Lecture 2 August 27th

Model of Stochastic dynamical systems (SDS)

(A) General Model

(1) $x_{t+1} = f_t(x_t, u_t, w_t)$ - System State Sequence

(2) $y_t = h_t(x_t, v_t)$ — Observation Sequence

$t = 0, 1, 2, \ldots$ $\mathbb{Z}_+$ (whole numbers)

u_t - Control action

$x_0, w_0, w_1, \ldots, v_0, v_1, \ldots$ Primitive random variables

(Their nature is not under our control.)

- Queueing example

$A_0, A_1, \ldots$ - arrival sequence of jobs
unit amount of work.

$A_i \in \{0, 1\}$

$w_0^1, w_1^1, w_2^1, \ldots$ Service sequence at server 1

$w_0^2, w_1^2, w_2^2, \ldots$ " 2

$w_j^i \in \{0, 1\}$

System state - number of jobs in each queue

$$x_{t+1}^1 = \max\left(x_t^1 + \mathbb{1}\{u_t = Q_1, A_t = 1\} - w_t^1, 0\right)$$

$$x_{t+2}^2 = \max\left(x_t^2 + \mathbb{1}\{u_t = Q_2, A_t = 1\} - w_t^2, 0\right)$$

$$y_t^1 = x_t^1, \quad y_t^2 = x_t^2 \quad \text{(Perfect observations)}$$

- Spacecraft example

$$x_{t+1} = x_t + u_t + w_t$$

$$y_t = C x_t + v_t$$

Ⓑ Classify Control Strategies / Laws / Policies

① <u>OPEN-LOOP CONTROL</u>

Fix some $u_0, u_1, \ldots$ $\quad \{u_t\}_{t \geq 0}$

Consider system evolution

$$x_1 = f_0(x_0, u_0, w_0)$$

$$x_2 = f_1(x_1, u_1, w_1)$$

$$= f_1(f_0(x_0, u_0, w_0), u_1, w_1)$$

$$= f_{1,0}(x_0, u_0, u_1, w_0, w_1)$$

(For all)

$$\vdots$$

$$\forall t \quad x_t = f_{t-1,0}(x_0, u_0, u_1, \ldots, u_{t-1}, w_0, w_1, \ldots, w_{t-1})$$

Furthermore, Starting from time $m \leq t$, we have

$$\forall\, t \geq m \qquad x_t = f_{t-1,m}(x_m, u_m, \dots, u_{t-1}, w_m, \dots, w_{t-1})$$

Similarly,
$$y_0 = h_0(x_0, v_0)$$
$$y_1 = h_1(x_1, v_1)$$
$$= h_1(f_0(x_0, u_0, w_0), v_1)$$
$$= h_{1,0}(x_0, u_0, w_0, v_1)$$
$$\vdots$$

Different.

$$\forall\, t \qquad y_t = h_{t,0}(x_0, u_0, \dots, u_{t-1}, w_0, \dots, w_{t-1}, v_t)$$

(2) What about the corresponding deterministic system?

(a) No $\{w_t\}_{t\geq 0}$ & $\{v_t\}_{t \geq 0}$

(b) or Set to constants w.p. 1.

$$x_{t+1} = f_t(x_t, u_t) \qquad (y_t = h_t(x_t))$$

x_0 is given & $\{u_t\}_{t\geq 0}$ is the control sequence

Fixing $\{u_t\}_{t\geq 0}$, $x_t = f_{t-1,0}(x_0, u_0, \dots, u_{t-1})$

(3) Feedback policies

Fix $g = (g_0, g_1, g_2, \dots) = \{g_t\}_{t \geq 0}$

$\forall t \qquad u_t = g_t(y_{0:t}, u_{0:t-1})$

$y_{0:t} = (y_0, y_1, \dots, y_t)$ & $u_{0,t-1} = (u_0, u_1, \dots, u_{t-1})$

$$x_1 = f_0(x_0, u_0, w_0)$$

$$= f_0(x_0, g_0(y_0), w_0)$$

$$= f_0(x_0, g_0(h_0(x_0, v_0)), w_0)$$

$$= f_{0,0}(x_0, v_0, w_0, g_0)$$

$$x_2 = f_1(x_1, u_1, w_1)$$

$$= f_1(f_{0,0}(x_0, v_0, w_0, g_0), g_1(y_0, y_1, u_0), w_1)$$

$$= f_{1,0}(x_0, w_0, w_1, v_0, v_1, g_0, g_1)$$

$\Rightarrow \forall t \qquad x_t = f_{t-1,0}(x_0, w_0, w_1, \dots, w_{t-1}, v_0, \dots, v_{t-1}, g_0, \dots, g_{t-1})$

Similarly,

$$y_t = h_{t,0}(x_0, w_0, \dots, w_{t-1}, v_0, \dots, v_t, g_0, \dots, g_{t-1})$$

(4) Specialize to deterministic systems

(Perfectly observed too)

Consider $g = (g_0, g_1, \dots, g_t, \dots)$ is fixed

$\Rightarrow \quad u_t = g_t(x_{0:t}, u_{0:t-1})$

x_0 is given

$x_t = f_{t-1}(x_0, g_0, \ldots, g_{t-1})$

Claim: For deterministic systems (as specified above) feedback control laws and open-loop control laws are equivalent.

Proof: ($\Rightarrow$) Consider an open loop control law

$\hat{u}_0, \hat{u}_1, \ldots$

Define $\quad g = (g_0, g_1, \ldots)$ such that (s.t.)

$g_t(x_{0:t}, u_{0:t}) = \hat{u}_t \quad \forall\, x_{0:t}, u_{0:t-1}$

(constant)

$\Rightarrow \quad x^g_t = f_{t-1,0}(x_0, g_0, \ldots, g_{t-1})$

$= f_{t-1,0}(x_0, \hat{u}_0, \ldots, \hat{u}_{t-1}) = x^{OL}_t$!

($\Leftarrow$) Consider an arbitrary feedback control law

$\hat{g} = (\hat{g}_0, \ldots, \hat{g}_t, \ldots)$.

$\hat{u}_0 = \hat{g}_0(x_0), \quad \hat{u}_1 = \hat{g}_1(x_0, x_1, \hat{u}_0), \ldots,$

$\hat{u}_t = \hat{g}_t(x_0, \dots, x_t, \hat{u}_0, \dots, \hat{u}_{t-1}), \dots$

are known at $t=0$! (Can be computed)

Why? $\hat{u}_0 = \hat{g}_0(x_0)$

$\Rightarrow \quad x_1 = f_1(x_0, \hat{u}_0) = f_{0,0}(x_0)$

$\hat{u}_1 = \hat{g}_1(x_{0:1}, \hat{u}_0)$

$= \hat{g}_{1;0}(x_0)$

$\vdots$

Same logic holds for the whole sequence.

Choose open-loop control law $\hat{u}_0, \hat{u}_1, \dots$

Then, $x_t^{OL} = x_t^{\hat{g}} \quad \forall t$! $\blacksquare$

---

(5) SDS - Open-loop versus feedback control laws.

Feedback control laws can give a richer set of system trajectories.

Example linear 1-D with perfect observations

$$x_{t+1} = x_t + u_t + w_t$$

$$y_t = x_t$$

Consider $g = (g_0, g_1, \dots)$ s.t.

$\forall t \quad u_t = g_t(x_{0:t}, u_{0:t-1})$

$$= g_t(x_t) = -x_t$$

Then $\quad x^g_{t+1} = x_t + u_t + w_t$

$$= x_t + g_t(x_t) + w_t$$

$$= w_t \qquad \forall t$$

$$x^g_0 = x_0, \quad x^g_{t+1} = w_t \quad \forall t \geq 1$$

Consider any OL control sequence

$$\bar{u}_0, \ldots, \bar{u}_t, \ldots$$

We get $\quad x^{OL}_t = x_0 + \sum_{l=0}^{t-1} \bar{u}_l + \sum_{l=0}^{t-1} w_l$

$$= \sum_{l=0}^{t-1} \bar{u}_l + x_0 + \sum_{l=0}^{t-1} w_l$$

Is it possible that $x^{OL}_t = x^g_t \quad \forall t \geq 0$? NO!

Assume that $x_0 \sim \mathcal{N}(0, Q)$ & $w_t \sim \mathcal{N}(0, R)$
iid $t \geq 0$

Mutually independent too!

$$x^{OL}_t = \sum_{l=0}^{t-1} \bar{u}_l + \underbrace{x_0 + \sum_{l=0}^{t-1} w_l}_{T_H \text{ Gaussian random variables}}$$

mutually independent.

$\mathcal{N}(0, Q+tR)$

$$X^{OL}_t \sim \mathcal{N}\left(\sum_{\ell=0}^{t-1} \bar{u}_\ell,\ Q+tR\right)$$

$t \geq 1$ $\quad X^g_t = w_{t-1} \sim \mathcal{N}(0, R)$

$t = 0$ $\quad x^g_t = x_0 \sim \mathcal{N}(0, Q)$

We cannot match!

SDS are different from deterministic system.

---

⑥ Other key difference

DDS

Remember, $\forall t \geq m$

$$x_t = f_{t-1,m}(x_m, u_m, \ldots, u_{t-1})$$

Knowing the state at time m suffices to determine future evolution.

Specifically, $x_{m+1} = f_m(x_m, u_m)$

Knowledge of $x_0, \ldots, x_{m-1}, u_0, \ldots, u_{m-1}$

gives no additional advantage.

No time for a SDS.

<u>Question</u>: $\mathbb{P}(X^g_{t+1} \in A \mid x_t, u_t) \stackrel{?}{=} \mathbb{P}(X^g_{t+1} \in A \mid x_0, \dots, x_t, u_0, \dots, u_t)$

(LHS) (RHS)

<u>Conditional expectation</u>

A event set.

LHS
$$\mathbb{P}(X^g_{t+1} \in A \mid x_t, u_t)$$
$$= \mathbb{P}(f_t(x_t, u_t, W_t) \in A \mid x_t, u_t)$$
$$= \mathbb{P}(\{\omega : f_t(x_t, u_t, \omega) \in A\} \mid x_t, u_t)$$

RHS
$$\mathbb{P}(X^g_{t+1} \in A \mid x_{0:t}, u_{0:t})$$
$$= \mathbb{P}(f_t(x_t, u_t, W_t) \in A \mid x_0, \dots, x_t, u_0, \dots, u_t)$$

X^g_t is a function of

$x_0, W_0, \dots, W_{t-1}, V_0, \dots, V_{t-1}$

U^g_t is a function of

$x_0, W_0, \dots, W_{t-1}, V_0, \dots, V_{t-1}, V_t$

If there is dependence of w_t on any of these random variables, then LHS may not equal RHS.

However, if w_t is independent of

$x_0, w_0, \ldots, w_{t-1}, v_0, \ldots, v_{t-1}, v_t,$

then LHS = RHS!

Core assumption

$$\boxed{\text{A1.} \quad x_0, \{w_t\}_{t\geq 0}, \{v_t\}_{t\geq 0} \text{ are mutually independent}}$$

Under A1

$$\mathbb{P}(x^g_{t+1} \in A \mid x_{0:t}, u_{0:t})$$

$$= \mathbb{P}(x^g_{t+1} \in A \mid x_t, u_t)$$

$$= \mathbb{P}(f_t(x_t, u_t, w_t) \in A \mid x_t, u_t)$$

Not a function of g!

Policy independence!

$f_t(x_t, u_t, w_t) = x_t + u_t + w_t$ $\qquad$ $w_t \sim \mathcal{N}(0, R)$

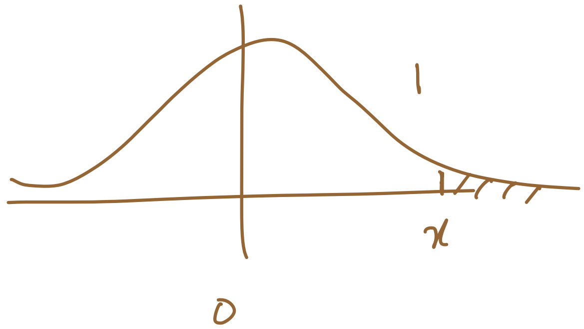
$A = \{x \geq 0\}$ $\qquad$ and A1 was true

$\mathbb{P}(f_t(x_t, u_t, w_t) \in A \mid x_t, u_t)$

$$= \mathbb{P}(w_t \geq -x_t - u_t \mid x_t, u_t)$$

$w_t \sim \mathcal{N}(0, R)$

$Q(x) = \mathbb{P}(\mathcal{N}(0,1) \geq x)$



$$= Q\left(\frac{-x_t - u_t}{\sqrt{R}} \right)$$

Revise Probability

Basic Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$

Ω – Sample Space

$\mathcal{F}$ – Event Space $\quad$ <u>σ-algebra</u>

Collection of subsets of Ω.

1. Non-empty - either Ω or $\phi \in \mathcal{F}$
2. Closed under complements

$$A \in \Omega, \text{ then } A^c = \Omega \setminus A \in \mathcal{F}.$$

3. Closed under countable unions

$$A_1, A_2, \ldots \in \Omega$$

$$\bigcup_{i=1}^{\infty} A_i \in \Omega$$

$$\lim_{n \to \infty} \bigcup_{i=1}^{n} A_i \triangleq \bigcup_{i=1}^{\infty} A_i$$

$\Omega = \{1, 2, \ldots\} = \mathbb{N}$ $\quad \mathcal{F} = 2^{\Omega} = \mathcal{P}(\Omega)$

(all subsets of Ω)

$A_i = \{i\} \quad i = 1, 2, \ldots$

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{N} \quad \left(\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\} \right)$$

$\mathbb{P}$ - Probability

$\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ $\quad$ $\times$ Only events in $\mathcal{F}$ have a probability associated with them!

1. $\mathbb{P}(\Omega) = 1$

2. (Countable additivity)

$A_1, \ldots, A_n, \ldots \in \mathcal{F}$

& mutually disjoint $\quad i \neq j \quad A_i \cap A_j = \phi$

(mutually exclusive)

$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \stackrel{?}{=} \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$