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# Teaching time-series analysis. I. Finite Fourier analysis of ocean waves

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The introduction of students to methods of time-series analysis is a pedagogical challenge, since the availability of easily manipulated computer software presents an attractive alternative to an understanding of the computations, as well as their assumptions and limitations. A two-part pedagogical tutorial exercise is offered as a hands-on laboratory to complement classroom discussions or as a reference for students involved in independent research projects. The exercises are focused on the analysis of ocean waves, specifically wind-generated surface gravity waves. The exercises are cross-disciplinary in nature and can be extended to any other field dealing with random signal analysis. The first exercise introduces the manual arithmetic steps of a finite Fourier analysis of a wave record, develops a spectrum, and compares these results to the results obtained using a fast Fourier transform (FFT). The second part of the exercise, described in the subsequent article, takes a longer wave record and addresses the theoretical and observed wave probability distributions of wave heights and sea surface elevations. These results are then compared to a FFT, thus linking the two pedagogical laboratory exercise parts for a more complete understanding of both exercises.

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## I. INTRODUCTION

A defining topic for undergraduate and graduate students majoring in ocean engineering or physical oceanography is the study and analysis of wind-generated surface gravity waves. All students have seen and most have interacted with surface waves in various recreational activities, and therefore surface waves are a physical oceanographic phenomenon much more appreciated than less visible oceanographic phenomena such as internal waves or thermohaline circulation. The appropriateness of innovatively using ocean surface waves to teach Fourier analysis is apparent when one considers the dramatic decline in students majoring in mathematics and science.[1] As educators, we must make our topics interesting and stimulating. These exercises are cross-disciplinary and ''hands-on'' in nature and thus can benefit students in many other fields involved in signal analysis.

The study of lake or ocean surface gravity waves generally commences with basic definitions of wave properties (e.g., height, length, period, frequency, and steepness) and continues through the development and formulations of monochromatic (i.e., single wave frequency) Airy, or linear, wave theory. However, at this point the study becomes much more difficult for students as they shift their studies from monochromatic waves to the more realistic and complex situation of a water surface composed of a large variety of waves with different periods, heights, phases, and propagation directions (Fig. 1). Analysis of this realistic and more complex system is accomplished with the use of finite Fourier analysis, wave frequency spectra, and wave height and sea surface elevation probability distributions.

This paper describes the first of two personal computer-based laboratory exercises which assist students in understanding finite Fourier analysis, wave spectra, and sea surface elevation and wave height probability distributions. Previous investigators and educators have provided a tutorial account of how a fast Fourier transform (FFT) algorithm works and its relationship to Fourier series;[2] have described the design of an electronic device to demonstrate Fourier synthesis to students;[3] have described an undergraduate experiment to illustrate chaotic motion and subsequent analysis using power spectra;[4] and have described how Fourier analysis can be taught in a microcomputer laboratory to treat simple and nontrivial responses of resonance circuits.[5]

The two pedagogical laboratory exercises described in this and the next article are appropriate for upper-level undergraduate or lower-level graduate students and would follow a thorough classroom discussion of the theory. Personal computer spreadsheet software is used because it is easily manipulated, commonly available, and pedagogically desirable. The use of spreadsheets is appropriate for the pedagogical objectives of this exercise, since the purpose is to lead the student in constructing the building blocks of the Fourier problem. In our experience, students can often obtain correct answers to Fourier problems, either analytically or by pressing the correct button or typing the correct one-line code (i.e., ''y=fft(x)'') in a software program like MATLAB; yet the overwhelming majority do not truly *understand* the analysis until they execute this spreadsheet approach. Thus more advanced research-oriented software that executes FFTs and other spectral operations at the click of a button are not suitable for this elementary pedagogical approach.

The first exercise takes approximately 2 h and introduces the manual arithmetic steps of a finite Fourier analysis applied to a relatively short time record of sea surface elevation. It continues with a comparison to the ''miracle'' of a FFT, a numeric tool included as an integral part of most mathematical software, and concludes with developing a simple wave spectrum. The second exercise, described in the subsequent article,[6] takes a longer sea surface, or water,

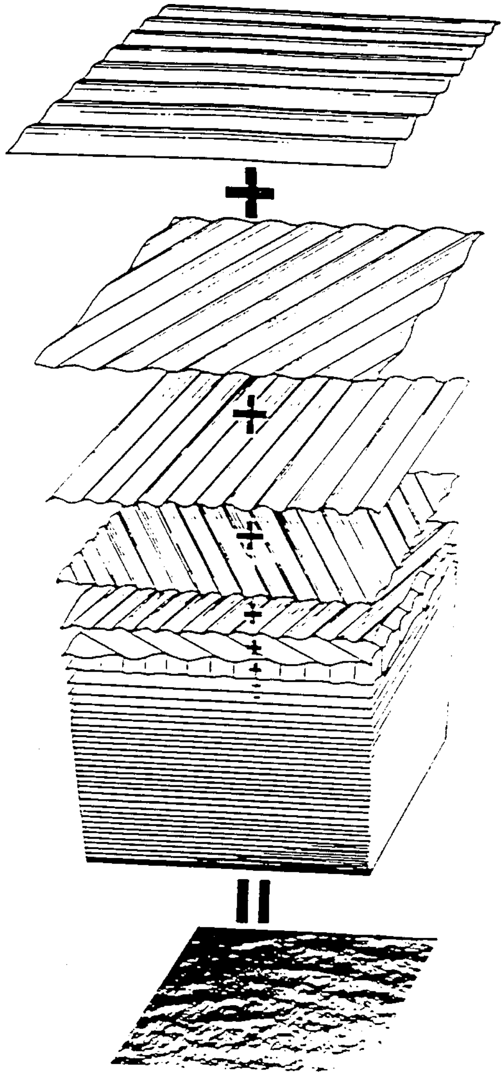

Fig. 1. A sum of many simple sine waves makes a sea (reprinted from W. J. Pierson, Jr., G. Neuman, and R. W. James, *Practical Methods for Observing and Forecasting Ocean Waves by Means of Wave Spectra and Statistics* (Hydrographic Office Pub. No. 603, U.S. Naval Hydrographic Office, Washington, DC, 1955, p. 24). Note that this paper's analysis deals with data observed at only one point in the ocean surface. Wavelength, wave speed, and directionality are not at issue.

elevation time series, applies the zero-crossing method of wave record analysis to determine wave heights, and calculates and compares theoretical and observed wave height and sea surface elevation probability distributions. These probability distributions are also a difficult concept for undergraduate students. The second exercise continues by comparing the probability distribution results to a FFT calculation, thus providing an essential link between the two exercises.

The exercises assume that the undergraduate students have had courses in college physics and mathematics which have already introduced them to sinusoidal wave characteristics of infinite Fourier series.

## II. FINITE FOURIER ANALYSIS OF OCEAN WAVES

Most undergraduate students have been introduced in their previous mathematics courses to the expression for an

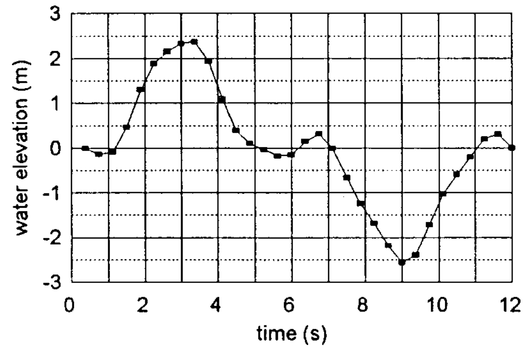


Fig. 2. The exercise's wave record of sea surface elevations, i.e., a time series observed at one spatial point on the ocean surface.

infinite Fourier series. However, fewer students are familiar with the corresponding equation for a finite Fourier series:[7–9]

$$\eta(t)=A_0+\sum_{q=1}^{N/2} A_q\cos(q\sigma_1 t)+\sum_{q=1}^{N/2-1} B_q\sin(q\sigma_1 t), \quad (1)$$

where $\eta$=sea surface elevation (m), $t$=time (s), $A_0$=record mean (m), $N$=total number of sampling points, $A_q$ and $B_q$=Fourier coefficients (m), $q$=harmonic component index (in the frequency domain), $\sigma_1$=fundamental radian frequency$=2\pi/T_R=2\pi f_1$ (rad s$^{-1}$), $f_1$=fundamental frequency$=1/T_R$ (Hz), $n$=data record index (in the time domain), $\Delta t$=sampling interval (s), $T_R$=record length$=N\Delta t$ (s),

$$A_0=\frac{1}{N}\sum_{n=1}^{N}\eta_n=\overline{\eta}, \quad (2)$$

$$A_q=\frac{2}{N}\sum_{n=1}^{N}\eta_n\cos\left(\frac{2\pi qn}{N}\right), \quad q=1,2,3,\ldots,\frac{N}{2}-1, \quad (3)$$

$$B_q=\frac{2}{N}\sum_{n=1}^{N}\eta_n\sin\left(\frac{2\pi qn}{N}\right), \quad q=1,2,3,\ldots,\frac{N}{2}-1, \quad (4)$$

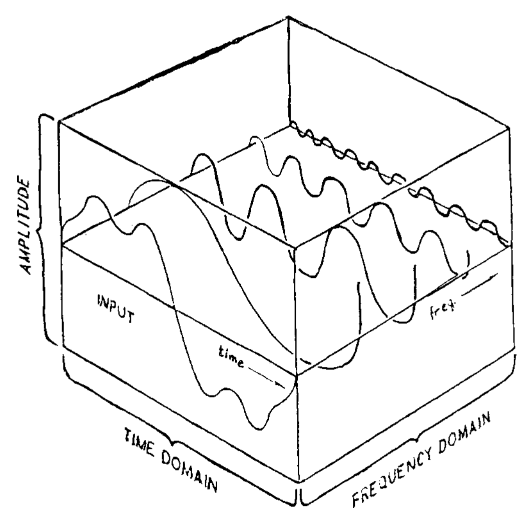


Fig. 3. Time domain and frequency domain representations of a periodic irregular wave. [Reprinted with permission from T. C. Gillmer and B. Johnson, *Introduction to Naval Architecture* (U.S. Naval Institute, Annapolis, MD, 1982), p. 261.]

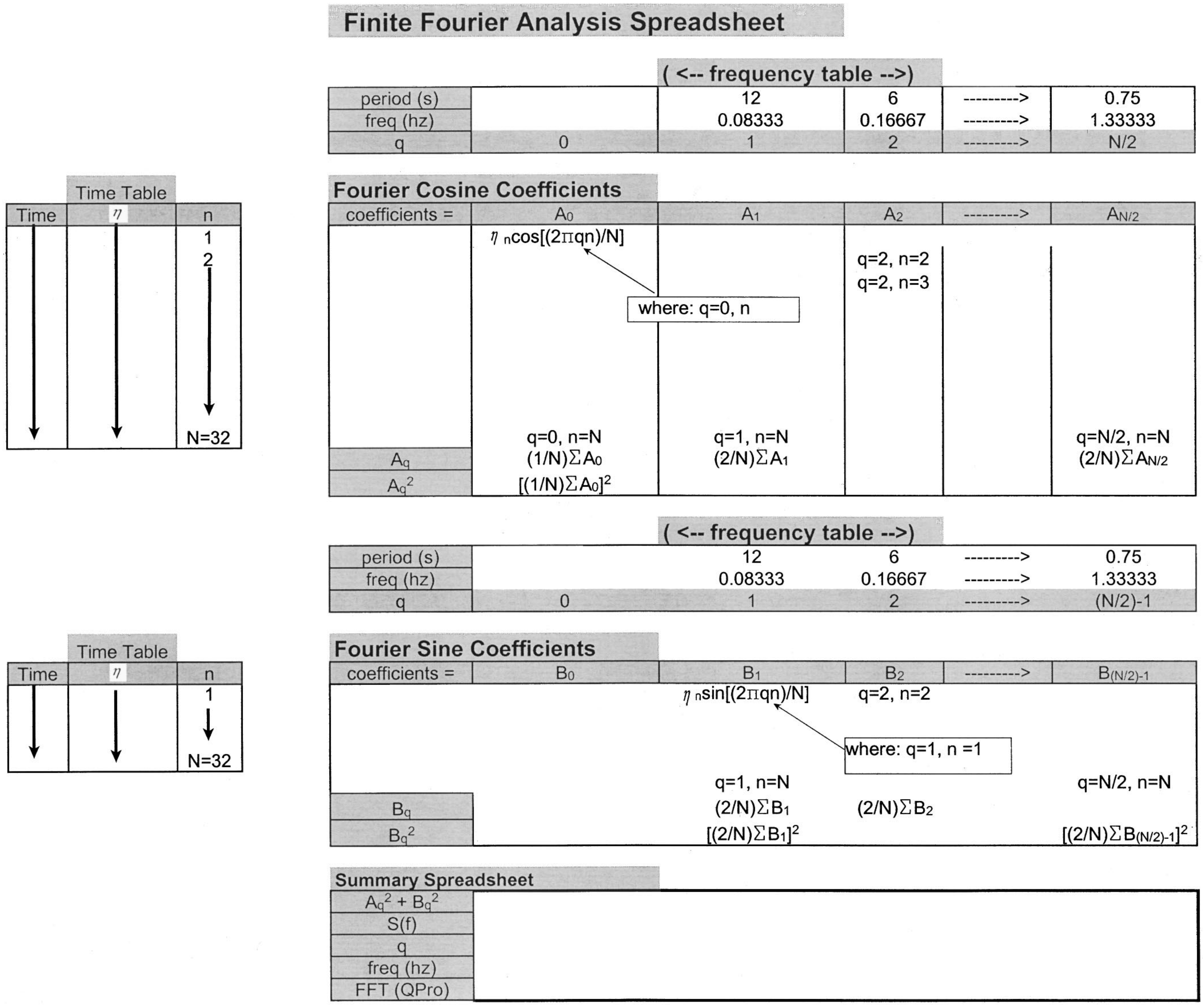


Fig. 4. Finite Fourier Analysis spreadsheet. A template spreadsheet is provided to the students prior to commencing the exercise. The template only provides the information shown with a gray background.

$$A_{N/2}=\frac{1}{N}\sum_{n=1}^{N}\eta_n\cos(n\pi). \tag{5}$$

Approaching wave analysis from a practical viewpoint, we ask the students to analyze a given wave record. This particular record has a duration of 12 s with a sampling interval of 0.375 s which provides a total of 32 data points. Unbeknown to the student, the record is the sum of three sine waves with periods of 12.0, 4.0, and 1.6 s and amplitudes of 1.60, 0.85, and 0.15 m, respectively.

We have the students plot the exercise's digital data so that they may view the record in an analog form and then we ask them to estimate, by observation only, what signals are ''hidden'' inside. Figure 2 illustrates the analog sea level data record provided to the students for this exercise.[10] It was specifically chosen because most students can visually discern some periodicity, but cannot quite explain or quantify the periodicity. This intrigue is used to keep them interested in the exercise results.

Additionally, this time series is an example of what a record of sea level would look like if taken by an instrument at a location in space, i.e., data observed at one spatial point on the ocean surface of Fig. 1. It must be emphasized that the point of the exercise is to use Fourier techniques to identify the frequencies present in the signal shown in Fig. 2, and that there is no attempt to deal with matters such as wave speed, wave length, or directionality, since these concepts would have been covered in classroom discussions.

Since a finite Fourier series is an analysis in both the time and frequency domain (Fig. 3), the analysis lends itself to a spreadsheet application. We provide the students with a spreadsheet file which has only the column and row titles shown in grey in Fig. 4.

The cosine Fourier coefficients are calculated in the upper portion of the spreadsheet and the sine Fourier coefficients are calculated in the lower portion, followed by a spreadsheet summary. Analogous to Fig. 3, the time domain is depicted in the vertical and is represented by identical time tables to the left of both Fourier coefficient spreadsheets, and the frequency domain is depicted in the horizontal across the top of both Fourier coefficient spreadsheets. Using inputs of $\eta$, $q$, $n$, and $N$, the cells calculate:

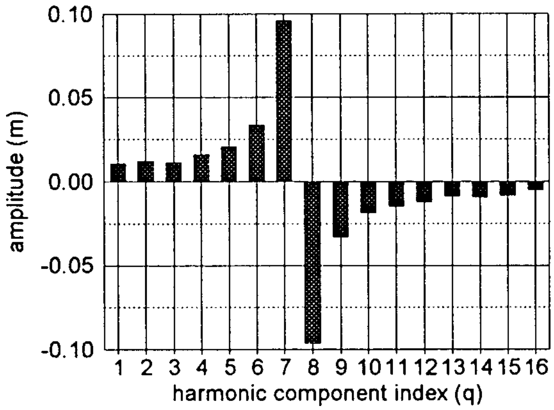


Fig. 5. An amplitude spectrum of Fourier cosine coefficient amplitudes vs frequency in harmonic component index ($q$) units.

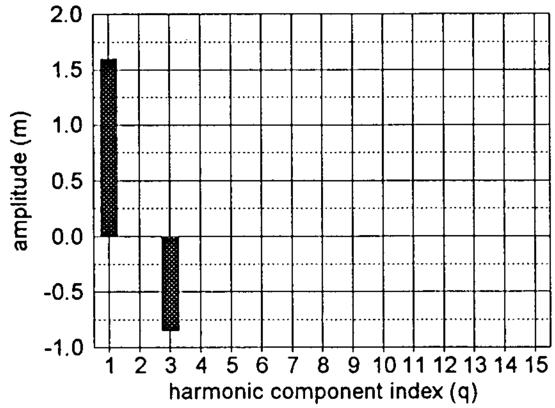


Fig. 6. An amplitude spectrum of Fourier sine coefficient amplitudes vs frequency in harmonic component index ($q$) units. (Note that there is a difference in amplitude scale from the previous figure.)

$$\eta_n \cos\left(\frac{2\pi q n}{N}\right), \tag{6}$$

$$\eta_n \sin\left(\frac{2\pi q n}{N}\right), \tag{7}$$

which are the summation arguments of Eqs. (3) and (4).

These cell values are subsequently summed from $n=1$ to $N$ and multiplied by the appropriate factor of $1/N$ or $2/N$ to determine $A_q$ and $B_q$ at each frequency. It is the actual spreadsheet coding of Eqs. (6) and (7), subsequent copying to other cells with different $q$ and $n$ values, and eventual summation, that provides the learning environment for understanding finite Fourier analysis.

The harmonics ($f_q$) of the fundamental frequency are the (discrete) Fourier frequencies determined from

$$f_q = q f_1, \quad q=1,2,3,\ldots,\frac{N}{2}. \tag{8}$$

Their relevance is addressed by illustrating that the original record is only resolved into $N/2$ Fourier frequencies and no others.

## III. SPECTRA

To increase the students' understanding of their analyses, we now charge them with creating the following three spectra from their spreadsheet values. These three different spectra types are used extensively in oceanographic literature, which can cause significant confusion for the students.

### A. Amplitude spectra

The students first produce an amplitude spectrum (Fig. 5) of cosine amplitudes (ordinate) versus frequency in terms of ''$q$,'' the harmonic component index (abscissa), in bar graph form. Then they produce an amplitude spectrum (Fig. 6) of sine amplitudes (ordinate) versus frequency in terms of ''$q$'' in bar graph form.

These graphs serve several purposes: to acquaint the students with the basic concept of a frequency spectrum plotted with respect to $q$, or the frequency itself, and to show order of magnitude as well as sign differences which can exist for Fourier coefficients.

### B. Energy spectrum

Students should be familiar with the concept that the energy contained in a sinusoidal signal is proportional to the square of its amplitude multiplied by the factor 0.5.[11] In the case of surface wave analysis, which is the immediate object of this exercise, we direct the students to small amplitude, or Airy, wave theory,[12] where one may derive the average total energy ($E$) of a progressive wave per unit horizontal area in units of $\mathrm{J\,m^{-2}}$ as:[11,13]

$$E = \tfrac{1}{2}\rho g A^2, \tag{9}$$

where $\rho$ is density ($\mathrm{kg\,m^{-3}}$) of the fluid, $g$ is gravitational acceleration ($\mathrm{m\,s^{-2}}$), and $A$ is the wave amplitude (m).[14] We point out to the students that energy is thus proportional to wave amplitude squared, while the other two parameters, $\rho$ and $g$, are just constants that contribute to the correct units. Hence, in practical terms, the energy is measured in units of length squared, in this case $\mathrm{m^2}$, as a surrogate for the full units of $\mathrm{J\,m^{-2}}$.

Thus to determine energy spectra, we need to consider spectra of squared amplitudes. Therefore, the students continue their exercise by entering their summary spreadsheet and calculating the sum of the squared Fourier coefficient amplitudes, i.e., $A_q^2+B_q^2$, for each frequency. They then produce an energy spectrum [Fig. 7(a)] composed of the sum of the squared amplitudes (using a logarithmic ordinate to make visible the small values, in units of $\mathrm{m^2}$, at the higher frequencies) versus frequency (linear abscissa). For the first time, the students are now able to discern where the principal energy is located.

In this example, the principal energy is located at $f_1 = 0.083\,33\ \mathrm{Hz}$ ($q=1$) and $f_3 = 0.25\ \mathrm{Hz}$ ($q=3$), with some additional energy at $f_7 = 0.583\,33\ \mathrm{Hz}$ ($q=7$) and $f_8 = 0.666\,67\ \mathrm{Hz}$ ($q=8$). The highest frequency that can be detected with data sampled at interval $\Delta t$ is determined [see Eq. (8)] as the $N/2$ harmonic of the fundamental frequency ($f_1$):

$$f_{N/2} = \frac{N}{2} f_1 = \frac{N}{2}\frac{1}{T_R} = \frac{N}{2}\frac{1}{N\Delta t} = \frac{1}{2\Delta t}. \tag{10}$$

This frequency is known as the Nyquist frequency ($f_c$):[15]

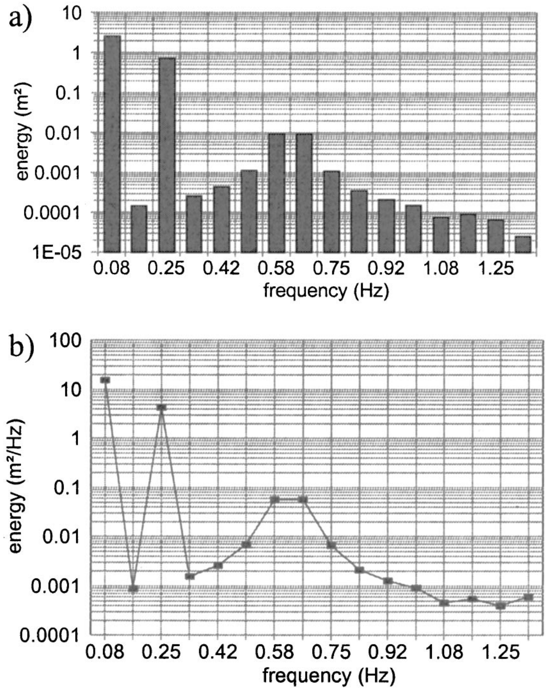


Fig. 7. (a) An energy spectrum and (b) renormalized as an energy density spectrum, both using a log-linear plot.

$$f_c = \frac{1}{2\Delta t} \tag{11}$$

and is a result of general applicability.[16] Therefore in this example, $f_c = f_{N/2} = f_{16} = 1.3333\,\mathrm{Hz}$ ($q=16$).

### C. Energy density spectrum

There are several equation variations which can be derived to describe an energy density spectrum,[17,18] $S(f)$ or $S(q)$, from a discrete data series. Emery and Thompson[19] provide excellent background on this topic. We use the following equation, where $\Delta f$ is the (constant) difference in frequency between any adjacent Fourier frequencies (i.e., $\Delta f = 1/[N\Delta t]$), to express $S(q)$ in $\mathrm{m}^2\,\mathrm{Hz}^{-1}$:

$$S(q) = \frac{1}{2}\frac{(A_q^2 + B_q^2)}{\Delta f} \quad \text{for } q = 1,2,3,\ldots,\frac{N}{2} \tag{12}$$

because it lends itself to an intuitive explanation of an energy density spectrum in terms of a ''sum of the squared amplitudes.'' Note that the left-hand side of Eq. (12) could just as well read $S(f)$ since each $q$ represents a distinct Fourier frequency. We take great care to explain to the students that the energy allocated to each Fourier frequency really belongs to a frequency band of width $\Delta f$ centered at that frequency. Hence the idea of ''density,'' i.e., energy (in $\mathrm{m}^2$) spread over a bandwidth $\Delta f$, yielding units of $\mathrm{m}^2\,\mathrm{Hz}^{-1}$. Note that the $\Delta f$ in the denominator of Eq. (12) represents a normalization of the energy spectrum; i.e., Fig. 7(b) is a plot of energy per unit frequency increment $\Delta f$. This particular normalization factor is correct in the case of surface waves; other normalization factors may be appropriate in time series of other types of random oscillatory phenomena.

Two clarifications must be made at this point. First, to be a *true* energy density spectrum, we should multiply the amplitude-squared spectrum by ''$\rho\, g$'' from Eq. (9), which yields energy density at a particular frequency:

$$E(q) = \rho g S(q), \tag{13}$$

where $E(q)$ is in units of $\mathrm{J\,m^{-2}\,Hz^{-1}}$. However many texts still refer to the amplitude-squared spectrum as the ''energy spectrum.'' We do so in this article. A second clarification is that adding Eq. (13) over all frequencies yields

$$E = \sum_{q=1}^{N/2} E(q)\Delta f, \tag{14}$$

where $E$ is again in units of $\mathrm{J\,m^{-2}}$ as it was in Eq. (9). The energy density spectrum is used extensively because the area under the spectrum curve is a measure of the total energy in the wave field.

Thus the students produce an energy density spectrum [Fig. 7(b)] calculated from Eq. (12). Again, a log-linear scale is used to make visible the small energetic contents at the higher frequencies. The points on the graph are now connected by a continuous line to stress the idea of a continuum of energy density from one Fourier frequency centered bandwidth to the next. Note that it would not be appropriate to connect the points on the graph of Fig. 7(b) with some kind of polynomial or spline interpolation curve since there is *no* information on what happens to the energy between adjacent Fourier frequencies. Hence we use a straight line connection between the energy concentrated within each frequency band $\Delta f$. The ultimate importance of this spectrum is that the area under the curve is a measure of the total energy in the wave signal. This exposure will be pursued when discussing wave forecasting techniques with the students at a later time.

We then ask the following questions:

(1) Identify the frequencies that contribute the most energy to the original signal. [*Answer*. The largest amount of energy is found at $q=1$ ($f_1 = 0.083\,33\,\mathrm{Hz}$), followed by an order of magnitude less energy at $q=3$ ($f_3 = 0.25\,\mathrm{Hz}$). There is less energy found, and evenly split, between $q=7$ ($f_7 = 0.583\,33\,\mathrm{Hz}$) and $8$($f_8 = 0.666\,67\,\mathrm{Hz}$).] What are the amplitudes of these significant waves? [*Answer*. 1.6 m for the wave at $f_1$ and 0.85 m for the wave at $f_3$.]

(2) How do these frequencies compare to your (the student's) guess when you first observed the original signal?

(3) Frequencies 0.583 33 ($q=7$) and 0.666 67 Hz ($q=8$) exhibit just about the same amount of energy. One interpretation would be that these two adjacent frequencies happen to contain similar amounts of energy. What might be another possible interpretation? [*Answer*. There is energy found at a frequency between these two frequencies but which is not resolved by this Fourier analysis with a fundamental frequency of $f_1 = 1/12\,\mathrm{s}$ or its harmonics.]

(4) Would you be able to resolve the above ambiguity by collecting data with a different $\Delta t$? [*Answer*. Yes, but you must be careful on how to do this. To obtain increased resolution, we wish to decrease $\Delta f$ where $\Delta f = 1/(N\Delta t)$. Therefore we could increase $\Delta t$ while keeping $N$ constant, or we could increase $N$ while keeping $\Delta t$ constant. In this particular case, doubling $\Delta t$ would resolve the problem. A shorter $\Delta t$ would result in increasing the upper frequency limit of resolvable Fourier frequencies, but not increased resolution, per Eq. (11). Changing $N$ is addressed in the next question.]

(5) Suppose that you maintain the same $\Delta t$ but the data record goes on for a longer time, i.e., you have more data

**Finite Fourier Analysis Spreadsheet**

( <-- frequency table --> )

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| period (s) | 0 | 12 | 6 | 4 | 3 | 2.4 | 2 | 1.714286 | 1.5 | 1.333333 | 1.2 | 1.090909 | 1 | 0.923077 | 0.857143 | 0.8 | 0.75 |
| freq (hz) | 0 | 0.083333 | 0.166667 | 0.25 | 0.333333 | 0.416667 | 0.5 | 0.583333 | 0.666667 | 0.75 | 0.833333 | 0.916667 | 1 | 1.083333 | 1.166667 | 1.25 | 1.333333 |
| q | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

**Time Table**

| Time (s) | Elev (m) | n(counter) |
|---|---|---|
| 0.375 | -0.01 | 1 |
| 0.75 | -0.14 | 2 |
| 1.125 | -0.09 | 3 |
| 1.5 | 0.47 | 4 |
| 1.875 | 1.3 | 5 |
| ↓ | ↓ | ↓ |
| 10.5 | -0.59 | 28 |
| 10.875 | -0.2 | 29 |
| 11.25 | 0.2 | 30 |
| 11.625 | 0.31 | 31 |
| 12 | 0 | 32 |

**Fourier Cosine Coefficients**

| coefficient | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | $A_{13}$ | $A_{14}$ | $A_{15}$ | $A_{16}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | -0.01 | -0.00981 | -0.00924 | -0.00831 | -0.00707 | -0.00556 | -0.00383 | -0.00195 | -6.1E-19 | 0.001951 | 0.003827 | 0.005556 | 0.007071 | 0.008315 | 0.009239 | 0.009808 | 0.01 |
| | -0.14 | -0.12934 | -0.09899 | -0.05358 | -8.6E-18 | 0.053576 | 0.098995 | 0.129343 | 0.14 | 0.129343 | 0.098995 | 0.053576 | 2.57E-17 | -0.05358 | -0.09899 | -0.12934 | -0.14 |
| | -0.09 | -0.07483 | -0.03444 | 0.017558 | 0.06364 | 0.088271 | 0.083149 | 0.050001 | 1.65E-17 | -0.05 | -0.08315 | -0.08827 | -0.06364 | -0.01756 | 0.034442 | 0.074832 | 0.09 |
| | 0.47 | 0.33234 | 2.88E-17 | -0.33234 | -0.47 | -0.33234 | -8.6E-17 | 0.33234 | 0.47 | 0.33234 | 1.44E-16 | -0.33234 | -0.47 | -0.33234 | -2E-16 | 0.33234 | 0.47 |
| | 1.3 | 0.722241 | -0.49749 | -1.27502 | -0.91924 | 0.253617 | 1.201043 | 1.08091 | 3.98E-16 | -1.08091 | -1.20104 | -0.25362 | 0.919239 | 1.275021 | 0.497488 | -0.72224 | -1.3 |
| | | ↓ | | | | ↓ | | | | | ↓ | | | | | ↓ | |
| | -0.59 | -0.41719 | 2.53E-16 | 0.417193 | 0.59 | 0.417193 | 2.89E-16 | -0.41719 | -0.59 | -0.41719 | 2.31E-15 | 0.417193 | 0.59 | 0.417193 | -4.9E-15 | -0.41719 | -0.59 |
| | -0.2 | -0.16629 | -0.07654 | 0.039018 | 0.141421 | 0.196157 | 0.184776 | 0.111114 | -1.6E-20 | -0.11111 | -0.18478 | -0.19616 | -0.14142 | -0.03902 | 0.076537 | 0.166294 | 0.2 |
| | 0.2 | 0.184776 | 0.141421 | 0.076537 | -5.4E-16 | -0.07654 | -0.14142 | -0.18478 | -0.2 | -0.18478 | -0.14142 | -0.07654 | 1.96E-16 | 0.076537 | 0.141421 | 0.184776 | 0.2 |
| | 0.31 | 0.304043 | 0.286403 | 0.257756 | 0.219203 | 0.172227 | 0.118632 | 0.060478 | -1.1E-15 | -0.06048 | -0.11863 | -0.17223 | -0.2192 | -0.25776 | -0.2864 | -0.30404 | -0.31 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_q$ | 0.005 | 0.010583 | 0.012105 | 0.011523 | 0.016017 | 0.020886 | 0.0336 | 0.096309 | -0.09625 | -0.03303 | -0.01866 | -0.01451 | -0.01227 | -0.00868 | -0.00954 | -0.00808 | -0.01 |
| $A_q^2$ | 2.5E-05 | 0.000112 | 0.000147 | 0.000133 | 0.000257 | 0.000436 | 0.001129 | 0.009275 | 0.009264 | 0.001091 | 0.000348 | 0.000211 | 0.00015 | 7.53E-05 | 9.11E-05 | 6.53E-05 | 0.0001 |


( <-- frequency table --> )

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| period (s) | 0 | 12 | 6 | 4 | 3 | 2.4 | 2 | 1.714286 | 1.5 | 1.333333 | 1.2 | 1.090909 | 1 | 0.923077 | 0.857143 | 0.8 | 0.75 |
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| q | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

**Time Table**

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|---|---|---|
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| 1.875 | 1.3 | 5 |
| ↓ | ↓ | ↓ |
| 10.5 | -0.59 | 28 |
| 10.875 | -0.2 | 29 |
| 11.25 | 0.2 | 30 |
| 11.625 | 0.31 | 31 |
| 12 | 0 | 32 |


**Fourier Sine Coefficients**

| coefficient | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ | $B_9$ | $B_{10}$ | $B_{11}$ | $B_{12}$ | $B_{13}$ | $B_{14}$ | $B_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| | | -0.00195 | -0.00383 | -0.00556 | -0.00707 | -0.00831 | -0.00924 | -0.00981 | -0.01 | -0.00981 | -0.00924 | -0.00831 | -0.00707 | -0.00556 | -0.00383 | -0.00195 |
| | | -0.05358 | -0.09899 | -0.12934 | -0.14 | -0.12934 | -0.09899 | -0.05358 | -1.7E-17 | 0.053576 | 0.098995 | 0.129343 | 0.14 | 0.129343 | 0.098995 | 0.053576 |
| | | -0.05 | -0.08315 | -0.08827 | -0.06364 | -0.01756 | 0.034442 | 0.074832 | 0.09 | 0.074832 | 0.034442 | -0.01756 | -0.06364 | -0.08827 | -0.08315 | -0.05 |
| | | 0.33234 | 0.47 | 0.33234 | 5.76E-17 | -0.33234 | -0.47 | -0.33234 | -1.2E-16 | 0.33234 | 0.47 | 0.33234 | 1.73E-16 | -0.33234 | -0.47 | -0.33234 |
| | | 1.08091 | 1.201043 | 0.253617 | -0.91924 | -1.27502 | -0.49749 | 0.722241 | 1.3 | 0.722241 | -0.49749 | -1.27502 | -0.91924 | 0.253617 | 1.201043 | 1.08091 |
| | | ↓ | | | | ↓ | | | | | ↓ | | | | | ↓ |
| | | 0.417193 | 0.59 | 0.417193 | -5.1E-16 | -0.41719 | -0.59 | -0.41719 | 1.01E-15 | 0.417193 | 0.59 | 0.417193 | 5.78E-16 | -0.41719 | -0.59 | -0.41719 |
| | | 0.111114 | 0.184776 | 0.196157 | 0.141421 | 0.039018 | -0.07654 | -0.16629 | -0.2 | -0.16629 | -0.07654 | 0.039018 | 0.141421 | 0.196157 | 0.184776 | 0.111114 |
| | | -0.07654 | -0.14142 | -0.18478 | -0.2 | -0.18478 | -0.14142 | -0.07654 | 1.08E-15 | 0.076537 | 0.141421 | 0.184776 | 0.2 | 0.184776 | 0.141421 | 0.076537 |
| | | -0.06048 | -0.11863 | -0.17223 | -0.2192 | -0.25776 | -0.2864 | -0.30404 | -0.31 | -0.30404 | -0.2864 | -0.25776 | -0.2192 | -0.17223 | -0.11863 | -0.06048 |
| | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $B_q$ | | 1.600162 | 0.000442 | -0.8499 | 0.000625 | -0.00022 | 0.000442 | 0.000644 | 1.56E-16 | -0.00027 | -0.00044 | -0.00028 | -0.00062 | 0.000271 | -0.00044 | -0.00052 |
| $B_q^2$ | | 2.560517 | 1.95E-07 | 0.722333 | 3.91E-07 | 4.81E-08 | 1.95E-07 | 4.15E-07 | 2.44E-32 | 7.17E-08 | 1.95E-07 | 7.82E-08 | 3.91E-07 | 7.36E-08 | 1.95E-07 | 2.67E-07 |

**Summary Spreadsheet**

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| freq (hz) | 0 | 0.083333 | 0.166667 | 0.25 | 0.333333 | 0.416667 | 0.5 | 0.583333 | 0.666667 | 0.75 | 0.833333 | 0.916667 | 1 | 1.083333 | 1.166667 | 1.25 | 1.333333 |
| q | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $A_q^2+B_q^2$ | 2.5E-05 | 2.560629 | 0.000147 | 0.722466 | 0.000257 | 0.000436 | 0.001129 | 0.009276 | 0.009264 | 0.001091 | 0.000348 | 0.000211 | 0.000151 | 7.54E-05 | 9.13E-05 | 6.55E-05 | 0.0001 |
| S(q) | **0.0006** | **15.36378** | **0.00088** | **4.334796** | **0.001542** | **0.002618** | **0.006775** | **0.055655** | **0.055584** | **0.006546** | **0.002091** | **0.001264** | **0.000905** | **0.000452** | **0.000548** | **0.000393** | **0.0006** |
| FFT S(q) | 0.0006 | 15.36377 | 0.00088 | 4.334788 | 0.001542 | 0.002618 | 0.006775 | 0.055655 | 0.055584 | 0.006546 | 0.002091 | 0.001264 | 0.000905 | 0.000452 | 0.000548 | 0.000393 | 0.0006 |

Fig. 8. Completed finite Fourier analysis spreadsheet.

points. What is gained by a longer data record? [*Answer*. A longer record with the same $\Delta t$ will yield more sampling points, or a higher $N$, resulting in more terms in the Fourier series. More frequency components result in more resolution in the obtained energy spectrum.[20] Another way of looking at this problem is as follows: As $N$ increases for the same $\Delta t$, $\Delta f(=1/(N\Delta t))$ decreases, thus providing increased frequency resolution.]

(6) For the same data record length ($T_R$), contrast the results of more data points and a shorter $\Delta t$ versus fewer data points and longer $\Delta t$. [*Answer*. From Eq. (11), we see that $\Delta t$ is inversely proportional to the Nyquist frequency. Therefore a smaller $\Delta t$ results in a higher upper-frequency limit on the resultant spectrum.]

(7) Suppose that you are designing an experiment where you want to investigate the spectral distribution of energy in wind speed at a certain location. You expect the wind record to contain frequencies as high as 1 Hz (wind gusts) and you want to be able to discriminate Fourier frequencies no more than 0.005 Hz apart. What is the shortest length of time that you must turn on your anemometer to measure the wind speed and what would the sampling interval $\Delta t$ have to be? [*Answer*. $f_1=1/T_R$, hence $0.005\ \text{Hz}=1/T_R$ or $T_R=200\,\text{s}$; $f_c=1/(2\Delta t)$, hence $1\ \text{Hz}=1/(2\Delta t)$ or $\Delta t=0.5\,\text{s}$.]

## IV. FAST FOURIER TRANSFORM

Now that the students have an appreciation for the arithmetic formulation of finite Fourier analysis, we introduce them to the FFT introduced in 1965 by Cooley and Tukey.[21] FFTs are very efficient algorithms capable of computing Fourier coefficients very quickly and very efficiently. FFTs are used in the ''real world'' of oceanography, meteorology, seismic and geophysical sciences, economics, electrical engineering, cryptology, etc.

Using the same spreadsheet software, we select our original 32 values of sea surface elevation and calculate the FFT values using the FFT subroutine provided with the spreadsheet software. Note that the FFT's ''$a$'' and ''$b$'' correspond to the summations of Eqs. (3) and (4) from 1 to $N$, but without the $2/N$ term. We can now easily generate an energy density spectrum $[S(q)]$ at each frequency (or $q$) from zero to the Nyquist frequency using an equation similar to Eq. (12):

$$S(q)=\frac{1}{2}\left(\frac{2}{N}\right)^2\frac{(\sqrt{a_q^2+b_q^2})^2}{\Delta f}=\frac{2}{N^2}\,\frac{a_q^2+b_q^2}{\Delta f},\qquad q=1,2,3,\ldots,\frac{N}{2}. \tag{15}$$

The students place the FFT-produced spectrum values on their spreadsheet. The spectrum produced by this process is identical to the laborious one produced earlier. The final completed spreadsheet is provided as Fig. 8.

It must be noted that no attempt has been made to detrend or filter the data or to window the resulting spectral estimate. These concepts are essential in time series analysis, of course, but it is felt that it is not appropriate to invoke them when the objective is simply a thorough understanding of the finite Fourier series and its application to a practical situation. It was deliberate that the signal given to the students for this exercise was generated by the addition of three simple sinusoids; no trend was added, or noise. It was sought to facilitate as much as possible the student's job: to identify through the Fourier method the component frequencies in the given signal. After the exercise is complete, the benefits and problems associated with filtering and windowing are qualitatively discussed.

## V. SUMMARY

This pedagogical exercise provides a practical application and illustrates the power of a finite Fourier series analysis. The student is initially given discrete data from a short wave record. After plotting it to observe the signal shape, the student is directed to a computer spreadsheet where the Fourier coefficients are generated by student-written algorithms which are a function of the data record index $(n)$ and harmonic component index $(q)$. The significance of the index $q$ as surrogate for multiples of the fundamental frequency is emphasized.

The energy spectrum is generated and the frequencies containing the most energy are identified. One of the frequencies cannot be resolved exactly, demonstrating the limitations of this method. Finally the same data are analyzed with the fast Fourier transform routine provided with the spreadsheet software and its results are compared to the student's analysis. The fact that they are identical illustrates the practical application of the FFT to the student, who by now understands the theory and limitations of the Fourier analysis method.

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[7]Note that the truncation of the infinite Fourier series is intentionally chosen such that the number of Fourier terms [i.e., $A_0$, $(N/2)$ cosine terms, and $((N/2)-1)$ sine terms] is equal to the number of sampling points $(N)$ in the data set. For more information on finite Fourier series, see G. M. Jenkins and D. G. Watts, *Spectral Analysis and its Applications* (Holden–Day, Oakland, CA, 1968), pp. 18–19.
[8]J. S. Bendat and A. G. Piersol, *Random Data: Analysis and Measurement Procedures* (Wiley-Interscience, New York, 1971), pp. 299–300.
[9]B. Kinsman, *Ocean Waves and the Directional Spectrum: An Engineering Tool* (Marine Sciences Research Center, State University of New York, Stony Brook, 1978), pp. 103–104.
[10]The digitized data (in m) used for this figure and the subsequent exercise are: −0.01, −0.14, −0.09, 0.47, 1.3, 1.89, 2.16, 2.34, 2.37, 1.93, 1.09, 0.39, 0.1, −0.03, −0.17, −0.15, 0.15, 0.32, −0.01, −0.67, −1.24, −1.68, −2.18, −2.56, −2.39, −1.72, −1.03, −0.59, −0.2, 0.2, 0.31, and 0.00.
[11]W. C. Elmore and M. A. Heald, *Physics of Waves* (McGraw–Hill, New York, 1969), p. 203. Note that in some physics texts this energy is considered an energy density per unit area, whereas in this paper we have used the oceanographic convention of reserving the term ''density'' to refer to the energy density spectrum as described in the next section. Also, the factor of ''1/2'' converts between peak and rms amplitudes in oscillatory phenomena such as elementary ac circuit formulas.
[12]G. B. Airy, ''On Tides and Waves,'' *Encyclopaedia Metropolitana* (B. Fellowes, London, 1845), pp. 241–396.
[13]M. Sorensen, *Basic Wave Mechanics for Coastal and Ocean Engineers* (Wiley, New York, 1993), pp. 18–20.
[14]Note that the wave amplitude is equal to one-half the wave height.
[15]The Nyquist frequency is also called the cutoff or folding frequency. For more details on the Nyquist frequency, see C. Chatfield, *The Analysis of Time Series, An Introduction* (Chapman and Hall, London, 1984), 3rd ed., pp. 131–132, or J. S. Bendat and A. G. Piersol, *Random Data: Analysis and Measurement Procedures* (Wiley-Interscience, New York, 1971), pp. 228–231.
[16]Students might appreciate the following ''real'' example. Let's say that a compact disk (CD) is required to reproduce an audio frequency of 20 KhZ (the high frequency limit for normal human hearing). Then $f_c = 20$ kHz and by solving Eq. (11) for $\Delta t$, we have $\Delta t = 0.000\,025$ s, i.e., the digitized data for this audio signal on the CD must be spaced no further apart than 1/40 of a millisecond.
[17]R. G. Dean and R. A. Dalrymple, *Water Wave Mechanics for Engineers and Scientists* (World Scientific, Singapore, 1991), pp. 193–194.
[18]M. J. Tucker, *Waves in Ocean Engineering* (Ellis Horwood, New York, 1991), pp. 41–43.
[19]W. J. Emery and R. E. Thomson, *Data Analysis Methods in Physical Oceanography* (Pergamon, New York, 1998), pp. 380–387 and 422.
[20]Although this larger data set yields a smaller $\Delta f$, the output spectrum tends to have a greater random error, or ''noise,'' associated with it. Fourier transforms of long records almost always require some form of data smoothing of the output. For a more thorough discussion of this topic, see Ref. 8, pp. 170–213.
[21]R. J. W. Cooley and J. W. Tukey, ''An algorithm for the machine calculation of complex Fourier series,'' Math. Comput. **19** (90), 297–301 (1965).

**INERT FACTS**

Nothing in education is so astonishing as the amount of ignorance it accumulates in the form of inert facts.

Henry Adams, *The Education of Henry Adams* (The Modern Library, New York, 1931—Originally published by the Massachusetts Historical Society, 1918), p. 379.