Question 3 : Given the transformation matrix M, we need to find the image of hyperbola ($y = 1/x$) under the projective transform.

Let us express the hyperbola in parametric form $(t, 1/t)$. Now in 3D homogeneous coordinate system we can express it as

$$\begin{bmatrix} t \\ 1/t \end{bmatrix} \rightarrow \begin{bmatrix} t \\ 1/t \\ 1 \end{bmatrix} \tag{1}$$

Taking the projective transform under the transformation matrix M

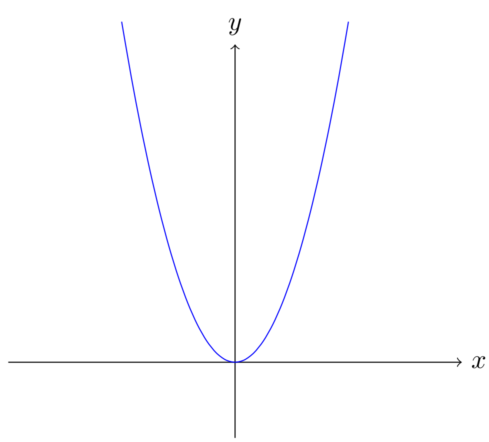
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1/t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/t \\ t \end{bmatrix} \tag{2}$$

Converting 3D homogeneous coordinates to 2D cartesian coordinates we get

$$\begin{bmatrix} 1 \\ 1/t \\ t \end{bmatrix} \rightarrow \begin{bmatrix} 1/t \\ 1/t^2 \end{bmatrix} \tag{3}$$

Converting the parametric form to implicit form we get

$$y = x^2 \tag{4}$$



(5)

Thus, we know that this is an upward parabola with the vertex at $(0, 0)$

Question 4 : We model the intrinsic parameters of the camera using the following matrix

$$K = \begin{bmatrix} f \times s & 0 & x_T \\ 0 & f \times s & y_T \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

This is because the aspect ratio (1+m) is given to be 1. Also, the shear is assumed to be 0.
Here, f is the focal length and s is the resolution.

For camera 1, let l_1 be the direction corresponding to the vanishing point (p_{1x}, p_{1y}).

$$\begin{bmatrix} p_{1x} \\ p_{1y} \\ 1 \end{bmatrix} = K_1 \times R_1 \times \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 0 & t_z \end{bmatrix} \times \begin{bmatrix} l_{11} \\ l_{12} \\ l_{13} \\ 0 \end{bmatrix} \tag{7}$$

$$p_1 = K_1 \times R_1 \times l_1 \qquad \textit{where } p_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \\ 1 \end{bmatrix} \tag{8}$$

Similarly

$$p_2 = K_1 \times R_1 \times l_2 \tag{9}$$

$$p_3 = K_1 \times R_1 \times l_3 \tag{10}$$

Since l_1 and l_2 are perpendicular

$$\begin{aligned} l_1^T \times l_2 = 0 \\ \implies (R_1^{-1} \times K_1^{-1} \times p_1)^T \times (R_1^{-1} \times K_1^{-1} \times p_2) = 0 \\ \implies p_1^T \times (K_1^{-1})^T \times (R_1^{-1})^T \times R_1^{-1} \times K_1^{-1} \times p_2 = 0 \\ \implies p_1^T \times (K_1^{-1})^T \times K_1^{-1} \times p_2 = 0 \end{aligned}$$

Because R_1^{-1} is an orthogonal matrix

Similarly,

$$\begin{aligned} \implies p_2^T \times (K_1^{-1})^T \times K_1^{-1} \times p_3 = 0 \\ \implies p_3^T \times (K_1^{-1})^T \times K_1^{-1} \times p_1 = 0 \end{aligned}$$

Using the above three equations, we can solve for the three variables ($f_1 \times s_1$, x_T and y_T). However f_1 and s_1 are clubbed together, so their individual values cannot be calculated.

Similarly, for camera 2, $f_2 \times s_2$ can be calculated, but not f_2 or s_2.

Now, consider vanishing point corresponding to l_1 on both the images. We have

$$p_1 = K_1 \times R_1 \times l_1 \tag{11}$$

$$q_1 = K_2 \times R_2 \times l_1 \tag{12}$$

This gives us

$$q_1 = K_2 \times (R_2 R_1^{-1}) \times K_1^{-1} \times p_1 \tag{13}$$

$R_2 R_1^{-1}$ is the rotation matrix (R) between the two camera.

$$\implies q_1 = K_2^{-1} \times R \times K_1^{-1} \times p_1$$

Similarly

$$q_2 = K_2^{-1} \times R \times K_1^{-1} \times p_2$$
$$q_3 = K_2^{-1} \times R \times K_1^{-1} \times p_3$$

Now, since R has 3 degrees of freedom and we have 3 equations, it can be solved to get R.

Hence,

1. R can be inferred.
2. t can never be inferred.
3. f_p, f_q, s_p and s_q cannot be individually inferred. But, $f_p \times s_p$ and $f_q \times s_q$ can be inferred.

Question 5a : Prove that the projections (in image plane) of any two parallel lines L_1, L_2 in $\mathbb{R}^3$ have an intersection point, the vanishing point.

Let us consider the parametric form of 2 parallel lines L_1, L_2 in $\mathbb{R}^3$,

$$L_1 = \begin{bmatrix} at+k1 \\ bt+k2 \\ ct+k3 \end{bmatrix} \qquad L_2 = \begin{bmatrix} at \\ bt \\ ct \end{bmatrix} \tag{14}$$

Also, let us consider the transformation martix for single point transform M

$$M = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{15}$$

The projective transform of the lines is as follows

$$trans(L_1) = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} at+k1 \\ bt+k2 \\ ct+k3 \\ 1 \end{bmatrix} = \begin{bmatrix} d(at+k1) \\ d(bt+k2) \\ d(ct+k3) \\ ct+k3 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{d(at+k1)}{ct+k3} \\ \frac{d(bt+k2)}{ct+k3} \\ 1 \end{bmatrix} \tag{16}$$

$$trans(L_2) = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} at \\ bt \\ ct \\ 1 \end{bmatrix} = \begin{bmatrix} d(at) \\ d(bt) \\ d(ct) \\ ct \end{bmatrix} \rightarrow \begin{bmatrix} \frac{da}{c} \\ \frac{db}{c} \\ 1 \end{bmatrix} \tag{17}$$

Now for $trans(L_1), trans(L_2)$ as $t \rightarrow \infty$ we get

$$\lim_{t \to \infty} trans(L_1) = \begin{bmatrix} \frac{da}{c} \\ \frac{db}{c} \\ 1 \end{bmatrix} \qquad \lim_{t \to \infty} trans(L_2) = \begin{bmatrix} \frac{da}{c} \\ \frac{db}{c} \\ 1 \end{bmatrix} \tag{18}$$

Hence we can say that as $t \rightarrow \infty$ both the lines intersect at the vanishing point.

Question 5b : Prove that the vanishing points corresponding to three (different) sets of parallel lines on a 3D plane are collinear in the image plane.

Let us consider the equations of 3 sets of parallel lines,

$$L_{p1} = \begin{bmatrix} a_1 t + k1 \\ b_1 t + k2 \\ c_1 t + k3 \end{bmatrix} \qquad L_{p2} = \begin{bmatrix} a_1 t \\ b_1 t \\ c_1 t \end{bmatrix} \tag{19}$$

$$L_{q1} = \begin{bmatrix} a_2 t + k4 \\ b_2 t + k5 \\ c_2 t + k6 \end{bmatrix} \qquad L_{q2} = \begin{bmatrix} a_2 t \\ b_2 t \\ c_2 t \end{bmatrix} \tag{20}$$

$$L_{r1} = \begin{bmatrix} a_3 t + k7 \\ b_3 t + k8 \\ c_3 t + k9 \end{bmatrix} \qquad L_{r2} = \begin{bmatrix} a_3 t \\ b_3 t \\ c_3 t \end{bmatrix} \tag{21}$$

Also, let us consider the transformation martix for single point transform M

$$M = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{22}$$

The intersection points of the lines (within same set) can be found directly using using the part a's result. They are as follows,

$$P_p = \begin{bmatrix} \frac{da_1}{c_1} \\ \frac{db_1}{c_1} \\ 1 \end{bmatrix} \qquad P_q = \begin{bmatrix} \frac{da_2}{c_2} \\ \frac{db_2}{c_2} \\ 1 \end{bmatrix} \qquad P_r = \begin{bmatrix} \frac{da_3}{c_3} \\ \frac{db_3}{c_3} \\ 1 \end{bmatrix} \tag{23}$$

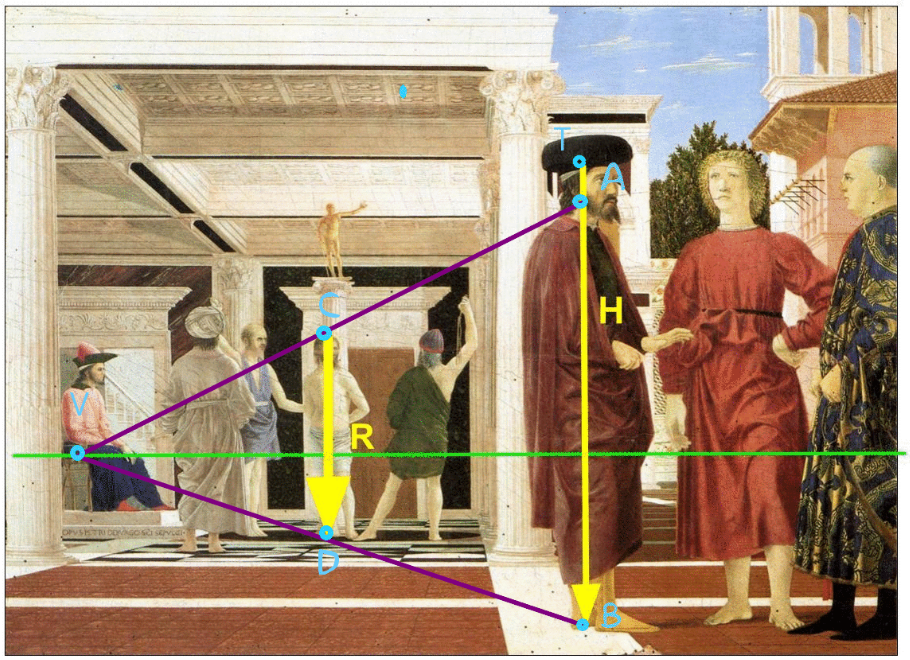
Now to show that they are colinear we need that $(P_p \times P_q) \circ P_r = 0$

$$\begin{vmatrix} \frac{da_1}{c_1} & \frac{da_2}{c_2} & \frac{da_3}{c_3} \\ \frac{db_1}{c_1} & \frac{db_2}{c_2} & \frac{db_3}{c_3} \\ 1 & 1 & 1 \end{vmatrix} = d^2 \begin{vmatrix} \frac{a_1}{c_1} & \frac{a_2}{c_2} & \frac{a_3}{c_3} \\ \frac{b_1}{c_1} & \frac{b_2}{c_2} & \frac{b_3}{c_3} \\ 1 & 1 & 1 \end{vmatrix} = \frac{d^2}{c_1 c_2 c_3} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{24}$$

This determinant is already 0 as the the given linens are coplanar. Hence we can say that the points P_p, P_q, P_r are coplanar.

Question 6 : Assuming that the green line on the image is the horizon line and also that the height of Christ, $R = 180cm$. We need to find the the height H, of the person on the right.

We need to find out the vanishing point along the horizontal for height estimation. We know that D and B are on the same horizontal level (as both are on ground). Also we are given that the green line is horizontal line. The intersection point of Line BD and Green Horizontal line gives the vanishing point V.



We have used MATLAB to find the following coordinates -
$T = (740, 210)$
$A = (740, 260)$
$B = (740, 790)$

We know that $AB = CD$. This is because AC and BD are parallel lines and intersect at vanishing point V. Using this information and the coordinates -

$$\frac{TB}{AB} = \frac{T_y - B_y}{A_y - B_y} = \frac{210 - 790}{260 - 790} = 1.094 \tag{25}$$

Also we know that -

$$\frac{TB}{AB} = \frac{H}{R} \rightarrow H = 180 * 1.094 = 197cm \tag{26}$$