Computational Complexity: Section 1

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In this article I'll try to introduce you to the area of computation complexity. The article will be a bit long before we get to the actual formal definitions because I feel that the rationale behind these definitions needs to be explained as well - and that understanding the rationale is even more important than the definitions alone.

Why is it important?

Example 1. Suppose you were assigned to write a program to process some records your company receives from time to time. You implemented two different algorithms and tested them on several sets of test data. The processing times you obtained are in Table 1.

# of records	10	20	50	100	1000	5000
algorithm 1	0.00s	0.01s	0.05s	0.47s	23.92s	47min
algorithm 2	0.05s	0.05s	0.06s	0.11s	0.78s	14.22s

Table 1. Runtimes of two fictional algorithms.

In praxis, we probably could tell which of the two implementations is better for us (as we usually can estimate the amount of data we will have to process). For the company this solution may be fine. But from the programmer's point of view, it would be much better if he could estimate the values in Table 1 **before** writing the actual code - then he could only implement the better algorithm.

The same situation occurs during programming contests: The size of the input data is given in the problem statement. Suppose I found an algorithm. Questions I have to answer before I start to type should be: Is my algorithm worth implementing? Will it solve the largest test cases in time? If I know more algorithms solving the problem, which of them shall I implement?

This leads us to the question: How to compare algorithms? Before we answer this question in general, let's return to our simple example. If we extrapolate the data in Table 1, we may assume that if the number of processed records is larger than 1000, algorithm 2 will be substantially faster. In other words, if we consider all possible inputs, algorithm 2 will be better for almost all of them.

It turns out that this is almost always the case - given two algorithms, either one of them is almost always better, or they are approximately the same. Thus, this will be our definition of a better algorithm. Later, as we define everything formally, this will be the general idea behind the definitions.

A neat trick

If you thing about Example 1 for a while, it shouldn't be too difficult to see that there is an algorithm with runtimes similar to those in Table 2:

# of records	10	20	50	100	1000	5000
algorithm 3	0.00s	0.01s	0.05s	0.11s	0.78s	14.22s

Table 2. Runtimes of a new fictional algorithm.

The idea behind this algorithm: Check the number of records. If it is small enough, run algorithm 1, otherwise run algorithm 2.

Similar ideas are often used in praxis. As an example consider most of the sort() functions provided by various libraries. Often this function is an implementation of QuickSort with various improvements, such as:

- if the number of elements is too small, run InsertSort instead (as InsertSort is faster for small inputs)
- if the pivot choices lead to poor results, fall back to MergeSort

What is efficiency?

Example 2. Suppose you have a concrete implementation of some algorithm. (The example code presented below is actually an implementation of MinSort - a slow but simple sorting algorithm.)

```
for (int i=0; i<N; i++)
  for (int j=i+1; j<N; j++)
    if (A[i] > A[j])
      swap( A[i], A[j] );
```

If we are given an input to this algorithm (in our case, the array A and its size N), we can exactly compute the number of steps our algorithm does on this input. We could even count the processor instructions if we wanted to. However, there are too many possible inputs for this approach to be practical.

And we still need to answer one important question: What is it exactly we are interested in? Most usually it is the behavior of our program in the **worst possible case** - we need to look at the input data and to determine an upper bound on how long will it take if we run the program.

But then, what is the worst possible case? Surely we can always make the program run longer simply by giving it a larger input. Some of the more important questions are: What is the worst input with 700 elements? **How fast** does the maximum runtime grow when we increase the input size?

Formal notes on the input size

What exactly is this "input size" we started to talk about? In the formal definitions this is the size of the input written in some fixed finite alphabet (with at least 2 "letters"). For our needs, we may consider this alphabet to be the numbers 0..255. Then the "input size" turns out to be exactly the size of the input file in bytes.

Usually a part of the input is a number (or several numbers) such that the size of the input is proportional to the number.

E.g. in Example 2 we are given an int N and an array containing N ints. The size of the input file will be roughly 5N (depending on the OS and architecture, but always linear in N).

In such cases, we may choose that this number will represent the size of the input. Thus when talking about problems on arrays/strings, the input size is the length of the array/string, when talking about graph problems, the input size depends both on the number of vertices (N) and the number of edges (M), etc.

We will adopt this approach and use N as the input size in the following parts of the article.

There is one tricky special case you sometimes need to be aware of. To write a (possibly large) number we need only logarithmic space. (E.g. to write 123456, we need only roughly $log_{10}(123456)$ digits.) This is why the naive primality test does not run in polynomial time - its runtime is polynomial in the **size** of the number, but not in its **number of digits**! If you didn't understand the part about polynomial time, don't worry, we'll get there later.

How to measure efficiency?

We already mentioned that given an input we are able to count the number of steps an algorithm makes simply by simulating it. Suppose we do this for all inputs of size at most N and find the worst of these inputs (i.e. the one that causes the algorithm to do the most steps). Let $f(N)$ be this number of steps. We will call this function the time complexity, or shortly the runtime of our algorithm.

In other words, if we have any input of size N, solving it will require at most $f(N)$ steps.

Let's return to the algorithm from Example 2. What is the worst case of size N? In other words, what array with N elements will cause the algorithm to make the most steps? If we take a look at the algorithm, we can easily see that:

- the first step is executed exactly N times
- the second and third step are executed exactly $N(N - 1)/2$ times
- the fourth step is executed at most $N(N - 1)/2$ times

Clearly, if the elements in A are in descending order at the beginning, the fourth step will always be executed. Thus in this case the algorithm makes $3N(N - 1)/2 + N = 1.5N^2 - 0.5N$ steps. Therefore our algorithm has $f(N) = 1.5N^2 - 0.5N$.

As you can see, determining the exact function f for more complicated programs is painful. Moreover, it isn't even necessary. In our case, clearly the $-0.5N$ term can be neglected. It will usually be much smaller than the $1.5N^2$ term and it won't affect the runtime significantly. The result "$f(N)$ is roughly equal to $1.5N^2$" gives us all the information we need. As we will show now, if we want to compare this algorithm with some other algorithm solving the same problem, even the constant 1.5 is not that important.

Consider two algorithms, one with the runtime N^2, the other with the runtime $0.001N^3$. One can easily see that for N greater than $1\ 000$ the first algorithm is faster - and soon this difference becomes apparent. While the first algorithm is able to solve inputs with $N = 20\ 000$ in a matter of seconds, the second one will already need several minutes on current machines.

Clearly this will occur always when one of the runtime functions grows **asymptotically faster** than the other (i.e. when N grows beyond all bounds the limit of their quotient is zero or infinity). Regardless of the constant factors, an algorithm with runtime proportional to N^2 will always be better than an algorithm with runtime proportional to N^3 **on almost all inputs**. And this observation is exactly what we base our formal definition on.

Finally, formal definitions

Let f, g be positive non-decreasing functions defined on positive integers. (Note that all runtime functions satisfy these conditions.) We say that $f(N)$ is $O(g(N))$ *(read: f is big-oh of g)* if for some c and N_0 the following condition holds:

$$\forall N > N_0;\ f(N) < c.g(N)$$

In human words, $f(N)$ is $O(g(N))$, if for some c almost the entire graph of the function f is below the graph of the function $c.g$. Note that this means that f grows at most as fast as $c.g$ does.

Instead of "$f(N)$ is $O(g(N))$" we usually write $f(N) = O(g(N))$. Note that this "equation" is **not symmetric** - the notion " $O(g(N)) = f(N)$" has no sense and " $g(N) = O(f(N))$" doesn't have to be true (as we will see later). (If you are not comfortable with this notation, imagine $O(g(N))$ to be a set of functions and imagine that there is a $\in$ instead of $=$.)

What we defined above is known as the big-oh notation and is conveniently used to specify upper bounds on function growth.

E.g. consider the function $f(N) = 3N(N - 1)/2 + N = 1.5N^2 - 0.5N$ from Example 2. We may say that $f(N) = O(N^2)$ (one possibility for the constants is $c = 2$ and $N_0 = 0$). This means that f doesn't grow (asymptotically) faster than N^2.

Note that even the exact runtime function f doesn't give an exact answer to the question "How long will the program run on my machine?" But the important observation in the example case is that the runtime function is quadratic. If we double the input size, the runtime will increase approximately to four times the current runtime, no matter how fast our computer is.

The $f(N) = O(N^2)$ upper bound gives us almost the same - it guarantees that the growth of the runtime function is at most quadratic.

Thus, we will use the O-notation to describe the time (and sometimes also memory) complexity of algorithms. For the algorithm from Example 2 we would say "The time complexity of this algorithm is $O(N^2)$" or shortly "This algorithm is $O(N^2)$".

In a similar way we defined O we may define Ω and Θ.

We say that $f(N)$ is $\Omega(g(N))$ if $g(N) = O(f(N))$, in other words if f grows at least as fast as g.

We say that $f(N) = \Theta(g(N))$ if $f(N) = O(g(N))$ and $g(N) = O(f(N))$, in other words if both functions have approximately the same rate of growth.

As it should be obvious, Ω is used to specify lower bounds and Θ is used to give a tight asymptotic bound on a function. There are other similar bounds, but these are the ones you'll encounter most of the time.

Some examples of using the notation

- $1.5N^2 - 0.5N = O(N^2)$.

- $47N \log N = O(N^2)$.
- $N \log N + 1\,000\,047N = \Theta(N \log N)$.
- All polynomials of order k are $O(N^k)$.
- The time complexity of the algorithm in Example 2 is $\Theta(N^2)$.
- If an algorithm is $O(N^2)$, it is also $O(N^5)$.
- Each comparision-based sorting algorithm is $\Omega(N \log N)$.
- MergeSort run on an array with N elements does roughly $N \log N$ comparisions. Thus the time complexity of MergeSort is $\Theta(N \log N)$. If we trust the previous statement, this means that MergeSort is an asymptotically optimal general sorting algorithm.
- The algorithm in Example 2 uses $\Theta(N)$ bytes of memory.
- The function giving my number of teeth in time is $O(1)$.
- A naive backtracking algorithm trying to solve chess is $O(1)$ as the tre of positions it will examine is finite. (But of course in this case the constant hidden behind the $O(1)$ is unbelievably large.)
- The statement "Time complexity of this algorithm is at least $O(N^2)$" is meaningless. (It says: "Time complexity of this algorithm is at least at most roughly quadratic." The speaker probably wanted to say: "Time complexity of this algorithm is $\Omega(N^2)$.")

When speaking about the time/memory complexity of an algorithm, instead of using the formal $\Theta(f(n))$-notation we may simply state the class of functions f belongs to. E.g. if $f(N) = \Theta(N)$, we call the algorithm *linear*. More examples:

- $f(N) = \Theta(\log N)$: logarithmic
- $f(N) = \Theta(N^2)$: quadratic
- $f(N) = \Theta(N^3)$: cubic
- $f(N) = O(N^k)$ for some k: polynomial
- $f(N) = \Omega(2^N)$: exponential

For graph problems, the complexity $\Theta(N + M)$ is known as "linear in the graph size".

Determining execution time from an asymptotic bound

For most algorithms you may encounter in praxis, the constant hidden behind the O (or Θ) is usually relatively small. If an algorithm is $\Theta(N^2)$, you may expect that the exact time complexity is something like $10N^2$, not 10^7N^2.

The same observation in other words: if the constant is large, it is usually somehow related to some constant in the problem statement. In this case it is good practice to give this constant a name and to include it in the asymptotic notation.

An example: The problem is to count occurences of each letter in a string of N letters. A naive algorithm passes through the whole string once for each possible letter. The size of alphabet is fixed (e.g. at most 255 in C), thus the algorithm is linear in N. Still, it is better to write that its time complexity is $\Theta(|S|.N)$, where S is the alphabet used. (Note that there is a better algorithm solving this problem in $\Theta(|S| + N)$.)

In a TopCoder contest, an algorithm doing $1\,000\,000\,000$ multiplications runs barely in time. This fact together with the above observation and some experience with TopCoder problems can help us fill the following table:

complexity	maximum N
$\Theta(N)$	$100\,000\,000$
$\Theta(N \log N)$	$40\,000\,000$
$\Theta(N^2)$	$10\,000$
$\Theta(N^3)$	500
$\Theta(N^4)$	90
$\Theta(2^N)$	20
$\Theta(N!)$	11

Table 3. Approximate maximum problem size solvable in 8 seconds.

A note on algorithm analysis

Usually if we present an algorithm, the best way to present its time complexity is to give a Θ-bound. However, it is common practice to only give an O-bound - the other bound is usually trivial, O is much easier to type and better known. Still, don't forget that O represents only an upper bound. Usually we try to find an O-bound that's as good as possible.

Example 3. Given is a sorted array A. Determine whether it contains two elements with the difference D. Consider the following code solving this problem:

```
int j=0;
for (int i=0; i<N; i++) {
  while ( (j<N-1) && (A[i]-A[j] > D) )
    j++;
  if (A[i]-A[j] == D) return 1;
}
```

It is easy to give an $O(N^2)$ bound for the time complexity of this algorithm - the inner while-cycle is called N times, each time we increase j at most N times. But a more careful analysis shows that in fact we can give an $O(N)$ bound on the time complexity of this algorithm - it is sufficient to realize that during the **whole execution** of the algorithm the command "`j++;`" is executed no more than N times.

If we said "this algorithm is $O(N^2)$", we would have been right. But by saying "this algorithm is $O(N)$" we give more information about the algorithm.

Conclusion

We have shown how to write bounds on the time complexity of algorithms. We have also demonstrated why this way of characterizing algorithms is natural and (usually more-or-less) sufficient.

The next logical step is to show how to estimate the time complexity of a given algorithm. As we have already seen in Example 3, sometimes this can be messy. It gets really messy when recursion is involved. We will address these issues in the second part of this article.

Computational Complexity: Section 2

In this part of the article we will focus on estimating the time complexity for recursive programs. In essence, this will lead to finding the order of growth for solutions of recurrence equations. Don't worry if you don't understand what exactly is a recurrence solution, we will explain it in the right place at the right time. But first we will consider a simpler case - programs without recursion.

Nested loops

First of all let's consider simple programs that contain no function calls. The rule of thumb to find an upper bound on the time complexity of such a program is:

- estimate the maximum number of times each loop can be executed,
- **add** these bounds for cycles following each other.
- **multiply** these bounds for nested cycles/parts of code,

Example 1. Estimating the time complexity of a random piece of code.

```
int result=0;                           //  1
for (int i=0; i<N; i++)                 //  2
  for (int j=i; j<N; j++) {             //  3
    for (int k=0; k<M; k++) {           //  4
      int x=0;                          //  5
      while (x<N) { result++; x+=3; }   //  6
    }                                   //  7
    for (int k=0; k<2*M; k++)           //  8
      if (k%7 == 4) result++;           //  9
  }                                     // 10
```
The time complexity of the while-cycle in line 6 is clearly $O(N)$ - it is executed no more than $N/3 + 1$ times.

Now consider the for-cycle in lines 4-7. The variable k is clearly incremented $O(M)$ times. Each time the whole while-cycle in line 6 is executed. Thus the total time complexity of the lines 4-7 can be bounded by $O(MN)$.

The time complexity of the for-cycle in lines 8-9 is $O(M)$. Thus the execution time of lines 4-9 is $O(MN + M) = O(MN)$.

This inner part is executed $O(N^2)$ times - once for each possible combination of i and j. (Note that there are only $N(N + 1)/2$ possible values for $[i, j]$. Still, $O(N^2)$ is a correct upper bound.)

From the facts above follows that the total time complexity of the algorithm in Example 1 is $O(N^2.MN) = O(MN^3)$.

From now on we will assume that the reader is able to estimate the time complexity of simple parts of code using the method demonstrated above. We will now consider programs using recursion (i.e. a function occasionally calling itself with different parameters) and try to analyze the impact of these recursive calls on their time complexity.

Using recursion to generate combinatorial objects

One common use of recursion is to implement a *backtracking* algorithm to generate all possible solutions of a problem. The general idea is to generate the solution incrementally and to step back and try another way once all solutions for the current branch have been exhausted.

This approach is not absolutely universal, there may be problems where it is impossible to generate the solution incrementally. However, very often the set of all possible solutions of a problem corresponds to the set of all combinatorial objects of some kind. Most often it is the set of all permutations (of a given size), but other objects (combinations, partitions, etc.) can be seen from time to time.

As a side note, it is always possible to generate all strings of zeroes and ones, check each of them (i.e. check whether it corresponds to a valid solution) and keep the best found so far. If we can find an upper bound on the size of the best solution, this approach is finite. However, this approach is everything but fast. Don't use it if there is **any** other way.

Example 2. A trivial algorithm to generate all permutations of numbers 0 to $N - 1$.

```
vector<int> permutation(N);
vector<int> used(N,0);

void try(int which, int what) {
  // try taking the number "what" as the "which"-th element
  permutation[which] = what;
  used[what] = 1;

  if (which == N-1)
    outputPermutation();
  else
    // try all possibilities for the next element
    for (int next=0; next<N; next++)
      if (!used[next])
        try(which+1, next);

  used[what] = 0;
}

int main() {
  // try all possibilities for the first element
  for (int first=0; first<N; first++)
    try(0,first);
}
```

In this case a trivial **lower** bound on the time complexity is the number of possible solutions. Backtracking algorithms are usually used to solve hard problems - i.e. such that we don't know whether a significantly more efficient solution exists. Usually the solution space is quite large and uniform and the algorithm can be implemented so that its time complexity is close to the theoretical lower bound. To get an upper bound it should be enough to check how much additional (i.e. unnecessary) work the algorithm does.

The number of possible solutions, and thus the time complexity of such algorithms, is usually exponential - or worse.

Divide&conquer using recursion

From the previous example we could get the feeling that recursion is evil and leads to horribly slow programs. The contrary is true. Recursion can be a very powerful tool in the design of effective algorithms. The usual way to create an effective recursive algorithm is to apply the *divide&conquer paradigm* - try to split the problem into several parts, solve each part separately and in the end combine the results to obtain the result for the original problem. Needless to say, the "solve each part separately" is usually implemented using recursion - and thus applying the same method again and again, until the problem is sufficiently small to be solved by brute force.

Example 3. The sorting algorithm MergeSort described in pseudocode.

```
MergeSort(sequence S) {
  if (size of S <= 1) return S;
  split S into S_1 and S_2 of roughly the same size;
  MergeSort(S_1);
  MergeSort(S_2);
  combine sorted S_1 and sorted S_2 to obtain sorted S;
  return sorted S;
}
```

Clearly $O(N)$ time is enough to split a sequence with N elements into two parts. (Depending on the implementation this may be even possible in constant time.) Combining the shorter sorted sequences can be done in $\Theta(N)$: Start with an empty S. At each moment the smallest element not yet in S is either at the beginning of S_1 or at the beginning ofS_2. Move this element to the end of S and continue.

Thus the total time to MergeSort a sequence with N elements is $\Theta(N)$ plus the time needed to make the two recursive calls.

Let $f(N)$ be the time complexity of MergeSort as defined in the previous part of our article. The discussion above leads us to the following equation:

$$f(N) = f(\lfloor N/2 \rfloor) + f(\lceil N/2 \rceil) + p(N)$$

where p is a linear function representing the amount of work spent on splitting the sequence and merging the results.

Basically, this is just a *recurrence equation*. If you don't know this term, please don't be afraid. The word "recurrence" stems from the latin phrase for "to run back". Thus the name just says that the next values of f are defined using the previous (i.e. smaller) values of f.

Well, to be really formal, for the equation to be complete we should specify some initial values - in this case, $f(1)$. This (and knowing the implementation-specific function p) would enable us to compute the exact values of f.

But as you hopefully understand by now, this is not necessarily our goal. While it is theoretically possible to compute a closed-form formula for $f(N)$, this formula would most probably be really ugly... and we don't really need it. We only want to find a Θ-bound (and sometimes only an O-bound) on the growth of f. Luckily, this can often be done quite easily, if you know some tricks of the trade.

As a consequence, we won't be interested in the exact form of p, all we need to know is that $p(N) = \Theta(N)$. Also, we don't need to specify the initial values for the equation. We simply assume that all problem instances with small N can be solved in constant time.

The rationale behind the last simplification: While changing the initial values does change the solution to the recurrence equation, it usually doesn't change its asymptotic order of growth. (If your intuition fails you here, try playing with the equation above. For example fix p and try to compute $f(8)$, $f(16)$ and $f(32)$ for different values of $f(1)$.)

If this would be a formal textbook, at this point we would probably have to develop some theory that would allow us to deal with the floor and ceiling functions in our equations. Instead we will simply neglect them from now on. (E.g. we can assume that each division will be integer division, rounded down.)

A reader skilled in math is encouraged to prove that if p is a polynomial (with non-negative values on N) and $q(n) = p(n + 1)$ then $q(n) = \Theta(p(n))$. Using this observation we may formally prove that (assuming the f we seek is polynomially-bounded) the right side of each such equation remains asymptotically the same if we replace each ceiling function by a floor function.

The observations we made allow us to rewrite our example equation in a more simple way:

$$f(N) = 2\, f(N/2) + \Theta(N) \tag{1}$$

Note that this is not an equation in the classical sense. As in the examples in the first part of this article, the equals sign now reads "is asymptotically equal to". Usually there are lots of different functions that satisfy such an equation. But usually all of them will have the same order of growth - and this is exactly what we want to determine. Or, more generally, we want to find the smallest upper bound on the growth of **all possible** functions that satisfy the given equation.

In the last sections of this article we will discuss various methods of solving these "equations". But before we can do that, we need to know a bit more about logarithms.

Notes on logarithms

By now, you may have already asked one of the following questions: If the author writes that some complexity is e.g. $O(N \log N)$, what is the base of the logarithm? In some cases, wouldn't $O(N \log_2 N)$ be a better bound?

The answer: The base of the logarithm does not matter, all logarithmic functions (with base > 1) are asymptotically equal. This is due to the well-known equation:

$$\log_a N = \frac{\log_b N}{\log_b a} \tag{2}$$

Note that given two bases a, b, the number $1/\log_b a$ is just a constant, and thus the function $\log_a N$ is just a constant multiple of $\log_b N$.

To obtain more clean and readable expressions, we always use the notation $\log N$ inside big-Oh expressions, even if logarithms with a different base were used in the computation of the bound.

By the way, sadly the meaning of $\log N$ differs from country to country. To avoid ambiguity where it may occur: I use $\log N$ to denote the decadic (i.e. base-10) logarithm, $\ln N$ for the natural (i.e. base-e) logarithm, $\lg N$ for the binary logarithm and $\log_b N$ for the general case.

Now we will show some useful tricks involving logarithms, we will need them later. Suppose a, b are given constants such that $a, b > 1$.

From (2) we get:

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$$

Using this knowledge, we can simplify the term $a^{\log_b N}$ as follows:

$$a^{\log_b N} = a^{\log_a N / \log_a b} = \left(a^{\log_a N}\right)^{1/\log_a b} = \left(a^{\log_a N}\right)^{\log_b a} = N^{\log_b a} \tag{3}$$

The substitution method

This method can be summarized in one sentence: Guess an asymptotic upper bound on f and (try to) prove it by induction.

As an example, we will prove that if f satisfies the equation (1) then $f(N) = O(N \log N)$.

From (1) we know that

$$\forall N;\ f(N) \le 2f(N/2) + cN$$

for some c. Now we will prove that if we take a large enough (but constant) d then for almost all N we have $f(N) \le dN \lg N$. We will start by proving the induction step.

Assume that $f(N/2) \le d(N/2)\lg(N/2)$. Then

$$\begin{aligned} f(N) &\le 2f(N/2) + cN \\ &\le 2d(N/2)\lg(N/2) + cN \\ &= dN(\lg N - \lg 2) + cN \\ &= dN\lg N - dN + cN \end{aligned}$$

In other words, the induction step will hold as long as $d > c$. We are always able to choose such d.

We are only left with proving the inequality for some initial value N. This gets quite ugly when done formally. The general idea is that if the d we found so far is not large enough, we can always increase it to cover the initial cases.

Note that for our example equation we won't be able to prove it for $N = 1$, because $lg\ 1 = 0$. However, by taking $d > 2(f(1) + f(2) + f(3) + c)$ we can easily prove the inequality for $N = 2$ and $N = 3$, which is more than enough.

Please note what exactly did we prove. Our result is that if f satisfies the equation (1) then for almost all N we have $f(N) \le dN\ lg\ N$, where d is some fixed constant. Conclusion: from (1) it follows that $f(N) = O(N\ lg\ N)$.

The recursion tree

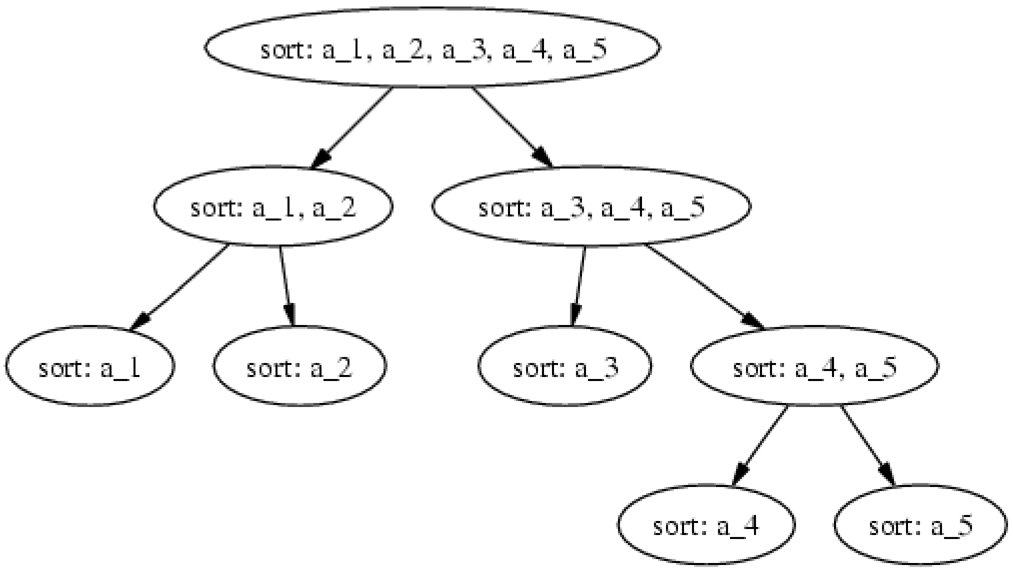
To a beginner, the previous method won't be very useful. To use it successfully we need to make a good guess - and to make a good guess we need some insight. The question is, how to gain this insight? Let's take a closer look at what's happening, when we try to evaluate the recurrence (or equivalently, when we run the corresponding recursive program).

We may describe the execution of a recursive program on a given input by a rooted tree. Each node will correspond to some instance of the problem the program solves. Consider an arbitrary vertex in our tree. If solving its instance requires recursive calls, this vertex will have children corresponding to the smaller subproblems we solve recursively. The root node of the tree is the input of the program, leaves represent small problems that are solved by brute force.

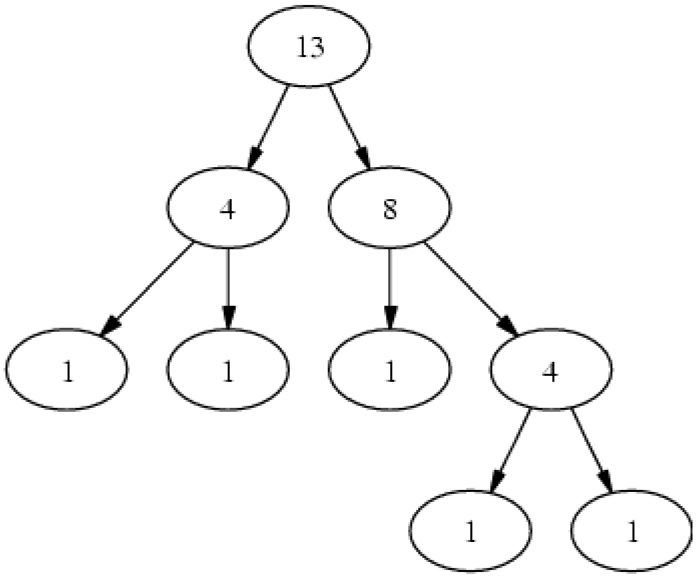
Now suppose we label each vertex by the amount of work spent solving the corresponding problem (excluding the recursive calls). Clearly the runtime is exactly the sum of all labels.

As always, we only want an asymptotic bound. To achieve this, we may "round" the labels to make the summation easier. Again, we will demonstrate this method on examples.

Example 4. The recursion tree for MergeSort on 5 elements.

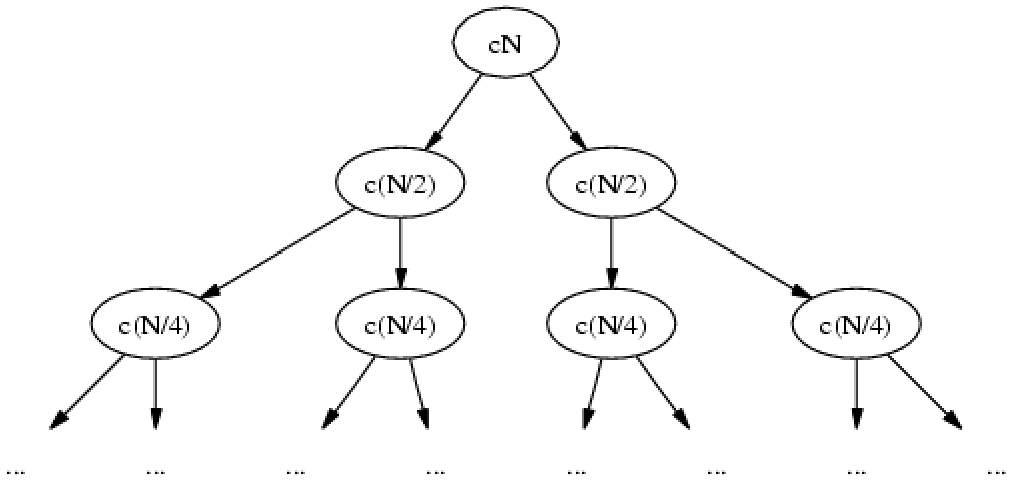


The recursion tree for the corresponding recurrence equation. This time, the number inside each vertex represents the number of steps the algorithm makes there.



Note that in a similar way we may sketch the general form of the recursion tree for any recurrence. Consider our old friend, the equation (1). Here we know that there is a number c such that the number of operations in each node can be bound by (c times the current value of N). Thus the tree in the example below is indeed the worst possible case.

Example 5. A worst-case tree for the general case of the recurrence equation (1).



Now, the classical trick from combinatorics is to sum the elements in an order different from the order in which they were created. In this case, consider an arbitrary level of the tree (i.e. a set of vertices with the same depth). It is not hard to see that the total work on each of the levels is cN.

Now comes the second question: What is the number of levels? Clearly, the leaves correspond to the trivial cases of the algorithm. Note that the size of the problem is halved in each step. Clearly after $lg\ N$ steps we are left with a trivial problem of size 1, thus the number of levels is $\Theta(\log N)$.

Combining both observations we get the final result: The total amount of work done here is $\Theta(cN \times \log N) = \Theta(N \log N)$.

A side note. If the reader doesn't trust the simplifications we made when using this method, he is invited to treat this method as a "way of making a good guess" and then to prove the result using the substitution method. However, with a little effort the application of this method could also be upgraded to a full formal proof.

More recursion trees

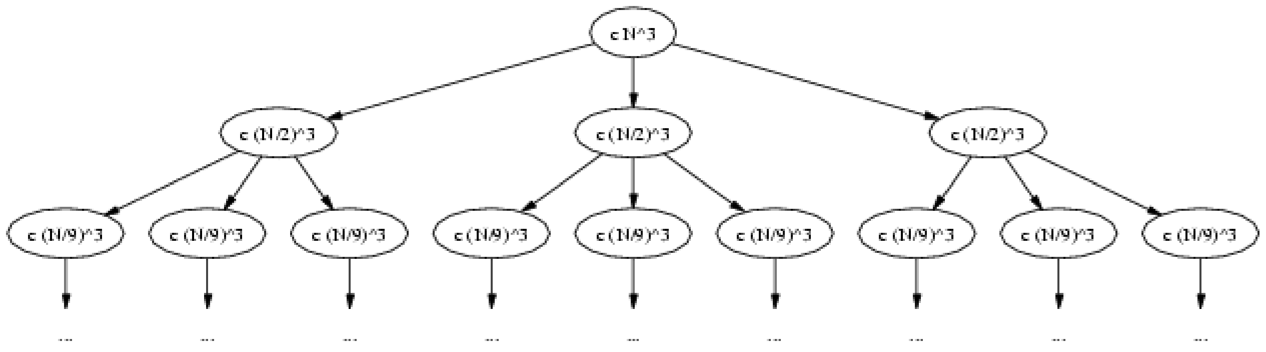
By now you should be asking: Was it really only a coincidence that the total amount of work on each of the levels in Example 5 was the same?

The answer: No and yes. No, there's a simple reason why this happened, we'll discover it later. Yes, because this is not always the case - as we'll see in the following two examples.

Example 6. Let's try to apply our new "recursion tree" method to solve the following recurrence equation:

$$f(N) = 3f(N/2) + \Theta(N^3)$$

The recursion tree will look as follows:



Let's try computing the total work for each of the first few levels. Our results:

level	1	2	3	...
work	cN^3	$\frac{3}{8}cN^3$	$\frac{3^2}{8^2}cN^3$	...

Clearly as we go deeper in the tree, the total amount of work on the current level decreases. The question is, how fast does it decrease? As we move one level lower, there will be three times that many subproblems. However, their size gets divided by 2, and thus the time to process each of them decreases to one eighth of the original time. Thus the amount of work is decreased by the factor $3/8$.

But this means that the entries in the table above form a geometric progression. For a while assume that this progression is infinite. Then its sum would be

$$S = \frac{cN^3}{1-\frac{3}{8}} = \frac{8}{5}cN^3 = \Theta(N^3)$$

Thus the total amount of work in our tree is $\Omega(N^3)$ (summing the infinite sequence gives us an upper bound). But already the first element of our progression is $\Theta(N^3)$. It follows that the total amount of work in our tree is $\Theta(N^3)$ and we are done.

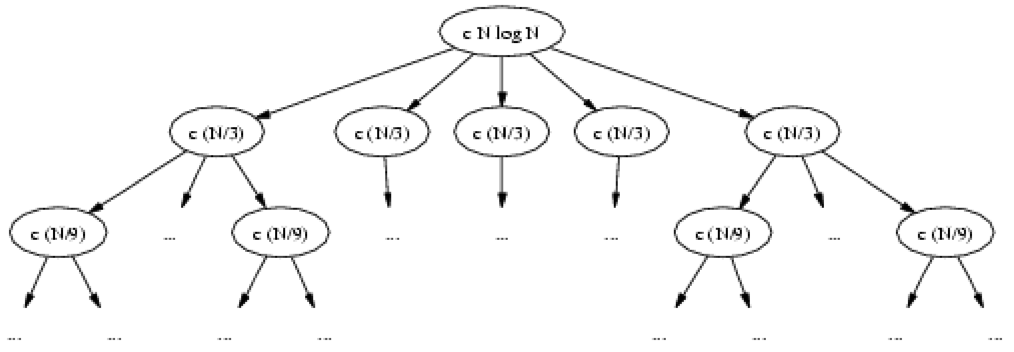
The important generalization of this example: If the amounts of work at subsequent levels of the recursion tree form a **decreasing geometric progression**, the total amount of work is asymptotically the same as the amount of work done in the root node.

From this result we can deduce an interesting fact about the (hypothetical) algorithm behind this recurrence equation: The recursive calls didn't take much time in this case, the most time consuming part was preparing the recursive calls and/or processing the results. (I.e. this is the part that should be improved if we need a faster algorithm.)

Example 7. Now let's try to apply our new "recursion tree" method to solve the following recurrence equation:

$$f(N) = 5f(N/3) + \Theta(N)$$

The recursion tree will look as follows:



Again, let's try computing the total work for each of the first few levels. We get:

level	1	2	3	...
work	cN	$\frac{5}{3} cN$	$\frac{5^2}{3^2} cN$	...

This time we have the opposite situation: As we go deeper in the tree, the total amount of work on the current level increases. As we move one level lower, there will be five times that many subproblems, each of them one third of the previous size, the processing time is linear in problem size. Thus the amount of work increased by the factor $5/3$.

Again, we want to compute the total amount of work. This time it won't be that easy, because the most work is done on the lowest level of the tree. We need to know its depth.

The lowest level corresponds to problems of size 1. The size of a problem on level k is $N/3^k$. Solving the equation $1 = N/3^k$ we get $k = log_3 N$. Note that this time we explicitly state the base of the logarithm, as this time it will be important.

Our recursion tree has $log_3 N$ levels. Each of the levels has five times more vertices than the previous one, thus the last level has $5^{\log_3 N}$ levels. The total work done on this level is then $c5^{\log_3 N}$.

Note that using the trick (3) we may rewrite this as $cN^{\log_3 5}$.

Now we want to sum the work done on all levels of the tree. Again, this is a geometric progression. But instead of explicitly computing the sum, we now **reverse** it. Now we have a **decreasing** geometric progression...and we are already in the same situation as in the previous example. Using the same reasoning we can show that the sum is asymptotically equal to the largest element.

It follows that the total amount of work in our tree is $\Theta(N^{\log_3 5}) \approx \Theta(N^{1.465})$ and we are done.

Note that the base-3 logarithm ends in the exponent, that's why the base is important. If the base was different, also the result would be asymptotically different.

The Master Theorem

We already started to see a pattern here. Given a recurrence equation, take the corresponding recurrence tree and compute the amounts of work done on each level of the tree. You will get a geometric sequence. If it decreases, the total work is proportional to work done in the root node. If it increases, the total work is proportional to the number of leaves. If it remains the same, the total work is (the work done on one level) times (the number of levels).

Actually, there are a few ugly cases, but almost often one of these three cases occurs. Moreover, it is possible to prove the statements from the previous paragraph formally. The formal version of this theorem is known under the name Master Theorem.

For reference, we give the full formal statement of this theorem. (Note that knowing the formal proof is not necessary to **apply** this theorem on a given recurrence equation.)

Let $a \geq 1$ and $b > 1$ be integer constants. Let p be a non-negative non-decreasing function. Let f be any solution of the recurrence equation

$$f(N) = af(N/b) + p(N)$$

Then:

1. If $p(N) = O(N^{(\log_b a)-\varepsilon})$ for some $\varepsilon > 0$ then $f(N) = \Theta(N^{\log_b a})$
2. If $p(N) = \Theta(N^{\log_b a})$, then $f(N) = \Theta(p(N)\log N)$.
3. If $p(N) = \Omega(N^{(\log_b a)+\varepsilon})$ for some $\varepsilon > 0$, and if $ap(N/b) \leq cp(N)$ for some $c < 1$ and for almost all N, then $f(N) = \Theta(p(N))$.

Case 1 corresponds to our Example 7. Most of the time is spent making the recursive calls and it's the number of these calls that counts.

Case 2 corresponds to our Example 5. The time spent making the calls is roughly equal to the time to prepare the calls and process the results. On all levels of the recursion tree we do roughly the same amount of work, the depth of the tree is always logarithmic.

Case 3 corresponds to our Example 6. Most of the time is spent on preparing the recursive calls and processing the results. Usually the result will be asymptotically equal to the time spent in the root node.

Note the word "usually" and the extra condition in Case 3. For this result to hold we need p to be somehow "regular" - in the sense that for each node in the recursion tree the time spent in the node must be greater than the time spent in its chidren (excluding further recursive calls). This is nothing to worry about too much, most probably all functions$p$ you will encounter in practice will satisfy this condition (if they satisfy the first condition of Case 3).

Example 8. Let $f(N)$ be the time Strassen's fast matrix multiplication algorithm needs to multiply two $N \times N$ square matrices. This is a recursive algorithm, that makes 7 recursive calls, each time multiplying two $(N/2) \times (N/2)$ square matrices, and then computes the answer in $\Theta(N^2)$ time.

This leads us to the following recurrence equation:

$$f(N) = 7f(N/2) + \Theta(N^2)$$

Using the Master Theorem, we see that Case 1 applies. Thus the time complexity of Strassen's algorithm is $\Theta(N^{\log_2 7}) \approx \Theta(N^{2.807})$. Note that by implementing the definition of matrix multiplication we get only a $\Theta(N^3)$ algorithm.

Example 9. Occasionally we may encounter the situation when the problems in the recursive calls are not of the same size. An example may be the "median-of-five" algorithm to find the k-th element of an array. It can be shown that its time complexity satisfies the recurrence equation

$$f(N) = f(N/5) + f(7N/10 + 6) + \Theta(N)$$

How to solve it? Can the recursion tree be applied also in such asymmetric cases? Is there a more general version of Master Theorem that handles also these cases? And what should I do with the recurrence $f(N) = 4f(N/4) + \Theta(N \log N)$, where the Master Theorem doesn't apply?

We won't answer these questions here. This article doesn't claim to be the one and only reference to computational complexity. If you are already asking these questions, you understand the basics you need for programming contests - and if you are interested in knowing more, there are good books around that can help you.

Thanks for reading this far. If you have any questions, comments, bug reports or any other feedback, please use the Round tables. I'll do my best to answer.