ASSIGMENT - I

1. (a) $\lim_{z\to i} \dfrac{(z^2+1)^7}{(z^6+1)} = \lim_{z\to i} \dfrac{f(z)}{g(z)}$ $\left[\lim_{z\to i} f(z) = \lim_{z\to i} g(z) = 0 \quad \therefore \text{L'Hopital can be used}\right]$

By L'Hopitals Rule

$$L = \lim_{z\to i} \frac{(z^2+1)^7}{(z^6+1)} = \lim_{z\to i} \frac{7(z^2+1)^6(2z)}{6z^5} = \frac{7(0)(2i)}{6i}$$

$$= \boxed{0}$$

(b) $L = \lim_{z\to i} \dfrac{[z^3+(1-3i)z^2+(i-3)z+(2+i)]}{(z-i)} = \lim_{z\to i} \dfrac{f(z)}{g(z)}$

$\lim_{z\to i} f(z) = \cancel{i^3} + \cancel{i\ldots}$

$= i^3 + 3i - 1 - 1 - 3i + 2 + i$

$= -\cancel{i} + \cancel{3i} - \cancel{2} - \cancel{3i} + \cancel{2} + \cancel{i} = \boxed{0}$

$\lim_{z\to i} g(z) = i - i$

$= \boxed{0}$

$\therefore$ L'Hopital rule can be used

$$L = \lim_{z\to i} \frac{f'(z)}{g'(z)} = \lim_{z\to i} \frac{3z^2 + 2z(1-3i) + (i-3)}{1}$$

$= 3i^2 + 2i(1-3i) + (i-3)$

$= -\cancel{3} + 2i + \cancel{6} + i - \cancel{3}$

$= \boxed{3i}$

2. $f(z) = ze^{z}$ ~~is analytic / if $\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}$~~

$= (x+iy)\, e^{(x+iy)}$

$= xe^{x}[\cos y + i\sin y] + iye^{x}[\cos y + i\sin y]$

$= [xe^{x}\cos y - ye^{x}\sin y] + i[xe^{x}\sin y + ye^{x}\cos y]$

$= u(x,y) + iv(x,y)$

$f(z)$ is analytic if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1)$ (AND) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$

[cauchy-Reimann eqn]

$$\frac{\partial u}{\partial x} = \cos y[e^{x} + xe^{x}] - y\sin y\, e^{x}$$

$$\frac{\partial v}{\partial y} = xe^{x}\cos y + e^{x}[\cos y - y\sin y]$$

$$= xe^{x}\cos y + e^{x}\cos y - y\sin y\, e^{x} = \frac{\partial u}{\partial x}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = -xe^{x}\sin y - e^{x}[\sin y + y\cos y]$$

$$\frac{\partial v}{\partial x} = \sin y[xe^{x} + e^{x}] + e^{x}y\cos y$$

$$= xe^{x}\sin y + e^{x}[\sin y + y\cos y] = -\frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \rightarrow (2)$$

$\therefore f(z)$ is analytic.

$$\boxed{f'(z) = e^{z} + ze^{z}}$$

3. (a) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

$u(x,y)$ is harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$

here, $\boxed{v = 0}$

$\therefore$ ~~$\frac{\partial^2 u}{\partial x^2} = 6xy + 4x$~~ $\quad \boxed{\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[6xy + 4x\right] = 6y + 4$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[3x^2 - 3y^2 - 4y\right] = -6y - 4$$

(+)

$$\therefore \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\boxed{u(x,y) \text{ is HARMONIC.}}$

- for $u + iv$ to be analytic, [cauchy Riemann eqn]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x$$

$$\therefore \boxed{v = 3xy^2 + 4xy + f(x)} \rightarrow ①$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 4y + 3y^2 - 3x^2$$

$$\boxed{v = 4xy + 3xy^2 - x^3 + g(y)} \rightarrow ②$$

From ① & ②, we get

$$\boxed{v(x,y) = 3xy^2 + 4xy - x^3 + c}$$

(b) $u(x,y) = \ln(x^2+y^2)$

For u to be harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2}\right]$$

$$= \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2}$$

$$= \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right]$$

$$= \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

(+)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ $\boxed{u(x,y) \text{ is HARMONIC}}$

- For $u + iv$ to be analytic [Cauchy-Riemann eqn]

(A) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = \frac{2x}{x^2+y^2}$

$$\boxed{v = 2\tan^{-1}(y/x) + f(x)} \to (1)$$

(B) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = \frac{-2y}{x^2+y^2}$

$$\boxed{v = -2\tan^{-1}(x/y) + f(y)} \to (2)$$

$$\boxed{= \pi + 2\tan^{-1}(y/x) + f(y)}$$

$\therefore$ From ① & ②,

$$\boxed{V = 2\tan^{-1}(y/x) + C}$$

4. $f(z) = \dfrac{z^7}{(1+z^2)^3}$

The singularities are $z = \pm i$

$\lim_{z \to i} \dfrac{z^7}{(1+z^2)^3}$ does NOT exist $\lim_{z \to -i} \dfrac{z^7}{(1+z^2)^3}$ does NOT exist

$\therefore$ They are NOT ~~essential~~ removable singularities

~~There exists NO ...~~

~~&~~ $\lim_{z \to i} (z-i)^n \dfrac{z^7}{(z-i)^3(z+i)^3}$ to exist,

$\boxed{n = 3}$

Similarly $\lim_{z \to -i} (z+i)^n \dfrac{z^7}{(z-i)^3(z+i)^3}$ to exist,

$\boxed{n = 3}$.

$\therefore$ $z = \pm i$ are poles of order 3.

u $\qquad\qquad\qquad$ v

5. $f(z) = [x^2 + C_1 y^2 - 2xy] + i\,[C_2 x^2 - y^2 + 2xy]$

is analytic.

By Cauchy-Riemann eqn.

$$(1)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (2)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

① :

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x - 2y \\ \frac{\partial v}{\partial y} &= 2x - 2y\end{aligned}\right\}\ \text{① is satisfied}$$

② :

$$\frac{\partial u}{\partial y} = 2C_1 y - 2x$$

$$\frac{\partial v}{\partial x} = 2C_2 x + 2y$$

$$-\frac{\partial v}{\partial x} = -2C_2 x - 2y$$

comparing x & y coefficients,

$$\boxed{\begin{aligned}&2C_1 = -2 \Rightarrow C_1 = -1 \\ &-2 = -2C_2 \Rightarrow C_2 = +1 \\ &C_1 = -1,\ C_2 = 1 \\ &\cancel{\therefore C_1 = C_2 = 1}\end{aligned}}$$

$w = \therefore f(z) = [x^2 - y^2 - 2xy] + i\,[x^2 - y^2 + 2xy]$

$$\cancel{f'(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial z} + i\frac{\partial w}{\partial y}}$$

$$f'(z) = \frac{\partial u}{\partial x} + \frac{i\,\partial v}{\partial x} \qquad = \boxed{2[(x-y) + i(x+y)]}$$

$$= (2x - 2y) + i(2x + 2y) = \cancel{2i(x+y)}$$

$$\frac{\partial z}{\partial x} + \frac{\partial u}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial x}$$

$$= \frac{\partial x}{\partial z} - \frac{\partial u}{\partial \bar{z}}$$

6. $f(z) = r^n \left[\cos(n\theta) + i\sin(n\theta)\right] = r^n e^{in\theta}$

$$= (re^{i\theta})^n$$

$\cancel{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}} = nz^{n-1}}$

$\cancel{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} - \frac{\partial u}{\partial \bar{z}} = nz^{n-1}}$

$\cancel{\frac{\partial v}{\partial x} =}$

$\boxed{f(z) = z^n}$ [where $z = re^{i\theta}$]

$\boxed{f(z) = (x + iy)^n}$

$f'(z) = nz^{n-1}$ exists if $z \neq 0 \Rightarrow \boxed{r \neq 0}$

$\therefore f(z)$ is ANALYTIC at $r \neq 0$.

$\therefore$ when $r \neq 0$,

$$f'(z) = n\left[re^{i\theta}\right]^{n-1}$$

$$= nr^{n-1} \times e^{i[(n-1)\theta]}$$

$$= \boxed{nr^{n-1}\left[\cos((n-1)\theta) + i\sin((n-1)\theta)\right]}$$

7. $u = x^2 - y^2$

For u to be harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2x) = 2$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-2y) = -2$$

(+)

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\boxed{u \text{ is harmonic}}$

$\cancel{v = \frac{y}{x^2+y^2}}$

~~For v to be harmonic,~~

$\cancel{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$

$\cancel{\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)}$

$\cancel{= \frac{-4x^2 y}{(x^2+y^2)^3} - \left[\frac{2y(x^4-y^4)}{(x^2+y^2)^4}\right]}$

$\cancel{\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2xy}{(x^2+y^2)^2}\right)}$

$\cancel{= \frac{4x^2 y}{(x^2+y^2)^3}}$

$=$

$2xy$

$x^2 + y^2 - \cancel{x}\, y(2y) \quad \frac{-2x^2(2y)}{(x^2+y^2)^3}$

$2y(x^2+y^2)^2 - 2y^2(x^2+y^2)(2y)$

$(x^2+y^2)\left[2yx^2 + 2y^3 - 4y^3\right] \quad 2y(x^2+y^2)(x^2-y^2) - \frac{4x^2 y}{(x^2+y^2)^3}$

$$v = \frac{y}{x^2+y^2}$$

For v to be harmonic,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial v}{\partial x} = \frac{-y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(-2y)(x^2+y^2)^2 - (-2xy)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2yx^2-2y^3) + (x^2+y^2)(8x^2y)}{(x^2+y^2)^4}$$

$$= \boxed{\frac{\cancel{8x^2y} + 6yx^2 - 2y^3}{(x^2+y^2)^3}}$$

$-2yx^2 -$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{[-2y][x^2+y^2]^2 - [x^2-y^2][2][x^2+y^2][2y]}{(x^2+y^2)^4}$$

$$(4y^3 - 4yx^2)$$

$$= \frac{(x^2+y^2)(-2yx^2-2y^3) + (x^2+y^2)\cancel{(4x^2y - 4x^3)}}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-6yx^2+2y^3)}{(x^2+y^2)^4}$$

$$= \frac{-[6yx^2-2y^3]}{(x^2+y^2)^4} = -\frac{\partial^2 v}{\partial x^2}$$

$4x[y^2-x^2]$

$$\therefore \boxed{\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0}$$

$$-iy x^2 - 6y^3)(x^2 - y^2 + 2y)$$

For u & v to be harmonic conjugate,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{[Cauchy-Riemann eq]}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow$$ u & v are NOT harmonic conjugate

$(a^2-b^2+2abi)(a+ib)$
$6a^2b$

8. $z = x + iy$

let $\omega = a + ib$ st $z = \omega^3$

$$x + iy = [a^3 - 3ab^2] + i[3a^2b - b^3]$$

$$\therefore x = a^3 - 3ab^2$$
$$y = 3a^2b - b^3$$

$$\therefore x^2 - y^2 + 2y = a^6 + 9a^2b^4 - 6a^4b^2 - b^6 - 9a^4b^2 + 6a^2b^4 + 6a^2b - 2b^3$$

$$\therefore \frac{\partial^2 f}{\partial a^2} = \frac{\partial}{\partial a}\left[6a^5 + 30ab^4 - 60a^3b^2 + 12ab\right]$$

$$= 30a^4 + 30b^4 - 180a^2b^2 + 12b \rightarrow (1)$$

$$\frac{\partial^2 f}{\partial b^2} = \frac{\partial}{\partial b}\left[-6b^5 + 60a^2b^3 - 30a^4b + 6a^2 - 6b^2\right]$$

$$= -30b^4 - 30a^4 - 12b + 180a^2b^2 \rightarrow (2)$$

$$\therefore \frac{\partial^2 f}{\partial a^2} + \frac{\partial^2 f}{\partial b^2} = (1) + (2) = \boxed{0}$$

$\therefore$ The function remains harmonic

9. $f(z) = \frac{\cot(\pi z)}{(z-a)^2}$

$\Rightarrow$ At $z = a$,

$\lim_{z \to a} \frac{\cot(\pi z)}{(z-a)^2}$ does NOT exist.

$\therefore$ $z = a$ is NOT a removable singularity.

However, for

$\lim_{z \to a} (z-a)^n \frac{\cot(\pi z)}{(z-a)^2}$ to exist,

$$\boxed{n = 2}$$

$\therefore$ $z = a$ is a POLE of order 2.

$\Rightarrow$ At $z = \infty$

$\lim_{z \to \infty} \frac{\cot(\pi z)}{(z-a)^2}$ exists and is equal to 0.

$\therefore$ $z = \infty$ is a REMOVABLE SINGULARITY.

10. $f(z) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{\alpha x}{y}\right)$

For f(z) to be analytic, [cauchy-Riemann eqn]

(A) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{2x}{2(x^2+y^2)} = \frac{1}{1+\frac{\alpha^2 x^2}{y^2}} \times \frac{-\alpha x}{y^2}$$

$$\boxed{\frac{x}{x^2+y^2} = \frac{-\alpha x}{y^2+\alpha^2 x^2}} \to (1)$$

(B) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{y}{x^2+y^2} = -\frac{1}{1+\frac{\alpha^2 x^2}{y^2}} \times \frac{\alpha}{y}$$

$$\boxed{\frac{y}{x^2+y^2} = \frac{-\alpha y}{\alpha^2 x^2+y^2}} \to (2)$$

From (1) & (2) we get

$$\left.\begin{array}{l} -\alpha = 1 \\ \alpha^2 = 1 \end{array}\right\} \cancel{\alpha = 1} \quad \boxed{\alpha = -1}$$

11. $v(x,y) = e^{-2xy}\cos(x^2-y^2)$

By Cauchy-Riemann eqn,

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

① $\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \left[e^{-2xy} \times -\sin(x^2-y^2) \times -2y\right] + \left[-2xe^{-2xy}\cos(x^2-y^2)\right]$

$$= 2ye^{-2xy}\sin(x^2-y^2) - 2xe^{-2xy}\cos(x^2-y^2)$$

② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\left[\left(-e^{-2xy}\sin(x^2-y^2)(2x)\right) + \left(-2ye^{-2xy}\cos(x^2-y^2)\right)\right]$

$$= 2xe^{-2xy}\sin(x^2-y^2) + 2ye^{-2xy}\cos(x^2-y^2)$$

From ① & ② we get

~~$u(x,y) = e^{-2xy}\sin(x^2-y^2)$~~

~~$ye^{-2xy}\sin(x^2-y^2)$~~ $\boxed{u(x,y) = e^{-2xy}\sin(x^2-y^2)}$

$$\boxed{u(x,y) = e^{-2xy}\sin(y^2-x^2)}$$

$Ae^{-Ay}\sin(x^2-y^2) + 2ye^{-Ay}\cos(x^2-y^2)$

12. $f(z) = (4x+y) + i(4x-y)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{4(x+\Delta x) + (y+\Delta y) + i[4(x+\Delta x) - (y+\Delta y)] - 4x - y + i(y-4x)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{(4-i)(\Delta x + i\Delta y)}{(\Delta x + i\Delta y)}$$

$\therefore$ This limit is path independant [doesn't depend on sign of Δx & Δy]

$$= (4-i)$$

$$\boxed{\therefore \frac{df}{dz} \text{ exists}}$$

13. $f(z) = \begin{cases} \left[\dfrac{x^3+y^3}{x^2+y^2}\right] + i\left[\dfrac{y^3-x^3}{x^2+y^2}\right] & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h-0} = \frac{h^3/h^2 - 0}{h-0} = 1$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial u}{\partial y} = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h-0} = \frac{h^3/h^2 - 0}{h-0} = 1$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h-0} = \frac{-h^3/h^2 - 0}{h-0} = -1$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial v}{\partial y} = \lim_{h\to 0} \frac{v(0,h)-v(0,0)}{h-0} = \frac{h^3/h - 0}{h-0} = 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ f(z) satisfies Reimann-Cauchy eqn at (0,0)

$\boxed{\text{It is ANALYTIC}}$

$$f'(z) = \cancel{\lim_{\substack{h\to 0\\ k\to 0}} \frac{h^3+k^3}{h^2+k^2} - 0}$$

Limit doesnt exist
$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{\left(\dfrac{h^3+k^3}{h^2+k^2}\right) + i\left(\dfrac{h^3-k^3}{h^2+k^2}\right) - (1-i) - (1+i)}{\sqrt{h^2+k^2}}$$

$\therefore \boxed{f'(z) \text{ does NOT exist}}$

15.

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

(a) $$\frac{\partial \vec{E}}{\partial t} = i\omega \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

(b) $$\cancel{\nabla\cdot\vec{E} = Re\{\nabla E\}}$$

$$\cancel{= Re\left\{\frac{\partial}{\partial x}\left[\vec{E}_0 \cos(\omega t - k_x x - k_y y)\right] + i\frac{\partial}{\partial y}\left[\vec{E}_0 \sin(\omega t - k_x x \ldots\right.\right.}$$

$$= k_x \vec{E}_0 \cos(\omega t - \vec{k}\cdot\vec{r})$$

(b) $$\bar{\nabla}\cdot\bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left[E_0(x)\, e^{i(\omega t - k_x x - k_y y - k_z z)}\right]$$

$$= -i k_x E_0(x)\, e^{i(\omega t - \bar{k}\cdot\bar{r})}$$ [assuming $\vec{E}_0$ is constant]

Similarly,

$$\frac{\partial E_y}{\partial y} = -i k_y E_0(y)\, e^{i(\omega t - \bar{k}\cdot\bar{r})}$$

$$\frac{\partial E_z}{\partial z} = -i k_z E_0(z)\, e^{i(\omega t - \bar{k}\cdot\bar{r})}$$

$$\therefore \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -i e^{i(\omega t - \bar{k}\cdot\bar{r})}\left[k_x E_0(x) + k_y E_0(y) + k_z E_0(z)\right]$$

$$= \boxed{-i e^{i(\omega t - \bar{k}\cdot\bar{r})} (\bar{k}\cdot\bar{E}_0)}$$

16. $\vec{B} = 3z^2 + 4\bar{z}$

$= [3(x^2 - y^2) + 4x] + i[6xy - 4y]$

u   v

(a) $\nabla \bar{B} = \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + i\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$

$= \left[(6x+4) - (6x-4)\right] + i\left[(-6y) + (6y)\right]$

$= \boxed{8}$

(b) $\nabla \cdot \bar{B} = Re(\bar{\nabla} B)$

$= Re\left(\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + i\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\right)$

$= Re\left[(12x) + i(6y + 6y)\right]$

$= \boxed{12x}$

(c) $|\text{curl } \bar{B}| = |\text{Img}(\bar{\nabla} B)|$

$= 6y + 6y$

$= \boxed{12y}$

(d) Laplacian $\bar{B} = \bar{\nabla}(\nabla B) = \bar{\nabla}(8)$

$= \boxed{0}$