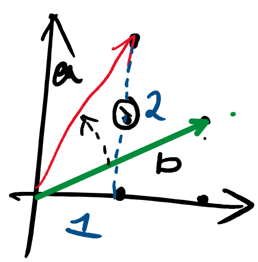
Dot product between two vectors

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow a \cdot b = 1*4 + 2*5 + 3*6$$
$$= 32$$

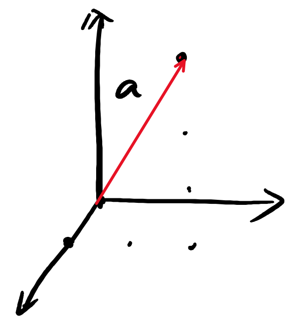
- What does the dot product tell us about the vectors?

- Geometric interpretation of a vector

$$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



- Geometric length of a vector $\|a\|_2 = \sqrt{a_1^2 + a_2^2}$

(L2 norm of a) $= \sqrt{1^2 + 2^2}$

- Angle between vectors $\left\{ -1 \leq \dfrac{a \cdot b}{\|a\| \|b\|} \leq 1 \right.$ (Cauchy-Schwarz inequality)

$$\boxed{\cos(\angle a, b) = \frac{a \cdot b}{\|a\| \|b\|}}$$

$\boxed{\begin{aligned} a \cdot b &= 0 \\ \angle a, b &= \pi/2 \end{aligned}}$

$a \cdot b = -ve$

$\cos(0) = 1$

$\cos(\pi) = -1$

$) = 0$

$\cos$

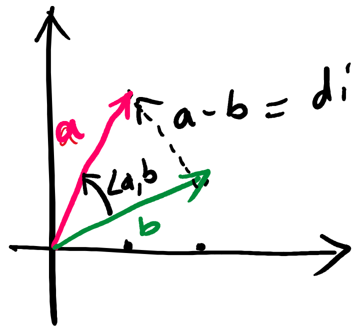
$\cos(\pi) = -1$

$\cos(\pi/2) = 0$

$\boxed{a \cdot b = +ve \quad \angle a,b < \pi/2}$

$\boxed{a \cdot b = -ve \quad \angle a,b > \pi/2}$

- Distance between vectors



$a - b$ = difference vector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

How big is the difference vector $= \|a - b\|_2$

$= \left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 1^2}$

$= \sqrt{2}$

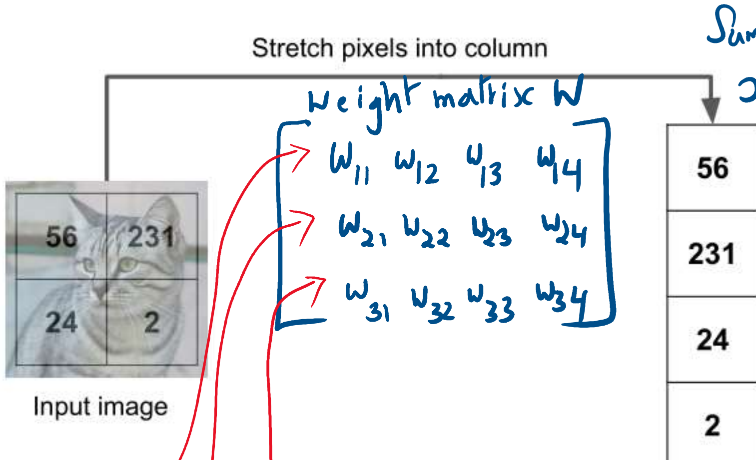
- An important relationship : $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$\|a\|^2 = a \cdot a$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\left(\sqrt{a_1^2 + a_2^2}\right)^2 \Longleftrightarrow a_1^2 + a_2^2$$

Example with an image with 4 pixels, and 3 classes (cat/dog/ship)



Weight matrix W

$$\begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \end{bmatrix}$$

Sample vector x

Weights for cats

$w_1 = \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{bmatrix}$

weights for dogs

$w_2 = \begin{bmatrix} w_{21} \\ w_{22} \\ w_{23} \\ w_{24} \end{bmatrix}$

weights for ship

$\begin{bmatrix} w_{31} \end{bmatrix}$

Cat score $= w_1 \cdot x$

Dog score $= w_2 \cdot x$

$$\boxed{\begin{aligned}&\text{Dog score} = w_2 \cdot x\\ &\text{ship score} = w_3 \cdot x\end{aligned}}$$

weights for ship

$$w_3 = \begin{bmatrix} w_{31} \\ w_{32} \\ w_{33} \\ w_{34} \end{bmatrix} \qquad \begin{bmatrix} w_{23} \\ w_{24} \end{bmatrix}$$

$\hookrightarrow$ matrix-vector product Wx

$W \to$ 3×4-vector

$x \hookrightarrow$ 4-vector

Why did we not do this?

$$\begin{matrix} w_1 \cdot x \\ w_2 \cdot x \\ w_3 \cdot x \end{matrix} \overset{?}{=} \underbrace{\begin{bmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \\ w_{14} & w_{24} & w_{34} \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \begin{bmatrix} w_1 \cdot x & w_2 \cdot x & w_3 \cdot x \end{bmatrix}$$

A matrix-vector product Wx = sequence of dot products

= dot products between the rows of matrix W (seen as vectors) and the vector x

Tensor-vector product	Matrix-vector product
E.g. $T \to (4, 3, 2)$, $x \to (2,)$	E.g. $A \to (3, 4)$, $x \to (4,)$

Linear combination of columns of a matrix A

Linear combination of columns of a matrix A

1st row of A

2nd row of A

3rd row of A

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \begin{cases} \begin{bmatrix} a^{(1)}\cdot x \\ a^{(2)}\cdot x \\ a^{(3)}\cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \\ x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = \begin{bmatrix} \; \end{bmatrix} \end{cases}$$

1st column of A

2nd column of A

3rd column of A

4th column of A

$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 =$ linear combination of columns of A using the elements of x as multipliers

Matrix-vector product

E.g. $A = \underset{2\times 3}{\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}}, \quad x = \underset{3}{\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}} = \begin{bmatrix} 1\times 7 + 2 * 8 + 3 * 9 \\ 4 \times 7 + 5 \times 8 + 6 \times 9 \end{bmatrix}$

$\parallel$ 2

Ax is defined as $= \begin{bmatrix} (\text{1st row of } A)\cdot x \\ (\text{2nd row of } A)\cdot x \end{bmatrix} = \begin{bmatrix} a^{(1)}\cdot x \\ a^{(2)}\cdot x \end{bmatrix}$

Tensor-Vector product

Tensor-Vector product

E.g. $T = \left[\begin{array}{c} \boxed{\begin{bmatrix} 76 & 120 \\ 74 & 124 \\ 78 & 136 \end{bmatrix}} \\ \boxed{\begin{bmatrix} 78 & 136 \\ 80 & 124 \\ 70 & 120 \end{bmatrix}} \end{array} \right]$, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Timestamp-0

Timestamp-1

2 × 3 × 2

Timestamps patients features

2

$$\begin{bmatrix} 76 & 120 \\ 74 & 124 \\ 78 & 136 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 76 \\ 74 \\ 78 \end{bmatrix}$$

$$\begin{bmatrix} 78 & 136 \\ 80 & 124 \\ 70 & 120 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 80 \\ 70 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 76 & 74 & 78 \\ 78 & 80 & 70 \end{bmatrix}$$

2 × 3

Matrix-Matrix Product

E.g. $A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}$. $B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$ $= \begin{bmatrix} a^{(1)} \cdot b_1 & a^{(1)} \cdot b_2 \\ a^{(2)} \cdot b_1 & a^{(2)} \cdot b_2 \end{bmatrix}$

2 × 3

3 × 2

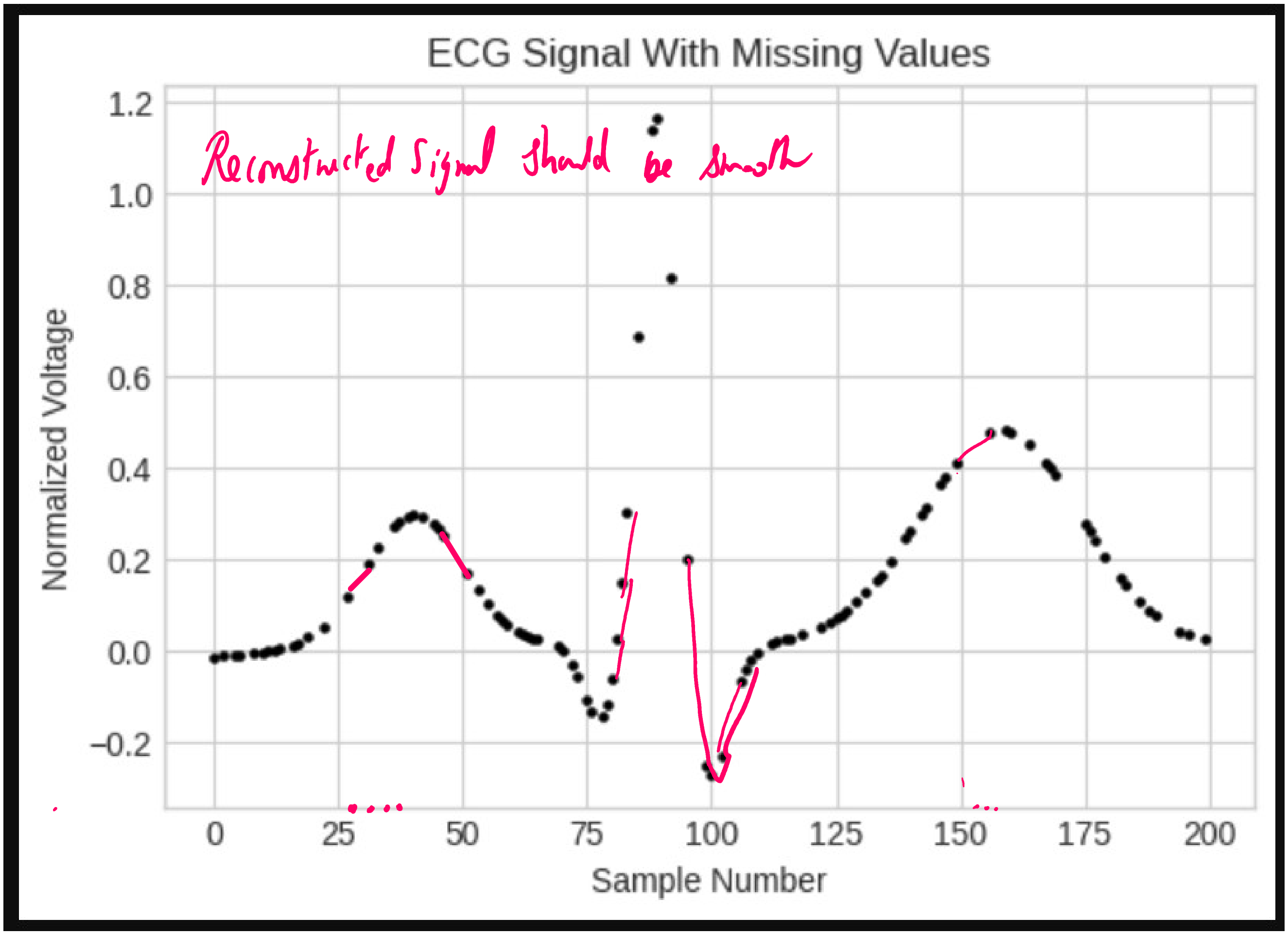
2 × 2

1st row of A

1st column of B

2nd row of A

1st column of B



$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}. \quad x = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ x_4 \\ v_2 \\ v_3 \end{bmatrix}.$$

missing ECG values

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & ? & ? & ? & ? & ? \\ ? & 1 & ? & ? & ? & ? \\ ? & ? & ? & 1 & ? & ? \end{bmatrix}}_{\text{Filter matrix for known ECG values}} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ x_4 \\ v_2 \\ v_3 \end{bmatrix} .
$$

Transposes

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

ECG vector with unknown values turned to zeros

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}}_{\text{vector of known ECG values}} . = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix}
$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{S_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}}_{\Downarrow\ x_{known}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S_2} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\Downarrow\ x_{unknown}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ x_4 \\ v_2 \\ v_3 \end{bmatrix}}_{\Downarrow\ x}$$

x Full ECG vector with missing values

$$\Downarrow \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \Downarrow \begin{bmatrix} 0 \\ 0 \\ v_1 \\ 0 \\ v_2 \\ v_3 \end{bmatrix}$$

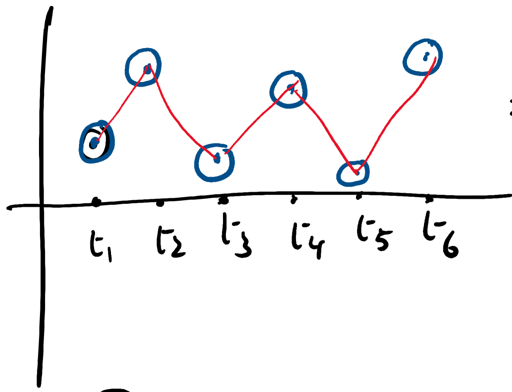
$$\Rightarrow \boxed{\underbrace{x}_{\Downarrow\ \text{ECG vector}} = \underbrace{S_1 x_{known}}_{\text{Known part of the ECG vector}} + \underbrace{S_2 x_{unknown}}_{\text{unknown part of the ECG vector}}}$$

Aside

$$\underbrace{\begin{bmatrix} \boxed{1 & -2 & 1} & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}}_{D} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} \boxed{y_1 - 2y_2 + y_3} \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ y_4 - 2y_5 + y_6 \end{bmatrix}$$

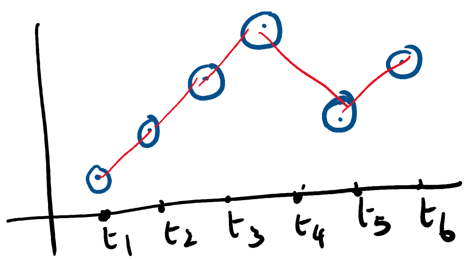
$$(y_1 - y_2) - (y_2 - y_3)$$

$$y = \begin{bmatrix} 15 \\ 30 \\ 10 \\ 25 \\ 5 \\ 30 \end{bmatrix} \Rightarrow Dy = \begin{bmatrix} 15 - 60 + 10 \\ 30 - 20 + 25 \\ 10 - 50 + 5 \\ 25 - 10 + 30 \end{bmatrix} = \begin{bmatrix} -35 \\ 35 \\ -35 \\ 45 \end{bmatrix}$$

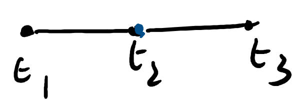


$\Rightarrow$ oscillatory signal

$$y = \begin{bmatrix} 15 \\ 20 \\ 25 \\ 30 \\ 20 \\ 25 \end{bmatrix}, \quad Dy = \begin{bmatrix} 15 - 40 + 25 \\ 20 - 50 + 30 \\ 25 - 60 + 20 \\ 30 - 40 + 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -15 \\ 15 \end{bmatrix}$$



Discrete version of second derivative



(Approx)

$$\frac{y_2 - y_1}{t_2 - t_1} = \frac{\text{change in temperature}}{\text{change in time}} = \frac{{}^\circ c}{Hr}$$

Sensitivity of temperature w.r.t. time

$y_2 - y$ ... $= {}^\circ c$

$$\frac{y_3 - y_2}{t_3 - t_2} = \frac{{}^\circ C}{Hr}$$

Detour

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boxed{\frac{\|x\|}{\sqrt{n}}} = \frac{\sqrt{1^2+1^2+1^2+1^2}}{\sqrt{4}} = \frac{\sqrt{4}}{\sqrt{4}} = 1$$

$\rightarrow Rms(x)$

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \frac{\|x\|}{\sqrt{n}} = 1$$

Reconstruction goal { the reconstructed ECG signal should be as smooth as possible

$$RMS\Big(D\big(\underbrace{S_1 x_{known} + S_2 x_{unknown}}_{\text{ECG vector}}\big)\Big)$$ to be as small as possible $\Rightarrow$ $$\frac{\|D(S_1 x_{known} + S_2 x_{unknown})\|_2}{\sqrt{n}}$$

$\sqrt{n}$ → constant

Should be as small as possible

$\Rightarrow$ Find the vector $x_{unknown}$ s.t. $\boxed{\|D(S_1 x_{known} + S_2 x_{unknown})\|_2^2}$ is as small as possible.

is as small as possible.

b = 4-vector $\quad$ A = 4x3-matrix $\quad$ unknown vector $\cancel{x}$ = 3-vector

$$\Rightarrow \boxed{\underbrace{\underbrace{D}_{4\times 6}\underbrace{S_1}_{6\times 3}}_{4\times 3}\underbrace{x_{\text{known}}}_{3\times 1}} + \boxed{\underbrace{D}_{4\times 6}\underbrace{S_2}_{6\times 3}}\boxed{\underbrace{x_{\text{unknown}}}_{3\times 1}} \Rightarrow \|\cancel{A}b + Ax\|_2^2$$

Not all linear systems of equations $Ax = b$ are solvable

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

D matrix

Standard least squares problem

Find a vector x s.t. $\|Ax - b\|_2^2$ is minimized

A: known matrix $\quad$ b: known vector

In Python, we say linalg.lstsq(A,b) to get the solution for the least squares problem.

For the ECG data, recall that we wanted to minimize

$$\| \underbrace{DS_1 x_{known}}_{-b} + \underbrace{DS_2}_{A} \underbrace{x_{unknown}}_{x} \|_2^2$$

Match A and b in the expression above with $\|Ax - b\|_2^2$

Solving systems of equations

Consider the following model for opinion formation among n individuals, each of whom interact with a certain number of individuals in the group. The numerical value of the ith person's opinion is denoted as x_i. The value of x_i is influenced by the following:

- The ith person's self opinion denoted as s_i
- The opinions of the remaining individuals x_j, where $j = 1, 2, \ldots, n$ and $j \neq i$.

Assuming that the ith person gives a weightage w_{ij} to the jth person's opinion, we can compute x_i as follows:

$$x_i = \frac{s_i + \sum_{j \neq i} w_{ij} x_j}{1 + \sum_{j \neq i} w_{ij}}, \quad i = 1, \ldots, n.$$

1. From the equation above, what do you see is the weightage that a person gives to their own opinion?
2. The equation above can be written as $(A + I)x = s$, where A is an $n \times n$-matrix and I represents the identity matrix. What are the elements of the matrix A, vectors x and s?

Scenario 4 people in a network

Self-opinion vector $= s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$ Can be calculated based on individual history.

Opinion vector $= x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that is built after interaction with others

1 → weightage that the 1st individual gives to the 2nd individual's opinion

Weights matrix $= W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$

$$x_i = \frac{s_i + \sum_{j \neq i} w_{ij} x_j}{1 + \sum_{j \neq i} w_{ij}},$$

Suppose $i = 1$, the 1st individual

Opinion of 1st individual $x_1 = \underline{s_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4}$

$$=1 \leftarrow w_{11} + w_{12} + w_{13} + w_{14}$$

$$\Rightarrow x_1 \overset{=1}{w_{11}} + \boxed{x_1 w_{12}} + x_1 w_{13} + x_1 w_{14} = s_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4$$

$$\Rightarrow \underbrace{x_1}_{\text{unknown}}(1 + \underbrace{w_{12}+w_{13}+w_{14}}_{\text{Known}}) + (\underbrace{-w_{12}}_{\text{Known}})\underbrace{x_2}_{\text{unknown}} + (\underbrace{-w_{13}}_{\text{Known}})\overset{\text{unknown}}{x_3} + (\underbrace{-w_{14}}_{\text{Known}})\overset{\text{unknown}}{x_4} = \underbrace{s_1}_{\text{Known}}$$

$$\Rightarrow \begin{cases} w_{12}(x_1 - x_2) + w_{13}(x_1 - x_3) + w_{14}(x_1 - x_4) + 1 \cdot x_1 = s_1 \\ w_{21}(x_2 - x_1) + 1 \cdot x_2 + w_{23}(x_2 - x_3) + w_{24}(x_2 - x_4) = s_2 \\ w_{31}(x_3 - x_1) + w_{32}(x_3 - x_2) + 1 \cdot x_3 + w_{34}(x_3 - x_4) = s_3 \\ w_{41}(x_4 - x_1) + w_{42}(x_4 - x_2) + w_{43}(x_4 - x_3) + 1 \cdot x_4 = s_4 \end{cases}$$

Given the weights matrix W and the self-opinion vector s, we get 4 equations in 4 unknowns

Recap of the matrix-vector product

$A=$
```
[[ 1  2 -1 -1]
 [ 2  4 -2  3]
 [-1  1 -2  4]]
```
, $x = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$Ax = \begin{bmatrix} a^{(1)} \cdot x \\ a^{(2)} \cdot x \\ a^{(3)} \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$a^{(1)} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Linear combination of the columns of A using the elements of x as multipliers

$$a^{(1)} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} a^{(2)} \cdot x \\ a^{(3)} \cdot x \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Same answer

as multipliers

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}, a_4 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4$$

$$= -1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} + (0) \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In general, if

A: $m \times n$-matrix

x: n-vector

Then

$$Ax = \begin{cases} \begin{bmatrix} a^{(1)} \cdot x \\ a^{(2)} \cdot x \\ \vdots \\ a^{(m)} \cdot x \end{bmatrix} & m \text{ dot products} \\ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n & \text{linear combination} \end{cases}$$

System of equations with

(1) no solution: $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 5 \end{cases}$ no solution

Solution exists means the system is consistent. otherwise the system is inconsistent

(2) unique solution: $\begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{cases} \Rightarrow \boxed{\begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix}}$ unique solution

(3) Infinitely many solutions: $\{ x_1 + x_2 = 1 \Rightarrow \boxed{\begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix}}, \begin{bmatrix} x_1 = 0.5 \\ x_2 = 0.5 \end{bmatrix}, \begin{bmatrix} x_1 = 0.4 \\ x_2 = 0.6 \end{bmatrix}$ infinitely many solutions

Elementary row operations

1. Divide/multiply a row by a non zero constant.
2. Subtract a scalar multiple of one row from another row.
3. Exchange two rows.

$$\boxed{\begin{matrix} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{matrix}} \Longrightarrow \begin{matrix} 2(x_1 + x_2) = 2 \\ x_1 + 2x_2 = 1 \end{matrix} \Rightarrow \boxed{\begin{matrix} 2x_1 + 2x_2 = 2 \\ x_1 + 2x_2 = 1 \end{matrix}}$$

$$\boxed{\begin{matrix} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{matrix}} \Longrightarrow \boxed{\begin{matrix} x_1 + 2x_2 = 1 \\ x_1 + x_2 = 1 \end{matrix}}$$

$$\boxed{\begin{matrix} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{matrix}} \xrightarrow{Eq① = Eq① + 2 * Eq②} \boxed{\begin{matrix} 3x_1 + 5x_2 = 3 \\ x_1 + 2x_2 = 1 \end{matrix}}$$

Reduced row echelon form (RREF)

1st column

2nd column

4th column

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 6 \\ 0 & 1 & -1 & 1 & 3 \\ -1 & -2 & 1 & 1 & -1 \end{array}\right]}_{\text{Original augmented matrix}} \longrightarrow \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{rref}}$$

$\Downarrow$

x_1, x_2, x_4 as the pivot variables and x_3 is the free variable

$\Downarrow$

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 + x_4 = 6 \\ x_2 - x_3 + x_4 = 3 \\ -x_1 - 2x_2 + x_3 + x_4 = -1 \end{cases} \Longrightarrow \begin{cases} x_1 + x_3 = 1 \\ x_2 - x_3 = 1 \\ x_4 = 2 \\ \cancel{0 = 0} \end{cases}$$

anything about x_3 here? Nothing!

x_3 can be anything

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 1 + x_3 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{matrix} \vdots \\ = \\ = \\ = \end{matrix} \begin{bmatrix} \vdots \\ 1 + x_3 \\ x_3 \\ 2 \end{bmatrix} \Rightarrow \underset{\substack{\Downarrow \\ \text{unknown} \\ \text{vector}}}{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

such that

$$\Rightarrow \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

<u>No solution case</u>

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 5$$

$$\Rightarrow \text{Augmented matrix} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{\text{Row 2} = \text{Row 2} - \text{Row 1}} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

<u>Infinitely many solutions case</u>

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$

$$\Rightarrow \text{Augmented matrix} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right]$$

$$\text{RREF} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

<u>Example</u> A b , Solving $Ax = b$

Example

A , b , Solving $Ax=b$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 2 & 4 & -2 & 3 & 3 \\ -1 & 1 & -2 & 4 & 2 \end{array}\right]$$

Original augmented matrix

```
(Matrix([
[1, 0,  1, 0, -2/5],
[0, 1, -1, 0,  4/5],
[0, 0,  0, 1,  1/5]]), (0, 1, 3))
```

RREF using Sympy

Solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - \frac{2}{5} \\ x_3 + \frac{4}{5} \\ x_3 \\ \frac{1}{5} \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{solution to } Ax=0} + \underbrace{\begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 0 \\ \frac{1}{5} \end{bmatrix}}_{\text{particular solution}}$$

Solution vector x

RHS vector b

$$A\left(x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix}\right) \stackrel{?}{=} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$A\left(x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) + A \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix}$$

$$= x_3 * A \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + A \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix}$$

$$x_3 * \begin{bmatrix} 0 \\ 0 \\ \ldots \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ \ldots \end{bmatrix}$$

✓

$$x_3 * \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

ALA Sessional-2 Review

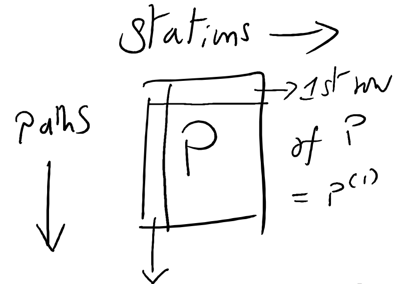
(i) Matrices : matrix-vector and matrix-matrix products

Questions will be scenario-based

E.g. 5 stations and 8 paths

$S_1 \; S_2 \; S_3 \; S_4 \; S_5$

$$\left\{ \begin{matrix} S_1 \; S_4 \; S_5 \\ S_2 \; S_3 \; S_4 \\ S_1 \; S_2 \; S_3 \; S_4 \end{matrix} \right\}$$



1st column of P = P_1

P : 8×5-matrix

P_1 : 8-vector , E.g $P_1 = \begin{array}{c} \text{Station-1} \\ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array}$

station-1 is in the first path

station-2 is in the second path

station-1 is not in the third path

$P^{(1)}$: 5-vector, E.g $P^{(1)} = \begin{array}{c} \text{Path-1} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \end{array}$

Path-1 contains station-1

Path-1 does not contain station-2

E.g.

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 7 \\ 8 \\ 9 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 1 \times 7 + 2 * 8 + 3 * 9 \\ 4 * 7 + 5 * 8 + 6 * 9 \end{bmatrix}$$

The 8x5-matrix P

$\mathbf{1}$-vector $= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $P\mathbf{1}$ is a special matrix-vector product

$$\begin{array}{c} \\ \end{array} P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

P_1, $P^{(1)T}$

$$\Rightarrow P\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \\ 3 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

3 → No. of stations in path-1

2 → No. of stations in path-2

E.g. What about the matrix-vector product $P^T\mathbf{1}$?

P^T → 5x8-matrix, $\mathbf{1}$ → 8-vector

$(\underbrace{P^T\mathbf{1}}_{\text{vector}})_1$ = 1st component of the vector

= How many paths in which station-1 shows up

$(\underbrace{P\mathbf{1}}_{\text{vector}})_3$ = 3rd component of the vector

= How many stations are in the 3rd path

Recall the unit vectors: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$Pe_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0 * \cdots$$

= Paths in which station-1 shows up $= P_1 =$ 1st column of P

Recall example

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad x = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$Px = \begin{cases} \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \\ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \end{bmatrix} & \text{Dot-product version} \\ 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} & \text{linear-combination version} \end{cases}$$

$$\underbrace{e_1^T \underbrace{P e_3}}= ?$$

matrix-vector product $= P_3 =$ 3rd column of matrix P

$=$ Paths in which station-3

matrix-vector produc... $_3$

= Paths in which station-3 shows up

$$\underbrace{e_1^T P_3}_{\text{matrix-vector product}} = P_3^T e_1 = \underbrace{P_3 \cdot e_1 = e_1 \cdot P_3}_{\text{Dot-product}} = \text{1st component of the vector } P_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad P_3 \cdot e_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

In general
$$\begin{cases} Pe_i = P_i = \text{ith column of matrix } P \\ e_i^T P e_j = (i,j)\text{th element of matrix } P \\ e_i^T P = p^{(i)} = \text{ith row of matrix } P \end{cases}$$

Properties of matrix-vector product: $(AB)^T = B^T A^T$, $(A^T)^T = A$

$$e_i^T P = (P^T e_i)^T = e_i^T (P^T)^T = e_i^T P$$

$A = P^T$, $B = e_i$; $B^T = e_i^T$, $A^T = (P^T)^T$

$P^T e_i$ → ith column of P^T = ith row of P

Build a matrix-vector product from a description

E.g. a signal sampled at 6 time stamps

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

. Samples at only the even timestamps

$$= \underbrace{S}_{\text{Sampling matrix}} \underbrace{x}_{\text{Signal}} = \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix}$$

What should be the matrix S such that we weight three successive timestamps equally to generate a new value.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$\frac{1}{3} \cdot x_1 + \frac{1}{3} \cdot x_2 + \frac{1}{3} \cdot x_3$

$\frac{1}{3} x_2 + \frac{1}{3} x_3 + \frac{1}{3} x_4$

?

?

$$\underbrace{\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}}_{\Downarrow \; S} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{\Downarrow \; x}$$

Matrix-Matrix Product

Matrix-Matrix Product

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$$

2×3 $\qquad$ 3×2

matrix-vector product $\qquad$ matrix-vector product

$$\left[\begin{bmatrix} * \\ * \end{bmatrix} \quad \begin{bmatrix} * \\ * \end{bmatrix} \right]$$

2×2

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

8×5 - matrix $\qquad$ 5×8 - matrix

Which matrix-matrix product makes sense: P^TP ✓ or PP^T ✓ or P^2?

P^TP: 5×8, 8×5 $\Rightarrow$ 5×5 - matrix

PP^T: 8×5, 5×8 $\Rightarrow$ 8×8 - matrix

$P^2 = P \cdot P$: 8×5, 8×5 — inner dimensions do not match

$$[AB]_{i,j} = \underset{\substack{\Downarrow \\ i\text{th row}}}{a^{(i)}} \cdot \underset{\substack{\Downarrow \\ j\text{th column}}}{b_j} = a^{(i)T} b_j$$

$$\Downarrow \qquad\qquad \Downarrow$$

ith row of A jth column of B

$$[P^TP]_{ij} = (\text{ith row of } P^T) \cdot (\text{jth column of } P)$$

$$\Downarrow$$

$$(\text{ith column of } P) \cdot (\text{jth column of } P)$$

$$= P_i \cdot P_j = P_i^T P_j$$

for e.g. $i=1, j=2 \Rightarrow [P^TP]_{1,2} = P_1 \cdot P_2$

$$P = \begin{array}{ccccc} P_1 & P_2 & & & \\ \end{array}$$
$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$P_1 \cdot P_2 = 1$ = no. of paths common to station-1 and station-2

How about $P_1 - P_2$? $\quad P_1 - P_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -2/5 \\ 0 & 1 & -1 & 0 & 4/5 \\ 0 & 0 & 0 & 1 & 1/5 \end{array}\right]$$

Always do this first

$\boxed{\text{Is the system consistent or inconsistent?}}$

... not possible ⇒ System is

$0 = 1/5$, not possible $\Rightarrow$ System is inconsistent

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & \\
\end{array}
$$

```
[1, 0,  1, 0, -2/5],
[0, 1, -1, 0,  4/5],
[0, 0,  0, 1,  1/5]]
```

x_1, x_2, x_4 = pivot variables

x_3 = free variable

$$x_1 + x_3 = -2/5$$
$$x_2 - x_3 = 4/5$$
$$x_4 = 1/5$$

$$\Rightarrow \begin{bmatrix} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \end{bmatrix} \begin{bmatrix} -2/5 - x_3 \\ 4/5 + x_3 \\ x_3 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \text{ where } x_3 \in \mathbb{R}$$

This system has infinitely many solutions, but there is an easy choice of x_3 which is equal to 0.

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix} \left(\begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Coefficient matrix A

Solution vector x

Right hand side vector b

$$\begin{bmatrix} \\ \end{bmatrix} \begin{bmatrix} -2/5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Matrix-vector product

$$= \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

matrix-vector product

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$