Math 501: Intro to Real Analysis
Homework 1

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Problem

Let A, B, C be subsets of a set $\mathbf{S}$. Prove the following statements and illustrate them with diagrams.

a $A^c \cup B^c = (A \cap B)^c$

b $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

c $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution

(a)

$$
\begin{aligned}
&\text{Let } x \in S \text{ be an element such that } x \in A^c \cup B^c.\\
\Rightarrow & x \notin A \quad or \quad x \notin B.\\
\Rightarrow & x \notin (A \cap B).\\
\therefore & x \in (A \cap B)^c.\\
\Rightarrow & A^c \cup B^c \subseteq (A \cap B)^c.\\
&\text{Now, let } y \in S \text{ be an element such that } y \in (A \cap B)^c\\
\Rightarrow & y \notin (A \cap B)\\
\Rightarrow & y \notin A \quad or \quad y \notin B\\
\Rightarrow & y \in A^c \quad or \quad y \in B^c\\
\Rightarrow & y \in A^c \cup B^c\\
\Rightarrow & (A \cap B)^c \subseteq A^c \cup B^c\\
& Since, \quad A^c \cup B^c \subseteq (A \cap B)^c \quad and \quad (A \cap B)^c \subseteq A^c \cup B^c\\
& A^c \cup B^c = (A \cap B)^c
\end{aligned}
$$

(b)

Let $x \in S$ be an element such that $x \in A \cap (B \cup C)$.

$\Rightarrow x \in A \quad and \quad x \in B \cup C$

$\Rightarrow x \in A \quad and \quad x \in B \qquad or \qquad x \in A \quad and \quad x \in C$

$\Rightarrow x \in A \cap B \quad or \quad x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Now, Let $x \in S$ be an element such that $x \in (A \cap B) \cup (A \cap C)$.

$\Rightarrow x \in (A \cap B) \quad or \quad x \in (A \cap C)$

$\Rightarrow x \in A \quad and \quad x \in B \qquad or \qquad x \in A \quad and \quad x \in C$

$\Rightarrow x \in A \quad and \quad x \in B \cup C$

$\Rightarrow x \in A \cap (B \cup C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$\therefore (A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

(c)

Let $x \in S$ be an element such that $x \in A \cup (B \cap C)$.

$\Rightarrow x \in A \quad or \quad x \in B \cap C$

$\Rightarrow x \in A \quad or \quad x \in B \qquad and \qquad x \in A \quad or \quad x \in C$

$\Rightarrow x \in A \cup B \quad and \quad x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Now, Let $x \in S$ be an element such that $x \in (A \cup B) \cap (A \cup C)$.

$\Rightarrow x \in (A \cup B) \quad and \quad x \in (A \cup C)$

$\Rightarrow x \in A \quad or \quad x \in B \qquad and \qquad x \in A \quad or \quad x \in C$

$\Rightarrow x \in A \quad or \quad x \in B \cap C$

$\Rightarrow x \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

$\therefore (A \cup B) \cap (A \cup C) = A \cup (B \cap C)$