+z)\,dV$$

$$I = 2\Bigg[\underbrace{\iiint_V dV}_{I_1} + \underbrace{\iiint_V z\,dV}_{I_2}\Bigg]$$

$$I_1 = V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$$

$$I_2 = \iiint_V z\,dV = \int_0^{\pi}\int_0^{1}\int_0^{2\pi} (r\cos\theta)(r^2\sin\theta\, d\phi\, dr\, d\theta)$$

$$= \pi\int_0^{\pi}\int_0^{1} r^3 \sin 2\theta\, dr\, d\theta$$

$$= \frac{\pi}{4}\int_0^{\pi} \sin 2\theta\, d\theta$$

$$= \frac{\pi}{4}\left[-\frac{\cos 2\theta}{2}\Big|_0^{\pi}\right]$$

$$= \boxed{0}.$$

$$\therefore I = 2\left[\frac{4}{3}\pi + 0\right] = \boxed{\frac{8\pi}{3}}$$

$\boxed{3}$

5. $\oint_C (-y\,dx + x\,dy)$:

$M = -y$

$N = x.$

$$\oint (M\,dx + N\,dy) = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \iint (1-(-1))\,dy\,dx$$

$$= 2\iint dy\,dx$$

$= 2 \times \text{Area}$ $\quad = 2 \times 1$

$= \cancel{2lb}$ $\quad = \boxed{2}$

~~(l & b are length & breadth of rectangle)~~