Algebraic Topology

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The Fundamental Group

Results

The Fundamental Group of the Circle

Theorem 1.1. $\pi_1(S^1) \cong \mathbb{Z}$

Remark. We have not mentioned a basepoint because, since the circle is path connected, the same fundamental group will be associated regardless of the choice of point.

Proof. Define $\gamma_n : I \to S^1, t \mapsto e^{2\pi int}$. Each γ_n is a loop which goes around the circle n times. Now, define $\Phi : \mathbb{Z} \to \pi_1(S^1), n \mapsto [\gamma_n]$. We claim that Φ is a group isomorphism.

- Homomorphism: We wish to show that $\Phi(m+n) = \Phi(m) * \Phi(n) \iff [\gamma_{m+n}] = [\gamma_m] * [\gamma_n] \iff [\gamma_{m+n}] = [\gamma_m * \gamma_n]$.
  $\gamma_{m+n}(s) = e^{2\pi i(m+n)s}$
  $\gamma_m * \gamma_n(s) = \begin{cases} e^{2\pi im2s} & s \in [0, \frac{1}{2}] \\ e^{2\pi in(2s-1)} & s \in [\frac{1}{2}, 1] \end{cases}$
  Clearly, the two are not equal as maps. We shall have to construct a nontrivial homotopy between them.
  For this, it suffices to show that $\gamma_n \sim \underbrace{\gamma_1 * (\gamma_1 * ...(\gamma_1 * \gamma_1))}_{\text{n times}}$.
  We shall prove this by induction. $n = 1, 2$ are obvious from definition. Suppose this is true for $n = k$. We now need to show that $\gamma_{k+1} \sim \gamma_1 * \gamma_k$.
  The $k+1$ divisions on the upper horizontal represent the number of loops run by γ_{k+1}.

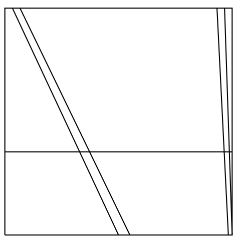

Figure 1: The Square of Homotopy

On the lower horizontal, which shows us $\gamma_1 * \gamma_k$, we run one loop in half the time, and k loops in the remaining half. The straight lines connecting the divisions show us the most natural homotopy between the two.
Consider the horizontal line drawn at a height t. By Thales' theorem, this will intersect the first line at a distance $f(t) = \frac{1}{k+1} + \frac{1}{2}(1-t)(1-\frac{2}{k+1})$. So at t, we want to complete our first loop by $s = f(t)$. So for $t < f(t)$, we write $h(s,t) = e^{2\pi i \frac{s}{f(t)}}$.

The remaining k loops will each take $\frac{1-f(t)}{k}$ time. So, for $t \geq f(t)$, we write $h(s,t) = e^{2\pi ik(s-f(t))}$.

Thus, the required homotopy is $h(s,t) = \begin{cases} e^{2\pi i \frac{s}{f(t)}} & t \in [0, f(t)] \\ e^{2\pi ik(s-f(t))} & t \in [f(t), 1] \end{cases}$

- Surjective:

Definition (Path Lifting). *Let $p : E \to B$ be a map and $f : X \to B$ be a continuous map. A lifting of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$. The following diagram represents the situation:*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Given $\gamma : I \to S^1, \tilde{\gamma} : I \to \mathbb{R}$ is a path lift of γ if $\exp \circ \tilde{\gamma} = \gamma$, where $\exp(t) = e^{2\pi it}$.

Definition (Evenly Covered Neighbourhood). *Let $f : X \to Y$ be a continuous surjective map. An open set $U \subseteq Y$ is said to be evenly covered by f (or an evenly covered neighbourhood) if $f^{-1}(U) = \sqcup V_\alpha$ such that for each α, V_α is open in X and $f|_{V_\alpha} \to U$ is a homeomorphism.*

Lemma 1.2. *The following will help us establish surjectivity.*

1. *Given $\gamma : I \to S^1$, there exists $0 = t_0 < t_1 ... < t_k = 1$ such that $\gamma([t_i, t_{i+1}])$ lies in some evenly covered neighbourhood of* $\exp$.
2. *Given $\gamma : I \to S^1$, there exists a lift $\tilde{\gamma} : I \to \mathbb{R}$ which is unique if $\tilde{\gamma}(0) \in \exp^{-1}(\gamma(0))$ is fixed.*

Proof. We prove the statements in order.

1. Let $t \in I$, and consider $U_t \ni \gamma(t)$ such that U_t is evenly covered by exp. We can always find such a U_t because, observe that we can write, for $U = S^1 \backslash \{-e^{i\theta}\}, \exp^{-1}(U) = \mathbb{R} \backslash \exp^{-1}(e^{-i\theta}) = \mathbb{R} \backslash (\mathbb{Z} + \theta) = \sqcup_{k \in \mathbb{Z}} V_k$ for any θ.
   S^1 is a compact metric space, and $\{U_t\}$ will form an open cover of it. Therefore, by the Lebesgue number lemma, there will exist a $\delta > 0$ such that any set with diameter $< \delta$ will lie in some U_t.
   We can now simply choose a partition $\gamma([0, t_1], \gamma([t_1, t_2])..., \gamma([t_{k-1}, 1])$ such that each set has diameter $< \delta$; then, each will lie in some evenly covered neighbourhood.
2. Now, we know there exists $0 = t_0 < t_1 ... < t_k = 1$ such that $\gamma([t_i, t_{i+1}])$ lies in some evenly covered neighbourhood. So consider $\gamma|_{[t_0, t_1]} \subseteq U_0, \exp^{-1}(U_0) = \sqcup_{k \in \mathbb{Z}} V_k^0$. Taking $\tilde{\gamma}(0)$ as given, pick a k such that $\tilde{\gamma}(0) \in V_k^0$, and define $\tilde{\gamma} : [t_0, t_1] \to \mathbb{R}, s \mapsto (\exp|_{V_k^0})^{-1}(\gamma(s))$. (This is well-defined and continuous because the restriction of exp is a homeomorphism.)
   We can now use the same method to define $\tilde{\gamma}$ on the remaining intervals and use the pasting lemma to put together $\tilde{\gamma} : I \to \mathbb{R}$.

$\square$

Now, to show that Φ is surjective, we need to show that an arbitrary loop $\gamma : I \to S^1$ is homotopic to some γ_n; that is, $\exp(\tilde{\gamma}) \sim \exp(\tilde{n})$, where $\tilde{\gamma}, \tilde{n}$ are the unique path lifts of γ, γ_n. (Note that we set $\gamma(0) = \gamma(1) = 1, \tilde{\gamma}(0) = 0$.) Now this will follow if we show that $\tilde{\gamma}$ and $\tilde{n}$ are based homotopic as paths.
To see this, set $n = \tilde{\gamma}(1) - \tilde{\gamma}(0)$. (This will obviously be an integer.) Clearly, $\tilde{n}(t) = (\exp)^{-1} e^{2\pi i n t} = nt = t\tilde{\gamma}(1) + (1-t)\tilde{\gamma}(0)$. This will be path homotopic to $\tilde{\gamma}$ rather trivially, thanks to the fact that $\mathbb{R}$ is convex: $h(s,t) = s\tilde{\gamma}(t) + (1-s)\tilde{n}(t)$.

- Injective:

Definition (Path homotopy lifting). *Let $\gamma, \gamma' : I \to S^1$ be homotopic via $h : I \times I \to S^1$, and $\tilde{\gamma} : I \to \mathbb{R}$ be a path lifting of γ. Then, the unique $\tilde{h} : I \times I \to \mathbb{R}$ is called the path homotopy lift of h, and is such that $\tilde{h}(-,0) = \tilde{\gamma}, \exp \circ \tilde{h} = h$.*

We shall defer the proof of the existence and uniqueness of $\tilde{h}$ to the next section, where a more general version of this statement is proven.

To show that Φ is injective, we can show its kernel is 0, which is that $[\gamma_n] = [\gamma_0] \implies n = 0$; in other words, $\gamma_n \sim \gamma_0 \implies n = 0$.
Now, let $h : I \times I \to S^1$ be a homotopy of based loops between $\gamma_0, \gamma_n : I \to S^1$ with $h(0,-) = h(1,-) = 1$. By homotopy lifting, we have a unique $\tilde{h} : I \times I \to \mathbb{R}$ such that $\tilde{h}(-,0) = \tilde{\gamma}_n$. By the uniqueness of $\tilde{h}$, this will also fix $\tilde{h}(-,1) = \tilde{\gamma}_0$.
Now, we have fixed $\tilde{h}(1,1) = 0$, and $\tilde{h}(1,-) = \exp^{-1}(h(1,-)) = \exp^{-1}(1)$ must be an integer. Since $\tilde{h}(1,t)$ is a continuous function and I is path connected, we have $\tilde{h}(1,-) = 0$ throughout.
But also, $\tilde{h}(1,0) = n \implies n = 0$, and we are done.

□

The Homotopy Lifting Property

Definition (Covering). *A map $\pi : E \to B$ is called a covering map, and E a covering space of B, if for every $x \in B$ there is an evenly covered neighbourhood U such that $x \in U$.*

Definition (Homotopy lifting). *We say that $\pi : E \to B$ has the homotopy lifting property if the following diagram admits an $\tilde{h}$ such that it commutes for all $Y, h, \tilde{h}_0$:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

$\tilde{h}$ is called the homotopy lift of h.

Remark. If we put $Y = \{*\}$, we get back path lifting; if $Y = I$, we get back path homotopy lifting.

Theorem 2.3. *Covering maps have unique homotopy liftings for each Y.*

Proof. Let $\pi : E \to B$ be a covering map and $h : Y \times I \to B$ be a homotopy. We need to show that there exists a lift $\tilde{h} : Y \times I \to E$ of h, and further, that if $\tilde{h}(-, 0)$ is given, $\tilde{h}$ is unique.

Consider, to begin with, for a given homotopy $h : Y \times I \to B, h_y : \{y\} \times I \to B$. This is just a path, and so we know it can be lifted to a unique path $\tilde{h}_y$ in E (given the starting point).

We claim that $\tilde{h} := \tilde{h}_y(t) \forall y \in Y, t \in I$ is the homotopy lift. That such an $\tilde{h}$ would make the above diagram commute follows from the fact that each path lift would. So, what remains to be checked is:

- Uniqueness: Since each path lift is unique, so will the whole thing be, and this is done.
- Continuity:

$\square$

Applications

The Homotopy Lifting Property

Theorem 2.4. *Let $\pi : \tilde{X} \to X$ be a covering map and X be path connected. Then, all fibers (inverse image of singletons) will have the same cardinality.*

Proof. Let $x_1, x_2, \in X$, and γ be a path from x_1 to x_2. Define a map $\varphi_\gamma : \pi^{-1}(x_1) \to \pi^{-1}(x_2), \tilde{x} \mapsto \tilde{\gamma}_{\tilde{x}}(1)$, where $\tilde{\gamma}$ is the path lift of γ which starts at $\tilde{x}$. Uniqueness of the path lift forces this map to be injective. The analogous map defined on $\bar{\gamma}$ will be injective for the same reason.
Now, $\varphi_{\bar{\gamma}}(\varphi_\gamma(\tilde{x}_1)) = \varphi_{\bar{\gamma}}(\tilde{\gamma}_{\tilde{x}_1}(1)) = \varphi_{\bar{\gamma}}(\tilde{x}_2) = \tilde{\bar{\gamma}}_{\tilde{x}_2}(1) = \tilde{x}_1 \implies \varphi_{\bar{\gamma}} \circ \varphi_\gamma = Id_{\pi^{-1}(x_1)}$. Similarly, $\varphi_\gamma \circ \varphi_{\bar{\gamma} = Id_{\pi^{-1}(x_2)}}$.
Thus, the maps are isomorphisms, proving the required. $\square$

Theorem 2.5. *Let $\pi_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ be the map induced by the covering. Then, π_* is injective.*

Proof. We shall show that $\ker(\pi_*) = 0 \iff [\pi(\gamma)] = 0 \implies [\gamma] = 0$, where γ is a based loop in the covering space.
Let $h : S^1 \times I \to X$ be a homotopy of based loops between $\pi(\gamma)$ and C_{x_0}. We would like to find another homotopy of based loops $\tilde{h}$ between γ and $C_{\tilde{x}_0}$ using h.
We can apply the homotopy lifting property with $Y = S^1, h$ as above and $\tilde{h}(-, 0) = \gamma$ to conclude the existence of a homotopy lift $\tilde{h}$ which will further satisfy $\pi \circ \tilde{h}(-, 1) = h(-, 1) = C_{x_0} \implies \tilde{h}(-, 1) = \pi^{-1}(x_0)$. Now, note that, since π is continuous and $\{x_0\}$ is a connected set, its preimage will also be connected.
This forces it to be a constant, because fibers are discrete under the subspace topology: For let U be an openly covered neighbourhood around x_0, so that $\pi^{-1}(U)$ is the disjoint union of open sets U_i homeomorphic to U. $U_i \cap \pi^{-1}(x_0)$ is forced to be a singleton for each i simply because $\pi|_{U_i}$ is a homeomorphism and so injective.
Similarly, $\tilde{h}(1, -)$ is constant. But we already know that $\tilde{h}(1, 0) = \tilde{x}_0$. Therefore, $\tilde{h}(-, 1) = C_{\tilde{x}_0}$. $\square$

Theorem 2.6 (Degree of a cover)**.** *The size of a fiber equals the index of $\pi_1(\tilde{X}, \tilde{x}_0)$ in $\pi_1(X, x_0)$.*

Proof. Not done in class. $\square$

The Fundamental Group of the Circle

Theorem 2.7 (Brouwer's Fixed Point Theorem)**.** *Any continuous $h : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.*

Proof. Suppose no fixed point exists. Conveniently center the disk at the origin and consider $r : \mathbb{D}^2 \to S^1, x \mapsto (1 - t_0)h(x) + t_0 x$, where t_0 is such that $||r(x)|| = 1$. What this map 'does' is extend the line joining $h(x), x$ to the boundary of the disk.
Observe that r is a deformation retract: Continuity follows trivially from that of h, and $r|_{\partial \mathbb{D}^2} = Id_{S^1}$. This itself is a contradiction, because unfortunately, there can be no retract of $\mathbb{D}^2$ to S^1. The argument is as follows:
Let i be the inclusion map $S^1 \hookrightarrow \mathbb{D}^2$. We will have induced maps $i_* : \pi_1(S^1) \to \pi_1(\mathbb{D}^2), r_* : \pi_1(\mathbb{D}^2) \to \pi_1(S^1)$; the induced map takes (the equivalence class of) a loop in the domain parent space to the (equivalence class of) the loop given by the image of the original loop under the original map.
Now, $r \circ i = Id_{S^1} \implies (r \circ i)_* = Id_{\pi_1(S^1)}$. But also, since $\pi_1(\mathbb{D}^2) = 0, i_*$ must be the trivial map, so that $(r \circ i)_* = r_* \circ i_* = 0$, which brings out the required contradiction.

$\square$

Theorem 2.8 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$.*

Proof. Suppose $p(z) = z^n + a_1 z^{n-1} + ...a_{n-1}z + a_n$ is a polynomial in $\mathbb{C}$ with no roots. We will show that this forces $n = 0$.
Consider $H_r(s) : I \to S^1, H_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, r \geq 0$. For each value of t, this is a loop on S^1 based at 1 ($H_r(1) = 1 \forall r$). Furthermore, $H_0(s) = 1$, so that H_0 is the constant loop. But note that H_r will be homotopic to H_0 for any r (simply define $h(s,t) : I \times I \to S^1, h(s,t) = H_{tr}$). Thus, $[H_r] = 0 \in \pi_1(S^1)$.
Now, choose $r > \max\{1, |a_1| + ... + |a_n|\}$. Then, for $|z| = r, |z|^n = r \cdot r^{n-1} > (|a_1| + ...|a_n|)|z^{n-1}|) \geq |a_1 z^{n-1} + ...a_n z^{n-1}| \geq |a_1 z^{n-1} + ...a_n|$. From this inequality, we may conclude that the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + ...a_n)$ has no roots on the circle $|z| = r$ for $t \in I$.
Now, for this fixed r, define the homotopy $F(s,t) = \frac{p_{1-t}(re^{2\pi i s})/p_{1-t}(r)}{|p_{1-t}(re^{2\pi i s})/p_{1-t}(r)|}$. $F(1,t) = 1, F(s,0) = H_r(s) \sim \varphi_0$, and $F(s,1) = e^{2\pi i n s} = \varphi_n \implies \varphi_0 \sim \varphi_n \implies n = 0$. $\square$

Theorem 2.9 (Borsuk-Ulam Theorem)**.** *If $f : S^2 \to \mathbb{R}^2$ is a continuous map, there exist a pair of points $\{x, -x\} \subset S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose no such pair exists for some continuous function f, and consider $g : S^2 \to S^1, x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}, \eta : I \to S^2, s \mapsto (\cos 2\pi s, \sin 2\pi s, 0)$, and finally, $\gamma := g \circ \eta : I \to S^1$. It is easy to see that γ is continuous, and since $\gamma(0) = \gamma(1)$, we see that γ is a loop based at

$\frac{f(e_1)-f(-e_1)}{|f(e_1)-f(-e_1)|} = x_0$, where $e_1 = (1, 0, 0)$.

- γ is not nullhomotopic: Lift γ to $\tilde{\gamma} : I \to \mathbb{R}$. Since $\gamma(s + \frac{1}{2}) = -\gamma(s), s \in [0, \frac{1}{2}]$, we also have $\tilde{\gamma}(s + \frac{1}{2}) = -\tilde{\gamma}(s) + \frac{q}{2}$, where q is an odd integer. [Argument incomplete.]
- γ is nullhomotopic: First, note that η is nullhomotopic, via $h(s, t) = ((1 - t^2)\cos 2\pi s, (1 - t^2)\sin 2\pi s, t)$. Then, $\gamma \sim C_{x_0}$ via $g \circ h$.

This brings out the required contradiction, and we conclude that such a pair of points must exist. $\square$

Interlude: Connectedness

Definition. *A topological space X is **connected** if it cannot be written as the union of two disjoint non-empty open sets.*

Definition. *A topological space X is **path connected** if for any $x, y \in X$, there exists a continuous $\gamma : I \to X$ such that $\gamma(0) = x, \gamma(1) = y$.*

Definition. *A topological space X is **simply connected** if it is path connected and it has trivial fundamental group.*

Definition. *A topological space X is **locally path connected** if every $x \in X$ has a path connected open neighbourhood.*

Definition. *A topological space X is **locally simply connected** if every $x \in X$ has a neighbourhood U which is simply connected, that is, every loop in U is nullhomotopic in U.*

Definition. *A topological space X is **semilocally simply connected** if every $x \in X$ has a neighbourhood U such that every loop in U is nullhomotopic in X.*

Covering Space Theory

The Universal Cover

Let X be a path-connected topological space. We want to find a cover $\tilde{X}$ which is simply connected. Such a cover shall be called a *universal cover* of X for reasons which will become clear.

First of all, suppose such a cover exists. Then, X must be semilocally simply connected: This amounts to saying that every $x \in X$ has a neighbourhood U such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is the zero map.
Choose an evenly covered neighbourhood and let $\pi^{-1}(x_0) \ni \tilde{x_0} \in \tilde{U}$ such that $\pi|_{\tilde{U}} \cong U$.
We have the following commutative diagram:

$$\begin{tikzcd} \tilde{U} \arrow[r, hook, "\tilde{i}"] \arrow[d, "\pi"] & \tilde{X} \arrow[d, "\pi"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd}$$

This will induce maps between the fundamental groups:

$$\begin{tikzcd} \pi_1(\tilde{U}, \tilde{x}) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & \pi_1(\tilde{X}, \tilde{x}) \arrow[d, "\pi_*"] \\ \pi_1(U, x) \arrow[r, hook, "i_*"] & \pi_1(X, x) \end{tikzcd}$$

But since $\tilde{X}$ is simply connected, its fundamental group will be trivial. This forces i_* to be the zero map.

Now, suppose γ, η are two paths in X based at x_0, and they are based homotopic via h. By the homotopy lifting property, $\tilde{\gamma}, \tilde{\eta}$ are based paths in $\tilde{X}$ which are *based* homotopic via $\tilde{h}$. They obviously agree at the starting point; we will also have $\tilde{\gamma}(1) = \tilde{\eta}(1)$, because both of them lie in $p^{-1} \circ (h(-, 1)) = \tilde{h}(-, 1)$, which is a connected discrete set (connected by virtue of being the continuous pre-image of a connected set, and discrete by virtue of being the pre-image of a fiber).

It is obvious that if two paths are based homotopic in the covering space, they will be based homotopic in the base space.

Therefore, $\gamma \sim_b \eta \iff \tilde{\gamma} \sim_b \tilde{\eta}$.

We now have enough heuristic constraints to furnish a final construction of the universal cover. Observe how *any* $\tilde{x} \in \tilde{X}$ is associated with an equivalence class of paths $[\gamma]$, where γ is a path in X and two paths are related if they are based homotopic as paths: Given a γ, lift it and see the lift's endpoint. This will remain the same for all elements of the class. Conversely, for any element in the cover, we will obviously have a path from the basepoint to it; projecting this down will give us the path whose equivalence class will identify the point (due to path connectedness of X).

Definition. *The universal cover of a path-connected & semilocally simply connected space X is $\tilde{X} = \mathcal{P}_{x_0}X/\sim$, where $f \sim g \iff f(1) = g(1)$ and the two are based homotopic as paths. The topology on it is the one generated by the basis consisting of sets of the form $U_\gamma = \{[\gamma * \alpha] \in \tilde{X} | \gamma \in \mathcal{P}_{x_0}X, \gamma(1) \in U, U \text{ is open and } \alpha \text{ is a path in } U\}$.*

Theorem 3.1. *$\tilde{X}$ is simply connected, and* $\mathrm{ev} : \tilde{X} \to X, [\gamma] \mapsto \gamma(1)$ *is a covering map.*

Intermediate Covers: Bottom-up

Consider the space $X_H := \mathcal{P}_{x_0}X/\sim_H$, where $\gamma \sim_H \eta \iff \gamma(1) = \eta(1)$ and $[\gamma\bar{\eta}] \in H \leq \pi_1(X, x_0)$.

Notice how we get back the universal cover for $H = \{e\}$, and the base space for $H = \pi_1(X, x_0)$ (for in the latter case, the equivalence constraint merely says that two paths are identified simply if they form a loop without requiring said loop to be in any particular subgroup of the fundamental group—thereby associating any $x \in X$ uniquely with the equivalence class of paths which end at it).

So:

- $\widetilde{X} = \widetilde{X}/\{e\}$
- $X_H = \widetilde{X}/H$
- $X = \widetilde{X}/\pi_1(X, x_0)$

We have a kind of 'tower' of topological spaces ordered by the size of the subgroup quotiented by. The universal cover justifies its name by being that which has to be quotiented by a different subgroup each time for a new cover to be produced.

All this suggests the following theorem:

Theorem 3.2. *X_H is a connected cover of X, and $\pi_1(X_H) \cong H$.*

Definition. *Two covers are called equivalent if there exists a homeomorphism φ such that the following diagram commutes:*

$$\begin{tikzcd}
(\hat{X}, \hat{x_0}) \arrow[rr, "\varphi"] \arrow[dr, "\hat{p}"] & & (X', x_0') \arrow[dl, "p'"'] \\
& (X, x_0) &
\end{tikzcd}$$

If, in particular, we set $X' = \hat{X}$, the homeomorphisms form a group under the composition operation. This is called the group of deck transformations of the covering space, denoted by $G(\hat{X})$.

Definition. *A cover is called* ***regular*** *if there exists a deck transformation between any two pre-images of the base point.*
A cover is called ***normal*** *if it is connected and its image under p_* in the fundamental group of the base space is a normal subgroup.*

We shall now rattle off a number of results.

Theorem 3.3. *Let (X, x_0) be a based space, $(\tilde{X}, \tilde{x}_0)$ a cover of it with map p, and Y be some path-connected and locally path-connected based space. The following hold:*

1. *For any continuous $f : (Y, y_0) \to (X, x_0)$, a lift of f exists $\iff$ $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

2. *$(\tilde{X}, \tilde{x}_0)$ is a connected regular cover $\iff$ It is a normal cover.*

3. *$G(X_H) \cong N(H)/H$.*

Proof. This will be long and painful.

1. We prove each implication in turn.
$\implies$: This one is easy. Suppose a lift $\tilde{f}$ exists. Then, $p \circ \tilde{f} = f \implies p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \supseteq p_* \circ \tilde{f}_*(\pi_1(Y, y_0)) = f_*(\pi_1(Y, y_0)) \implies f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

   $\impliedby$: Given that the condition holds, we need to define a lift of f.
Let $\tilde{f}(y) := \widetilde{f(y)}$, the endpoint of the path lift of $f(\gamma)$ initialized at an $\tilde{x}_0 \in p^{-1}(x_0)$, where γ is a path from y_0 to y. We claim that this is a lift of f.

2. The following lemma will help us out.

   Lemma 3.4. *Two connected based covers are equivalent $\iff$ Their induced images are equal.*

   $\implies$: We begin by showing that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H_0$ and $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = H_1$ are conjugates of each other. Consider a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then, $p(\tilde{\gamma}) = g \in \pi_1(X, x_0)$. Now, for any loop $\tilde{f}$ based at $\tilde{x}_0, \tilde{\gamma}^{-1} * \tilde{f} * \tilde{\gamma}$ will be a loop based at $\tilde{x}_1 \implies g^{-1} * H_0 * g \subseteq H_1$. Similarly, $H_0 = g * H_1 * g^{-1}$, thus proving that the two are conjugates of one another.

   Now, let $(\tilde{X}, \tilde{x}_0)$ be a connected regular cover, and let $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$. We want to show that $g^{-1} * H * g = H$ for all $g \in \pi_1(X, x_0)$. Let $g = [\gamma]$, and consider the endpoint $\tilde{x}_1 \in p^{-1}(x_0)$ of the lift of γ initialized at $\tilde{x}_0$. We know by the above argument that $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = g^{-1} * H * g$. But also, by assumption of their equivalence, $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = H$. This proves that H is normal.

   $\impliedby$: Next, we suppose $(\tilde{X}, \tilde{x}_0)$ is a normal cover. Then, $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = g^{-1} * H * g$ for all $g \equiv [\gamma] \in \pi_1(X, x_0)$ Consider any $\tilde{x}_1 \in p^{-1}(x_0)$. By the above, $p_*((\tilde{X}, \tilde{x}_1)) = H\prime$ is a conjugate of H. But since H is normal, $H = H'$; and so by the lemma, $(\tilde{X}, \tilde{x}_1)$ is equivalent to $(\tilde{X}, \tilde{x}_0)$. Since this is true for all the pre-images of x_0, this proves regularity.

3. Define $\Phi : N(H) \to G(X_H), [\gamma] \mapsto \varphi_{\tilde{\gamma}}$, the deck transformation taking $\tilde{\gamma}(1) = \tilde{x}_1$ to $\tilde{x}_0$. We claim that this is a surjective group homomorphism with kernel H; the result will then follow from the first isomorphism theorem.

□

Corollary 3.4.1. *The following bijective correspondences hold:*

- *Subgroups $\longleftrightarrow$ Isomorphism classes of connected based covers*
- *Subgroups up to conjugation $\longleftrightarrow$ Isomorphism classes of connected covers*

Theorem 3.5. *Let X_H be a normal cover of X. Then, $X_H/G(X_H) \cong X$.*

Intermediate Covers: Top-down

Lemma 3.6. *Let X be a topological space, and G be a group which acts properly discontinuously on it (that is, for all $x \in X$, there exists an open set $U \ni x$ such that $g \cdot U \cap U = \emptyset$ for all $g \neq e$).*

Given a (reasonable) based space (X, x_0), a discrete group G, and a group homomorphism $\rho : \pi_1(X, x_0) \to G$ (which induces a natural (left) group action of π_1 on G, $[\gamma] \cdot g := \rho([\gamma])g$), we wish to construct a covering $Y \to Y/G \cong X$, where G will act properly discontinuously on Y.

Theorem 3.7. *Let $X_\rho := (\tilde{X} \times G)/\sim$, where $(\tilde{x}, g) \sim (\tilde{y}, g') \iff \exists [\gamma] \in \pi_1(X, x_0)$ such that $[\gamma] \cdot g = g', [\gamma * \alpha]_{\tilde{X}} = \tilde{x}$ (where $\tilde{x} = [\alpha]_{\tilde{X}}$, if we recall the universal cover's construction).*

Then, the (right) action of G on X_ρ given by $(\tilde{x}, g) \cdot g_1 \mapsto (\tilde{x}, gg_1)$ is properly discontinuous. Furthermore, $X_\rho/G \cong X$.

Lemma 3.8. *ρ is surjective $\iff$ X_ρ is connected.*

Lemma 3.9. *Let (X, x_0) be a path-connected space with universal cover. Then, for any group G, the following sets stand in bijective correspondence:*

$$\mathrm{Hom}(\pi_1(X, x_0), G) \leftrightarrow \{\text{Based } G\text{-regular covers of } (X, x_0)\}/G - \text{equivalence}$$

Here, two covers are G-equivalent if they are equivalent as covers, and if the homeomorphism satisfies $\varphi(g \cdot z) = g \cdot \varphi(z)$.

Seifert-Van Kampen theorem

Definition. *The amalgamated product of two free groups, denoted by $F_1 *_H F_2$ (where $H \leq F_1, H \leq F_2$), is their free product quotiented by the normal subgroup generated by$\{\varphi_1(h)\varphi_2(h^{-1}), h \in H\}$, where φ_1, φ_2 are the inclusion maps of H into F_1, F_2 respectively.*

Theorem 3.10. *Let $X = U \cup V$ be such that U, V are non-empty open sets, $X, U, V, U \cap V$ all have a universal cover, and $U, V, U \cap V \ni x_0$ are all path-connected. Then, there exists an isomorphism $\varphi : \pi_1(U, X_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$.*

Theorem 3.11. *$\mathcal{F}_2$ is isomorphic to a subgroup of $\mathcal{F}_3$. $\mathcal{F}_3$ is isomorphic to a subgroup of $\mathcal{F}_2$. Also, $\mathcal{F}_2, \mathcal{F}_3$ are not isomorphic.*

Proof. There is an obvious inclusion map $\mathcal{F}_2 \to \mathcal{F}_3$. For the converse, we shall construct two topological spaces with fundamental group $\mathcal{F}_2, \mathcal{F}_3$, such that the latter is a cover of the former. The first result will then follow from theorem 2.5.
We will use the fact that the k-bouquet of circles has fundamental group $\mathcal{F}_k$.

- $\bigcirc\!\bigcirc\!\bigcirc$ covers $\bigcirc\!\bigcirc$:
- $\pi_1(\bigcirc\!\bigcirc\!\bigcirc)= \mathcal{F}_3$: It suffices to show homotopy equivalence of the same with the 3-bouquet of circles.

□