# Differential Geometry

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## Smooth Manifolds

**Definition** (Manifold). *A* ***topological manifold*** *$M$ is a second countable Hausdorff space which is* ***locally Euclidean****, that is, given a point $p \in M$, there exists a neighbourhood $p \in U$ such that $U$ is homeomorphic to some open subset of $\mathbb{R}^n$.*

**Definition** (Atlas). *An* ***atlas*** *is a collection of ordered pairs $(U, \phi)$ called coordinate charts such that $U \subseteq M$ and $\phi : U \to \tilde{U} \subseteq \mathbb{R}^n$ is a homeomorphism.*

**Definition.** *Two coordinate charts $(U, \phi), (V, \psi)$ are said to be* ***smoothly compatible*** *if either $U \cap V = \emptyset$ or $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism.*

**Definition.** *A smooth atlas for a manifold is an atlas such that all the charts are smoothly compatible with one another.*

**Lemma 1.1.** *Given a smooth atlas $A$, there is a unique maximal smooth atlas $\tilde{A}$ containing $A$. We call this the* ***smooth structure*** *on $M$.*

*Remark.* For $n \neq 4, \mathbb{R}^n$ has a unique smooth structure (up to diffeomorphism). However, $\mathbb{R}^4$ has uncountably many distinct smooth structures.

**Definition** (Smoothness). *A function $f : M \to N$ is called* ***smooth*** *at $p$ if there exist coordinate charts $(U, \phi) \subseteq M, (V, \psi) \subseteq N$ such that $f(U) \subseteq V$, and $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth at $\phi(p)$.*

## Tangent Spaces

**Definition** (Derivation). *Given a smooth manifold $M$ of dimension $n$, a* ***derivation*** *at a point $p \in M$ is a linear map $X : C^\infty(M) \to \mathbb{R}$ such that $X(fg) = X(f)g(p) + f(p)X(g)$.*

**Definition** (Tangent Space). *The tangent space of a manifold at a point $p$, $T_pM$, is the vector space (over $\mathbb{R}$) of all derivations at $p$.*

**Definition** (Pushforward). *Given a smooth map $F : M \to N$, the **differential** of $F$ at $p$, denoted by $dF_p$ or $F_{*,p} : T_pM \to T_{F(p)}N$, is defined by $dF_p(X)(f) = X(f \circ F)$.*

**Theorem 2.1.** *$\{\frac{\partial}{\partial e^i}|_a\}$ form a basis for $T_a\mathbb{R}^n$.*

**Lemma 2.2.** *Let $\frac{\partial}{\partial x^i|_p} := d(\phi^{-1})_{\phi(p)}(\frac{\partial}{\partial e^i}|_{\phi(p)})$, where $\phi : U \to \tilde{U} \subseteq \mathbb{R}^n$ is a homeomorphism (making $d\phi_p : T_pU \cong T_pM \to T_{\phi(p)}U \cong T_{\phi(p)}\mathbb{R}^n$ an isomorphism). Then, $\{\frac{\partial}{\partial x^i}|_p\}$ forms a basis for $T_pM$.*

**Lemma 2.3.** *For a smooth map $F : M \to N$, the matrix representation of $dF_p$ is given by $a_{ij} = \frac{\partial F^i}{\partial x^j}|_p := \frac{\partial(\psi \circ F \circ \phi^{-1})^i}{\partial e^j}|_{\phi(p)}$.*

**Definition.** *Given a curve $\gamma : I \to M$ such that $\gamma(0) = p$, define $\gamma_1 \sim \gamma_2 \iff (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for all $f \in C^\infty(M)$.*

**Lemma 2.4.** *Given $\gamma : I \to M$, define $\gamma'(0) = d\gamma_0(\frac{d}{dt}|_0) \in T_{\gamma(0)}M$. Then, given $X \in T_pM$, there exists a smooth curve such that $\gamma'(0) = X$.*

**Theorem 2.5.** *Let $V_pM$ be the vector space constituted by the equivalence classes of curves $[\gamma]$. Then, $V_pM \cong T_pM$ through the mapping $[\gamma] \mapsto \gamma'(0)$.*

*Remark.* The above theorem offers a more geometric perspective on the tangent space. It is constituted by the set of 'curves' (initialized at the same point) on $M$ pointing in all possible directions (wherein two curves are said to point in the same 'direction' if any function has the same derivative along them).

**Definition** (Tangent bundle). *$TM = \sqcup_p T_pM$ is known as the **tangent bundle** of a manifold $M$.*

**Theorem 2.6.** *$TM$ is a smooth manifold of dimension $2n$.*

*Remark.* $n$ from the manifold $M$, and $n$ more from $T_pM$.

## Vector Fields

**Definition** (Vector field)**.** *A **vector field** on a smooth manifold is a map* $X : M \to TM$ *such that* $\pi \circ X = id_M$*. If* $X$ *is a smooth map, we call it a smooth vector field.*

*Remark.* $X$ assigns a vector to each point on the manifold. The set of all smooth vector fields on a manifold, denoted by $\chi(M)$, forms a vector space over $\mathbb{R}$. The following theorem characterizes smooth vector fields.

**Theorem 3.1.** *The following statements are equivalent:*

1. $X$ *is a smooth vector field*
2. $Xf \in C^\infty(M) \forall f \in C^\infty(M)$*, where* $(Xf)(p) := (X_p)f$
3. $\{f_i\}$ *are smooth, where* $X = \sum f_i \frac{\partial}{\partial x^i}$*.*

*Remark.* The expression in (3) is obtained by setting $f_i = X^i$, where $X_p = \sum X^i(p) \frac{\partial}{\partial x^i}|_p$ for a vector field $X$. The expression will be local, since the basis refers to a coordinate neighbourhood of $p$.
The next theorem offers an identification of elements of the vector space $\chi(M)$ with elements of the ring $C^\infty(M)$. Note that, as such, the former forms a module over the latter.

**Theorem 3.2.** *Let* $X$ *be any smooth vector field and* $y : C^\infty(M) \to C^\infty(M)$ *be any derivation. Then, the following hold:*

1. $X \in C^\infty(M)$
2. *There exists a* $Y \in \chi(M)$ *such that* $Yf = yf$ *for all* $f \in C^\infty(M)$

**Definition** (F-related)**.** *Let* $F : M \to N$ *be a smooth map. Then,* $X \in \chi(M)$ *is said to be **F-related** to* $Y \in \chi(N)$ *if* $dF_p(X_p) = Y_{F(p)}$ *for all* $p \in M$*.*

*Remark.* The next lemma offers an equivalent characterization of F-relatedness.

**Lemma 3.3.** *Let* $F : M \to N$ *be smooth. Then,* $X \in \chi(M), Y \in \chi(N)$ *are F-related* $\iff \forall f \in C^\infty(N), X(f \circ F) = (Yf) \circ F$*.*

*Remark.* The next theorem essentially tells us when the pushforward of a vector field $F_*X$ (defined as $F_*(X)(p) = dF_p(X_p)$) is smooth. The condition is slightly stronger than just having $F$ be smooth.

**Theorem 3.4.** *If* $F : M \to N$ *is a diffeomorphism, for every* $X \in \chi(M)$*, there is a unique* $Y \in \chi(N)$ *that is F-related to* $X$*.*

## Submersions, Immersions, Submanifolds

**Definition** (Rank)**.** *The rank of* $F : M \to N$ *at* $p \in M$ *is the rank of the linear map* $dF_p : T_pM \to T_{F(p)}N$*.*

**Definition** (Submersion/Immersion)**.** *A smooth map* $F : M \to N$ *is said to be a submersion (immersion resp.) if it has constant rank* $n$ *(*$m$ *resp.).*

**Definition** (Smooth embedding)**.** *A smooth embedding is an immersion homeomorphic on its image.*

**Theorem 4.1** (Rank theorem)**.** *Let* $F : M \to N$ *be of constant rank* $k$*. Given* $p \in M$*, there exists coordinate neighbourhoods around* $p, F(p)$ *with* $F(U) \subseteq V$ *such that* $\hat{F}(p)(u_1, ...u_m) = (u_1, ...u_k, 0, ...0)$*.*

**Definition** (Submanifold)**.** *Let* $M$ *be a smooth* $n$*-manifold and* $S \subseteq M$ *be a subset with the subspace topology. Furthermore, suppose* $S$ *is covered by charts* $(U, \phi)$ *such that* $\phi(S \cap U)$ *is a* $k$*-slice in* $\mathbb{R}^n$*. Then,* $S$ *is a topological manifold with dimension* $k$*, and we call it an* ***embedded submanifold****.*

**Lemma 4.2.** $i : S \to M$ *is a smooth embedding.*

**Lemma 4.3.** *If* $F : M \to N$ *is a smooth embedding,* $F(M)$ *is an embedded submanifold of* $N$*.*

**Definition.** *Let* $\phi : M \to N$ *be a map.*

1. *For any* $c \in N$*, the set* $\phi^{-1}(c) \subseteq M$ *is called a* ***level set*** *of* $\phi$*.*
2. *If* $p \in M$ *is such that* $d\phi_p$ *is surjective, we call* $p$ *a* ***regular point*** *of* $\phi$*.*
3. *If* $c \in N$ *is such that every* $p \in \phi^{-1}(c)$ *is a regular point, we call* $c$ *a* ***regular value*** *of* $\phi$*.*

*If something is not a regular point (value resp.), we call it a critical point (value resp.).*

**Theorem 4.4** (Constant rank level set theorem)**.** *Let* $\Phi : M \to N$ *be a smooth map of constant rank* $k$*. Then, each level set of* $\Phi$ *is an embedded submanifold of* $M$ *with codimension* $k$*..*

**Lemma 4.5.** *Let $\phi : M \to N$ be a smooth map of constant rank and $S$ be any level set of $\phi$. Then, $T_pS = \ker(d\phi_p)$ for all $p \in S$.*

## Differential Forms

**Definition** (Cotangent space)**.** *A **cotangent vector** is an element of the **cotangent space** $(T_pM)^* = T_p^*M$. The **cotangent bundle** is the collection $T^*M = \sqcup_p T_p^*M$.*

**Definition** (Differential 1-form)**.** *A **differential one-form** is a map $\omega$ from $M \to T^*M$ such that $\pi \circ \omega = Id_M; p \mapsto \omega_p \in T_p^*M$.*

**Definition** (Differential of a map)**.** *For $f \in C^\infty(M)$, the **differential** of $f$ at $p \in M$ is the cotangent vector $df_p : T_pM \to \mathbb{R}$ defined by $df_p(X_p) := X_pf$.*

*Remark.* We have already used the phrase 'differential of a map' in section 2 to refer to something slightly different. If we use the fact that $T_a\mathbb{R} \cong \mathbb{R}$, it can be seen that they actually amount to the same thing.
The cotangent bundle will be a manifold of dimension $2n$ with constructions largely identical to that of the tangent bundle.

**Lemma 5.1.** *The dual basis of $\{\frac{\partial}{\partial x_i}|_p\}$ is given by $\{dx_{ip}\}$.*

*Remark.* Much like the case of vector fields, we can, in some coordinate neighbourhood $U$, write a one-form as $\omega = \sum_i a_i dx_i$.
The next theorem characterizes smooth one-forms. Observe how the third statement allows us to recast one-forms as maps from $\chi(M)$ to $C^\infty(M)$.

**Theorem 5.2.** *The following statements are equivalent:*

1. *$\omega$ is a smooth one-form*
2. *$\{a_i\}$ are all smooth*
3. *$\omega X \in C^\infty(M)$ for all $X \in \chi(M)$, where $\omega X(p) := \omega_p(X_p)$.*

**Definition** (Tensors)**.** *A **k-tensor** is a multilinear map $T : \underbrace{V \times .. \times V}_{k \text{ times}} \to \mathbb{R}$.*
*An **alternating** $k$-tensor is a $k$-tensor such that $T(v_{\sigma(1)}, ...v_{\sigma(k)}) = sgn(\sigma)T(v_1, ...v_k)$, where $\sigma \in S_k$.*
*The set of all alternating $k$-tensors on a vector space $V$ forms a vector space $\Lambda^k(V)$.*

**Lemma 5.3.** *If $V$ has dimension $n$, $\Lambda^k(V)$ has dimension $nC_k$.*

**Definition** (Exterior bundle)**.** *We call $\Lambda^k(M) = \sqcup_{p\in M}\Lambda^k(T_pM)$ the **exterior bundle** on $M$.*

*Remark.* The exterior bundle of a manifold is itself a smooth manifold of dimension $n + nC_k$ ($n$ from the manifold $M$, and $nC_k$ more from $\Lambda^k(T_pM)$).

**Definition** (Differential k-form)**.** *A **differential k-form** is a map $s : M \to \Lambda^k(M)$ such that $\pi \circ s = Id_M$.*

*Remark.* Observe how $\Lambda^1(M) = T^*M$, making this consistent with our earlier understanding of one-forms.
We denote the space of smooth $k-$forms on $M$ by $\Omega^k(M)$, and $\Omega^*(M) = \sqcup_k\Omega^k(M)$.

**Definition** (Pullback)**.** *Let $F : M \to N$ be a smooth map. Then, $F^{*,p} : \Lambda^k(T_{F(p)}N) \to \Lambda^k(T_pM)$, is defined by $(F^{*,p}s)(v_1, ..., v_k) = s(F_{*,p}v_1, ...F_{*,p}v_k)$ (where $s \in \Lambda^k(T_{F(p)}N), v_i \in T_pM$).*

*The **pullback** of $F, F^* : \Omega^*(N) \to \Omega^*(M)$, is defined by $(F^*\omega)_p = F^{*,p}\omega_{F(p)}$.*

**Definition** (Wedge product)**.** *Let $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$. Then, $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$, where the right-hand side is the antisymmetrization of $\omega_p \cdot \eta_p$.*

*Remark.* One could imagine the pullback as in contrast with the pushforward, which is as $F_* : \chi(M) \to \chi(N), (F_*X)_p = F_{*,p}X_p$.
The wedge product and pullback of smooth forms is smooth.
$(\Omega^*(M), \wedge)$ forms a graded algebra, and on it, $F^*$ is a graded algebra homomorphism.

**Lemma 5.4.** *If $\{v_1, ..., v_n\}$ is a basis for $V$ and $\{e_1, ..., e_n\}$ is the dual basis, then $\{e_{i_1} \wedge ... \wedge e_{i_k}\}_{1\le i_1...\le i_k\le n}$ is a basis for $\Lambda^k(V)$, and $(e_{i_1} \wedge ... \wedge e_{i_k})(v_1, ..., v_k) = \det[e_{i_l}(v_j)]_{l,j}$.*

*Remark.* The formula will follow from the definition of the wedge product and the Leibniz formula for determinants. With it, we can write $df_1 \wedge ... \wedge df_k = \sum_{1\le i_1\le...i_k\le n} a_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k}$, where $a_{i_1,...,i_k} \in C^\infty(M)$.
From the same argument, one can write, for $f_i \in C^\infty(M), (df_1 \wedge ... \wedge df_k)(\frac{\partial}{\partial x_{\mu_1}}, ..., \frac{\partial}{\partial x_{\mu_k}}) = \det[\frac{\partial f_i}{\partial x_{\mu_j}}]_{i,j=1}^k$.
In the particular case of top forms, we have the following change-of-coordinates formula: $dx_1 \wedge ... \wedge dx_n = hdy_1 \wedge ... \wedge dy_n, h = \det[\frac{\partial x_i}{\partial y_j}]_{i,j=1}^k$.

**Definition** (Anti-derivation)**.** *An **anti-derivation** (of degree 1) $D : \Omega^*(M) \to \Omega^*(M)$ is an $\mathbb{R}$-linear map such that:*

1. *$D(\omega \wedge \eta) = (D\omega) \wedge \eta + (-1)^k\omega \wedge (D\eta), \omega \in \Omega^k(M), \eta \in \Omega^l(M)$*
2. *$D(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$.*

**Definition** (Exterior derivative)**.** *An* ***exterior derivative*** *on $M$ is an antiderivation $D$ on $\Omega^*(M)$ such that:*

1. $D \circ D = 0$
2. $(Df)(X) = Xf$

**Theorem 5.5.** *Given a smooth manifold $M$, a unique exterior derivative $d$ exists.*

*Remark.* We can write out the action of $d$ explicitly.
For $\omega \in \Omega^k(M), dw = d(\sum_{1 \le i_1 \le ... i_k \le n} a_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k})$
$= \sum_{1 \le i_1 \le ... i_k \le n, 1 \le j \le n} \frac{\partial a_{i_1,...,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}$.

**Lemma 5.6.** *$F^*$ commutes with $d$.*

## Orientation

**Definition** (Orientation on a vector space)**.** *Let $V$ be an $n$-dimensional vector space. $\beta \in \Lambda^n(V)$ induces the orientation $[v_1, ..., v_n]$ if $\beta(v_1, .., v_n) > 0$, where $(v_1, ..., v_n) \sim (u_1, ..., u_n) \iff$ the two basis sets are related by a matrix of positive determinant.*

*Remark.* Alternatively, we can define an equivalence class on $\Lambda^n(V)$ (which, recall, is one-dimensional) by deeming $\beta \sim \eta \iff \beta = c\eta, c > 0$. This allows us to define orientation as an equivalence class of covectors.

**Definition** (Orientation on a manifold)**.** *A pointwise/rough orientation on $M$ is a collection of orientations $[\mu_p]$ on $T_pM$.*

**Definition** (Frame)**.** *A* ***frame*** *for an $n$-dimensional manifold is a collection of vector fields $X_1, ..., X_n$ such that for all $p \in M, \{X_{1,p}, ..., X_{n,p}\}$ forms a basis for $T_pM$.*

**Definition** (Smooth orientation)**.** *A pointwise orientation $\mu$ is said to be smooth at $p$ if there exists a frame $X_1, ...X_n$ smooth at $p$ such that $[Y_{1,p}, ..., Y_{n,p}] \sim \mu_p$.*

**Definition** (Orientable)**.** *A manifold $M$ is* ***orientable*** *if it admits a smooth orientation.*

**Lemma 6.1.** *A connected orientable manifold has exactly two orientations.*

**Theorem 6.2.** *Let $M$ be a smooth $n$-dimensional manifold. Then, the following statements are equivalent:*

1. *$M$ is orientable*
2. *$M$ admits a nowhere vanishing smooth $n$-form*
3. *The transition maps on $M$ all have positive Jacobian*

*Remark.* An orientation can be defined on an orientable manifold by making a choice of nowhere-vanishing top form; top forms can be partitioned into two elements through the equivalence relation $\omega \sim \omega' \iff \omega = f\omega'$ for some $f \in C^\infty(M)$ such that $f > 0$.

**Definition** (Orientation-preserving maps)**.** *Let $(M, [\omega_M]), (N, [\omega_N])$ be two oriented smooth manifolds, and $F : M \to N$ be a smooth map. We say $F$ is **orientation-preserving** if $[F^*\omega_N] = [\omega_M]$.*

*Remark.* It can be shown that the third statement in the above theorem amounts to saying precisely that all the transition maps of the manifold are orientation-preserving.

**Definition** (Contraction)**.** *The map $i_v : \Lambda^k(V) \to \Lambda^{k-1}(V), \omega \mapsto i_v\omega$, called interior multiplication or contraction with $v$, is defined as $i_v\omega(v_1, ..., v_{k-1}) = \omega(v, v_1, ..., v_{k-1})$.*

## Postscript

### Lie groups & Lie algebras