1. Gauss Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots a_{nn}x_n = b_n$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Augmented Matrix : $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$

Part I : Forward Elimination

[we reduce A to upper Δr matrix]

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{12} & a_{22} & a_{13} & \\ \vdots & \vdots & a_{23} & \\ a_{1n} & a_{2n} & a_{3n} & \dots \end{bmatrix}$$

- Pivot
- Eliminated.

Step 1a (a_{11} is pivot) :

Factor : $\frac{a_{21}}{a_{11}}$

Row 2 transf. : $R_2 \to R_2 - (\text{Factor} \times R_1)$

1b (a_{11} is pivot)

Factor : $\frac{a_{31}}{a_{11}}$

Loop(steps)
↳ Loop(rows) // Pivot is set
  ↳ Loop(cols) // factor is set

Part 2 : Back Substitution

Gauss. Jordan Elimination :

→ Reduce A to identity matrix ←

L.U. Decomposition

$$[A] = [L][U]$$

$$[L] = \begin{bmatrix} l_{11} & & & \\ l_{12} & l_{22} & & \\ l_{13} & l_{23} & l_{33} & \\ l_{1n} & l_{2n} & \dots & l_{nn} \end{bmatrix}$$

$$[U] : \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & \vdots \\ & 1 & u_{23} & u_{24} & \vdots \\ & & 1 & a_{34} & \vdots \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$[L][U][X] = [B]$$

$$[L][Y] = [B] \Rightarrow [U][X] = [Y]$$

↳ Forward Subst. ↳ Back Subst.

[L] :

$$l_{i1} = a_{i1}$$

$$l_{i2} = a_{i2} - l_{i1}u_{12}$$

$$l_{i3} = a_{i3} - [l_{i1}u_{13} + l_{i2}u_{23}]$$

$$l_{i4} = a_{i4} - [l_{i1}u_{14} + l_{i2}u_{24} + l_{i3}u_{34}]$$

$$\boxed{l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}}$$

[U] :

$$u_{1j} = \frac{a_{1j}}{l_{11}}$$

$$u_{2j} = \frac{a_{2j} - l_{21}u_{1j}}{l_{22}}$$

$$u_{3j} = \frac{a_{3j} - [l_{31}u_{1j} + l_{32}u_{2j}]}{l_{33}}$$

$$\boxed{u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}}{l_{ii}}}$$

→ way to find L & U :

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

Step 1

Row 2 : factor $= \frac{p}{a}$

$$\begin{bmatrix} a & b & c \\ p/a & q - \frac{bp}{a} & r - cp/a \\ x & y & z \end{bmatrix}$$

Row 3 : Factor = $\frac{x}{a}$

$$\begin{bmatrix} a & b & c \\ \frac{p}{a} & q-\frac{pb}{a} & r-\frac{cp}{a} \\ \frac{x}{a} & y-\frac{bx}{a} & z-\frac{cx}{a} \end{bmatrix}$$

Step 2 :

Row 3 : Factor = $\frac{\left(y-\frac{bx}{a}\right)}{\left(q-\frac{bp}{a}\right)}$

$$\begin{bmatrix} a & b & c \\ p/a & q-bp/a & r-cp/a \\ x/a & \frac{(y-bx/a)}{(q-bp/a)} & \cdots \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ p/a & 1 & \\ x/a & \frac{(y-bx/a)}{(q-bp/a)} & 1 \end{bmatrix} \quad U = \begin{bmatrix} a & b & c \\ & q-bp/a & r-cp/a \\ & & \cdots \end{bmatrix}$$

(ex) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 7 & 2 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 7 & -12 & -19 \end{bmatrix}$

factor = 12

$\begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 7 & -12 & 5 \end{bmatrix}$

$\hookrightarrow \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 7 & 12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ & -1 & -2 \\ & & 5 \end{bmatrix}$

Tridiagonal systems :

$$\begin{bmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & a_{32} & a_{33} & a_{n-2,n-1} & \\ & & a_{43} & \cdot \; a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{nn} \end{bmatrix}$$

$\Downarrow$

$$\begin{bmatrix} d_1 & u_1 & & & \\ l_2 & d_2 & u_2 & & \\ & l_3 & d_3 & \ddots & \\ & & \ddots & \ddots & u_{n-1} \\ & & & l_n & d_n \end{bmatrix}$$

$$\boxed{\begin{aligned} x_i &= \frac{b_i - u_i x_{i+1}}{d_i} \\ x_n &= b_n / d_n \end{aligned}}$$

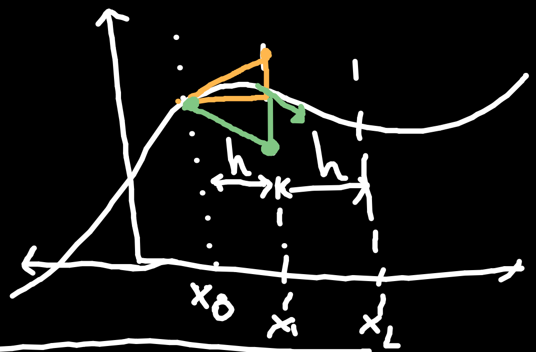
(Check lec 29)

→ Ordinary Differential Eqns:

⇒ First-order ODEs :

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

(1) Eulers Explicit & Implicit Method:



Explicit

$$y_1 = y_0 + hf(x_0, y_0)$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$\vdots$$
$$y_{i+1} = y_i + hf(x_i, y_i)$$

Implicit

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

→ for implicit, we take

$$g(y_{i+1}) = y_{i+1} - y_i - hf(x_{i+1}, y_{\_})$$

and equate it to 0 ⇒ then we Newton-Raphson method.

$$y_{i+1}^{new} = y_{i+1}^{old} - \frac{g(y_{i+1}^{old})}{g'(y_{i+1}^{old})}$$

↳ Modified eulers method $[y_{i+1}^{e} = y_i + hf(x_i, y_i)]$

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{e})\right]$$

⇒ Runge-Kutta Methods :

- $$y_{i+1} = y_i + h\frac{dy}{dx}\Big|_{x_i} + \frac{h^2}{2}\frac{d^2y}{dx^2}\Big|_{x_i} + O(h^3)$$

$$\frac{dy}{dx}\Big|_{x_i} = f(x_i, y_i) = m_0$$

$$\therefore \frac{d^2y}{dx^2}\Big|_{x_i, y_i} = \frac{\partial f}{\partial x} + \frac{dy}{dx}\cdot\frac{\partial f}{\partial y}$$

$$= \left(\frac{\partial f}{\partial x} + m_0\frac{\partial f}{\partial y}\right)\Big|_{x_i, y_i}$$

$$\rightarrow \boxed{y_{i+1} = y_i + hm_0 + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + m_0\frac{\partial f}{\partial y}\right)}$$

$$= y_i + h(w_1 m_1 + w_2 m_2)$$ [2nd order RK

$$\rightarrow y_{i+1} = y_i + h(w_1 m_1 + w_2 m_2)$$

$$m_1 = f(x_1, y_1^p)$$

$$= f((x_i + a_1 h), (y_i + b_1 m_0 h))$$

$$x_1 = x_i + a_1 h$$

$$y_1^p = y_i + (a_1 h) m_0$$

$$= y_i + b_1 h m$$

$$= \left. f(x_i, y_i) + (a_1 h) \frac{\partial f}{\partial x} \right|_{x_i} + \left. (b_1 h m_0) \frac{\partial f}{\partial y} \right|_{x_i}$$

$$m_2 = m_0 + \left. (a_2 h) \frac{\partial f}{\partial x} \right|_{x_i} + \left. (b_2 h m_0) \frac{\partial f}{\partial y} \right|_{x_i}$$

We get,

$$y_{i+1} = y_i + h m_0 (w_1 + w_2) + \left(h^2 \left. \frac{\partial f}{\partial x} \right|_{x_i} \right) (a_1 w_1 + a_2 w_2)$$

$$+ \left(h^2 m_0 \left. \frac{\partial f}{\partial x} \right|_{x_i} \right) (b_1 w_1 + b_2 w_2) + O(h^3)$$

Acc to Taylor series,

$$y_{i+1} = y_i + h m_0 + \frac{h^2}{2} \left. \frac{\partial f}{\partial x} \right|_{x_1} + \frac{h^2}{2} m_0 \left. \frac{\partial f}{\partial y} \right|_{x_i}$$

Comparing,

$$w_1 + w_2 = 1$$

$$a_1, b_1 = 0$$

$$a_2 w_2, b_2 w_2 = \frac{1}{2}$$

In modified Eulers,

$$w_1 = w_2 = \frac{1}{2}$$

$$a_2 = b_2 = 1.$$

$$\therefore y_{i+1} = y_i + h\left(\frac{1}{2} m_1 + \frac{1}{2} m_2\right)$$

$$m_1 = f(x_i, y_i) = m_0$$

$$m_2 = f(x_i + h, y_i + h m_0)$$

↳ Midpoint Method :

$$w_1 = 0, \quad w_2 = 1$$

$$a_2 = b_2 = \frac{1}{2}.$$

$$y_{i+1} = y_i + h(m_2)$$

$$m_2 = f((x_i + a_2 h), (y_i + b_2 h m_0))$$

$$= f\left(\left(x_i + \frac{h}{2}\right), \left(y_i + \frac{h m_0}{2}\right)\right)$$

↳ Ralton's Method :

$$w_1 = \frac{1}{3}, \quad w_2 = \frac{2}{3}$$

$$a_2 = \frac{3}{4}, \quad b_2 = \frac{3}{4}$$

$$y_{i+1} = y_i + h\left(\frac{m_1}{3} + \frac{2m_2}{3}\right)$$

$$m_1 = f((x_i),(y_i)) = m_0$$

$$m_2 = f\left(\left(x_i + \frac{3h}{4}\right), \left(y_i + \frac{3m_0h}{4}\right)\right)$$

↳ 3rd order R-K methods :

$$y_{i+1} = y_i + h(w_1m_1 + w_2m_2 + w_3m_3)$$

$$m_1 = f(x_i, y_i) = m_0$$

$$m_2 = f((x_i + a_2h), (y_i + b_{21}m_0h))$$

$$m_3 = f((x_i + a_3h), (y_i + b_{31}m_1h + b_{32}m_2h))$$

Classical RK-3 :

$$w_1 = \frac{1}{6}, \quad w_2 = \frac{4}{6}, \quad w_3 = \frac{1}{6}$$

$$a_2 = \frac{1}{2} = b_{21}$$

$$a_3 = 1, \quad b_{31} = -1, \quad b_{32} = 2$$

$$y_{i+1} = y_i + h\left(\frac{m_1}{6} + \frac{4}{6}m_2 + \frac{1}{6}m_3\right)$$

$$m_1 = m_0$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{hm_0}{2}\right)$$

$$m_3 = f(x_i + h, y_i - hm_1 + 2hm_2)$$

⇒ Classical RK-4 :

$$y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

⇒ System of ODEs :

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots y_n) \Rightarrow y_1(x_0) = y_{10}$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2 \dots y_n) \Rightarrow y_{n0}$$

Point A $[x_A = x_i]$:

$y_{1A} = y_{1A}$, $y_{nA} = y_{ni}$

$m_{nA} = f(x_i, y_{1A}, y_{2A}, \dots y_{nA})$

Point B $[x_B = x_i + h/2]$

$y_{1B} = y_{1i} + \frac{1}{2} h m_{1A}$

$y_{nB} = y_{ni} + \frac{1}{2} h m_{nA}$

$m_{1B} = f_1(x_B, y_{1B}, y_{2B} \dots y_{nB})$

$m_{nB} = f_n(x_B, y_{1B}, \dots y_{nB})$

Point C $[x_C = x_i + h/2]$

$y_{1C} = y_{1i} + \frac{1}{2} h m_{1B}$

$m_{nC} = f_n(x_C, y_{1C}, y_{2C} \dots y_{nC})$

Point D $[x_D = x_i + h]$

$y_{1D} = y_{1i} + h m_{1C}$

$y_{nD} = y_{ni} + h m_{nC}$

$m_{1D} = f_1(x_D, y_{1D}, y_{2D} \dots y_{nD})$

$m_{nD} = f_n(x_D, y_{1D}, \dots y_{nD})$

$$\boxed{\begin{aligned} y_{1,i+1} &= y_{1,i} + \frac{h}{6}(m_{1A} + 2m_{1B} + 2m_{1C} + m_{1D}) \\ y_{n,i+1} &= y_{n,i} + \frac{h}{6}(m_{nA} + 2m_{nB} + 2m_{nC} + m_{nD}) \end{aligned}}$$

System of ODEs :

$y' = f(x, y, z)$ $\quad z' = g(x, y, z)$

$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)$

$K_1 = hf(x_i, y_i, z_i) = h m_y$

$K_2 = hf(x_i + h, y_i + h m_y, z_i + h m_z)$

$z_{i+1} = z_i + \frac{1}{2}(L_1 + L_2)$

$L_1 = hg(x_i, y_i, z_i) = h m_z$

$L_2 = hg(x_i + h, y_i + h m_y, z_i + h m_z)$

Using 4^{th} order R-K:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Same as before.

$$\begin{cases} k_1 = hf(x_i, y_i) \\ k_2 = hf(x_i + h/2, y_i + \frac{1}{2}k_1) \\ k_3 = hf(x_i + h/2, y_i + \frac{1}{2}k_2) \\ k_4 = hf(x_i + h, y_i + k_3) \end{cases}$$

R.K. Summary:

⇒ Second Order:

$$\text{Taylor}: y_{i+1} = y_i + h\overbrace{f(x_i, y_i)}^{m_0} + \frac{h^2}{2}\left(\frac{\partial f}{\partial x}\bigg|_{x_i} + m_0 \frac{\partial f}{\partial y}\bigg|_{x_i}\right)$$

$$\text{RK}: y_{i+1} = y_i + hm_0(w_1 + w_2) + (a_1 w_1 + a_2 w_2)\frac{h^2}{2}\frac{\partial f}{\partial x} + (b_1 w_1 + b_2 w_2)\frac{h^2}{2} m_0 \frac{\partial f}{\partial y}$$

$w_1 + w_2 = 1$

$a_1 = b_1 = 0$

$a_2 w_2 = b_2 w_2 = 1/2$

Modified Eulers: $w_1 = 1/2$, $w_2 = 1/2$, $a_2 = b_2 = 1$.

$$y_{i+1} = y_i + h\left(\frac{m_1}{2} + \frac{m_2}{2}\right)$$

$$m_1 = f(x_i, y_i)$$

$$m_2 = f(x_i + h, y_i + m_0 h)$$

Third order:

$$y_{i+1} = y_i + \frac{h}{6}(m_1 + 4m_2 + m_3)$$

$$m_1 = m_0 : f(x_i, y_i)$$

$$m_2 : f\left(x_i + h/2, y_i + \frac{hm_0}{2}\right)$$

$$m_3 = f(x_i + h, y_i - m_1 h + 2m_2 h)$$

Fourth Order:

$$y_{i+1} = y_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$m_1 = m_0$

$m_2 = f(x_i + h/2,\ y_i + m_0 h/2)$

$m_3 = f(x_i + h/2,\ y_i + m_2 h/2)$

$m_4 = f(x_i + h,\ y_i + m_3)$

→ Shooting Methods: [higher order ODEs]

take $\frac{dy}{dx} = y_2$ @ Solve simultaneous linear ODEs. using modified Euler method.

(6) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ $y(0) = 1$, $y(1) = 0$

$\frac{dy_1}{dx} = y_2 = f_1(y_1, y_2, x) \Rightarrow y_1(0) = 1$

$\frac{dy_2}{dx} = 4y_2 - 3y_1 = f_2(x, y_1, y_2)$
$\Rightarrow y_2(0) = 0$

$h = 0.5$:

$y_1(0.5) = y_1(0) + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$m_1 = f_1(x_0, y_0, y_{20})$
$= 0$

$m_2 = f_1(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, y_{20} +$

b) $\frac{d^2y}{dx} - \frac{4dy}{dx} + 3y = 0$ $\quad y(0)=1$, $y(1)=0$

$\frac{dy_1}{dx} = y_2 = f_1(y_1, y_2, x) \Rightarrow y_1(0)=1$

$\frac{dy_2}{dx} = 4y_2 - 3y_1 = f_2(x, y_1, y_2)$
$\Rightarrow y_2(0)=0$

$\underline{h = 0.5}$:

$y_1(0.5) = y_1(0) + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$m_1 = f_1(x_0, y_0, y_{20})$
$= 0$

$m_2 = f_1\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, y_{20} + \frac{hM_1}{2}\right)$
$= f_1(0.25, 1, -0.75)$
$= -0.75$

$m_3 = f_1\left(x_0 + \frac{h}{2}, y_{10} + \frac{m_2 h}{2}, y_{20} + \frac{M_2 h}{2}\right)$
$= -0.5$

$y_2(0.5) = y_2(0) + \frac{1}{6}(M_1 + 2M_2 + 2M_3 + M_4)$

$M_1 = f_2(0, 1, 0)$
$= -3$

$M_2 = f_2\left(x_0 + \frac{h}{2}, y_{10} + \frac{m_1 h}{2}, y_{20} + \frac{M_1 h}{2}\right)$
$= f_2(0.25, 1, -0.75)$
$= 4(-0.75) - 3$
$= -6$

$$M_4 = f_1(x_0 + h,\ Y_{10} + m_3 h,\ Y_{20} + M_3 h)$$

$$= f_1(0.25,\ 0.\cancel{35},\ -4.21875)$$

$$= -4.21875$$

$$y_2(0.5) = 1 + \frac{0.5}{6}(-1.5 - 1 - 4.21875)$$

$$= \boxed{0.44}$$

$$M_3 = f_2\left(X_0 + \frac{h}{2},\ Y_{10} + \frac{m_2 h}{2},\ Y_{20} + \frac{M_2 h}{2}\right)$$

$$= f_2(0.25,\ 0.8125,\ -1.5)$$

$$= 4(-1.5) - 3(0.8125)$$

$$= -8.4375$$

$$M_4 = f_2(0.25,\ 0.5,\ -4.21875)$$

$$= -18.375$$