MML assignment 2:-

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Task 1:-

a) Given data :

- $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_K \end{bmatrix}$, also $n > K$

where $\vec{x}$ is the image vector and $\vec{s}$ is the secrete message.

- $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ is the modification vector

- We encode the message using $\vec{z}$
$$(\vec{x} + \vec{z})$$

- $\vec{z}$ must be small to ensure that original image $\vec{x}$ and $(\vec{x}+\vec{z})$ almost look the same.

- Original message is decoded by:
$$D(\vec{x}+\vec{z}) = y$$
$$\hat{s} = \text{sign}(y)$$
where D is a $K \times n$ matrix used for decoding.

Solution:-

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$\rightarrow$ $D(\vec{x}+\vec{z}) = \alpha \vec{s}$

$$D\vec{z} + D\vec{x} = \alpha \vec{s}$$

$$\boxed{Dz = (\alpha S - Dx)}$$

$\rightarrow$ Also $\vec{z}$ should be as small as possible

i.e $\vec{z} = \arg\min(||z||)$

This is least norm problem:

minimize $||z||^2$

subjected to $Dz = (\alpha S - Dx)$ $\quad$ let $(\alpha S - Dx) = E$

Lagrangian function is given by:

$$L(z, \lambda) = ||z||^2 + \lambda_1 (d_1^T z - e_1) + \dots + \lambda_K (d_i^T z - e_K)$$

Here, d_i^T are rows of D

$\lambda = [\lambda_1 \ \dots \ \lambda_K]^T$ is lagrange multipliers

$$\frac{\partial L(z, \lambda)}{\partial \lambda_i} = d_i^T z - e_i$$

$$d_i^T z - e_i = 0 \qquad i = 1 \dots K$$

$$d_i' z = e_i \quad \Rightarrow D' z = E \rightarrow (1)$$

$$\frac{\partial \mathcal{L}(z, \lambda)}{\partial z_i} = 2 \sum_{i=1}^{n} z_i + \sum_{i=1}^{K} \lambda_i (d_j)_i$$

$$\frac{\partial L(z, \lambda)}{\partial z_i} = 2z + D^T \lambda$$

$$2z + D^T \lambda = 0 \rightarrow (2)$$

from (1) & (2):

$$\begin{bmatrix} 2I & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ E \end{bmatrix}$$

$$\begin{bmatrix} 2I & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha S - Dx \end{bmatrix}$$

The KKT conditions are:

$$\begin{bmatrix} 2I & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha S - Dx \end{bmatrix}$$

$$2z + D^T \lambda = 0 \rightarrow (a)$$
$$Dz = \alpha S - Dx \rightarrow (b)$$

from (a):

$$z = -\frac{1}{2} D^T \lambda \rightarrow (c)$$

substituting ⓒ in ⓑ:

$$-\frac{1}{2} D D^T \lambda = \alpha S - Dx$$

$$\lambda = -2\underbrace{(DD^T)}_{}^{-1}(\alpha S - Dx) \rightarrow ⓓ$$

→ as D is a wide matrix, if all the rows are linearly independent then (DD^T) is invertable.

Substituting ⓓ in ⓐ:

$$2z - 2D^T(DD^T)^{-1}(\alpha S - Dx) = 0$$

$$z = D^T(DD^T)^{-1}(\alpha S - Dx)$$

$$z = D^+(\alpha S - Dx)$$

where D^+ is the pseudo inverse of the wide matrix D.

$$\boxed{\begin{array}{l} z = D^+(\alpha S - Dx) \\ \text{where } D^+ = D^T(DD^T)^{-1} \end{array}}$$