Unit IV FUNDAMENTAL OF FUZZY LOGIC

Definition: Fuzzy Set

A fuzzy set of set X is a pair, $\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A: X \rightarrow [0,1]$ is a function which map all values in X to [0,1] which is also called as **membership function**. The value $\mu_A(x)$ is called grade of membership of x.

Example: Let Ã be the fuzzy set of "smart" students, where "smart" is fuzzy term.

$$\tilde{A} = \{(g_1,0.4)(g_2,0.5)(g_3,1)(g_4,0.9)(g_5,0.8)\}$$

Here Ã indicates that the smartness of g_1 is 0.4 and so on

For Finite set, $X = \{x_1, x_2, x_3, \ldots, x_n\}$, the fuzzy , $\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\}$ for each $x \in X$ is often denoted by

$$\tilde{A} = \left\{\frac{\mu_X(x_1)}{x_1}, \frac{\mu_X(x_2)}{x_2}, \ldots, \frac{\mu_X(x_n)}{x_n}\right\} \text{ or } \tilde{A} = \sum_{i=1}^{n} \frac{\mu_X(x_i)}{x_i}$$

Operation on Fuzzy Sets

Let $\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\}$ and $\tilde{B} = \{(x, \mu_B(x)) \mid x \in X\}$

be any two fuzzy sets of $X = \{x_1, x_2, \ldots x_n\}$

① Union :

$$\tilde{A} \cup \tilde{B} = \{(x, \mu_{A \cup B}(x)) \mid x \in X\}$$

where $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$

② Intersection

$$\tilde{A} \cap \tilde{B} = \{(x, \mu_{A \cap B}(x)) \mid x \in X\}$$

where $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

③ Complement

$$\tilde{A}^c = \{ (x, \mu_{A^c}(x)) \mid x \in X \}$$

where $\mu_{A^c}(x) = 1 - \mu_A(x)$

④ Equality

$\tilde{A} = \tilde{B}$ iff $\mu_A(x) = \mu_B(x)$ for each $x \in X$

⑤ Algebraic product of fuzzy sets

$$\tilde{A} \cdot \tilde{B} = \{ (x, \mu_A(x) \cdot \mu_B(x)) \mid x \in X \}$$

⑥ Multiplication of fuzzy sets by a crisp number

$$d\tilde{A} = \{ (x, d\,\mu_A(x)) \mid x \in X, d \in [0,1] \}$$

⑦ Power of fuzzy set

$$(\tilde{A})^P = \{ (x, (\mu_A(x))^P) \mid x \in X, P \geq 0 \}$$

⑧ Algebraic sum of fuzzy sets

$$\tilde{A} + \tilde{B} = \{ (x, \mu_{A+B}(x)) \mid x \in X \}$$

where $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$

⑨ Bounded sum of fuzzy sets

$$\tilde{A} \oplus \tilde{B} = \{ (x, \mu_{A \oplus B}(x)) \mid x \in X \}$$

Where $\mu_{A \oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$

(10) Algebraic difference

$$\tilde{A} - \tilde{B} = \{(x, \mu_{A-B}(x)) \mid x \in X\}$$

Where
$$\mu_{A-B}(x) = \mu_{A \cap B^c}(x)$$
$$= \min\{\mu_A(x), \mu_{B^c}(x)\}$$
$$= \min\{\mu_A(x), 1 - \mu_B(x)\}$$

(11) Bounded Difference

$$\tilde{A} \ominus \tilde{B} = \{(x, \mu_{A \ominus B}(x)) \mid x \in X\}$$

Where,
$$\mu_{A \ominus B}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

EXAMPLES

Q.1 Let $\tilde{A} = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

& $\tilde{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.8), (x_4, 0.9)\}$

be any two fuzzy sets on set $X = \{x_1, x_2, x_3, x_4\}$

then find the following values

(1) $\tilde{A} \cup \tilde{B}$ (2) $\tilde{A} \cap \tilde{B}$ (3) $(\tilde{A})^c, (\tilde{B})^c$ (4) $\tilde{A} \cdot \tilde{B}$

(5) $0.7\tilde{B}$ (6) $(\tilde{A})^{10}$ (7) $\tilde{A} + \tilde{B}$ (8) $\tilde{A} \oplus \tilde{B}$

(9) $\tilde{A} - \tilde{B}$ (10) $\tilde{A} \ominus \tilde{B}$

MEMBERSHIP FUNCTION

* **Universe of Discourse** : It is domain of the membership function.

Def^n : It is a function that maps elements in the universe of discourse to a value in the interval set $[0,1]$ with each element having degree of membership.

Mathematically : Let X be any set then a map

$\mu_A : X \to [0,1]$ is called membership function

Universe of Discourse

Note : Membership function provides smooth (or gradual) transition from region outside the interval to region inside the interval.

Ex.

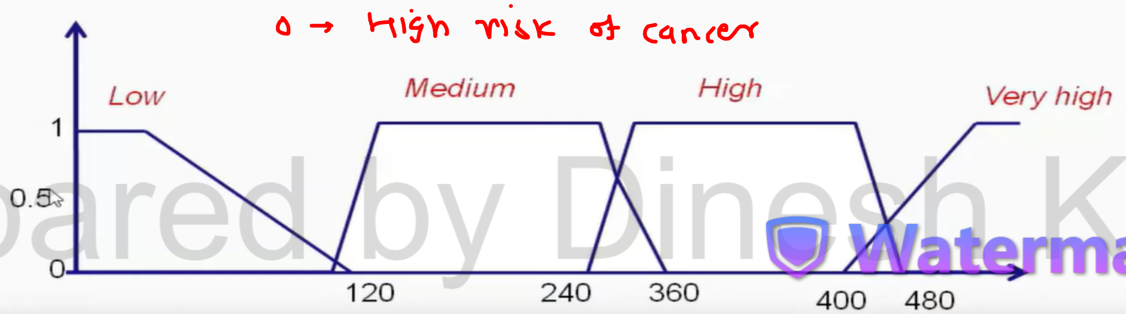
Consider a problem to identify the risk of cancer among a section of workforce employed in Lead industry based on no. of minutes a worker is exposed per day to the Lead processing unit.

So we identify 4 sets of risk of having cancer amongst the workers-

Low - [0 -120] ; Medium-[118- 360] ; High- [240-480] ;Very high=[400 or more]

1 → Less risk of cancer

0 → High risk of cancer



TYPES OF MEMBERSHIP FUNCTION

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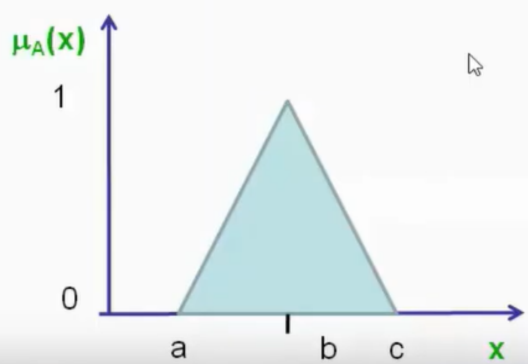
① Traingular membership function

② Trapezoidal membership function

③ Gaussian membership function

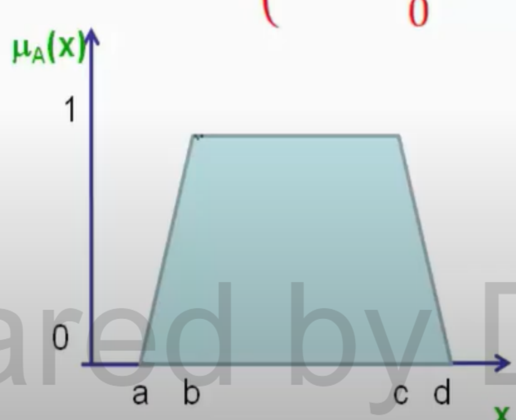
- Triangular: specified as

$$triangle(x{:}\,a,b,c) = \begin{cases} 0 & x<a \\ (x-a)/(b-a) & a\leq x\leq b \\ c-x/c-b & b\leq x\leq c \\ 0 & x>c \end{cases}$$



- Trapezoidal: specified as

$$= \begin{cases} 0 & x<a \\ x-a)/(b-a) & a\leq x<b \\ 1 & b\leq x<c \\ d-x/d-c & c\leq x<d \\ 0 & x>d \end{cases}$$

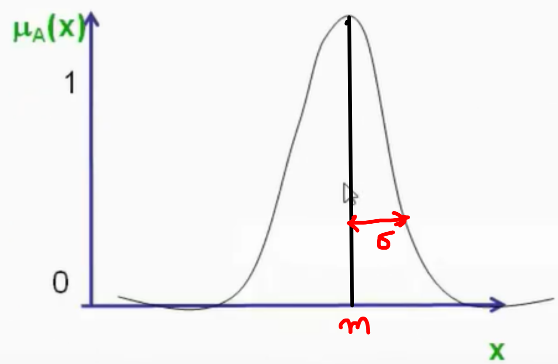


- Gaussian: specified as

$$gaussian(x: m, \sigma) = e^{-\left(\frac{(x-m)^2}{\sigma^2}\right)}$$

m = mean

σ = standard deviation.



FEATURES OF MEMBERSHIP FUNCTION

① core : set of $x \in X$ for which $\mu_A(x) = 1$

OR $\{x \in X \mid \mu_A(x) = 1\}$

② support : set of $x \in X$ for which $\mu_A(x) > 0$

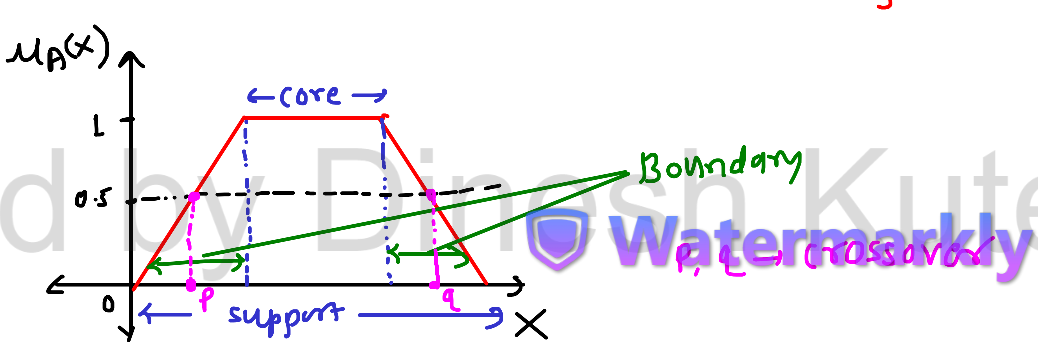
OR $\{x \in X \mid \mu_A(x) > 0\}$

③ crossover : set of $x \in X$ for which $\mu_A(x) = 0.5$

OR $\{x \in X \mid \mu_A(x) = 0.5\}$

④ Boundary : set of $x \in X$ for which $0 < \mu_A(x) < 1$

* Graphical representation of features of membership function.



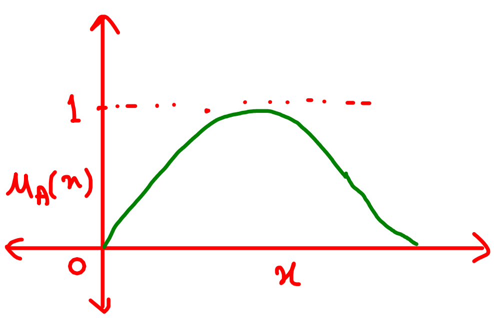
TYPES OF FUZZY SETS

a) CONVEX FUZZY SET

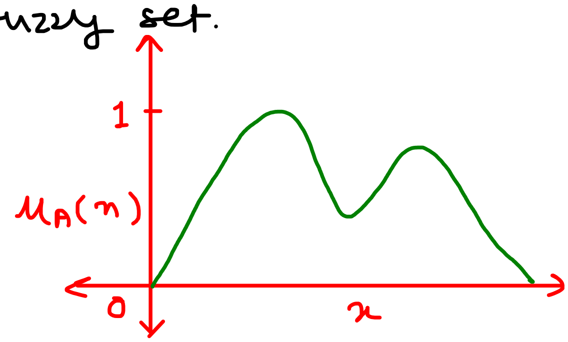
Defn: A fuzzy set $\tilde{A}$ on X is said to be convex if and only if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$ for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$

Note: It is described by membership function whose membership values are strictly monotonically increasing or monotonically decreasing or initially monotonically increasing and then monotonically decreasing.

* Non convex fuzzy set: A fuzzy set which is not convex is called non convex fuzzy set.



convex fuzzy set



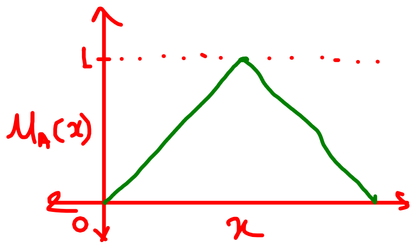
Non convex fuzzy set.

b) NORMAL FUZZY SETS

Defn: A fuzzy set $\tilde{A}$ is normal fuzzy set if there exist at one element whose membership value is one

i.e. there exist one $x \in X$ for which $\mu_A(x) = 1$

* Subnormal fuzzy set : A fuzzy set is not normal is called subnormal fuzzy set



"Normal fuzzy set"

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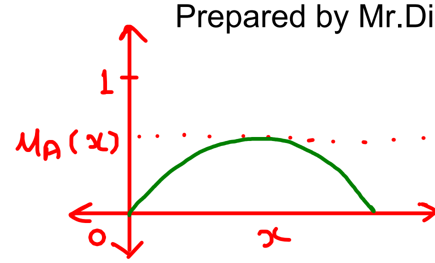


Not normal fuzzy set.
"subnormal fuzzy set"

ALPHA CUT (α-cut)

Defⁿ: The α-cut or α-level or cut worthy set of fuzzy set $\tilde{A}$ of set X is the following crisp set given by

$$\tilde{A}_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha ,\ \alpha \in [0,1] \}$$

STRONG ALPHA CUT

Defⁿ: The α-cut or α-level or cut worthy set of fuzzy set $\tilde{A}$ of set X is the following crisp set given by

$$\tilde{A}_\alpha = \{ x \in X \mid \mu_A(x) > \alpha ,\ \alpha \in [0,1] \}$$

HEIGHT OF FUZZY SET

Defⁿ: The height of fuzzy set $\tilde{A}$ is the largest membership value of element.

i.e. $h(\tilde{A}) = \max \{ \mu_A(x) \mid x \in X \}$

* Examples on Alpha cut and Strong Alpha cut

Q.1 If $\tilde{A} = \{(1, 0.2), (2, 0.4), (3, 0.6), (4, 0.8), (5, 1), (6, 0.8), (7, 0.6), (8, 0.4), (9, 0.2)\}$

then find (i) α cut of 0.4 ($A_{0.4}$)

(ii) Strong α cut of 0.6 ($A^{+}_{0.6}$)

Soln : (i)

$$A_{0.4} = \{x \in A \mid \mu_A(x) \geq 0.4\}$$
$$= \{2, 3, 4, 5, 6, 7, 8\} \quad \text{—— } 1$$

(ii)

$$A^{+}_{0.6} = \{x \in A \mid \mu_A(x) > 0.6\}$$
$$= \{4, 5, 6\}$$

* Example on core, support, height & boundary of Fuzzy set

Q.1 If $\tilde{A} = \{(a, 0), (b, 0.4), (c, 0.8), (d, 1), (e, 1), (f, 0.8), (g, 0.4), (h, 0)\}$

then find core($\tilde{A}$), support($\tilde{A}$), height($\tilde{A}$), boundary($\tilde{A}$) and crossover($\tilde{A}$)

Soln : $\text{core}(\tilde{A}) = \{x \in A \mid \mu_A(x) = 1\}$
$$= \{d, e\}$$

$$\text{support}(\tilde{A}) = \{ x \in A \mid \mu_A(x) > 0 \}$$
$$= \{ b, c, d, e, f, g \}$$

$$\text{height}(\tilde{A}) = \max\{ \mu_A(x) \mid x \in A \}$$
$$= 1 \quad (\because \text{largest membership value})$$

$$\text{boundary}(\tilde{A}) = \{ x \in A \mid 0 < \mu_A(x) < 1 \}$$
$$= \{ b, c, f, g \}$$

$$\text{crossover}(\tilde{A}) = \{ x \in A \mid \mu_A(x) = 0.5 \}$$
$$= \{ \phi \}$$

Prepared by Mr.Dinesh Kute

There are 3 types of cardinality of fuzzy set

a) Scalar cardinality

b) Relative cardinality

c) fuzzy cardinality

a) SCALAR CARDINALITY

For a fuzzy set $\tilde{A}$ defined on finite set X, it's scalar cardinality is denoted by $|\tilde{A}|$ and defined as

$$|\tilde{A}| = \sum_{x \in X} \mu_A(x)$$

b) RELATIVE CARDINALITY

For a fuzzy set $\tilde{A}$ defined on finite set X, it's Relative cardinality is denoted by $\|\tilde{A}\|$ and defined as

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{|X|} = \frac{\sum_{x \in X} \mu_A(x)}{|X|}$$

c) FUZZY CARDINALITY

For a fuzzy set $\tilde{A}$ defined on finite set X, it's Fuzzy cardinality is denoted by $|\tilde{A}|_F$ and defined as

$$|\tilde{A}|_F = \{ (|\tilde{A}_\alpha|, \alpha) \mid \text{for all } \alpha \}$$

EXAMPLE

Q. Find the scalar,relative and fuzzy cardinality for each of the following fuzzy sets

* Example on Fuzzy cardinality.

Q.1 If $\tilde{A} = \{(x_1, 0), (x_2, 0.3), (x_3, 0.6), (x_4, 0.9), (x_5, 1), (x_6, 0.9), (x_7, 0.6), (x_8, 0.3), (x_9, 0)\}$

then find scalar, relative and fuzzy cardinality of $\tilde{A}$.

Soln (1) scalar cardinality:

$$|\tilde{A}| = \sum_{x \in A} \mu_A(x)$$

$$= 0 + 0.3 + 0.6 + 0.9 + 1 + 0.9 + 0.6 + 0.3 + 0$$

$$= 4.6$$

(2) Relative cardinality

$$\|\tilde{A}\| = \frac{\sum_{x \in A} \mu_A(x)}{|X|} = \frac{|\tilde{A}|}{|X|} = \frac{4.6}{9} = \underline{\underline{0.51}}$$

(3) Fuzzy cardinality:

$$|\tilde{A}_\alpha| = \{(|\tilde{A}_\alpha|, \alpha) \mid \text{for all } \alpha\}$$

$\alpha = 0, 0.3, 0.6, 0.9, 1$

$\tilde{A}_0 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} \Rightarrow |\tilde{A}_0| = 9$

$\tilde{A}_{0.3} = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \Rightarrow |\tilde{A}_{0.3}| = 7$

$$\tilde{A}_{0.6} = \{x_3, x_4, x_5, x_6, x_7\} \Rightarrow |\tilde{A}_{0.6}| = 5$$

$$\tilde{A}_{0.9} = \{x_4, x_5, x_6\} \Rightarrow |\tilde{A}_{0.9}| = 3$$

$$\tilde{A}_1 = \{x_5\} \Rightarrow |\tilde{A}_1| = 1$$

$$\therefore \quad |\tilde{A}|_F = \{(9, 0), (7, 0.3), (5, 0.6), (3, 0.9), (1, 1)\}$$

X	A_1	A_2	A_3	A_4	A_5
Age	Infant	Kid	Young	Adult	Senior
5	0	1	0	0	0
15	0	0.3	0.2	0	0
25	0	0	0.8	0.7	0
35	0	0	1	0.9	0
45	0	0	0.6	1	0
55	0	0	0.4	1	0.8
65	0	0	0.1	1	0.9

FUZZY RELATION

Def^n: A fuzzy relation is a mapping from cartesian product of fuzzy sets to $[0,1]$

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets defined on set X and Y respectively then fuzzy relation $\tilde{R}$ is defined as

$$\tilde{R} = \tilde{A} \times \tilde{B}$$

where relation $\tilde{R}$ has membership function,

$\mu_R : \tilde{A} \times \tilde{B} \rightarrow [0,1]$ defined as

$$\mu_R(x,y) = \min \{ \mu_A(x), \mu_B(y) \}$$

* Fuzzy relation representation using matrix

Let $\tilde{A} = \{ (a_1, \mu_A(a_1)), (a_2, \mu_A(a_2)), \ldots (a_n, \mu_A(a_n)) \}$

$\tilde{B} = \{ (b_1, \mu_A(b_1)), (b_2, \mu_A(b_2)), \ldots (b_m, \mu_A(b_m)) \}$

then $\tilde{R} = \tilde{A} \times \tilde{B}$

$$= \begin{bmatrix} \mu_R(a_1, b_1) & \mu_R(a_1, b_2) & \cdots & \mu_R(a_1, b_m) \\ \mu_R(a_2, b_1) & \mu_R(a_2, b_2) & \cdots & \mu_R(a_2, b_m) \\ \vdots & & & \\ \mu_R(a_n, b_1) & \mu_R(a_n, b_2) & \cdots & \mu_R(a_n, b_m) \end{bmatrix}$$

* Example :

Q.1 Find fuzzy relation matrix of $\tilde{A} \times \tilde{B}$,

Given $\tilde{A} = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$

$\tilde{B} = \{(b_1, 0.5), (b_2, 0.6)\}$

Soln : $\tilde{R} = \tilde{A} \times \tilde{B}$

and $\mu_R(x, y) = \min\{\mu_A(x), \mu_R(y)\}$

$$\therefore \tilde{R} = \tilde{A} \times \tilde{B} = \begin{bmatrix} \mu_R(a_1, b_1) & \mu_R(a_1, b_2) \\ \mu_R(a_2, b_1) & \mu_R(a_2, b_2) \\ \mu_R(a_3, b_1) & \mu_R(a_3, b_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

OPERATIONS ON FUZZY RELATION

Let $\tilde{R}$ and $\tilde{S}$ be two fuzzy relation matrices then

① Union

$$\tilde{R} \cup \tilde{S} = \begin{bmatrix} \mu_{R\cup S}(x_1, y_1) & \mu_{R\cup S}(x_1, y_2) & \cdots & \mu_{R\cup S}(x_1, y_n) \\ \mu_{R\cup S}(x_2, y_1) & \mu_{R\cup S}(x_2, y_2) & \cdots & \mu_{R\cup S}(x_2, y_n) \\ \vdots & & & \vdots \\ \mu_{R\cup S}(x_m, y_1) & \mu_{R\cup S}(x_m, y_2) & \cdots & \mu_{R\cup S}(x_m, y_n) \end{bmatrix}$$

where $\mu_{R\cup S}(x_i, y_i) = \max\{\mu_R(x_i, y_i), \mu_S(x_i, y_i)\}$

② Intersection

$$\mu_{R\cap S}(x_i, y_i) = \min\{\mu_R(x_i, y_i), \mu_S(x_i, y_i)\}$$

③ Complement

$$\mu_{R^c}(x_i, y_i) = 1 - \mu_R(x_i, y_i)$$

④ Projection

$$\text{Projection of } \tilde{R} \text{ on } x_K = \max\{\mu_R(x_K, y_i) \mid \forall i\}$$

OR

$$\text{Projection of } \tilde{R} \text{ on } y_K = \max\{\mu_R(x_i, y_K) \mid \forall i\}$$

* **Examples**: Let $\tilde{R}$ and $\tilde{S}$ be two relation matrices defined by

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0.7 \\ 0.0 & 0.8 & 0.0 & 0.0 \\ 0.9 & 1.0 & 0.7 & 0.8 \end{bmatrix} \end{array} \qquad \tilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.0 & 0.9 & 0.6 \\ 0.9 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0.0 & 0.8 & 0.5 \end{bmatrix} \end{array}$$

then find

ⓐ $\tilde{R} \cup \tilde{S}$

ⓑ $\tilde{R} \cap \tilde{S}$

ⓒ Projection of $\tilde{R}$ on x_1, x_2, x_3

ⓓ Projection of $\tilde{S}$ on y_1, y_2, y_3.

Soln ① $\tilde{R} \cup \tilde{S} =$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.9 & 0.7 \\ 0.9 & 0.8 & 0.5 & 0.7 \\ 0.9 & 1 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

② $\tilde{R} \cap \tilde{S} =$

$$\begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.4 & 0 & 0.1 & 0.6 \\ x_2 & 0 & 0.4 & 0 & 0 \\ x_3 & 0.3 & 0 & 0.7 & 0.5 \end{array}$$

③ Projection of $\tilde{R}$ on $x_1 = \max\{0.8, 0.1, 0.1, 0.7\}$

$$\bar{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0.7 \\ 0.0 & 0.8 & 0.0 & 0.0 \\ 0.9 & 1.0 & 0.7 & 0.8 \end{bmatrix} \end{array} \qquad \bar{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.0 & 0.9 & 0.6 \\ 0.9 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0.0 & 0.8 & 0.5 \end{bmatrix} \end{array}$$

$= 0.8$

$x_2 = \max\{0, 0.8, 0, 0\} = 0.8$

$x_3 = \max\{0.9, 1, 0.7, 0.8\} = 1$

④ Projection of $\tilde{S}$ on $y_1 = \max\{0.4, 0.9, 0.3\} = 0.9$

$y_2 = \max\{0, 0.4, 0\} = 0.4$

$y_3 = \max\{0.9, 0.5, 0.8\} = 0.9$

$y_4 = \max\{0.6, 0.7, 0.5\} = 0.7$

Prepared by Mr.Dinesh Kute

Def^n: If $\tilde{R}$ is fuzzy relation on $X \times Y$ and $\tilde{S}$ is fuzzy relation on $Y \times Z$ then fuzzy composition $\tilde{T}$ is a fuzzy relation on $X \times Z$ and defined as

(A) Fuzzy max-min composition

$$\tilde{T} = \left\{ \left((x,z), \mu_T(x,z) \right) \mid (x,z) \in X \times Z \right\}$$

where $\mu_T(x,z) = \max\limits_{y \in Y} \left\{ \min \left(\mu_R(x,y), \mu_S(y,z) \right) \right\}$

OR

$$\tilde{T} = \begin{array}{c|c} & z_1 \quad z_2 \quad \cdots \\ \hline \begin{array}{c} x_1 \\ x_2 \\ \vdots \end{array} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

* Example:

Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$

consider the following fuzzy relations

$$\tilde{R} = \begin{array}{c|c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \hline \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.6 \\ 0.8 & \underline{0.3} \end{bmatrix} \end{array}, \quad \tilde{S} = \begin{array}{c|c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \hline \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & \underline{0.7} \end{bmatrix} \end{array}$$

Find fuzzy composition $\tilde{T} = \tilde{R} \circ \tilde{S}$ using fuzzy max-min composition.

Soln: $\tilde{R} : X \to Y$ and $\tilde{S} : Y \to Z$

$$\therefore \tilde{T} = \tilde{R} \circ \tilde{S} : X \to Z$$

$$\therefore \quad \tilde{T} = \tilde{R} \circ \tilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.6 \\ 0.8 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

$$\therefore \mu_T(x_1, z_1) = \max\{\min\{\underline{0.7}, 0.8\}, \min\{0.6, \underline{0.1}\}\} = \underline{\underline{0.7}}$$

$$\mu_T(x_1, z_2) = \max\{\min\{0.7, \underline{0.5}\}, \min\{0.6, \underline{0.6}\}\} = \underline{\underline{0.6}}$$

$$\mu_T(x_1, z_3) = \max\{\min\{0.7, \underline{0.4}\}, \min\{\underline{0.6}, 0.7\}\} = \underline{\underline{0.6}}$$

$$\mu_T(x_2, z_1) = \max\{\min\{0.8, \underline{0.8}\}, \min\{0.3, \underline{0.1}\}\} = \underline{\underline{0.8}}$$

$$\mu_T(x_2, z_2) = \max\{\min\{0.8, \underline{0.5}\}, \min\{\underline{0.3}, 0.6\}\} = \underline{\underline{0.5}}$$

$$\mu_T(x_2, z_3) = \max\{\min\{0.8, \underline{0.4}\}, \min\{\underline{0.3}, 0.7\}\} = \underline{\underline{0.4}}$$

(B) Fuzzy Max product composition :

$$\tilde{T} = \{((x, z), \mu_T(x, z)) \mid (x, z) \in X \times Z\}$$

Where $\mu_T(x, z) = \max\limits_{y \in Y} \{\mu_R(x, y) \cdot \mu_S(y, z)\}$

OR

$$\tilde{T} = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & \cdots \end{array} \\ \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \end{array}$$

* **Example :**

Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$

consider the following fuzzy relations

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 \\ 0.8 & 0.3 \end{bmatrix} \end{array}, \quad \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

Find Fuzzy composition $\tilde{T} = \tilde{R} \circ \tilde{S}$ using fuzzy max-Product composition.

Soln: $\tilde{R} : X \to Y$ & $\tilde{S} : Y \to Z$

$\therefore \tilde{R} \circ \tilde{S} : X \to Z$

$$\therefore \tilde{T} = \tilde{R} \circ \tilde{S} = \begin{array}{cc} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.56 & 0.36 & 0.42 \\ 0.64 & 0.4 & 0.32 \end{bmatrix} \end{array}$$

$$\therefore \mu_T(x_1, z_1) = \max\{0.7 \times 0.8,\ 0.6 \times 0.1\}$$
$$= \max\{0.56,\ 0.06\}$$
$$= 0.56$$

$$\mu_T(x_1, z_2) = \max\{0.7 \times 0.5,\ 0.6 \times 0.6\}$$
$$= \max\{0.35,\ 0.36\} = 0.36$$

$$\mu_T(x_1, z_3) = \max\{0.7 \times 0.4,\ 0.6 \times 0.7\}$$
$$= \max\{0.28,\ 0.42\}$$
$$= 0.42$$

$$\mu_T(x_2, z_1) = \max\{0.8 \times 0.8,\ 0.3 \times 0.1\}$$
$$= \max\{0.64,\ 0.03\}$$
$$= 0.64$$

$$\mu_T(x_2, z_2) = \max\{0.8 \times 0.5,\ 0.3 \times 0.6\}$$
$$= \max\{0.4,\ 0.18\} = 0.4$$

$$\mu_T(x_2, z_3) = \max\{0.8 \times 0.4,\ 0.3 \times 0.7\}$$

$$= \max\{0.32,\ 0.21\} = \underline{\underline{0.32}}$$

* **Example :**

If $\tilde{R}$ and $\tilde{S}$ are fuzzy relations define on the fuzzy sets, $\tilde{A} = \{(x_1, 0.1), (x_2, 0.6), (x_3, 0.7), (x_4, 0)\}$, $\tilde{B} = \{(y_1, 0.2), (y_2, 0.4), (y_3, 0.1)\}$ and $\tilde{C} = \{(z_1, 0.5), (z_2, 0.1)\}$such that $\tilde{R} = \tilde{A} \times \tilde{B}$ and $\tilde{S} = \tilde{B} \times \tilde{C}$. Find fuzzy composition $\tilde{T} = \tilde{R} \circ \tilde{S}$ using Max- Min fuzzy composition method	07

Soln : $\tilde{R} = \tilde{A} \times \tilde{B} : X \to Y$

$$\therefore \tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\tilde{S} : \tilde{B} \times \tilde{C} : Y \to Z$

$$\therefore \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.1 \\ 0.1 & 0.1 \end{bmatrix} \end{array}$$

$\therefore$ By Max - min

$\therefore \tilde{T} = \tilde{R} \circ \tilde{S} : X \to Z$

$$\therefore \tilde{T} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 \\ 0.4 & 0.1 \\ 0.4 & 0.1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\mu_T(x_1, z_1) = \max\{\min\{\underline{\underline{0.1}}, 0.2\}, \min\{\underline{\underline{0.1}}, 0.4\},$$

$$\min\{\underline{0.1}, 0.1\}\} = \underline{\underline{0.1}}$$

$$\mu_T(x_1, z_2) = \max\{\min\{\underline{\underline{0.1}}, 0.1\}, \min\{\underline{\underline{0.1}}, 0.1\},$$

$$\min\{\underline{0.1}, 0.1\}\} = \underline{\underline{0.1}}$$

$$\mu_T(x_2, z_1) = \max\{\min\{0.2, 0.2\}, \min\{0.4, 0.4\},$$

$$\min\{0.1, 0.1\}\} = \underline{\underline{0.4}}$$

$$\mu_T(x_2, z_2) = \max\{\min\{0.2, \underline{\underline{0.1}}\}, \min\{0.4, \underline{\underline{0.1}}\},$$

$$\min\{0.1, \underline{\underline{0.1}}\}\} = \underline{\underline{0.1}}$$

$$\mu_T(x_3, z_1) = \max\{\min\{0.2, 0.2\}, \min\{0.4, 0.4\},$$

$$\min\{0.1, 0.1\}\} = \underline{\underline{0.4}}$$

$$\mu_T(x_3, z_2) = \max\{\min\{0.2, \underline{\underline{0.1}}\}, \min\{0.4, \underline{\underline{0.1}}\},$$

$$\min\{0.1, \underline{\underline{0.1}}\}\} = \underline{\underline{0.1}}$$

$$\mu_T(x_4, z_1) = \max\{\min\{0, 0.2\}, \min\{0, 0.4\},$$

$$\min\{0, 0.1\}\} = 0$$

$$\mu_T(x_4, z_2) = \max\{\min\{0, 0.1\}, \min\{0, 0.1\},$$

$$\min\{0, 0.1\}\} = \underline{\underline{0}}$$

DEFUZZIFICATION

1. Fuzzification converts the crisp input (binary) into fuzzy value.
2. In general, fuzzy result generated can not be used in an application.
3. Any controller can only understand the crisp output. So it is necessary to convert fuzzy output into crisp value.
4. 'Defuzzification' converts fuzzy output into crisp value.
5. There is no systematic procedure for choosing a good defuzzification strategy. Selection of defuzzification procedure depends on the properties of the application.

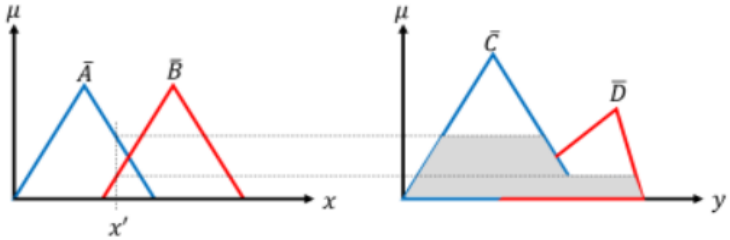
RULE BASE

Consider the following two rules in the fuzzy rule base.

R_1: If x is A then y is C

R_2: If x is B then y is D

A pictorial representation of the above rule base is shown in the following figures



What is the **crisp output** for an input say x' ?

DEFUZZIFICATION METHODS

① Lamda cut method (α - cut method)

$$\tilde{A}_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha , \alpha \in (0, 1] \}$$

* Example :

① $\tilde{A} = \{ (x_1, 1), (x_2, 0.5), (x_3, 0.3), (x_4, 0.4) \}$

Find $\tilde{A}_{0.5}$, $\tilde{A}_{0.4}$

② $\tilde{R} = \begin{bmatrix} 0.9 & 0 & 0.2 \\ 1 & 0.5 & 0 \\ 0.3 & 1 & 0 \end{bmatrix}$

Find $\tilde{R}_{0.8}$, $\tilde{R}_{0.3}$

Solution : $\tilde{R}_{0.8} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\tilde{R}_{0.3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

* Propertio of α-cut

Let $\tilde{A}$ and $\tilde{B}$ be any two fuzzy sets defined on same universe of discourse (X) then

ⓐ $(\tilde{A} \cup \tilde{B})_\alpha = \tilde{A}_\alpha \cup \tilde{B}_\alpha$

ⓑ $(\tilde{A} \cap \tilde{B})_\alpha = \tilde{A}_\alpha \cap \tilde{B}_\alpha$

(c) $(\tilde{A}^c)_\alpha \neq (\tilde{A}_\alpha)^c$

(d) for any $\alpha \geq \beta$ implies $\tilde{A}_\alpha \subseteq \tilde{A}_\beta$

(2) Maxima Methods:

Maxima methods are quite simple but not as trivial as lambda cut methods. Maxima methods relies on the position of maximum membership of element at particular position in fuzzy set.

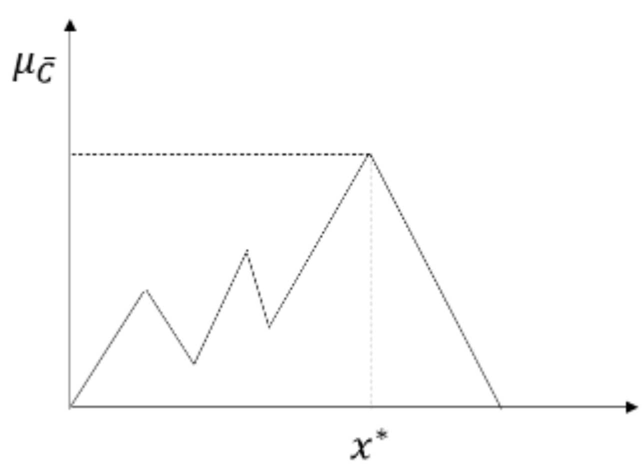
The set of methods under maxima methods we will be discussing here are:

- Height method
- First of maxima (FoM)
- Last of maxima (LoM)
- Mean of maxima (MoM)

Height method:

This method is based on **Max-membership principle**, and defined as follows.

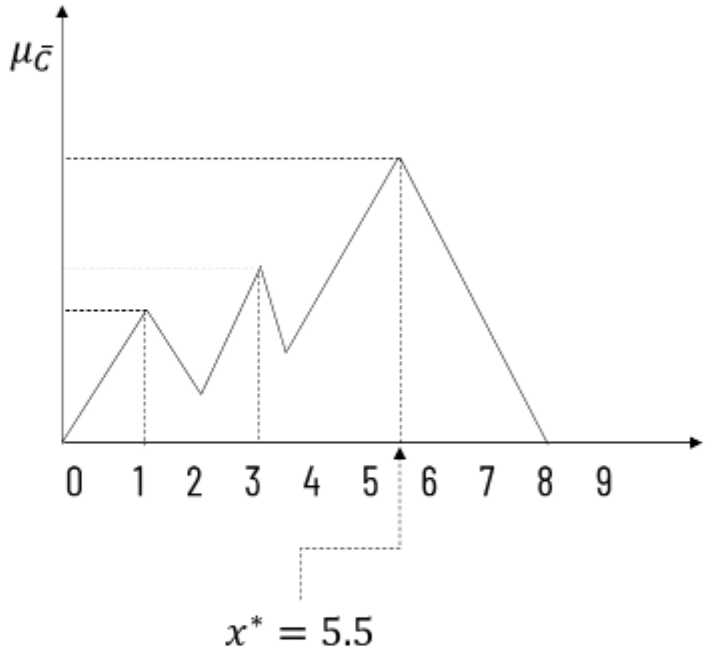
$$\mu_{\underline{C}}(x^*) \geq \mu_{\underline{C}}(x), \forall x \in X$$



Height method

Note: This method is applicable when **height is unique**.

Example:

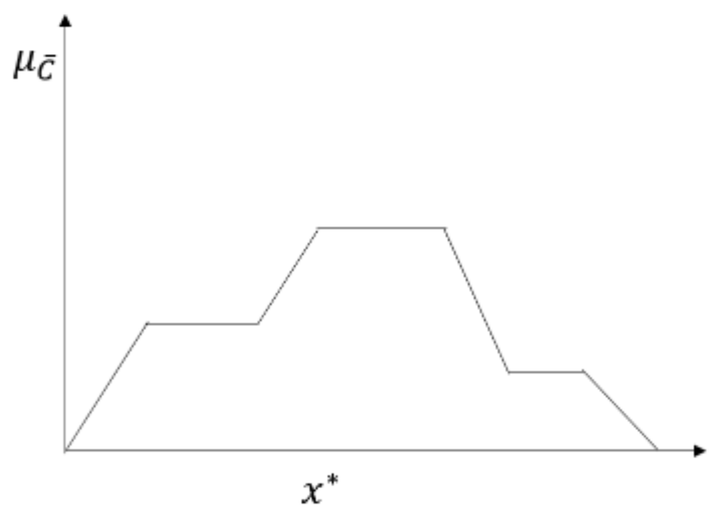


Example of height method

First of Maxima (FoM) method:

Determine the smallest value of the domain with maximized membership degree

FoM = First of Maxima: $x^* = \min\{ x \mid \boldsymbol{\mu}_{\underline{C}}(x) = h(\underline{C}) \}$

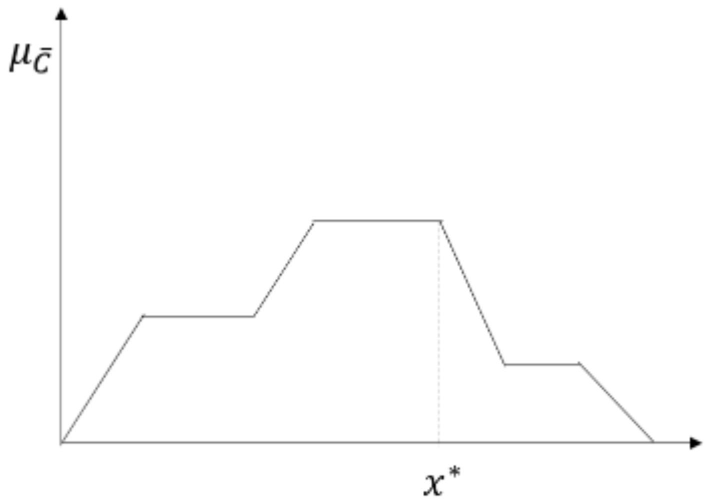


First of maxima

Last of Maxima (LoM) method:

Determine the largest value of the domain with maximized membership degree

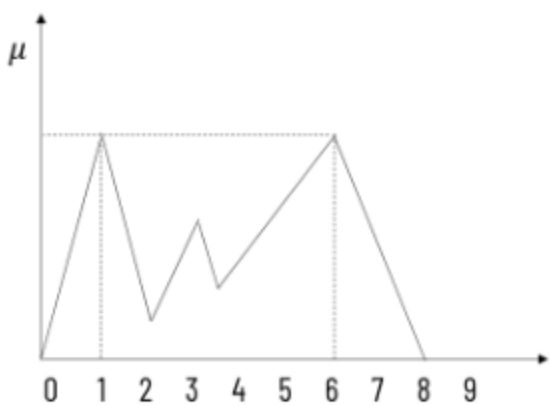
LoM = Last of Maxima: $x^* = \max\{ x \mid \mu_{\underline{C}}(x) = h(\underline{C}) \}$



Last of maxima

Example: First of Maxima and Last of Maxima

Find the defuzzification value for given fuzzy set



First of Maxima: $x^* = 1$

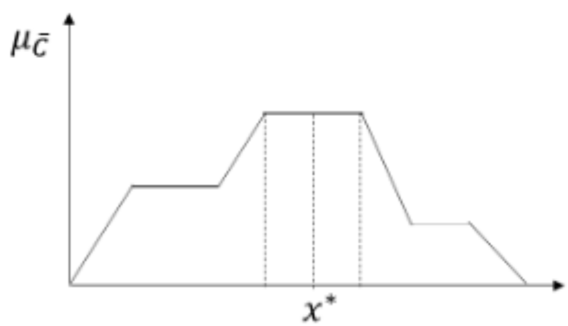
Last of Maxima: $x^* = 6$

Middle of Maxima (MoM) method:

In order to find middle of maxima, we have to find the “middle” of elements with maximum membership value

$$x^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

Where, M ={ x_i | $\mu_{\underline{C}}(x_i)$ = h($\underline{C}$) }, Or M is the set of points having highest membership value



Middle of maxima

Note: This method is applicable to **symmetric functions** only

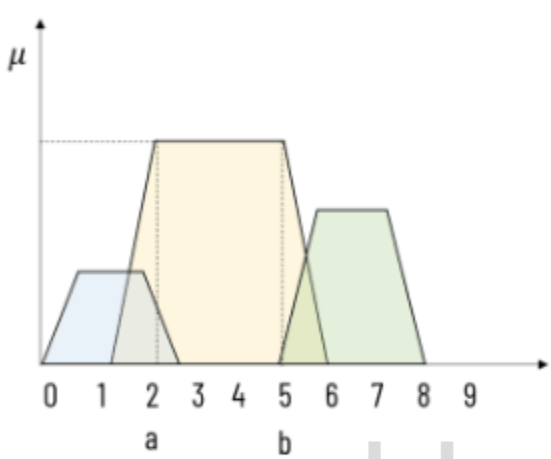
Example: Middle of maxima

Find the deffizified value for given fuzzy set using middle of maxima method:

$x^* = (a + b) / 2$

$x^* = (2 + 5) / 2$

$x^* = 3.5$



(B) Weighted Average Method

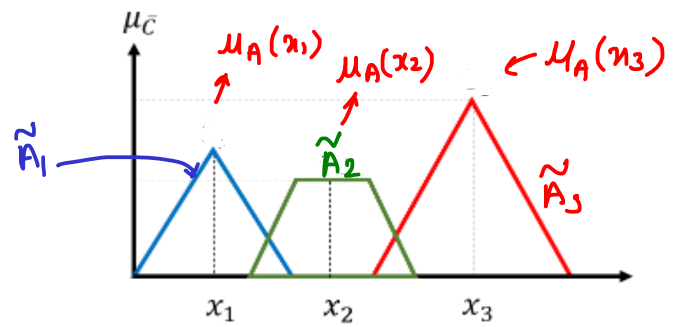
① It is one of the simplest and widely used defuzzification technique

② This method is also called "Sugeno defuzzification" method

③ Formed by weighting each functions in the output by it's respective maximum membership value

④ The crisp value according to this method is

$$x^* = \frac{\sum_i \mu_{A_i}(x_i) \cdot x_i}{\sum_i \mu_{A_i}(x_i)}$$



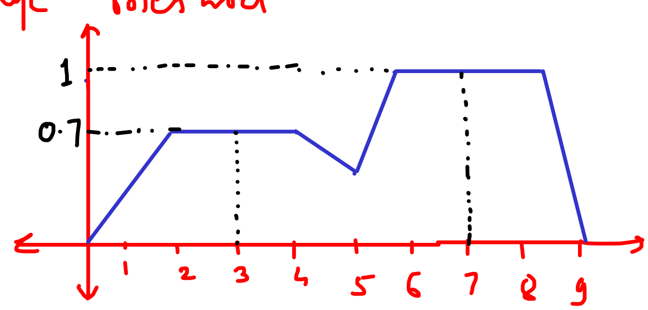
Aggregated fuzzy output

⑤ Less computationally intensive

$$x^* = \frac{x_1 \mu_{A_1}(x_1) + x_2 \mu_{A_2}(x_2) + x_3 \mu_{A_3}(x_3)}{\mu_{A_1}(x_1) + \mu_{A_2}(x_2) + \mu_{A_3}(x_3)}$$

* Example:

Find defuzzified value of given fuzzy output using weighted average method.



Solution:

$$x^* = \frac{(3 \times 0.7) + (7 \times 1)}{0.7 + 1}$$

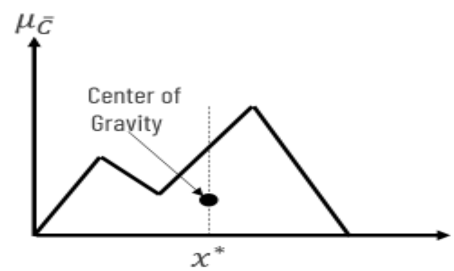
$$x^* = 5.941$$

(c) Center of Gravity (CoG)

① It is most prevalent and physically appealing method from all the defuzzification methods.

② The basic principle in CoG method is find point x^* where a vertical line would slice the aggregate into two equal masses

③ It is defined as



Defuzzification using CoG method

$$x^* = \frac{\sum_i \mu_A(x_i) \cdot x_i}{\sum_i \mu_A(x_i)}$$

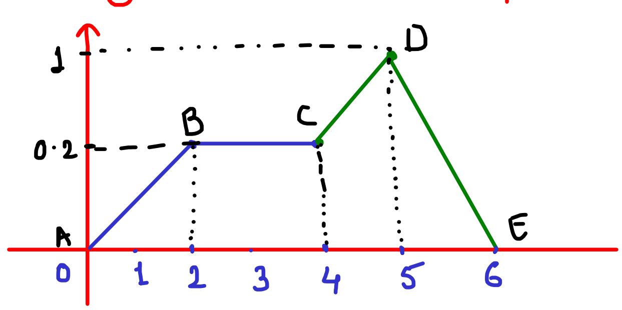
(for discrete values)

$$x^* = \frac{\int \mu_A(x)\, x\, dx}{\int \mu_A(x)\, dx}$$

(for continuous values)

* Example:

Find crisp value corresponding to following fuzzy output using center of gravity method



Solution: ⓐ for line AB

A(0,0), B(2,0.2)

Eq^n of line is

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\therefore \frac{y-0}{x-0} = \frac{0.2-0}{2-0} \Rightarrow \boxed{y = 0.1x}$$

(b) Eq^n of line BC

$$y = 0.2$$

(c) Eq^n of CD

C(4, 0.2), D(5, 1)

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-0.2}{x-4} = \frac{1-0.2}{5-4}$$

$$\Rightarrow \frac{y-0.2}{x-4} = \frac{0.8}{1} \Rightarrow y-0.2 = 0.8x-3.2$$

$$\boxed{\therefore y = 0.8x-3}$$

(d) Eq^n of line DE

D(5, 1) and E(6, 0)

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-1}{x-5} = \frac{0-1}{6-5}$$

$$\Rightarrow \frac{y-1}{x-5} = \frac{-1}{1} \Rightarrow y-1 = -x+5$$

$$\boxed{y = -x+6}$$

$$\therefore x^* = \frac{\int \mu_A(x)\, x\, dx}{\int \mu_A(x)\, dx}$$

$$= \frac{\int_A^B \mu_A(x)\, x\,dx + \int_B^C \mu_A(x)\, x\,dx + \int_C^D \mu_A(x)\, x\,dx + \int_D^E \mu_A(x)\, x\,dx}{\int_A^B \mu_A(x)\,dx + \int_B^C \mu_A(x)\,dx + \int_C^D \mu_A(x)\,dx + \int_D^E \mu_A(x)\,dx}$$

$$= \frac{\int_0^2 (0.1x)\,x\,dx \quad \int_2^4 (0.2)\,x\,dx + \int_4^5 (0.8x-3)\,x\,dx + \int_5^6 (-x+6)\,x\,dx}{\int_0^2 0.1x\,dx + \int_2^4 0.2\,dx + \int_4^5 (0.8x-3)\,dx + \int_5^6 (-x+6)\,dx}$$

$$= \frac{\frac{4}{15} + \frac{6}{5} + \frac{83}{30} + \frac{8}{3}}{\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{1}{2}} = \frac{69}{17} = \underline{\underline{4.059}}$$

(D) Centre of Sum (CoS)

In this method crisp value is calculated using area of individual traingle rather than overlapped region in centre of Gravity.

crisp value using centre of sum (CoS) is

$$x^* = \frac{A_1 x_1 + A_2 x_2 + \cdots A_n x_n}{A_1 + A_2 + \cdots + A_n}$$

where x_i = centre of Largest area of fuzzy set $\tilde{A}_i$

A_i = area of the fuzzy set $\tilde{A}_i$

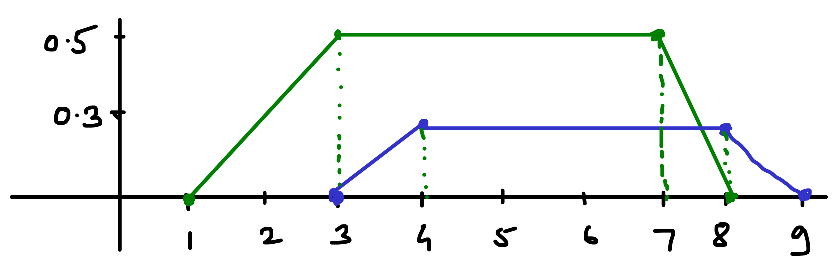
*** Example :**

Find crisp value corresponding to following fuzzy output using center of sum method

$$\tilde{A}_1 = \{(1,0), (3, 0.5), (7, 0.5), (8,0)\}$$

$$\tilde{A}_2 = \{(3,0), (4, 0.3), (8, 0.3), (9,0)\}$$

Soln



$$x^* = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}$$

Where, A_1 = Area of fuzzy set $\tilde{A}_1$

$$= \frac{1}{2}(2 \times 0.5) + (4 \times 0.5) + \frac{1}{2}(1 \times 0.5)$$

$= 2.75$

x_1 = centre of largest area of fuzzy set $\tilde{A}_1$

$= \frac{1}{2}(7+3) = 5$

A_2 = Area of fuzzy set $\tilde{A}_2$

$= \frac{1}{2}(1 \times 0.3) + (4 \times 0.3) + \frac{1}{2}(1 \times 0.3)$

$= 1.5$

x_2 = centre of largest area of fuzzy set $\tilde{A}_2$

$= \frac{1}{2}(4+8) = 6$

$$\therefore \quad x^* = \frac{(5 \times 2.75) + (6 \times 1.5)}{2.75 + 1.5} = \underline{\underline{5.353}}$$