EECS502 Stochastic Processes	Lecture 2 - 01/14/2020

Review of Probability Theory & Stochastic Process

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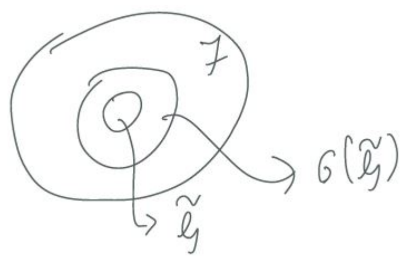
1 Probability Theory

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Definition 1 (Random variable)**.** *X is a random variable, $X : \Omega \mapsto \mathbb{R}$ that is measurable.*
Borel σ-algebra $\mathcal{B}(\mathbb{R})$
$\forall B \in \mathcal{B}(\mathbb{R})$, find $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ (Inverse map).
$\tilde{\mathcal{G}} = \left\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\right\}$ - Collection of subsets of Ω.
$\tilde{\mathcal{G}} \subseteq \mathcal{F}$ or not?
If yes, then X is measurable and a random variable.
In simpler terms, $\forall B \in \mathcal{B}(\mathbb{R})$, $X^{-1}(B) \in \mathcal{F}$.

$\sigma(\tilde{\mathcal{G}})$ - Smallest σ-algebra that contains $\mathcal{G}$.
X is a random variable if and only if $\sigma(\tilde{\mathcal{G}}) \subseteq \mathcal{F}$.



Induced distribution for a random variable is defined on each $B \in \mathcal{B}(\mathbb{R})$ by

$$\begin{aligned}\mathbb{P}_X(B) &= \mathbb{P}(X \in B)\\ &= \mathbb{P}(X^{-1}(B)), \text{ since } X^{-1}(B) \in \mathcal{F}.\end{aligned}$$

We have already defined a random variable X as

$$\begin{aligned}X : \Omega \mapsto& \mathbb{R}.\\ (\mathcal{F}, \mathbb{P}) \ \ & (\mathbb{B}(R))\end{aligned}$$

Now we generalize to random variables taking values in other probability spaces. For example,

$$\begin{aligned}X : \Omega_1 \mapsto& \Omega_2 \text{ (Examples are } \mathbb{R}^2 \text{ or } \mathbb{R}^3).\\ (\mathcal{F}, \mathbb{P}) \ \ & (F_2)\end{aligned}$$

$X : \Omega_1 \mapsto \Omega_2$ is a random variable.
If it is a measurable map,
$\forall B \in \mathcal{F}_2$, $X^{-1}(B) \in \mathcal{F}_1$ needs to hold.

Example 2 (Uniform distribution)**.** $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, $\mathbb{P}$: uniform distribution in $[0, 1]$, i.e., if $[a, b] \subseteq [0, 1]$, $\mathbb{P}([a, b]) = b - a$.

Then for $B \in \mathcal{B}([0,1])$, $\mathbb{P}(B) = \int_0^1 \mathbb{1}_B(x)\,dx$.

Example 3 (Binary expansion)**.** $\omega \in \Omega$, $\omega_i \in \{0,1\}$.

Binary expansion: $0.\omega_1\omega_2\omega_3\ldots$.

Suppose $0.1000\ldots$ with probability $\frac{1}{2}$, or $0.011111\ldots$ with probability $\frac{1}{2}$.

Define a random variable $X : \Omega \mapsto \{0,1\}$, $X(\omega) = \omega_1$, $2^{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$.

Induced measure $\mathbb{P}_X(X = 0) = \frac{1}{2} = \mathbb{P}_X(X = 1)$.

X is a Bernoulli random variable.

Example 4 (Decimal expansion)**.** $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P}$ uniform distribution.

$X : \Omega \mapsto \{0, 1, \ldots, 9\}$, $\omega \to 0.\omega_1\omega_2\omega_3\ldots$.

$X(\omega) = \omega_1$, $2^{\{0,\ldots,9\}}$ contains all subsets.

$\mathbb{P}_X(X = 0) = \mathbb{P}_X(X = 1) = \ldots = \mathbb{P}_X(X = 9) = \frac{1}{10}$ is the induced distribution.

Typically, for discrete random variables, power set (the set of all subsets) is a σ-algebra.

Example 5 (Exponential distribution)**.** $X(\omega) = -\log(\omega)$, $\Omega = [0,1]$.

Space of $X(\omega)$ is $\mathbb{R}_+$, $\mathcal{B}(\mathbb{R}_+)$ is the σ-algebra.

The cumulative distribution function is

$$\begin{aligned}
F_X(x) &= \mathbb{P}_X(X \in [0,x]) \\
&= \mathbb{P}(X^{-1}([0,x])) && (X(\omega) = -\log(\omega)) \\
&= \mathbb{P}([e^{-x}, 1]) && (\omega = e^{-X(\omega)}) \\
&= 1 - e^{-x},
\end{aligned}$$

which is the exponential distribution of parameter 1.

Example 6. $\Omega = \{0,1\}^2$. Elements are $(0,0), (0,1), (1,0)$ & $(1,1)$. (2-bit numbers uniformly chosen.)

$\mathcal{F} = 2^\Omega$ (Power set).

$\mathbb{P}$ uniform on Ω.

$X \to$ First bit; $Y \to$ Second bit; $Z = X \oplus Y$ (XOR), all take values in $\{0,1\} = \Omega_2$, and $\mathcal{F}_2 = 2^{\Omega_2} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$.

$$\begin{aligned}
X^{-1}(\{0\}) &= \{(0,0), (0,1)\},\ X^{-1}(\{1\}) = \{(1,0), (1,1)\} \\
Y^{-1}(\{0\}) &= \{(0,0), (1,0)\},\ Y^{-1}(\{1\}) = \{(0,1), (1,1)\} \\
Z^{-1}(\{0\}) &= \{(0,0), (1,1)\},\ Z^{-1}(\{1\}) = \{(0,1), (1,0)\}.
\end{aligned}$$

Let $\sigma(X)$ denote the smallest σ-algebra under which X is measurable.

$\forall B \in \mathcal{F}_2$, find $X^{-1}(B)$, collect as $\tilde{\mathcal{G}}$. Find $\sigma(\tilde{\mathcal{G}})$.

$\sigma(X) = \{\emptyset, \{(0,0), (0,1)\}, \{(1,0), (1,1)\}, \Omega\}$, which is strictly smaller than 2^Ω. (e.g. $\{(0,0)\} \in 2^\Omega$)

Example 7. $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$.

$$Z(\omega) = \begin{cases} 0 & \text{if } \omega \in [0, 0.5) \\ 1 & \text{if } \omega \in [0.5, 1] \end{cases}$$

$\sigma(Z)$ is a subset of $\mathcal{B}([0,1])$. $\sigma(Z) = \{\emptyset, [0, 0.5), [0.5, 1], [0,1]\}$, where $[0,1] = \Omega$.

Example 8. $\Omega \subseteq \mathbb{R}^2$. $\Omega = \{(x,y) : x^2 + y^2 \le 1\}$.

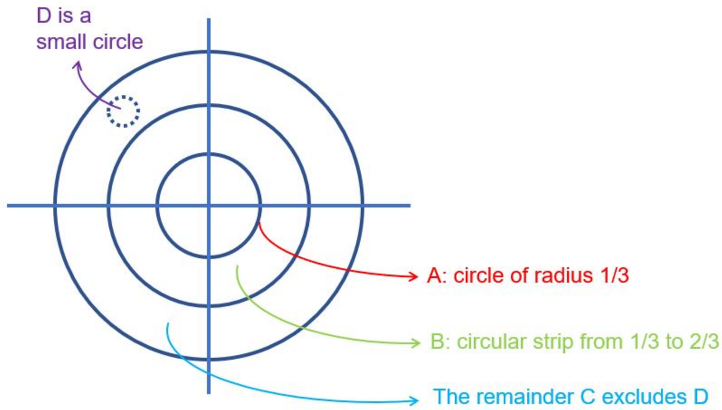
A, B, C, D are disjoint and union is Ω - partition

$\mathcal{F} = \{\emptyset, A, B, C \cup D, A \cup B, A \cup C \cup D, B \cup C \cup D, \Omega\}$.

$X : \Omega \mapsto \{1,2,3,4\} = \Omega_2$, and $\mathcal{F}_2 = 2^{\Omega_2}$.

We define X as:

$$X(\omega) = \begin{cases} 1, & \omega \in A, \\ 2, & \omega \in B, \\ 3, & \omega \in C, \\ 4, & \omega \in D. \end{cases}$$



Is X a random variable?

$\forall E \in \mathcal{F}_2$, check if $X^{-1}(E) \in \mathcal{F}$.
Choose $E = \{3\}$. What is $X^{-1}(E)$? - $X^{-1}(E) = C$.
Does C belong to $\mathcal{F}$? No!
Therefore, X is not a random variable.

Please review:

- Collection of random variables,
- Moments,
- Convergence,
- Independence.

2 Stochastic Processes

Definition 9 (Random process/Stochastic process)**.** *Given a fully ordered index set $\mathcal{I}$ (usually $\mathbb{N}, \mathbb{Z}_+, \mathbb{Z}, \mathbb{R}, \mathbb{R}_+$) and a probability space $(E, \mathcal{E})$, a stochastic process is a function $X : \mathcal{I} \times \Omega \mapsto E$, such that for each $i \in \mathcal{I}$ we have $X(i, \cdot) : \Omega \mapsto E$ is an E-valued random variable, i.e. $X(i, \omega) \in E$.*

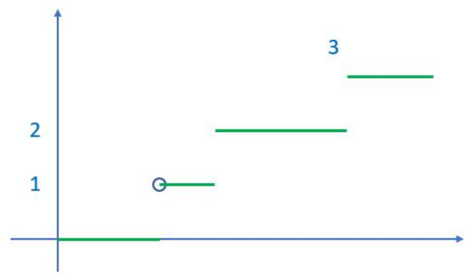
Example 10 (Binary expansion)**.** $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$.
$\omega = 0.\omega_1\omega_2\omega_3\omega_4 \ldots$
Now suppose ω above is a binary expansion.
$X_1(\omega) = \omega_1$, $X_2(\omega) = \omega_2$, ...
If $\{X_i(\omega)\}_{i=1}^{\infty}$, $\mathcal{I} = \mathbb{N}$, $X : \mathcal{I} \times \Omega \mapsto \{0, 1\}$(as E)
$X_1, X_2, \ldots$ are random variables. $\{X_i(\omega)\}_{i=1}^{\infty}$ is a stochastic process, which takes values in $\{0, 1\}$.
Independent and identically distributed (i.i.d.)
Bernoulli($\frac{1}{2}$) random variables
Example 11. $\{X_i\}_{i=1}^{\infty}$, $X_i \sim$ i.i.d. exp(1) random variables.
Example 12. $\{X_i\}_{i=1}^{\infty}$, $X_i \sim$ i.i.d. $\mathcal{N}(0, \sigma^2)$
$S_0 = 0, S_1 = X_1, S_2 = X_1 + X_2, S_3 = X_1 + X_2 + X_3, \ldots$
$S_n = \sum_{i=1}^{n} X_i$ ($n = 0$, then value=0 by definition).

$$\mathbb{E}[S_1 S_2] = \mathbb{E}[X_1^2 + X_1 X_2] = \mathbb{E}[X_1^2] + \mathbb{E}[X_1 X_2]$$
$$= \sigma^2 + \mathbb{E}[X_1]\mathbb{E}[X_2] = \sigma^2 \neq \mathbb{E}[S_1]\mathbb{E}[S_2] = 0.$$

$\{S_j\}_{j=0}^{\infty} = \{S_j\}_{j\in\mathbb{Z}_+}$, $S_i \sim \mathcal{N}(0, i\sigma^2)$

Example 13 (Poisson process)**.** $\{N(t)\}_{t\in\mathbb{R}_+}$, $N(0) = 0$, $N(t) \in \mathbb{Z}_+$(non-negative integers).

Counting process



$N(t) \sim \text{Poisson}(t)$, $\mathbb{P}(N(t) = k) = e^{-t}\frac{k^t}{k!}, k \in \mathbb{Z}_+$.

$N(s,t) = N(t) - N(s)$ - increment.

Independent increments. Jump process.

t_1, t_2, t_3, $(N(t_1) = N(t_1) - N(0), N(t_1, t_2), N(t_2, t_3))$ are all independent,

and $N(s,t) \sim \text{Poisson}((t-s))$, $t \geq s$.

Example 14 (Brownian motion)**.** - Wiener process

$\{W(t)\}_{t\in\mathbb{R}_+}, W(0) = 0, W(t) \sim \mathcal{N}(0,t)$ ($\sim$: distributed as).

Independent increments t_1, t_2, t_3.

$(W(t_1), W(t_1, t_2), W(t_2, t_3))$ independent, and

$W(s,t) = W(t) - W(s) \sim \mathcal{N}(0, t-s), t \geq s$.

Continuous sample-paths

$X : \mathcal{I} \times \Omega \mapsto E$,

$X(i, \cdot)$ is a random variable for every i.

$X(\cdot, \omega)$ - Sample path.