NOTE : where ever multiple cancellations are done, they are demarkated by different direction (\, /, X etc)

1. (a) $\bar{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$$\overline{\text{curl } A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\text{div}(\overline{\text{curl } A}) = \cancel{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} A_z\right)} - \cancel{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z} A_y\right)}$$

$$+ \cancel{\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} A_x\right)} - \cancel{\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} A_z\right)}$$

$$+ \cancel{\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} A_y\right)} - \cancel{\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} A_x\right)}$$

$$= \boxed{0}$$

1.(b) $\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$\nabla \times (\nabla \phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \hat{i}\left(\cancel{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right)} - \cancel{\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)}\right)$$

$$+ \hat{j}\left(\cancel{\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)} - \cancel{\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)}\right)$$

$$+ \hat{k}\left(\cancel{\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)} - \cancel{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)}\right)$$

$$= \boxed{\vec{0}}$$

2. $\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$

$\text{div}\, F = \vec{\nabla}\cdot\vec{F}$

$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-x^2y^2)$

$= \boxed{2xy + xz}$

3. $\overrightarrow{\text{grad}\, F} = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}$

$\overrightarrow{\text{curl}(\text{grad}\, F)} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ \partial F/\partial x & \partial F/\partial y & \partial F/\partial z \end{vmatrix}$

We know $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial y}\right)$

$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x}\right)$

$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$

$\therefore \nabla \times \nabla F = 0$

4. $\text{curl}\, F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz^2 & xy & yz \end{vmatrix}$

$= (z)\hat{i} - (-2yz)\hat{j} + (y - z^2)\hat{k}$

$\text{div}(\overrightarrow{\text{curl}\, f}) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(+2yz) + \frac{\partial}{\partial z}(y - z^2)$

$= 0 + 2z - 2z$

$= \boxed{0}$

5. A vector field A is a gradient field if $\overrightarrow{\text{curl} A} = \vec{0}$

$$\overrightarrow{\text{curl} F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-2y-2y)$$

$$\neq \boxed{\vec{0}}$$

$\therefore$ $\vec{F}$ is NOT a gradient field

6. $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$\vec{F} = \nabla f \times \nabla g$

$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

$$= \frac{\partial}{\partial x}\left(\cancel{\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial z}} - \cancel{\frac{\partial f}{\partial z} \cdot \frac{\partial g}{\partial y}}\right)$$

$$+ \frac{\partial}{\partial y}\left(\cancel{\frac{\partial f}{\partial z} \frac{\partial g}{\partial x}} - \cancel{\frac{\partial f}{\partial x} \frac{\partial g}{\partial z}}\right)$$

$$+ \frac{\partial}{\partial z}\left(\cancel{\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y}} - \cancel{\frac{\partial f}{\partial y} \frac{\partial g}{\partial x}}\right)$$

7. $\bar{F}$ is conservative if $\overline{\text{curl} F} = 0$

$\overline{\text{curl} F} =$ ~~$\hat{i} \quad \hat{j} \quad \hat{k}$~~

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(4y^2 + \frac{3x^2y}{z^2}\right) & \left(8xy + x^3/z^2\right) & \left(11 - \frac{2x^3y}{z^3}\right) \end{vmatrix}$$

$$= \hat{i}\left(-\frac{2x^3}{z^3} + \frac{2x^3}{z^3}\right) - \hat{j}\left(-\frac{6x^2y}{z^3} + \frac{6x^2y}{z^3}\right)$$

$-\frac{2}{x^3}$

$$+ \hat{k}\left(8y + \frac{3x^2}{z^2} - 8y - \frac{3x^2}{z^2}\right)$$

$$= \vec{0}$$

$\boxed{\therefore \bar{F} \text{ is CONSERVATIVE}}$

~~8/ (a) $\vec{\nabla} f = x^3 + 4y$ : for this to be true~~
~~$\text{curl}(\vec{\nabla} f) = 0$~~

~~$\hat{i} \quad \hat{j} \quad \hat{k}$~~
~~$\partial/\partial x \quad \partial/\partial y$~~

8. (a) is wrong since gradient is a VECTOR

(b) is wrong since divergence is a SCALAR

(c) is wrong since curl is a VECTOR but divergence is a SCALAR

$\therefore$ (a), (b), (c) are all <u>WRONG</u>

9. Volume $= \left| [\overline{OA} \quad \overline{OB} \quad \overline{OC}] \right|$

$$= \begin{vmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 5 & 4 \end{vmatrix}$$

$$= \left| 3(-6) + 1(-2) \right| = |-20| = \boxed{20}$$

10. (i) $\nabla [\ln(1+x^2+2y^2)] = \boxed{\left(\frac{2x}{1+x^2+2y^2}\right)\hat{i} + \left(\frac{4y}{1+x^2+2y^2}\right)\hat{j}}$

(ii) $\nabla \left(\frac{1}{x^2+y^2+z^2}\right)$

$$= \boxed{\frac{-2}{x^2+y^2+z^2} (x\hat{i} + y\hat{j} + z\hat{k})}$$

11. directional derivative

$$= \frac{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{3}$$

gradient

$$= \frac{\left[(y+z)(1) + (x+z)(2) + (x+y)(2)\right]}{3}$$

$$= [4x + 3y + 3z]/3$$

At $(1,2,0)$,

$$= \frac{(4+6+0)}{3} = \boxed{\frac{10}{3}}$$

12. $\overrightarrow{\text{gradient}}$ is the normal.

$\therefore \bar{\nabla}(x^2 - 8y^2 + z^2)$

$= (2x)\hat{i} - (16y)\hat{j} + (2z)\hat{k}$

$\therefore$ Unit Vector at $(8,1,4)$ is

$$\frac{(16)\hat{i} - (16)\hat{j} + 8\hat{k}}{24}$$

$$= \boxed{\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}}$$

13. $\vec{\nabla}(\bar{a} \cdot \bar{b} \times \vec{r})$

$\cancel{= \frac{\partial}{\partial x}(\vec{r} \cdot (\bar{a} \times \bar{b}))\hat{i}}$

Let $\bar{a} \times \bar{b} = \bar{c} = A\hat{i} + B\hat{j} + C\hat{k}$

$\therefore \vec{\nabla} = \frac{\partial}{\partial x}(Ax + By + Cz)\hat{i} + \frac{\partial}{\partial y}(Ax + By + Cz)\hat{j}$

$+ \frac{\partial}{\partial z}(Ax + By + Cz)\hat{k}$

$= \boxed{A\hat{i} + B\hat{j} + C\hat{k}} = \boxed{\bar{a} \times \bar{b}}$

14. $\operatorname{div}[\vec{r} f(r)] = \frac{\partial}{\partial x}(xf(r)) + \frac{\partial}{\partial y}(yf(r) + \frac{\partial}{\partial z}(zf(r))$

$$= 3f(r) + x \cdot \frac{\partial f(r)}{\partial x} + y \cdot \frac{\partial f(r)}{\partial y} + z \cdot \frac{\partial f(r)}{\partial z}$$

$$= 3f(r) + x \cdot \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial x} + y \cdot \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial y} + z \cdot \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{}$$

14. (contd): $\operatorname{div}[\vec{r} f(r)] = 3f(r) + \frac{x^2}{r} f'(r) + \frac{y^2}{r} f'(r)$

$+ \frac{z^2}{r} f'(r)$

$= \boxed{3f(r) + r f'(r)}$

15. $\text{curl}(f \vec{\nabla} f)$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$f \vec{\nabla} f = f\frac{\partial f}{\partial x}\hat{i} + \frac{f\partial f}{\partial y}\hat{j} + f\frac{\partial f}{\partial z}\hat{k}$$

$$\text{curl}(f\vec{\nabla} f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f\frac{\partial f}{\partial x} & f\frac{\partial f}{\partial y} & f\frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\cancel{\frac{\partial}{\partial y}\left(\frac{f\partial f}{\partial z}\right)} - \cancel{\frac{\partial}{\partial z}\left(\frac{f\partial f}{\partial y}\right)}\right)$$

$$+ \hat{j}\left(\cancel{\frac{\partial}{\partial z}\left(\frac{f\partial f}{\partial x}\right)} - \cancel{\frac{\partial}{\partial x}\left(\frac{f\partial f}{\partial z}\right)}\right)$$

$$+ \hat{k}\left(\cancel{\frac{\partial}{\partial x}\left(\frac{f\partial f}{\partial y}\right)} - \cancel{\frac{\partial}{\partial y}\left(\frac{f\partial f}{\partial x}\right)}\right)$$

$$= \boxed{0}$$

16. $\psi = \dfrac{\bar{m} \cdot \bar{r}}{r^3} - \bar{r}$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad [r = (x^2+y^2+z^2)^{1/2}]$

Let $\bar{m} = \bar{A}\hat{i} + B\hat{j} + C\hat{k}$

$\therefore \psi = \dfrac{Ax+By+Cz}{\cancel{r^3}\,(x^2+y^2+z^2)^{3/2}}$ $\cancel{(x\hat{i} + y\hat{j} + z\hat{k})}$

$\nabla \psi = \dfrac{\partial \psi}{\partial x}\hat{i} + \dfrac{\partial \psi}{\partial x}\hat{j} + \dfrac{\partial \psi}{\partial z}\hat{k}$

$= \left[\dfrac{A(x^2+y^2+z^2)^{3/2} - (Ax+By+Cz)(3/\cancel{2})(x^2+y^2+z^2)^{1/2}(\cancel{2}x)}{(x^2+y^2+z^2)^3}\right]$

$= \left[\dfrac{A}{(x^2+y^2+z^2)^{3/2}} - \dfrac{3(\bar{m}\cdot\bar{r})(x)}{(x^2+y^2+z^2)^{5/2}}\right]\hat{i}$

$+ \left[\dfrac{B}{(x^2+y^2+z^2)^{3/2}} - \dfrac{3(\bar{m}\cdot\bar{r})(y)}{(x^2+y^2+z^2)^{5/2}}\right]\hat{j}$

$+ \left[\dfrac{C}{(x^2+y^2+z^2)^{3/2}} - \dfrac{3(\bar{m}\cdot\bar{r})(y)}{(x^2+y^2+z^2)^{5/2}}\right]\hat{k}$

$= \dfrac{(A\hat{i} + B\hat{j} + C\hat{k})}{(x^2+y^2+z^2)^{3/2}} - \dfrac{3(\bar{m}\cdot\bar{r})(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{5/2}}$

$= \boxed{\dfrac{\vec{m}}{r^3} - \dfrac{3(\bar{m}\cdot\bar{r})\vec{r}}{r^5}}$

17. (i) $\text{div}\,\bar{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

$= 2xy - (0) + 0$

$= \boxed{2xy}$

$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & 3x-z^3 & 4y^2 \end{vmatrix}$

$= \hat{i}(8y+3z^2) - \hat{j}(0-0)$

$+ \hat{k}(3-x^2)$

$x^3y$

$= (8y+3z^2)\hat{i} + (3-x^2)\hat{k}$

(ii) $\text{div}\,\bar{F} = (3) + (x^3) - (1)$

$= \cancel{3+x^3-3}\ 2 + x^3$

$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (3x+2z^2) & \dfrac{x^3y^2}{\cancel{xy}} & 7x-z \end{vmatrix}$

$\cancel{= \hat{i}(0) - \hat{j}(7-4z) + \hat{k}(3x^2y)}$

$\cancel{= (4z-7)\hat{j} + 3x^2y\hat{k}}$

$= (4z-7)\hat{j} + 3x^2y\,\hat{k}$

18. $\bar{a} = f(r)\vec{r} = f(r)\,[x\hat{i} + y\hat{j} + z\hat{k}]$

For $\bar{a}$ to be solenoidal, $\text{div}\,\bar{a} = 0$

$$\text{div}\,\bar{a} = \frac{\partial \bar{a}}{\partial x} + \cancel{\frac{\partial \bar{a}}{\partial y}}$$

$$= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$= x.\frac{\partial f(r)}{\partial x} + f(r) + y.\frac{\partial f(r)}{\partial y} + f(r)$$

$$+ z.\frac{\partial f(r)}{\partial z} + f(r) = 0$$

$$\frac{\partial f(r)}{\partial x}x + \frac{\partial f(r)}{\partial y}y + \frac{\partial f(r)}{\partial z}.z = -3f(r)$$

From (14):

$$f'(r).r = -3f(r)$$

$$\int \frac{\partial f(r)}{f(r)} = \int -\frac{3\,dr}{r}$$

$$\ln[f(r)] = -3\ln(r) + \ln c$$

$$\therefore \boxed{f(r) = \frac{c}{r^3}}$$

19. $\text{div}\left(\bar{A} \times \frac{\vec{r}}{r^3}\right) = \bar{\nabla}\cdot\left(\bar{A} \times \frac{\vec{r}}{r^3}\right)$

$= \cancel{\frac{\vec{r}}{r^3}}\cdot(\cancel{\nabla}\times\bar{A}) - \bar{A}\cancel{\cdot}\left(\bar{\nabla}\cancel{\times\frac{\vec{r}}{r^3}}\right)$

Let $\bar{A} = L\hat{i} + M\hat{j} + N\hat{k}$ $\quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\cancel{\bar{\nabla}\cdot\left(\bar{A}\times\frac{\vec{r}}{r^3}\right) =}$ $\cancel{\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{(Mz-Ny)}{(x^2+y^2+z^2)^{3/2}} & & \end{vmatrix}}$

$\bar{\nabla}\cdot\left(\bar{A}\times\frac{\vec{r}}{r^3}\right) =$

$= \frac{\partial}{\partial x}\left(\frac{Mz-Ny}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{Nx-Lz}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{Ly-Mx}{(x^2+y^2+z^2)^{3/2}}\right)$

$= \frac{-3(Mz-Ny)x - 3(Nx-Lz)y - 3(Ly-Mx)z}{(x^2+y^2+z^2)^{5/2}}$

20. $\nabla e^{r^2} = \frac{\partial}{\partial x} e^{x^2+y^2+z^2}\hat{i} + \frac{\partial}{\partial y} e^{x^2+y^2+z^2}\hat{j}$

$+ \frac{\partial}{\partial z} e^{x^2+y^2+z^2}\hat{k}$

$= \cancel{e^{x^2+y^2+z^2}(2x)}$

$= \boxed{2e^{x^2+y^2+z^2}(x\hat{i} + y\hat{j} + z\hat{k})}$

$= \frac{3}{(x^2+y^2+z^2)^{5/2}}\left[\cancel{Nxy} - \cancel{Mxz} + \cancel{Lyz} - \cancel{Nxy} + \cancel{Mxz} - \cancel{Lyz}\right]$

$= \boxed{0}$