On the Origin of Symmetry

Symplectic geometry & Noether's theorem

Aditya Dwarkesh

DMS Day 2024

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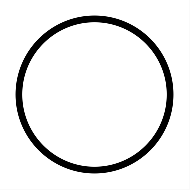
The Concept of a Manifold

Examples



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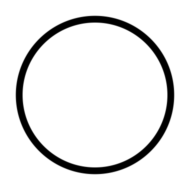
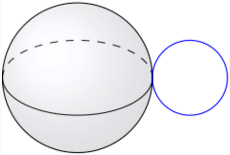


Non-examples



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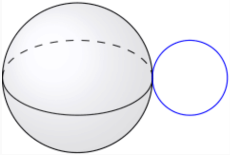
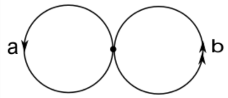
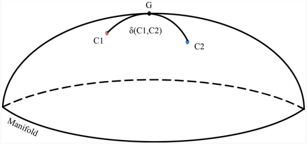




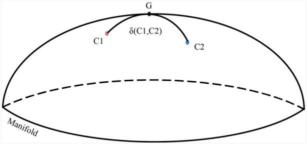
Structures on Manifolds

Lengths and Areas



A manifold with a metric allows us to measure lengths and angles, and is called a Riemannian manifold.

Lengths and Areas



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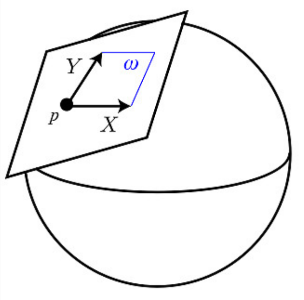

A manifold with a symplectic form allows us to measure oriented areas, and is called a symplectic manifold.

Classical mechanics on symplectic manifolds

Phase space: 3 position coordinates + 3 momentum coordinates + dynamical laws

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Generalization from $(\mathbb{R}^6)^n$: A smooth manifold M, with some additional structure encoding dynamics.

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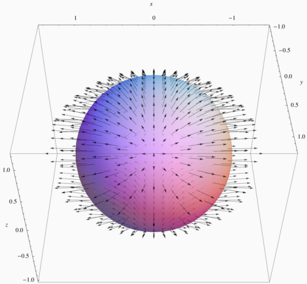
Generalization from $(\mathbb{R}^6)^n$: A smooth manifold M, with some additional structure encoding dynamics.

Question: How does a symplectic form produce dynamical laws on the manifold?

Dynamics

Vector fields



A vector field on a manifold.

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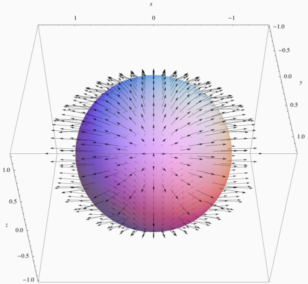

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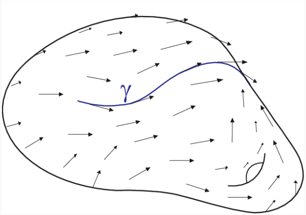
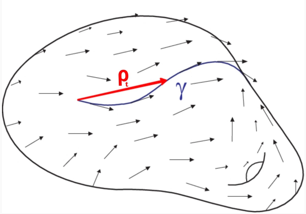


The integral curve of a vector field.

Flow



The flow of a vector field.

Hamiltonian vector field

Through the following equation, a given function $f : M \to \mathbb{R}$ is associated to a vector field X_f :

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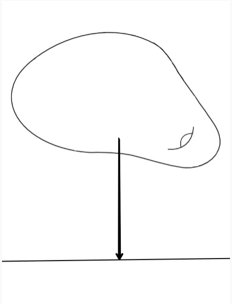
Question: How does a symplectic form produce dynamical laws on the manifold?

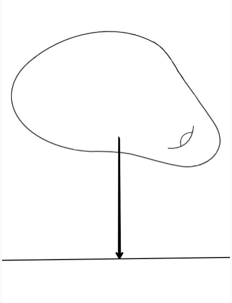
Answer: Given a Hamiltonian function f, the symplectic form ω associates to it a vector field X_f, whose flow describes the time-evolution of the system.

Mind map



Function.

Mind map



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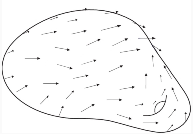

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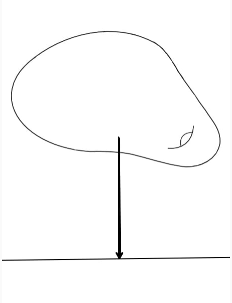

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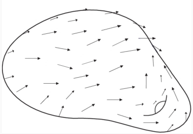

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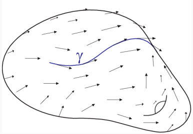
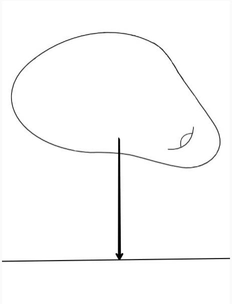


Integral curve.

Mind map



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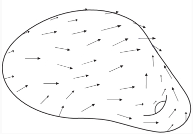

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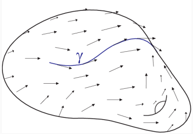

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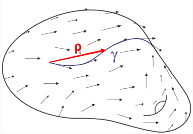

Flow.

Noether’s Theorem

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$$\boxed{H|_{\rho_t(M)} = C}$$

Theorem: Let (M, ω, H) be a Hamiltonian system. If f is a conserved quantity, its Hamiltonian vector field is an infinitesimal symmetry.

Emmy Noether, 1882-1935.

Thank you!