Hilbert’s paradox of the Grand Hotel

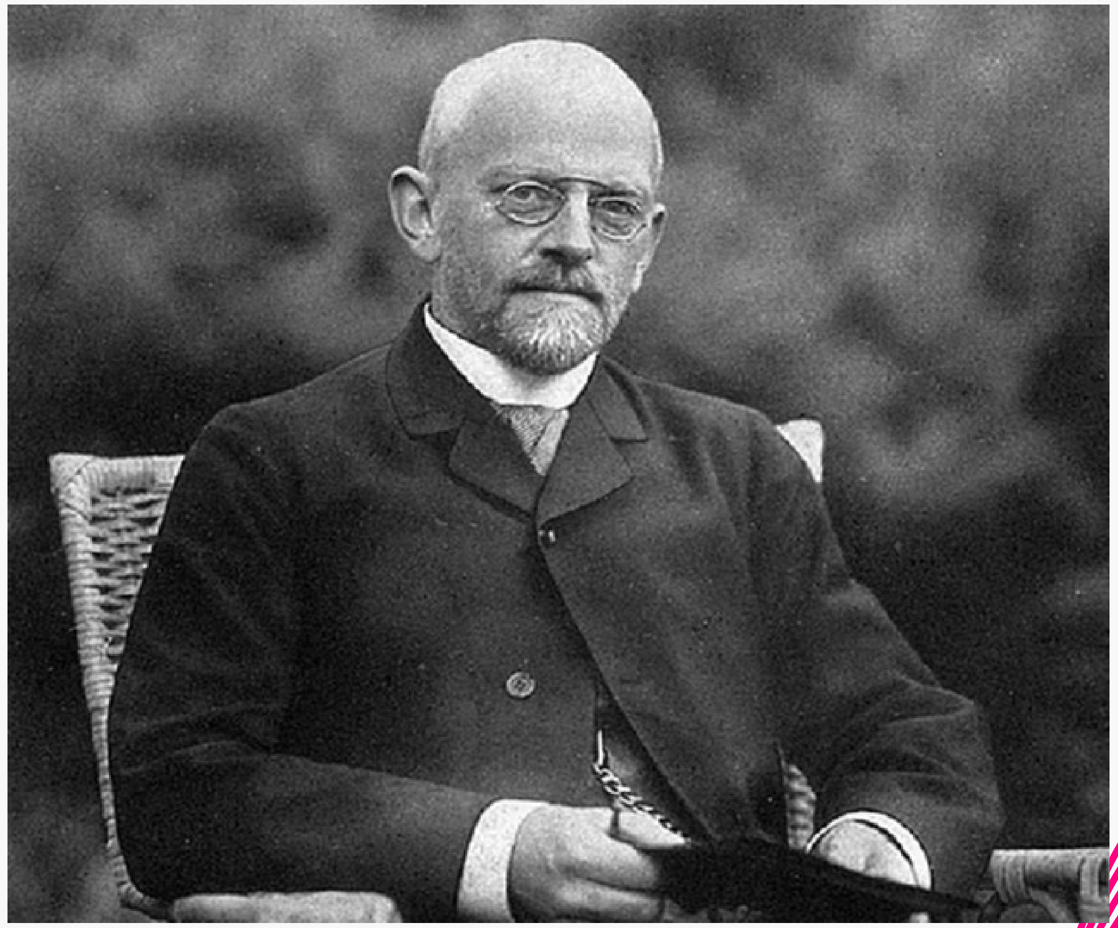
(Discrete Mathematics final project)

PRESENTED BY
Aditya Gupta , Anish , Himang Chandra Garg

ABOUT THE MATHEMATICIAN

David Hilbert was a German Mathematician and one of the most renowned and influential mathematicians of his time . His contributions in the world of mathematics is immense . Some of his fields of contribution are invariant theory , commutative algebra , algebric number theory , foundation of mathematics .

One of his famous paradox is the Hilbert's paradox of the Grand Hotel . Let us learn more about it .



ABSTRACT

Hilbert’s Paradox was proposed by David Hilbert in 1924, considers an infinite number of rooms in a hotel numbered in a sequence starting from 1,2,3...
and so on. It can also be thought of as an idea of Infinite layers. In his paradox, he explained how guests arriving at a hotel can be provided a room. Initially, all rooms are empty, but as soon as the guests arrive, each guest is provided a room by shifting all guests to one room. Suppose there are initially ”k” guests staying in a hotel, and then next ”n” guests arrive at the hotel asking for the rooms; we can shift all the existing guests by ”n.” So, the hotel cannot be fully occupied as there’s an assumption that there is an infinite number of rooms.

ABSTRACT

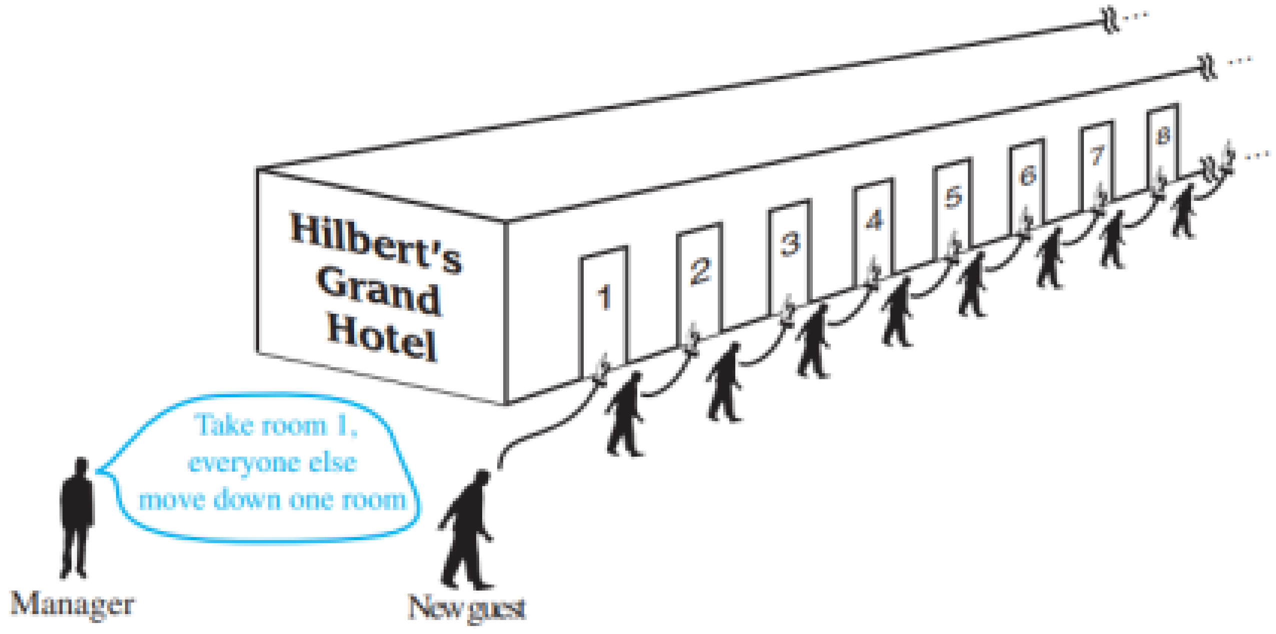


FIGURE 2 A New Guest Arrives at Hilbert's Grand Hotel.

Figure 2: A image from the TextBook

An interactive image for the same can be found in our textbooks also . (Shown as fig-2)

Now, let’s move to the depth of this famous Paradox and learn through coding problems and some of the well known proofs how we can do this . The coding problems is a great way to tell some of the patterns and logical arguments how you can fit guests in such a hotel . The Theoretical aspects of this tells us about more simpler and famous ways of fitting guests in a infinite hotel .

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

1) Suppose, currently there are ”n” guests staying at the hotel starting from ”1”, and a new guest arrive at then we can shift all the currently staying guests by ”1” room, i.e. room 1 guest to room 2, room 2 guest to room 3 and so on, therefore room n guest to room n+1. In a more general way,

Room n → Room n + 1

i.e. $f(n) : n \rightarrow n + 1$, where

$f(n)$ = allocated room to existing guest

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

2) Now, suppose instead of a single guest, an infinite number of guests arrive at the hotel, this time each guest is shifted by 2 rooms i.e. Room 1 guest is 2 moved to Room 2, Room 2 guest is moved to Room 4, and so on. In a more general way,

Room n → 2 * Room n

i.e. f(n) : n → 2 * n

f(n) = allocated room to existing guest

Now , By doing this we can simply put the the infinite number of guests in infinite number of odd numbered hotel rooms as they are also infinite ! This is a good application of the fact that number of odd and even positive integers are countably infinite .

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

3) Now, lets consider some more complex idea. Suppose, instead of infinite guests, an infinite number of buses arrive, with each bus containing an infinite number of guests. What to do next ? We can use the method of ”prime factorisation” i.e. ”Fundamental Theorem of Arithmetic” which states that ”every whole positive number greater than 1 can be written as a product of its prime factors and this product is unique”

e.g. 72 = 2 x 2 x 2 x 3 x 3

and, there is no way of writing ”72” as a product of different prime numbers.

TARGET/OBJECTIVE # 3

TARGET/OBJECTIVE # 4

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

We can use this to solve this problem. Lets assume, bus(b) and each seat(s), then the current guests are represented on bus(b) = 0, and now, each person is moved to room ” ($2^b * 3^s$)”.

e.g. guest coming from bus = 5 and seat = 4 is moved to room 2^5 * 3^4 = 2^5 *9^2.

i.e. f(n) : n → 2b * 3s

f(n) = allocated room to existing guest

Using this approach, there will be nobody without a room and there will be a lots of empty rooms also, and by Fundamental Theorem of Arithmetic, we can guarantee that each person must move to their own room and no room will be allocated twice.

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

4) Now, lets expand this, suppose we have our hotel inside an airport, and an infinite number of Aeroplanes arrive, each carrying infinite number of coaches, and each coach containing an infinite number of guests. And, we have to allocate rooms to those infinite guests, So we can expand our above ”Prime Factorisation” method by assuming Aeroplanes(A), Coaches(c) and each coast contains seats (n), so now, we can allocate a room using the approach

allocated room = $2^n * 3^c * 5^A$

i.e. $f(n) : n \rightarrow 2^n * 3^c * 5^A$

$f(n)$ = allocated room to existing guest

using this approach, each guest is allocated a room in our fully occupied hotel .

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

5)

We can extend this an infinite number of times saying Infinite layers of our fully occupied Hotel. Now let us view one of few cases where we cant give every person on a bus having infinitely many people a room . This can also be viewed as why Real numbers are not countably infinite .

WHY?

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

5)

We can extend this an infinite number of times saying Infinite layers of our fully occupied Hotel. Now let us view one of few cases where we cant give every person on a bus having infinitely many people a room . This can also be viewed as why Real numbers are not countably infinite .

WHY? DIAGNALIZATION!

INTRODUCTION

SOME INTERESTING MATHEMATICAL FACTS

5)

Consider the following case now: there is a bus carrying infinite number of people but this time every person has a unique name , lets suppose in the format of A's and B's . Now if you make a new name such that the ith position of the new name does not match with the ith position of the ith person you get a completely new name . This shows that we practically cant give every person a new room . This is somewhat like what we do with the real numbers when proving them uncountable . Interesting right! Making it no vacancy .

EXISTING LITERATURE

Hilbert’s Paradox is veridical paradox that i.e. a statement that seems true at first sense, but upon inspection, it seems out to be impossible. And it somehow relates with the famous ”Cantors set theory and infinity” that developed the concepts of ”arithmetic of infinite sets” called ”cardinals and ordinals” and extended the arithmetic of natural numbers. He denoted cardinal numbers using ”Hebrew letter (aleph)”. In a similar way, Hilberts Paradox works, that is it assumes the each and every guest can be accommodated in the hotel and there would be no room that is allocated to more than guest.

Now, lets see the proof of famous “Cantors Diagonlisation Theorem” in the next slide.

Cantor’s Diagonalisation Theorem

Consider the set of sequence of substrings.

S = { A1, A2, A3 ...... An } where,

A_1 = “w” w w w w w w w w w w w ....

A_2 = ww “ww” ww ww ww ww ww ....

A_3 = www www “www” www www ....

.

A_n = w ww www wwww ....

we notice that in sequence formed in “An” using the diagonal formed by each sequence doesn’t appear in any of the above “A_n-1” sequences proving the existence of uncountable sets.

Cantor's diagonal argument

$$
\begin{array}{lcl}
s_1 & = & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 & = & 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 & = & 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 & = & 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 & = & 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots \\
s_6 & = & 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots \\
s_7 & = & 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots \\
s_8 & = & 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots \\
s_9 & = & 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots \\
s_{10} & = & 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots \\
s_{11} & = & 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots \\
\vdots & & \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots
\end{array}
$$

$$
s = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\ldots
$$

EXISTING LITERATURE

But, both the theorems are kind of intuitive since Cantor says the size of sets can be infinite also (which the first theory that talks about infinity, and but at that time none of the theories were there to prove infinity) there are still arguments going on to extend the layers of infinity and some concepts of this paradox are also used in Quantum physics, and there is some theory related to Hilbert's paradox known as "The Quantum Hilbert Hotel", but we won't discuss it, since it is beyond the scope of this course

TOPIC OF STUDY

1) We studied various methods through which we can accommodate an infinite number of people in our grand hotel. Through this, we got an idea about how infinities work and we were able to connect this with the concept of countable and uncountable set theory which we learned throughout the course such as,

(a) cantors dialgonalisation theorem
(b) countable and uncountable sets
(c) nested layers of infinity
(d) Fundamental Theorem of Arithmetic (Number Theory)

TOPIC OF STUDY

2) Through this project, we studied some more unorthodox methods of filling the hotel with infinite guests through the coding problems through which we can observe patterns and arguments in such logical problems.

3) And, in the topic “Latest developments” we learned a couple of new concepts like
(a) Introduction to Room Rent
(b) Refund of Room Rent
(c) Realistic way to accomodate an infinite number of new people
(d) Mathematical and logical equations related to these topics

LATEST DEVELOPMENTS

Many new problems are variations of Hilbert Hotel

- **Introduction of Room Rent**
- **Refund of Room Rent**
- **Realistic way to accommodate an infinite number of new people**

INTRODUCTION OF ROOM RENT

Let us suppose Rooms in Hilbert Hotel were charged differently for each room. Room number 1 cost one coin, room number 2 cost 1/2 coin, and this pattern continues till infinity. Mathematically, Cost of room number n = 1/n So, when all rooms are occupied, during the calculation of revenue, we meet a divergent series (so the revenue should be infinite).

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent (so the revenue must be infinite).

REFUND OF ROOM RENT

Imagine some priority guest arrives at the Hotel suddenly, and the manager wants to give him the most expensive room, which is room number 1. Therefore, all the guests will move to the following room number, which costs less, and since our manager is honest, he should refund their money. Now, the refund calculation involves the sum of an Infinite series of differences in the room. It looks like the sum will be extensive, and the hotel manager will lose a lot of money, but upon calculation, it is one coin only! Now, if more than

$$\sum_{l=1}^{n} \left(\frac{1}{l} - \frac{1}{l+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

REFUND OF ROOM RENT

Now, if more than one special guest arrives in the hotel(let k special guests), we must shift all our guests in room number n to n+k. Now the calculation of refund becomes

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}$$

$\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+k})$ is a telescoping sum: $\sum_{l=1}^{n}(\frac{1}{l}-\frac{1}{l+k}) =(1-\frac{1}{1+k})+(\frac{1}{2}-\frac{1}{2+k})+\cdots+(\frac{1}{k}-\frac{1}{k+k})+(\frac{1}{k+1}-\frac{1}{(k+1)+k})+\cdots+(\frac{1}{n-1}-\frac{1}{(n-1)+k})+(\frac{1}{n}-\frac{1}{n+k})=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}-(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+k})$ for $n\geq k$, and $\lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}-(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+k}))=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}$.

REALISTIC WAY TO ACCOMMODATE AN INFINITE NUMBER OF NEW PEOPLE

The most suitable way to accommodate an infinite number of people involves partitioning the set N into sets Ak for k=1,2,3 and then relocating the guest from room number n to Ak to room number n+k.

$$A_k = \{n \mid (k-1)^2 + 1 \leq n \leq k^2\} \text{ for } = 1, 2, 3, \ldots.$$

Since the set Ak is disjoint and the Union of Ak from k = 1 to infinity = N. Now, let us assume there is a function f that is well-defined and is one-to-one from set N to N.

Let $f : N \rightarrow N$ be defined by $f(n) = n + k$ for $n \in Ak$ for $k = 1, 2, 3$

$$f(1) = 2,\ f(2) = 4,\ f(3) = 5,\ f(4) = 6,\ f(5) = 8,\ f(6) = 9, \ldots$$

REALISTIC WAY TO ACCOMMODATE AN INFINITE NUMBER OF NEW PEOPLE

So the range of the function is 2,4,5,6,8,9,10.... So, the person currently in room number n is moved to room f(n), and by doing so, we can accommodate infinitely many new guests. The reason behind this is that after this reallocation, room numbers in the set 1,3,7,13... will always be vacant. Now, the refund becomes

$$\sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \left(\frac{1}{n} - \frac{1}{n+k}\right) = \sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \frac{k}{n(n+k)}$$

$$\sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \frac{k}{n(n+k)} \leq \sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \frac{k}{((k-1)^2+1)\left((k-1)^2+1+k\right)}$$

$$= \sum_{k=1}^{\infty} \frac{(2k-1)k}{((k-1)^2+1)\left((k-1)^2+1+k\right)}.$$

CONCLUSION

There were so many new concepts we learnt though this project such as:

1. cantors dialgonalisation theorem
2. countable and uncountable sets
3. nested layers of infinity
4. Fundamental Theorem of Arithmetic (Number Theory)
5. Introduction to Room Rent
6. Refund of Room Rent
7. Realistic way to accomodate an infinite number of new people
8. Mathematical and logical equations related to these topics

References

Link of the coding problems:

1. Project Euler Problem 359: https://projecteuler.net/problem=359
2. Codeforces Contest 1344 Problem A: https://codeforces.com/contest/1344/problem/A

Latest developments:

1. https://ijrpr.com/uploads/V4ISSUE3/IJRPR10757.pdf
2. https://www.researchgate.net/publication/264089674Hilbert′ sGrandHotelwithaseriestwist

References

About the Mathematician:

1. David Hilbert - Wikipedia: https://en.wikipedia.org/wiki/
DavidHilbert DavidHilbert′ sContributionsinM athematics : https : //studiousguy.com/david – hilberts – contributions – in – mathematics/

Abstract and Introduction:

2.1 Hilbert’s Paradox of the Grand Hotel - Wikipedia: https://en.wikipedia.org/wiki/Hilbert

2. Hilbert’s Hotel - Plus Magazine: https://plus.maths.org/content/hilberts□hotel

3. Hilbert’s Paradox of the Grand Hotel - OwlCation: https://owlcation.com/stem/Hilberts Paradox-of-the-Grand-Hotel-Another-Look-at-I

4. Hilbert’s Paradox of the Grand Hotel - YouTube: https://www.youtube.com/watch?v=OxGsU8oIWjY

References

Existing Resources:

https://en.wikipedia.org/wiki/Georg_Cantor

https://en.wikipedia.org/wiki/Cantor%27s_first_set_theory_article

THANK YOU