IAAC 2022

Qualification Round Solutions By: **Aditya Ramdasi**

Problem A: The James Webb Space Telescope (5 Points)

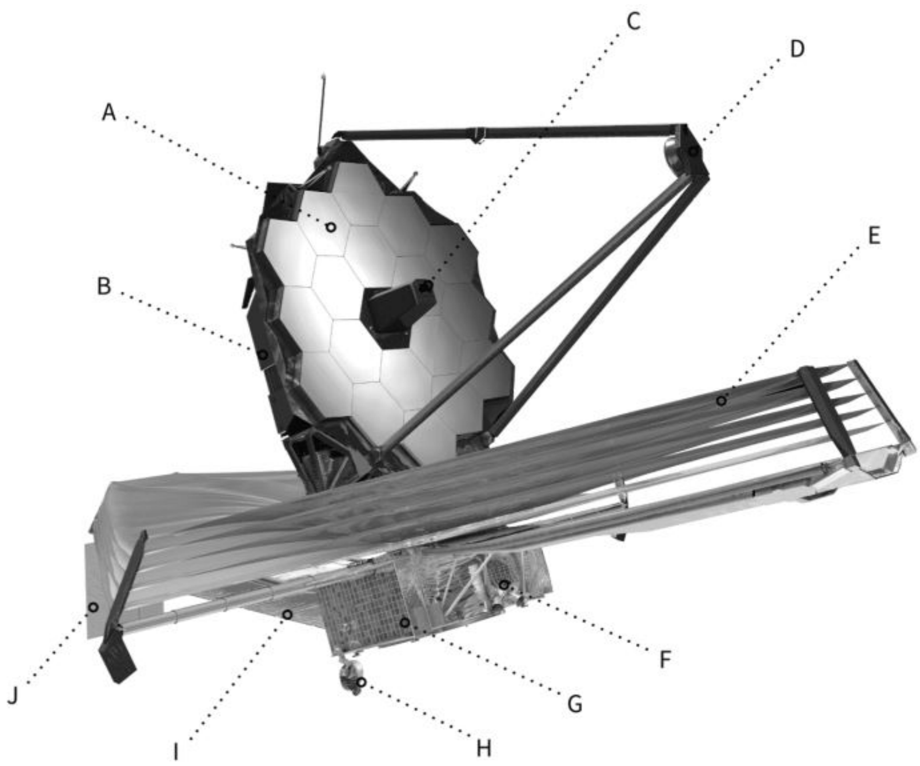

Figure 1: Components of JWST - Credits: IAAC 2022

(A) Primary Mirror

(B) **I.S.I.M** (Integrated Science Instrument Module)

(C) **O.T.E** (Optical Telescope Element)

(D) Secondary Mirror

(E) Multilayered Sun-shield

(F) Spacecraft Bus

(G) Solar Power Array

(H) Earth Pointing Antenna

(I) Star Tracker

(J) Momentum Trim Flap

Problem B: Very Dense Earth (5 Points)

Given:

- Average Density of a Neutron Star $(\rho_n) = 5 \times 10^{17}$ kg m^{-3}
- Total Mass of Earth $(M_e) = 5.97 \times 10^{24}$ kg

To find:

- Diameter of the Earth (D_e) if $\rho_e = \rho_n$

First, we find the expression for the Diameter of Earth (D_e) in terms of given quantities and constants -

$$\rho_e = \frac{M_e}{\frac{4}{3}\pi R_e^3}$$
$$\Rightarrow \rho_e = \frac{3M_e}{4\pi R_e^3}$$
$$\Rightarrow R_e^3 = \frac{3M_e}{4\pi\rho_e}$$
$$\Rightarrow R_e = \left(\frac{3M_e}{4\pi\rho_e}\right)^{\frac{1}{3}}$$
$$D_e = 2 \times R_e$$
$$\Rightarrow D_e = 2\left(\frac{3M_e}{4\pi\rho_e}\right)^{\frac{1}{3}}$$
$$\text{But given } \rho_n = \rho_e$$
$$\Rightarrow D_e = 2\left(\frac{3M_e}{4\pi\rho_n}\right)^{\frac{1}{3}}$$

We now have the final expression for calculating the diameter of the Earth (D_e) in the given condition. We substitute the provided values (M_e, ρ_n) into the right hand side of the equation to get -

$$D_e = 2\left(\frac{3 \times (5.97 \times 10^{24} kg)}{4\pi \times (5 \times 10^{17} kgm^{-3})}\right)^{\frac{1}{3}}$$
$$= 283.57\text{m}$$

Thus, we get that if the Earth had the density of a neutron star, its diameter would be -

$$\boxed{D_e = 283.57\text{m}}$$

Problem C: Asteroid Field (5 Points)

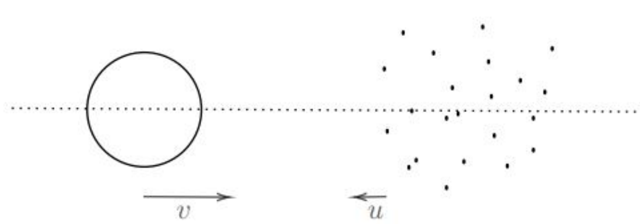

Figure 2: Collision with Asteroid Field - Credits: IAAC 2022

Given:

- Mass of Earth $= M$
- Radius of Earth $= R$
- Initial velocity of Earth $= +v$
- Initial velocity of asteroid field $= -u$
- Density of asteroid field $= \rho$ (asteroids/volume)
- Length of asteroid field $= d$
- Average mass of 1 asteroid $= m$

Assumptions:

- Final velocity of Earth $= v'$
- Total mass of asteroid field $= M_a$

To Prove:

$$\Delta v = v\left(1 - \frac{1}{1+\pi R^2 d\rho \frac{m}{M}}\right) \tag{1}$$

To solve this question, we use the **Law of Conservation of Linear Momentum**. Since there is **no external force** on the Earth-Asteroids system, we can say that the total momentum of this system remains conserved throughout the entire collision. Thus, we can say that -

$$\text{Total Momentum before collision} = \text{Total Momentum after collision}$$
$$\Rightarrow (P_i)_{\text{Earth}} + (P_i)_{\text{Asteroids}} = (P_f)_{\text{Earth}} + (P_f)_{\text{Asteroids}}$$

Now, we can say that the asteroids from the field **stick to the Earth** after they crash onto it. Thus, the **final velocity** of both the Earth and the crashed asteroids **is the same** i.e. (v'). It is $(+v')$ since Earth's momentum is very high as compared to the asteroid field. Thus, we have -

$$\begin{aligned} M(v) + M_a(-u) &= M(v') + M_a(v') \\ &= v'(M + M_a) \\ \Rightarrow v' &= \frac{Mv - M_a u}{M + M_a} \\ \Rightarrow v' &= \left(\frac{v - \frac{M_a}{M}u}{1 + \frac{M_a}{M}}\right) \\ \Rightarrow v' &= v\left(\frac{1 - \frac{M_a}{M}.\frac{u}{v}}{1 + \frac{M_a}{M}}\right) \end{aligned}$$

Now, given that $v >> u$, we can say that $\frac{u}{v} \to 0 \Rightarrow \frac{M_a}{M}.\frac{u}{v} \to 0$. Substituting this in the expression above, we get -

$$v' = v\left(\frac{1}{1 + \frac{M_a}{M}}\right) \tag{2}$$

Now, since Δv is the slow down of Earth's velocity due to collisions with the Asteroid field, we can say that $\Delta v = v - v'$. Therefore, we compute Δv using equation (2) to get -

$$\Delta v = v - v' \tag{3}$$
$$\Rightarrow \Delta v = v - v\left(\frac{1}{1 + \frac{M_a}{M}}\right) \tag{4}$$
$$\Rightarrow \Delta v = v\left(1 - \frac{1}{1 + \frac{M_a}{M}}\right) \tag{5}$$

Lastly, we need to compute the value of M_a in terms of given quantities. For computing the **total mass of the asteroid field** that collides with the Earth, we have -

$$\begin{aligned} M_a &= \text{Density of asteroids} \times \text{Volume of asteroids colliding} \times \text{Mass of 1 asteroid} \\ &= \rho \times (\text{Cross sectional area of Earth exposed} \times \text{Distance covered inside Asteroid field}) \times m \\ &= \rho \times (\pi R^2 \times d) \times m \\ &= \pi R^2 d\rho m \end{aligned}$$

Now, we substitute this value of M_a into equation (5) to get the final expression as -

$$\begin{aligned} \Delta v &= v\left(1 - \frac{1}{1 + \frac{\pi R^2 d\rho m}{M}}\right) \\ \Rightarrow \Delta v &= v\left(1 - \frac{1}{1 + \pi R^2 d\rho \frac{m}{M}}\right) \end{aligned}$$

Thus, we have proved our intended result as -

$$\boxed{\Delta v = v\left(1 - \frac{1}{1+\pi R^2 d\rho \frac{m}{M}}\right)} \tag{6}$$

Problem D: Position of the JWST (5 Points)

(a) Why is it important to position the JWST behind the Earth?
Answer -
The **James Webb Space Telescope** (JWST) primarily observes in **infra-red** frequencies. This can be detected both as light and heat. As the telescope is built to detect and map celestial objects that are very far away from us, it needs to be **shielded** from heat and infra-red light sources very near to it. This means, it needs to be shielded from the sun. Thus, the best position for the JWST is the **Second Lagrange Point (L2)**, which lies **behind the Earth** and along the line joining the Earth and the Sun. At this position, the JWST will be reasonable **well shielded** from the excess **infra-red noise** of the Sun by the Earth, and also remain in a relatively **stable orbit** without needing much interference by us.

(b) Angular Velocity ω of the JWST -
Answer -

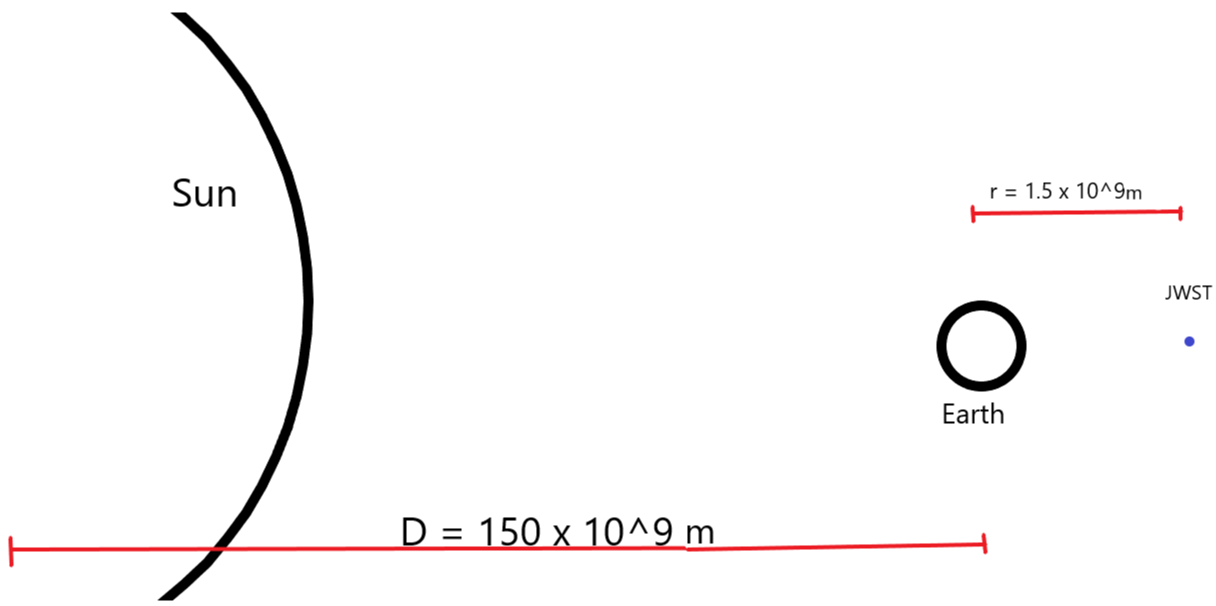

Figure 3: Thematic image to find out ω of JWST

Refer to Figure 3 for this solution. Note that the image is not to scale, and only for thematic purposes.
List of constants needed -

- M_t = Mass of the telescope (JWST)
- M_e = Mass of Earth = 5.97×10^{24} kg
- M_s = Mass of Sun = 1.989×10^{30} kg
- G = Universal Gravitational Constant = 6.674×10^{-11} S.I units

The **magnitude** of the **total gravitational force** on the telescope is given by -

$$|F|_g = \frac{GM_tM_e}{r^2} + \frac{GM_tM_s}{(r+D)^2} \tag{7}$$

The **total centrifugal force** on the telescope due to its circular motion (orbital velocity = v) is given by -

$$F_\omega = \frac{M_t v^2}{D+r} \tag{8}$$

For the telescope to be in a stable orbit, these two forces need to be more or less **equal in magnitude**. They are opposing each other in terms of direction, with F_g directed **towards the Sun** while F_ω **radially outward**. Both forces act along the line joining the centres of all three bodies.

To solve for the value of v, we equate these forces and get -

$$\begin{aligned}
F_\omega &= |F|_g \\
\Rightarrow \frac{M_t v^2}{D+r} &= \frac{GM_tM_e}{r^2} + \frac{GM_tM_s}{(r+D)^2} \\
\Rightarrow M_t\frac{v^2}{D+r} &= GM_t\left(\frac{M_e}{r^2} + \frac{M_s}{(r+D)^2}\right) \\
\Rightarrow \frac{v^2}{D+r} &= G\left(\frac{M_e}{r^2} + \frac{M_s}{(r+D)^2}\right) \\
\Rightarrow v^2 &= G\left(\frac{M_e(D+r)}{r^2} + \frac{M_s}{(r+D)}\right)
\end{aligned}$$

Now, we want to find **angular velocity** (ω). We know that -

$$\omega = \frac{v}{r+D} \tag{9}$$

Using the above equation, we take the **square root** of the expression obtained for v^2 and divide it by $r+D$, to get the final value of ω. Thus, we have -

$$\begin{aligned}
\omega &= \frac{v}{r+D} \\
\Rightarrow \omega &= \frac{\sqrt{G\left(\frac{M_e(D+r)}{r^2} + \frac{M_s}{(r+D)}\right)}}{r+D}
\end{aligned}$$

We now have our **final expression** to calculate ω, as the values of all the terms in the right hand side of the above **equation are known**. We **substitute the required values** of these terms (as given in Figure 3 and the list of constants above) to get -

$$\begin{aligned}
\omega &= \frac{\sqrt{6.674 \times 10^{-11}\,(4.0198 \times 10^{13} + 1.312 \times 10^{19})}}{151.5 \times 10^9} \\
&= \frac{\sqrt{6.674 \times 10^{-11}\,(1.312 \times 10^{19})}}{151.5 \times 10^9} \\
&= \frac{2.95821 \times 10^4}{151.5 \times 10^9} \\
&= 1.9526 \times 10^{-7}\text{rad/s}
\end{aligned}$$

Thus, we find that **angular velocity** (w) of the **JWST** is -

$$\boxed{\omega = 1.9526 \times 10^{-7}\text{rad/s}} \tag{10}$$

(c) Acceleration of the JWST -

Answer -

To find the acceleration of the **JWST**, we first need to find the total **net force** (F_{net}) on it. This is given by -

$$F_{\text{net}} = F_\omega - F_g \tag{11}$$

By Newton's second law of motion, we get that the **net acceleration** (a_{net}) is given by -

$$\begin{aligned}
a_{\text{net}} &= \frac{F_{\text{net}}}{M_t} \\
&= \frac{(F_\omega - F_g)}{M_t} \\
&= \frac{M_t\omega^2(r+D)}{M_t} - \frac{GM_tM_s}{(r+D)^2M_t} \\
&= \omega^2(r+D) - \frac{GM_s}{(r+D)^2}
\end{aligned}$$

Now, we substitute the value of ω (obtained in equation (10)) and other constants in this expression to get -

$$\begin{aligned}
a_{\text{net}} &= (1.9526 \times 10^{-7})^2(151.5 \times 10^9) - \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(151.5 \times 10^9)^2} \\
&= 0.00577615984 - 0.00578356631 \\
&= -7.406 \times 10^{-6}\text{m / sec}^2
\end{aligned}$$

The **negative sign** of the acceleration indicates that the **gravitational force** is **greater** than the **centrifugal force**, while the **magnitude** of the net acceleration shows how small the difference is.

Thus, there is a small, but not negligible net acceleration to the JWST, **towards the Sun**, whose value is given by -

$$\boxed{a_{\text{net}} = -7.406 \times 10^{-6} \text{m/s}^2} \tag{12}$$

(d) Stability of JWST's orbit -
Answer -

1. The **JWST** doesn't exactly orbit the sun by sitting **on L2**. Instead, it **orbits L2**.
2. JWST's large **primary mirror** and **sun-shield** experience significant amounts of **Solar Radiation Pressure (SRP)** along the radially outward direction.
3. This **radially outward** force helps to counter the **inward net force** calculated in part (c), thus **stabilizing the orbit** as a whole. It also provides a **torque** to the telescope, which helps in maintaining the telescope's orbit around L2.
4. Orbits around L2 are inherently unstable, as it is a **"saddle point"** solution to the **Lagrange equation**. So the orbit starts **decaying** slowly, and the rate of decay **increases** if not kept in check.
5. To keep this orbital decay under control, **periodic firing** of thrusters in appropriate directions (every 6 months) is done.
6. Since fast communication with the telescope is necessary for executing such **orbital maintenance manoeuvres**, the JWST's orbit is designed to **keep it out** of the Earth's or Moon's **Shadow**.

Problem E: Infrared Radiation (5 Points)

Answer -

1. **Infrared Radiation (IR)** is that part of the **electromagnetic spectrum** whose wavelength is **greater than** the **visible spectrum** of light and **lesser** than **microwaves**, at around **780 nm - 1 mm**.
2. Unlike visible light, IR is **invisible** to the human eye. All objects in the universe emit some kind of IR, but the Sun and fire are two major sources.
3. We mainly detect **infrared radiation** through the **heat energy** it carries, unlike visible light which can only be seen.
4. The JWST is mainly designed to detect **faint and very distant objects**.
5. The farther these are, the older they are. Thus, many of them are **too cool** to emit visible light that is enough to reach us. But since IR has **greater wavelengths**, it can traverse **large cosmic distances** with much greater efficiency.
6. Longer wavelengths of IR light also imply very **little scattering** due to interstellar dust, so detecting the direction of IR gives a **highly accurate location** of the source object.
7. Due to all these **advantages**, IR astronomy offers over visible light detection, the **JWST** observes **Infrared light.**