Hilbert's paradox of the Grand Hotel

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Project Work

This project is about the Hilbert's paradox of the Grand Hotel . This was proposed by David Hilbert in 1924 . Let use see more about the Mathematician David Hilbert .

1 About The Mathematician

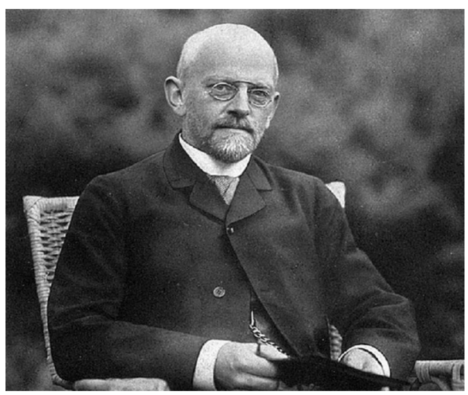


Figure 1: David Hilbert

David Hilbert was a German Mathematician and one of the most renowned and influential mathematicians of his time . His contributions in the world of mathematics is immense . Some of his fields of contribution are invariant theory , commutative algebra , algebric number theory , foundation of mathematics . One of his famous paradox is the Hilbert's paradox of the Grand Hotel . Let us learn more about it .

2 Abstract

Hilbert's Paradox was proposed by David Hilbert in 1924, considers an infinite number of rooms in a hotel numbered in a sequence starting from 1,2,3...

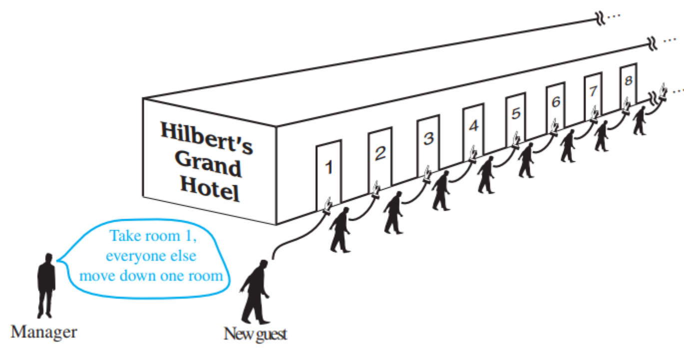


FIGURE 2 **A New Guest Arrives at Hilbert's Grand Hotel.**

Figure 2: A image from the TextBook

and so on. It can also be thought of as an idea of Infinite layers. In his paradox, he explained how guests arriving at a hotel can be provided a room. Initially, all rooms are empty, but as soon as the guests arrive, each guest is provided a room by shifting all guests to one room. Suppose there are initially "k" guests staying in a hotel, and then next "n" guests arrive at the hotel asking for the rooms; we can shift all the existing guests by "n." So, the hotel cannot be fully occupied as there's an assumption that there is an infinite number of rooms. Now, let's move to the depth of this famous Paradox and learn through coding problems and some of the well known proofs how we can do this . The coding problems is a great way to tell some of the patterns and logical arguments how you can fit guests in such a hotel . The Theoretical aspects of this tells us about more simpler and famous ways of fitting guests in a infinite hotel .

3 Introduction

Suppose, currently there are "n" guests staying at the hotel starting from "1", and a new guest arrive at then we can shift all the currently staying guests by "1" room, i.e. room_1 guest to room_2, room_2 guest to room_3 and so on, therefore room_n guest to room_n+1. In a more general way,

$$\text{Room\_n} \rightarrow Room\_n + 1$$

$$\text{i.e. f(n) : n} \rightarrow n + 1, where$$

$$\text{f(n)} = \text{allocated\_room\_to\_existing\_guest}$$

Now, suppose instead of a single guest, an infinite number of guests arrive at the hotel, this time each guest is shifted by 2 rooms i.e. Room_1 guest is moved to Room_2, Room_2 guest is moved to Room_4, and so on. In a more general way,

$$\text{Room\_n} \rightarrow 2 * Room\_n$$

$$\text{i.e. f(n) : n} \rightarrow 2 * n$$

$$\text{f(n)} = \text{allocated\_room\_to\_existing\_guest}$$

Now , By doing this we can simply put the the infinite number of guests in infinite number of odd numbered hotel rooms as they are also infinite ! This is a good application of the fact that number of odd and even positive integers are countably infinite .
Now, lets consider some more complex idea. Suppose, instead of infinite guests, an infinite number of buses arrive, with each bus containing an infinite number of guests. What to do next ? We can use the method of "prime factorisation" i.e. "Fundamental Theorem of Arithmetic" which states that "every whole positive number greater than 1 can be written as a product of its prime factors and this product is unique"

e.g. 72 = 2 x 2 x 2 x 3 x 3

and, there is no way of writing "72" as a product of different prime numbers.

We can use this to solve this problem. Lets assume, bus(b) and each seat(s), then the current guests are represented on bus(b) = 0, and now, each person is moved to room " $(2^b * 3^s)$".

e.g. guest coming from bus = 5 and seat = 4 is moved to room $2^5 * 3^4 = 2592$.

$$\text{i.e. f(n) : n} \rightarrow 2^b * 3^s$$

$$\text{f(n)} = \text{allocated\_room\_to\_existing\_guest}$$

Using this approach, there will be nobody without a room and there will be a lots of empty rooms also, and by Fundamental Theorem of Arithmetic, we can guarantee that each person must move to their own room and no room will be allocated twice.

Now, lets expand this, suppose we have our hotel inside an airport, and an infinite number of Aeroplanes arrive, each carrying infinite number of coaches, and each coach containing an infinite number of guests. And, we have to allocate rooms to those infinite guests, So we can expand our above "Prime Factorisation" method by assuming Aeroplanes(A), Coaches(c) and each coast contains seats (n), so now, we can allocate a room using the approach

$$\text{allocated\_room} = 2^n * 3^c * 5^A$$

$$\text{i.e. f(n) : n} \rightarrow 2^n * 3^c * 5^A$$

$$\text{f(n)} = \text{allocated\_room\_to\_existing\_guest}$$

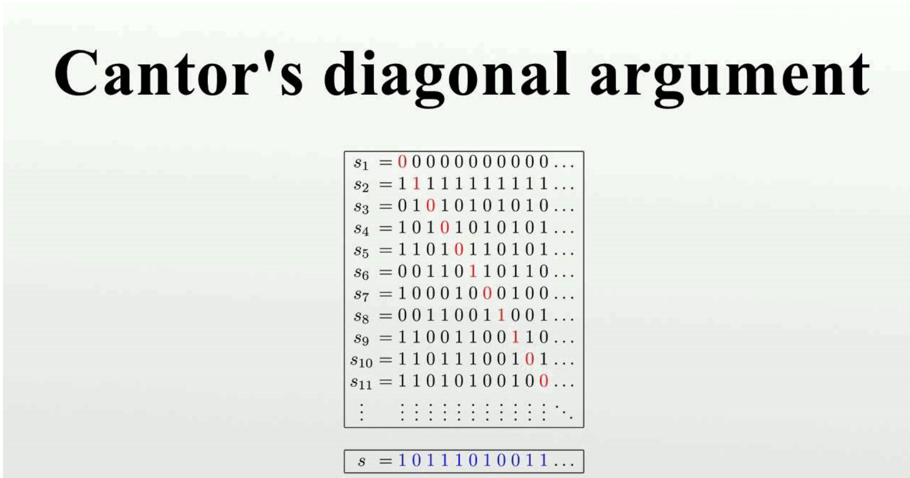
Figure 3: Enter Caption

Using this approach, each guest is allocated a room in our fully occupied hotel.

We can extend this an infinite number of times saying Infinite layers of our fully occupied Hotel. Now let us view one of few cases where we cant give every person on a bus having infinitely many people a room . This can also be viewed as why Real numbers are not countably infinite . DIAGNALIZATION! Consider the following case now: there is a bus carrying infinite number of people but this time every person has a unique name , lets suppose in the format of A's and B's . Now if you make a new name such that the ith position of the new name does not match with the ith position of the ith person you get a completely new name . This shows that we practically cant give every person a new room . This is somewhat like what we do with the real numbers when proving them uncountable . Interesting right! Making it no vacancy .

4 Existing Literature

Hilbert's paradox is a "veridical paradox" (a statement that seems contradictory at first sense, but upon inspection, turns out to be true or impossible) that is, it leads to a counter-intuitive result that is probably true. And, it's a thought experiment which illustrates a counter-intuitive property of infinite sets. Hilbert's paradox somehow relates with the famous "Cantors set theory and infinity" that developed the concepts of "arithmetic of infinite sets" called "cardinals and ordinals" which extended the arithmetic of natural numbers. He denoted cardinal numbers using "Hebrew letter (aleph)" with a natural number subscript. He also proved that real numbers are not countable using his famous "Diagonlisation Theorem" and the theorem also states that power set (set of all subsets) of an infinite set is always larger than the original set



which is kind of obvious as consider the set(S) = 1 , 2 , 3 of size = 3, will have the power set [P(S)] = phi, 1, 2, 3, 1,2, 2,3, 1,3 , note the size of power set is greater than the original set(S) and it also proved the existence of uncountable sets. Lets prove it,
Cantors Diagonals Theorem to prove that there exists uncountable sets also.
Consider the set(S) containing sequences of string literals i.e.
S = A1, A2, A3 ...., An where,

A1 = "a" a a a a a a a a a a ....

A2 = aa "aa" aa aa aa aa aa aa ....

A3 = aaa aaa "aaa" aaa aaa aaa aaa ....

...

...

...

...

Am = a aa aaa aaaa aaaaa aaaaaa ....

we can notice that sequence at the bottom i.e. "Am" doesn't appear anywhere from our set(S), since the sequence "Am" is formed from the diagonals of all the above sets proving the existence of uncountable sets. In a similar way, Hilberts Paradox works, that is it assumes the each and every guest can be accommodated in the hotel and there would be no room that is allocated to more than guest. This is also similar to Hilberts paradox that says allocating new guests to Hilberts Hotel maps to a larger set of possible arrangements in Cantors theorem.
But, both the theorems are kind of intuitive since Cantor says the size of sets can be infinite also (which the first theory that talks about infinity, and none

of the theories were there to prove infinity) and Hilbert's paradox that says a hotel cannot be full whether there are infinite number of guests. However, critics have argued that implementing Hilbert's paradox in real life is not practical and is only valid and true in thoughts and is purely theoretical. This is not the end, there are still arguments going on to extend the layers of infinity and some concepts of this paradox are also used in Quantum physics, and there is some theory related to Hilbert's paradox known as "The Quantum Hilbert Hotel", but we won't discuss it, since it is beyond the scope of this course.

5 The topic of study

The topics of our study are as follows:

1. We studied the various methods of how we can fit people in fully filled infinite hotel . This gave use many new theoretical ideas on how infinities work . We were able to connect this will the countable and uncountable set theory we learned throughout the course . Our findings and study is well summarised in the abstract and the introduction sections of our document for this . Some of the topics were :
   (a) cantors dialgonalisation theorem
   (b) countable and uncountable sets
   (c) nested layers of infinity
   (d) Fundamental Theorem of Arithmetic (Number Theory)
2. We studied the more unorthodox methods of filling the infinite hotel using coding problems . They taught us how we can observe patterns and logical arguments in such questions .
3. In the latest developments of the topic we also learned a couple of new things:
   (a) Introduction to Room Rent
   (b) Refund of Room Rent
   (c) Realistic way to accomodate an infinite number of new people
   (d) Mathematical and logical equations related to these topics

   These topics are discussed in a great detail in the latest developments part .

Many of these lead to our formulations of new ideas which we implement in questions and theory in general .

6 Latest developments

Many new problems are variations of Hilbert Hotel.

6.1 Introduction of Room Rent

Let us suppose Rooms in Hilbert Hotel were charged differently for each room. Room number 1 cost one coin, room number 2 cost 1/2 coin, and this pattern continues till infinity. Mathematically, Cost of room number n = 1/n So, when all rooms are occupied, during the calculation of revenue, we meet a divergent series (so the revenue should be infinite).

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent (so the revenue must be infinite).

6.2 Refund of Room Rent

Imagine some priority guest arrives at the Hotel suddenly, and the manager wants to give him the most expensive room, which is room number 1. Therefore, all the guests will move to the following room number, which costs less, and since our manager is honest, he should refund their money. Now, the refund calculation involves the sum of an Infinite series of differences in the room. It looks like the sum will be extensive, and the hotel manager will lose a lot of money, but upon calculation, it is one coin only! Now, if more than

$$\sum_{l=1}^{n} \left(\frac{1}{l} - \frac{1}{l+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

one special guest arrives in the hotel(let k special guests), we must shift all our guests in room number n to n+k. Now the calculation of refund becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+k}\right)$ is a telescoping sum: $\sum_{l=1}^{n} \left(\frac{1}{l} - \frac{1}{l+k}\right) = \left(1 - \frac{1}{1+k}\right) + \left(\frac{1}{2} - \frac{1}{2+k}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+k}\right) + \left(\frac{1}{k+1} - \frac{1}{(k+1)+k}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{(n-1)+k}\right) + \left(\frac{1}{n} - \frac{1}{n+k}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+k}\right)$ for $n \geq k$, and $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$.

6.3 Realistic way to accommodate an infinite number of new people

The most suitable way to accommodate an infinite number of people involves partitioning the set N into sets Ak for k=1,2,3 and then relocating the guest from room number n to Ak to room number n+k.

$$A_k = \{n | (k-1)^2 + 1 \leq n \leq k^2\} \text{ for} = 1, 2, 3, \ldots.$$

Since the set Ak is disjoint and the Union of Ak from k = 1 to infinity = N. Now, let us assume there is a function f that is well-defined and is one-to-one from set N to N.

Let f : N → N be defined by f (n) = n + k for n ∈ Ak for k = 1, 2, 3

$$f(1) = 2,\ f(2) = 4,\ f(3) = 5,\ f(4) = 6,\ f(5) = 8,\ f(6) = 9, \ldots.$$

So the range of the function is 2,4,5,6,8,9,10.... So, the person currently in room number n is moved to room f(n), and by doing so, we can accommodate infinitely many new guests. The reason behind this is that after this reallocation, room numbers in the set 1,3,7,13... will always be vacant. Now, the refund becomes,

$$\sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \left(\frac{1}{n} - \frac{1}{n+k}\right) = \sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \frac{k}{n(n+k)}$$

$$\sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \frac{k}{n(n+k)} \leq \sum_{k=1}^{\infty} \sum_{n=(k-1)^2+1}^{k^2} \frac{k}{((k-1)^2+1)\left((k-1)^2+1+k\right)}$$
$$= \sum_{k=1}^{\infty} \frac{(2k-1)k}{((k-1)^2+1)\left((k-1)^2+1+k\right)}.$$

This result is finite as the series is convergent.

7 Conclusion

It was a pretty interesting topic and a research work to do . There were many new learning for us which are :

1. cantors dialgonalisation theorem
2. countable and uncountable sets
3. nested layers of infinity
4. Fundamental Theorem of Arithmetic (Number Theory)
5. Introduction to Room Rent

6. Refund of Room Rent
7. Realistic way to accomodate an infinite number of new people
8. Mathematical and logical equations related to these topics

We Conclude our project work here .