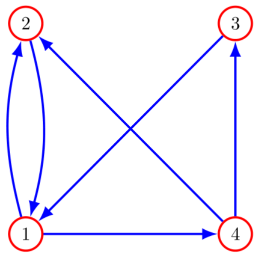
AML5101 | Applied Linear Algebra | Sessional-2 Solutions

1. [10 points] [CO 2, BT 3] Suppose we have four stations that are connected by train services as shown in the following graph:



The *adjacency matrix* associated with this graph has entries defined as

$$A_{ij} = \begin{cases} 1 & \text{if direct train service exists from station } j \text{ to station } i, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write the adjacency matrix A associated with this graph clearly showing its entries.

Solution:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) In plain English, explain what Ae_3, $(A\mathbf{1})_2$, and $\left(A^{\mathrm{T}}\mathbf{1}\right)_1$ mean in terms of stations and trains.

Solution:

- Ae_3 is the 3rd column of A which represents the destination stations for which there is a direct train from the 3rd station.
- $(A\mathbf{1})_2$, represents the number of stations from which we have a direct train to the 2nd station.
- $\left(A^{\mathrm{T}}\mathbf{1}\right)_1$ represents the number of stations for which there is a direct train from the 1st station.

2. [10 points] [CO 2, BT 3] The n-vector c gives the daily number of the COVID-19 cases over n days, with $n \geq 4$. The $(n-3)$-vector d gives the difference between the current number of cases and the average number of cases over the previous three days, starting from the fourth day. Specifically, for

$i = 1, \ldots, n-3$, we have

$$d_i = c_{i+3} - (c_i + c_{i+1} + c_{i+2})/3.$$

Note that d is an $(n-3)$-vector. For the specific case $n = 6$, write down the entries of matrix A clearly such that $d = Ac$.

Solution:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -1/3 & -1/3 & -1/3 & 1 & 0 & 0 \\ 0 & -1/3 & -1/3 & -1/3 & 1 & 0 \\ 0 & 0 & -1/3 & -1/3 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}.$$

3. [CO 2, BT 3] Consider the 5×5-matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) For a 5-vector x, how are Ax and x related?

Solution:

$$A^2 = A \times A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

which means that A^2x results in the vector x shifted downward by 2 places.

(b) Compute A^2 defined as the matrix-matrix product $A \times A$. Use the result to identify what A^5 would be without any further calculations.

Solution: Using the logic that A^2x results in the vector x shifted downward by 2 places, we see that A^5x shifts the vector x downward by 5 places which results in the original vector itself. That is, $A^5x = x$, and therefore A^5 is the 5×5-identity matrix.

4. [10 points] [CO 2, BT 4] We consider a collection of n people who participate in a social network in which pairs of people can be connected by *friending* each other. The $n \times n$-matrix F is the friend matrix, whose $(i, j)th$ entry is defined by

$$F_{ij} = \begin{cases} 1 & \text{if persons i and j are friends,} \\ 0 & \text{if not.} \end{cases}$$

We assume that the friend relationship is symmetric, i.e., person i and person j are friends means person j and person i are friends. We will also assume that $F_{ii} = 0$.

(a) Suppose t is the n-vector such that t_i represents the number of friends the ith person has. Express t as a matrix-vector product using an appropriate matrix and an appropriate vector.

> **Solution:** $t = F\mathbf{1}$, where $\mathbf{1}$ is the ones vector.

(b) Suppose C is the $n \times n$-matrix with C_{ij} equal to the number of friends persons i and j have in common. The diagonal entry C_{ii}, which is the total number of friends person i has in common with herself, is the total number of friends of person i. Express C as a product of two appropriate matrices.

> **Solution:** $C = F \times F = F^2$.

5. [10 points] [CO 2, BT 3] Consider solving the system of equations $Ax = b$ for the matrix A given below:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & -2 & 4 \\ -2 & -1 & -5 \end{bmatrix}.$$

The RREF of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & b_2 \\ 0 & 1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \\ 0 & 0 & 0 & -b_1 + 3b_2 + b_4 \end{array}\right].$$

(a) For one of the following right hand side vectors, the system is consistent. Which one is that:

$$\text{(i) } b = \begin{bmatrix} 5 \\ -9 \\ 1 \end{bmatrix} \text{ (ii) } b = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ (iii) } b = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix} ?$$

Solution: There was a typo in the problem. The components of vector b should satisfy:

$$\begin{aligned} -2b_1 - b_2 + b_3 &= 0 \\ -b_1 + 3b_2 + b_4 &= 0 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix}}_{\text{Augmented matrix}} \longrightarrow \underbrace{\begin{bmatrix} \boxed{1} & 0 & -3/7 & -1/7 \\ 0 & \boxed{1} & -1/7 & 2/7 \end{bmatrix}}_{\text{rref}}.$$

The corresponding solution is:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \left\{ b_3 \begin{bmatrix} 3/7 \\ 1/7 \\ 1 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 1/7 \\ -2/7 \\ 0 \\ 1 \end{bmatrix} \middle| \, b_3, b_4 \in \mathbf{R} \right\}.$$

(b) For the following right hand side vector, solve the system of equations and express the answer as a vector clearly showing the particular solution and the free variable(s) part, if any:

$$b = \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix}.$$

Solution: Any vector b that is of the form derived in the previous part will make the system consistent. For example, if we choose $b_3 = 1$ and $b_4 = 0$, we get

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 1/7 \\ 1 \\ 0 \end{bmatrix},$$

which will make the system consistent.