IYMC 2022

Qualification Round Solutions By: **Aditya Ramdasi**

Problem A

Given -

$$f(x) = (\log(3^x) - 2\log(3))(x^2 - 1) \tag{1}$$

To find the roots of given $f(x)$, we equate $f(x) = 0$. Thus, we have -

$$(\log(3^x) - 2\log(3))(x^2 - 1) = 0$$

If the product of quantities is zero, then *at least* one of them is equal to zero.
Equating first bracket to zero, we get -

$$\begin{aligned} \log(3^x) - 2\log(3) &= 0 \\ \implies \log(3^x) &= 2\log(3) \\ \implies x\log(3) &= 2\log(3) \\ \implies x &= 2 \end{aligned}$$

Equating second bracket to zero, we get -

$$\begin{aligned} x^2 - 1 &= 0 \\ \implies x^2 &= 1 \\ \implies |x| &= 1 \\ \implies x &= 1, -1 \end{aligned}$$

Thus, we get that the roots of given $f(x)$ are $\boxed{\text{x= -1,1,2}}$

Problem B

To find the value of the infinite sum -

$$1 + \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + \frac{3}{\pi^4} + ... \tag{2}$$

Let the value of this infinite sum be $1 + S$. Then, we have -

$$\begin{aligned} 1 + S &= 1 + \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + \frac{3}{\pi^4} + ... \\ \implies S &= \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + \frac{3}{\pi^4} + ... \\ \implies \pi S &= 3 + \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + ... && \text{(Multiplying both sides by } \pi\text{)} \\ \implies \pi S &= 3 + S && (S \text{ is an infinite sum}) \\ \implies S(\pi - 1) &= 3 \\ \implies S &= \frac{3}{(\pi - 1)} \\ \implies 1 + S &= 1 + \frac{3}{(\pi - 1)} \\ &= \frac{\pi + 2}{\pi - 1} && \text{(Numerical value} \approx 2.4\text{)} \end{aligned}$$

Thus, we get that -

$$\boxed{1 + \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + \frac{3}{\pi^4} + ... = \left(\frac{\pi + 2}{\pi - 1}\right)}$$

Problem C

To find the numerical value of -

$$N = \log\left[\log(3)\cdot\left(\log(2)\cdot\left(\frac{\sqrt{3}-2\sin(\pi/3)}{\pi^3+1}+1\right)\right)-\log(2)\log(3)+(-1)^{100}\right] \tag{3}$$

For ease of calculation, let's represent this entire term in the following manner -

$$N = \log\left[\log(3)\cdot\left(\log(2)\cdot\left(\frac{A}{\pi^3+1}+1\right)\right)-\log(2)\log(3)+B\right]$$

Now, we calculate A, B separately, and then substitute their values back in the expression to get our final numerical value.
To compute the value of A -

$$\begin{aligned} A &= \sqrt{3}-2\sin(\pi/3) \\ &= \sqrt{3}-2\left(\frac{\sqrt{3}}{2}\right) \\ &= \sqrt{3}-\sqrt{3} \\ &= 0 \\ \implies A &= 0 \end{aligned}$$

To compute the value of B -

$$\begin{aligned} (-1)^{100} &= ((-1)^2)^{50} \\ &= 1^{50} \\ &= 1 \\ \implies B &= 1 \end{aligned}$$

On substituting both these values in our expression, we have -

$$\begin{aligned} N &= \log\left[\log(3)\cdot\left(\log(2)\cdot\left(\frac{0}{\pi^3+1}+1\right)\right)-\log(2)\log(3)+1\right] \\ \implies N &= \log\left[\log(3)\cdot(\log(2)\cdot(1))-\log(2)\log(3)+1\right] \\ \implies N &= \log\left(\log(2)\log(3)-\log(2)\log(3)+1\right) \\ \implies N &= \log(1) \\ \implies N &= 0 \end{aligned}$$

Thus, we finally get that -

$$\boxed{\log\left[\log(3)\cdot\left(\log(2)\cdot\left(\frac{\sqrt{3}-2\sin(\pi/3)}{\pi^3+1}+1\right)\right)-\log(2)\log(3)+(-1)^{100}\right]=0}$$

Problem D

To find value of $\sigma(p^2)$:

Given: **n** is an integer of the form, $\mathbf{n} = \mathbf{p^2}$, where p is any prime number.
Since p is a prime number, the **prime factorization** of $p^2 = p^1 \times p^1$.
$\implies$ All factors of p^2 are of the form $p^m \times p^n$, where $m = 0, 1$ and $n = 0, 1$.
$\implies$ So the possible **combinations** for (m, n) are $(0, 0)$; $(0, 1)$ **OR** $(1, 0)$; $(1, 1)$, which translate to the factors $1, p, p^2$ respectively. Thus, we have -

$$\boxed{\sigma(p^2) = 1 + p + p^2} \text{ (by the given definition of } \sigma(n)\text{)}.$$

To prove that $\sigma(n) < 2n$:

Let's assume the contrary case to be true i.e. $\sigma(n) \geq 2n$. Thus, we have -

$$\begin{aligned} \sigma(p^2) &\geq 2p^2 \\ \implies 1 + p + p^2 &\geq 2p^2 \\ \implies 1 + p &\geq p^2 \\ \implies 1 + p - p^2 &\geq 0 \\ \implies p^2 - p - 1 &\leq 0 \\ \implies p \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] & \\ \implies p &< 2 \end{aligned}$$

But we know that p is a prime number. This is a direct contradiction, since all prime numbers are **greater than or equal to** 2. Thus, our **initial assumption is false.**

Thus, $\boxed{\sigma(n) < 2n}$
Hence Proved!

Problem E

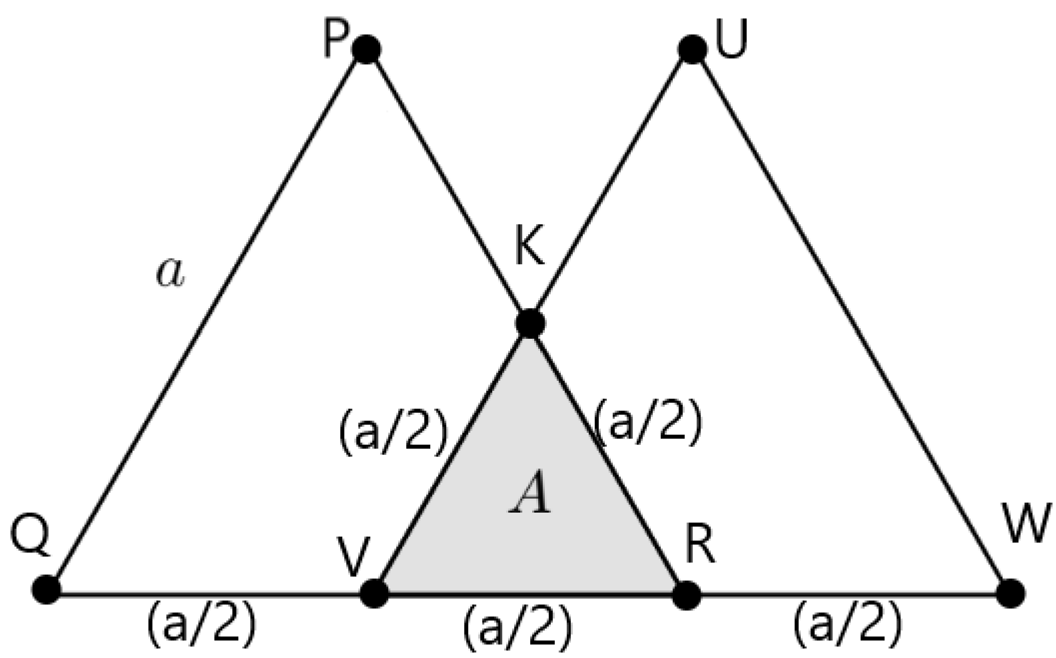

Figure 1: Side lengths after horizontal shift

Refer to the above figure for this solution.
Since the **equilateral triangles** are horizontally shifted by $(a/2)$, $\triangle$**KVR** is an **equilateral triangle** of side-length $(a/2)$. The shaded **grey area** "A" is equal to the area of $\triangle$**KVR.**
We know that the **area** of an **equilateral triangle** of side-length l is given by -

$$\text{Area} = \frac{\sqrt{3}}{4} \cdot l^2 \tag{4}$$

Thus, if we substitute $l = (a/2)$ in the above equation, we get -

$$\begin{aligned} A &= \frac{\sqrt{3}}{4} \cdot \left(\frac{a}{2}\right)^2 \\ &= \frac{\sqrt{3}a^2}{16} \end{aligned}$$

Thus, we get that the area "A" of the shaded **grey area** is -

$$\boxed{A = \frac{\sqrt{3}a^2}{16}}$$