ASSIGNMENT - 2

1. $r^n \vec{r} = \cancel{(x^2+y^2+z^2)^{n/2}\,(x\hat{i}+y\hat{j}+z\hat{k})}$

$\cancel{\vec{\nabla}\cdot(r^n\vec{r})}$

1. $$\vec{\nabla}\cdot(r^n\vec{r}) = \frac{1}{1\times r\times r\sin\theta}\frac{\partial}{\partial r}(r^{n+1}\times r^2\sin\theta)$$

For $\frac{\partial}{\partial r}(r^{n+1}\times r^2\sin\theta)$ to be 0,

$$r^{n+1} = r^{-2} \Rightarrow n+1=-2 \Rightarrow \boxed{n=-3}$$

2. (a) cartesian: $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{\nabla}\cdot(\vec{r}) = 1+1+1 = \boxed{3}$$

(b) cylindrical: $\vec{s} = \rho\hat{e}_\rho + z\hat{k}$

$$\vec{\nabla}\cdot(\vec{s}) = \frac{1}{1\times\rho\times 1}\left[\frac{\partial}{\partial\rho}[\rho^2] + \frac{\partial}{\partial\phi}[0] + \frac{\partial}{\partial z}[z\rho]\right]$$

$$= \frac{1}{\rho}[2\rho+\rho] = \boxed{3}$$

(c) spherical: $\vec{s} = r\hat{e}_r$

$$\vec{\nabla}\cdot(\vec{s}) = \frac{1}{1\times r\times r\sin\theta}\left[\frac{\partial}{\partial r}(r^3\sin\theta) + 0\right]$$

$$= \frac{3r^2\sin\theta}{r^2\sin\theta} = \boxed{3}$$

3. $\bar{B} = \bar{\nabla} \times (A_\phi(r,\theta))\hat{e}_\phi$

$$= \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & \cancel{rA_\phi}\,0 & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{\partial}{\partial \theta}(r\sin\theta A_\phi)\,\hat{e}_r - r\left(\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right)\hat{e}_\theta$$

$= r$

$V = \vec{\nabla} \times \vec{B}$

$$= \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{\partial}{\partial\theta}(r\sin\theta A_\phi) & -r\left(\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right) & 0 \end{vmatrix}$$

$$= \boxed{\begin{aligned} &\frac{\partial}{\partial\phi}\left[r\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right]\hat{e}_r \\ &+ r\left(\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)\right)\right)\hat{e}_\theta \\ &+ r\sin\theta\left[\frac{\partial}{\partial r}\left(-r\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right) - \frac{\partial^2(r\sin\theta A_\phi)}{\partial\theta^2}\right]\hat{e}_\phi \end{aligned}}$$

$$= r\sin\theta \left[\frac{\partial}{\partial r} \left(\cancel{r(\sin\theta}\; r\sin\theta \left(A_\phi + r \frac{\partial A_\phi}{\partial r} \right) \right) \right.$$

$$\left. - r\frac{\partial}{\partial \theta} \left(\cos\theta A_\phi + \sin\theta \frac{\partial A_\phi}{\partial \theta} \right) \right] \hat{e}_\phi$$

$$= r^2 \sin\theta \left[\sin\theta \left[\frac{\partial A_\phi}{\partial r} + \frac{\partial A_\phi}{\partial r} + r\frac{\partial^2 A_\phi}{\partial r^2} \right] \right.$$

$$\left. - \left[-\sin\theta A_\phi + \cos\theta \frac{\partial A_\phi}{\partial \theta} + \cos\theta \frac{\partial A_\phi}{\partial \theta} + \sin\theta \frac{\partial^2 A_\phi}{\partial \theta^2} \right] \right]$$

$$= \boxed{r^2 \sin\theta \left(2\sin\theta \frac{\partial A_\phi}{\partial r} + r\sin\theta \frac{\partial^2 A_\phi}{\partial r^2} + \sin\theta A_\phi - 2\cos\theta \frac{\partial A_\phi}{\partial \theta} - \sin\theta \frac{\partial^2 A_\phi}{\partial \theta^2} \right) \hat{e}_\phi}$$

4. $\nabla^2 \psi = 0 \Rightarrow \nabla . (\nabla \psi) = 0$

$$\nabla \psi = \frac{1}{1} \frac{\partial \psi}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \hat{e}_\phi + \frac{1}{1} \frac{\partial \psi}{\partial z} \hat{e}_z$$

$$= \boxed{\frac{\partial \psi}{\partial \rho} \hat{e}_\rho}$$

$$\therefore \nabla . (\nabla \psi) = \frac{1}{1 \times \rho \times 1} \left[\frac{\partial}{\partial \rho} \left(\frac{\partial \psi}{\partial \rho} \times \rho \times 1 \right) \right]$$

$$= \frac{1}{\rho} \times \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) = 0$$

$$\therefore \cancel{\frac{\partial^2 \psi}{\partial \rho^2}} \frac{1}{\rho} \left[\frac{\partial \psi}{\partial \rho} + \frac{\rho \partial^2 \psi}{\partial \rho^2} \right] = 0$$

(OR)

$$\cancel{\frac{1}{\rho} \frac{\partial \psi}{\partial \rho}}$$

$$\boxed{\frac{\partial^2 \psi}{\partial \rho^2} = \frac{-1}{\rho} \frac{\partial \psi}{\partial \rho}}$$

$$\int \frac{\frac{\partial}{\partial \rho} \left(\frac{\partial \psi}{\partial \rho} \right)}{\frac{\partial \psi}{\partial \rho}} = \int \frac{-1}{\rho} \partial \rho$$

$$\ln \left(\frac{\partial \psi}{\partial \rho} \right) = -\ln (c \rho)$$

$$\text{(or)} \quad \boxed{\rho \frac{\partial \psi}{\partial \rho} = c}$$

$$\int \frac{\partial \rho}{\rho} = \int k \, \partial \psi$$

$$\ln \rho = k\psi + c$$

$$\therefore \boxed{\psi = k_1 \ln \rho + k_2}$$

5. $$\vec{\nabla} f(r) = \frac{1}{1} \frac{\partial f(r)}{\partial r} \hat{e}_r + \cancel{\frac{1}{r} \frac{\partial f(r)}{\partial \theta}} \hat{e}_\theta^{\,=0}$$

$$+ \cancel{\frac{1}{r\sin\theta} \frac{\partial f(r)}{\partial \phi}} \hat{e}_\phi$$

$$= 0$$

$$= \boxed{\frac{df(r)}{dr} \hat{r}}$$

6. $$\vec{\nabla} r^n = \frac{df(r)}{dr} \hat{r} = \boxed{n r^{n-1} \hat{r}}$$

7.

(i) $X = 2z^2 \rho \cos\phi$ at $(1, \pi/2, -1)$

$$\vec{\nabla} X = \frac{1}{1} \frac{\partial (2z^2 \rho \cos\phi)}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial (2z^2 \rho \cos\phi)}{\partial \phi} \hat{e}_\phi$$

$$+ \frac{1}{1} \frac{\partial (2z^2 \rho \cos\phi)}{\partial z} \hat{e}_z$$

$$= \boxed{\begin{aligned} &2z^2 \cos\phi \, \hat{e}_\rho \; \cancel{- 2z^2 \sin\phi \, \hat{e}_\phi} \\ &- 2z^2 \sin\phi \, \hat{e}_\phi \\ &+ 4z\rho \cos\phi \, \hat{e}_z \end{aligned}}$$

$$= 2(0)\hat{e}_\rho - 2(1)\hat{e}_\phi \mp (-4)(0)\hat{e}_z$$

$$= \boxed{-2\hat{e}_\phi}$$

(ii) $\chi = 2r\sin\theta\cos\phi$ at $(1, \pi/2, \pi)$

$$\vec{\nabla}\chi = \frac{1}{1} \times \frac{\partial(2r\sin\theta\cos\phi)}{\partial r}\hat{e}_r$$

$$+ \frac{1}{r} \times \frac{\partial(2r\sin\theta\cos\phi)}{\partial\theta}\hat{e}_\theta$$

$$+ \frac{1}{r\sin\theta}\frac{\partial(2r\sin\theta\cos\phi)}{\partial\phi}\hat{e}_\phi$$

$$= 2\sin\theta\cos\phi\,\hat{e}_r + 2\cos\theta\cos\phi\,\hat{e}_\theta$$

$$- 2\sin\phi\,\hat{e}_\phi$$

$$= 2(1)(-1)\hat{e}_r + 2(0)(-1)\hat{e}_\theta$$

$$- 2(0)\hat{e}_\phi$$

$$= \boxed{-2\hat{e}_r}$$

8. (i) $\vec{\nabla}\cdot(\rho\sin\phi\,\hat{e} + \rho^2 z\,\hat{\phi} + z\cos\phi\,\hat{z})$

$$= \frac{1}{1\times\rho\times1}\left[\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(\rho z\cos\phi)\right]$$

$$= 2\sin\phi + 0 + \cancel{\rho}\cos\phi = \boxed{2\sin\phi + \cancel{\rho}\cos\phi}$$

(ii) $\vec{\nabla}\cdot\left(\frac{1}{r^2}\cos\theta\,\hat{r} + r\sin\theta\cos\phi\,\hat{\theta} + \cos\theta\,\hat{\phi}\right)$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{\cancel{r^2}}\cos\theta \times \cancel{r^2}\sin\theta\right) + \frac{\partial}{\partial\theta}(r^2\sin^2\theta\cos\phi)\right.$$

$$\left. + \frac{\partial}{\partial\phi}(r\cos\theta)\right]$$

$$= \frac{\cos\phi}{\sin\theta} \times 2\sin\theta\cos\theta = \boxed{2\cos\theta\cos\phi}$$

(iii) $\vec{\nabla} \cdot \left[(2\rho + \rho\sin^2\theta)\hat{\rho} + (\rho\sin\phi\cos\phi)\hat{\phi} + 3z\hat{z} \right)$

$$= \frac{1}{1 \times \rho \times 1} \left[\frac{\partial}{\partial \rho}(2\rho^2 + \rho^2\sin^2\theta) + \frac{\partial}{\partial \phi}(\rho\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z\rho) \right]$$

$$= \frac{1}{\rho} \left[4\rho + 2\rho\sin^2\theta + \frac{\rho}{\cancel{2}} \times \cancel{2}\cos(2\phi) + 3\rho \right]$$

$$= \frac{1}{\rho} \left[4\rho + 2\rho\sin^2\theta + \rho\cos(2\phi) + 3\rho \right]$$

$$= \boxed{2\sin^2\theta + \cos(2\phi) + 7}$$

(iv) $\vec{\nabla} \cdot ((\rho\sin\phi)\hat{\rho} + (\rho^2 z)\hat{\phi} + (z\cos\phi)\hat{z})$

$$= \frac{1}{1 \times \rho \times 1} \left[\frac{\partial}{\partial \rho}(\rho^2\sin\phi) + \frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\rho\cos\phi) \right]$$

$$= \frac{1}{\rho} \left[2\rho\sin\phi + \rho\cos\phi \right]$$

$$= \boxed{2\sin\phi + \cos\phi}$$

9. $G(\rho, \phi, z) = (-\cos^2\phi \tanh\rho)\,\hat{\rho} + \left[\sin 2\phi \left(\frac{\cosh\rho}{\cosh z}\right)^{1/\rho}\right]\hat{\phi}$

$$+ \; (\cos^2\phi \tanh\rho)\,\hat{z}$$

$$\vec{\nabla} \times \vec{G} = \frac{1}{1 \times \rho \times 1} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ G_\rho & \rho G_\phi & G_z \end{vmatrix}$$

$$\cancel{= \frac{1}{\rho}\left[\left(\frac{\partial}{\partial\phi}\left(\sin 2\phi \left(\frac{\cosh\rho}{\cosh z}\right)^{1/\rho}\right) - \right.\right.}$$

$$= \frac{1}{\rho}\Bigg[\left(\frac{\partial}{\partial \phi}\left(\cos^2\phi \tanh z\right) - \frac{\partial}{\partial z}\left(\sin 2\phi \ln\left(\frac{\cosh\rho}{\cosh z}\right)\right)\right)\hat{\rho}$$

$$- \rho\left(\frac{\partial}{\partial \rho}\left(\cos^2\phi \tanh z\right) - \frac{\partial}{\partial z}\left(-\cos^2\phi \tanh\rho\right)\right)\hat{\phi}$$

$$+ \left(\frac{\partial}{\partial \rho}\left(\sin 2\phi \ln\left(\frac{\cosh\rho}{\cosh z}\right)\right) - \frac{\partial}{\partial \phi}\left(-\cos^2\phi \tanh\rho\right)\right)\hat{z}\Bigg]$$

$$= \frac{1}{\rho}\Bigg[\left(+2\sin\phi\cos\phi \tanh z - \sin 2\phi \tanh z\right)\hat{\rho}$$

$$- \rho(0)\,\hat{\phi}$$

$$+ \left(\sin 2\phi \tanh\rho + \left(-2\sin\phi\cos\phi\tanh\rho\right)\right)\hat{z}\Bigg]$$

$$= \boxed{\vec{0}}$$

$\cancel{(\alpha^2+\beta^2)}$

10. $\vec{\nabla}\phi = \frac{1}{h_\alpha}\frac{\partial\phi}{\partial\alpha}\hat{\alpha} + \frac{1}{h_\beta}\frac{\partial\phi}{\partial\beta}\hat{\beta} + \frac{1}{h_\gamma}\frac{\partial\phi}{\partial\gamma}\hat{\gamma}$

$h_\alpha = h_\beta = \frac{1}{(\alpha^2+\beta^2)} \quad \left(\left|\frac{\partial\vec{r}}{\partial\alpha}\right| = \left|\frac{\partial\vec{r}}{\partial\beta}\right| = \frac{1}{\alpha^2+\beta^2} \right)$

$h_\gamma = 1$

$$\boxed{\therefore \vec{\nabla}\phi = (\alpha^2+\beta^2)\frac{\partial\phi}{\partial\alpha}\hat{\alpha} + (\alpha^2+\beta^2)\frac{\partial\phi}{\partial\beta}\hat{\beta} + \frac{\partial\phi}{\partial\gamma}\hat{\gamma}}$$

11. $F = \frac{x^2 y}{(x^2+y^2)^2}\hat{i} + \sqrt{x^2+y^2}\,\hat{j} + z(1-x^2-y^2)\hat{k}$

we know $x^2+y^2 = \rho^2$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$\therefore F = \frac{\cancel{\rho^3}\cos^2\phi\sin\phi}{\rho^4}\hat{i} + \rho\hat{j} + z(1-\rho^2)\hat{k}$

$\hat{i} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$

$\hat{j} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$

$$\therefore F = \frac{\cos^2\phi \sin\phi}{\rho}\left[\cos\phi\, \hat{\rho} - \sin\phi\, \hat{\phi}\right]$$

$$+ \rho\left[\sin\phi\, \hat{\rho} + \cos\phi\, \hat{\phi}\right] + z(1-\rho^2)\hat{k}$$

$$= \left[\frac{\cos^3\phi \sin\phi}{\rho} + \rho\sin\phi\right]\hat{\rho} + \left[\rho\cos\phi - \frac{\cos^2\phi\sin^2\phi}{\rho}\right]\hat{\phi}$$

$$+ z(1-\rho^2)\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{1\times\rho\times 1}\left[\frac{\partial}{\partial\rho}\left[\cos^3\phi\sin\phi + \rho^2\sin\phi\right] + \frac{\partial}{\partial\phi}\left[\rho\cos\phi - \frac{\cos^2\phi\sin^2\phi}{\rho}\right] + \frac{\partial}{\partial z}\left(z(\rho-\rho^3)\right)\right]$$

$$= \frac{1}{\rho}\left[2\rho\sin\phi - \rho\sin\phi - \frac{1}{\rho}\times\sin 2\phi\cos 2\phi + \rho - \rho^3\right]$$

$$= \sin\phi - \frac{\sin 2\phi\cos 2\phi}{\rho^2} + 1 - \rho^2$$

$$= \boxed{1 + \sin\phi - \rho^2 - \frac{\sin 4\phi}{2\rho^2}}$$

12. $\vec{A} = 2y\,\hat{i} - 3\hat{j} + 2z\hat{k}$

$$y = \rho\sin\phi$$
$$\hat{i} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$
$$\hat{j} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$

$$\therefore \vec{A} = 2\rho\sin\phi(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}) - 3(\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}) + 2z\hat{k}$$

$$= \boxed{[\rho\sin(2\phi) - 3\sin\phi]\hat{\rho} - [2\rho\sin^2\phi + 3\cos\phi]\hat{\phi} + 2z\hat{k}}$$

13.

$$x = 1$$
$$y = 1$$
$$z = 1$$

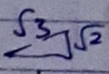
(A) cylindrical:

$$\left.\begin{aligned}\rho\cos\phi &= 1\\ \rho\sin\phi &= 1\end{aligned}\right\} \tan\phi = 1 \quad \boxed{\begin{aligned}\phi &= 45^\circ\\ \rho &= \sqrt{2}\end{aligned}}$$

$\therefore$ In cylindrical, coordinates are $\boxed{(\sqrt{2}, \pi/4, 1)}$

(B) spherical:

$$\left.\begin{aligned}r\sin\theta\cos\phi &= 1\\ r\sin\theta\sin\phi &= 1\end{aligned}\right\} \phi = 45^\circ$$
$$r\cos\theta = 1$$



$$\left.\begin{aligned}r\sin\theta &= \sqrt{2}\\ r\cos\theta &= 1\end{aligned}\right\} \tan\theta = \sqrt{2}$$
$$\theta = \tan^{-1}(\sqrt{2})$$
$$r = \sqrt{3}$$

$\therefore$ In spherical, coordinates are

$$\boxed{(\sqrt{3}, \tan^{-1}(\sqrt{2}), \pi/4)}$$

14. $F = \rho^3 \sin\phi$

(a) $\vec{\nabla} F = \frac{1}{1} \times \frac{\partial}{\partial \rho} (\rho^3 \sin\phi) \hat{\rho} + \frac{1}{\rho} \times \frac{\partial}{\partial \phi} (\rho^3 \sin\phi) \hat{\phi}$

$+ \frac{1}{1} \times \frac{\partial}{\partial z} (\rho^3 \sin\phi) \hat{z}$

$= (3\rho^2 \sin\phi) \hat{\rho} + (\rho^2 \cos\phi) \hat{\phi}$

$= \boxed{\rho^2 [3\sin\phi \, \hat{\rho} + \cos\phi \, \hat{\phi}]}$

(b) $\rho = (x^2+y^2)^{1/2}$

$\sin\phi = \frac{y}{(x^2+y^2)^{1/2}}$

$\therefore F = \rho^3 \sin\phi = (x^2+y^2)^{3/2} \frac{y}{(x^2+y^2)^{1/2}}$

$= y(x^2+y^2)$

$= \boxed{x^2 y + y^3}$

$\vec{\nabla} F = \frac{\partial}{\partial x} (x^2 y + y^3) \hat{i} + \frac{\partial}{\partial y} (x^2 y + y^3) \hat{j}$

$= \boxed{(2xy) \hat{i} + (x^2 + 3y^2) \hat{j}}$

$x = \rho \cos\phi \qquad \hat{i} = \cos\phi \, \hat{\rho} - \hat{\phi} \sin\phi$

$y = \rho \sin\phi \qquad \hat{j} = \sin\phi \, \hat{\rho} + \cos\phi \, \hat{\phi}$

$\therefore \vec{\nabla} F = 2\rho^2 \sin\phi \cos\phi [\cos\phi \, \hat{\rho} - \sin\phi \, \hat{\phi}]$

$+ [\rho^2 \cos^2\phi + 3\rho^2 \sin^2\phi]$
$[\sin\phi \, \hat{\rho} + \cos\phi \, \hat{\phi}]$

$$= \begin{bmatrix} 2\rho^2 \sin\phi \cos^2\phi & + \rho^2 \sin\phi \cos^2\phi \\ & + 3\rho^2 \sin^3\phi \end{bmatrix} \hat{\rho}$$

$$+ \begin{bmatrix} -2\rho^2 \sin^2\phi \cos\phi & + \rho^2 \cos^3\phi \\ & + 3\rho^2 \sin^2\phi \cos\phi \end{bmatrix} \hat{\phi}$$

$$= [3\rho^2 \sin\phi (\sin^2\phi + \cos^2\phi)] \hat{\rho} + [\rho^2 \cos\phi (\sin^2\phi + \cos^2\phi)] \hat{\phi}$$

$$= 3\rho^2 \sin\phi \, \hat{\rho} + \rho^2 \cos\phi \, \hat{\phi}$$

$$= \boxed{\rho^2 [3\sin\phi \, \hat{\rho} + \cos\phi \, \hat{\phi}]}$$

$\therefore$ 10 is same in both systems.