The Riemann-Stieltjes Integral

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Answer 2. If $f(x) = \epsilon > 0$ at some point $x = x_0$, then by continuity, there exists an $\delta > 0$ such that $f(x) > \frac{\epsilon}{2}$ for $x \in (x_0 - \delta, x_0 + \delta)$.
$\int_a^b f(x)dx \geq L(P, f) \geq \frac{\epsilon\delta}{2} > 0$ for any partition P, contradicting the fact that the integral equals 0.
Thus, $f(x) = 0$ for all $x \in [a, b]$.

Answer 4. It is clear that $L(P, f) = 0$ for all partitions P. We need to show that there exists an $\epsilon > 0$ such that $U(P, f) > \epsilon$ for all partitions P.
By the Archimedean property, sup $f(x) = 1$ for any interval. Thus, $U(P, f) = (b - a) > \frac{b-a}{2} > 0$, and we are done.

Answer 5. No. For a counterexample, we may construct $f(x) = -1$ for all irrational $x, f(x) = 1$ for all rational x. This is not Riemann integrable as shown above, but $f^2(x) = 1$ everywhere, and is thus Riemann integrable.
Taking $\phi(x) = x^{\frac{1}{3}}, h(x) = f^3(x) \implies \phi(h(x) = f(x)$. Since $\phi(x)$ is continuous and $h(x)$ is Riemann integrable, $f(x)$ is Riemann integrable.

Answer 8. Note that, since f is montonically decreasing, $f(n+1) \leq f(x) \leq f(n) \implies f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n)$.
Therefore, $\sum_{n=1}^{N} f(n) \leq \sum_{n=1}^{N} \int_n^{n+1} f(x)dx \leq \sum_{n=1}^{N} f(n) \implies \sum_{n=2}^{N+1} f(n) \leq \int_1^{N+1} f(x)dx \leq \sum_{n=1}^{N} f(n)$.
Letting $N \to \infty$ gives us the required bi-implication.