Aditya Ramdasi
Age: 19 years
Country: India

Pre-Final Round 2022

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Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend printing this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The eight problems are separated into three categories: 3x basic problems (A; four points), 3x advanced problems (B; six points), 2x special-creativity problems (C; eight points).
- You receive points for the correct solution as well as for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 46 points in total. You qualify for the final round if you reach at least 22 points (under 18 years) or 30 points (over 18 years).
- Please consider following notation that is used for the problems
  - $x, y \in \mathbb{R}$ denotes a real number, $n, k \in \mathbb{N}$ denotes a positive integer.
  - f, g, h denote functions. The domain and co-domain should follow from the context.
  - The *roots* of a function f are all x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
  - The natural logarithm of x is written as $\log(x)$.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/en/login`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no Word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sunday 20. November 2022, 23:59 UTC+0.**
- The results of the pre-final round will be announced on Monday 28. November 2022.

Good luck!

Problem A.1

Determine A, B, C such that all of the following functions intersect the point $(2, 2)$:

$$f_1(x) = Ax + 1 \qquad f_2(x) = Bx^2 + 2 \qquad f_3(x) = Bx^3 + 3$$

Solution :- *I assume $\boxed{f_3(x) = C.x^3 + 3}$ (printing mistake)

→ For $f_1(x)$, $f_2(x)$ and $f_3(x)$ to intersect at $(2,2)$, all of them will have to pass through $(2,2)$.

→ Thus, we have : $f_1(2) = 2$, $f_2(2) = 2$ and $f_3(2) = 2$

① $f_1(2) = 2A+1 = 2 \Rightarrow \boxed{A = \frac{1}{2}}$

② $f_2(2) = 4B+2 = 2 \Rightarrow \boxed{B = 0}$

③ $f_3(2) = 8C+3 = 2 \Rightarrow \boxed{C = -\frac{1}{8}}$

Thus, for all three functions to intersect at $(2,2)$, the values of A, B, C should be ⟶

$\boxed{A = 0.5}$ $\boxed{B = 0}$ $\boxed{C = -0.125}$

Problem A.2

Find all $x \in \mathbb{R}$ that are solutions to this equation: $0 = (1 - x - x^2 - ...) \cdot (2 - x - x^2 - ...)$

Solution :-

To find all $x \in R$ such that: $0 = (1-x-x^2-....)(2-x-x^2-....)$

$\therefore \left[1-(x+x^2+x^3+...)\right]\left[2-(x+x^2+x^3+...)\right] = 0$

→ Sum of a Geometric Progression (G.P): $a, ar, ar^2 .....$
(with infinite terms) $= \frac{a}{1-r}$ [for $|r|<1$]

Let us assume $|x|<1$ & use the above result.....
[If this is not true, we will find that out on solving. Then we may have to use a different method.]

$\therefore \left[1 - \frac{x}{1-x}\right]\left[2 - \frac{x}{1-x}\right] = 0$ [$a=r=x$]

$\therefore$ Either $1 - \frac{x}{1-x} = 0$ OR $2 - \frac{x}{1-x} = 0$

$\Rightarrow 1 = \frac{x}{1-x}$ $\Rightarrow 2 = \frac{x}{1-x}$

$\Rightarrow 1-x = x$ $\Rightarrow 2-2x = x$

$\Rightarrow \boxed{x = \frac{1}{2}}$ $\Rightarrow \boxed{x = \frac{2}{3}}$

We thus got values of $|x| < 1$. we are safe & correct in our assumption. Thus, $\boxed{x = 0.5, 0.\overline{66}}$

Problem A.3

Find the derivative $f'(x)$ of the following function with respect to x:

$$f(x) = \sin\left(\pi^{\sin x} + \pi^{\cos x}\right)$$

Solution :-

To find the derivative $f'(x)$ here, we use two properties $\longrightarrow$

(1) $\frac{d}{dx}(a^x) = a^x \ln(a)$ (2) Chain Rule : $f'(g(x))\, g'(x) = \frac{d}{dx}(f(g(x)))$

$$\therefore \frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left[\sin\left(\pi^{\sin x} + \pi^{\cos x}\right)\right]$$

$$= \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \frac{d}{dx}\left(\pi^{\sin x} + \pi^{\cos x}\right)$$

$$= \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \left[\pi^{\sin x} \ln(\pi) \cdot \frac{d}{dx}(\sin x) + \pi^{\cos x} \cdot \ln(\pi) \frac{d}{dx}(\cos x)\right]$$

$$= \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \ln(\pi) \left[\pi^{\sin x} \cos x \cancel{+} - \pi^{\cos x} \sin x\right]$$

Thus, we have $\longrightarrow$

$$\therefore \boxed{f'(x) = \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \ln(\pi) \left[\pi^{\sin x} \cos x - \pi^{\cos x} \sin x\right]}$$

Problem B.1

Let H_n define the sum of reciprocals of all integers from 1 to n:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

Prove the following identity:

$$H_{2n} - H_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \dots + \frac{1}{2n-1} - \frac{1}{2n}$$

Solution :-

Let us try to simplyfy the right hand side (RHS) of the given identity, and work our way towards the left hand side. (LHS)

$\therefore$ RHS $\longrightarrow$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots \dots \frac{1}{2n-1} - \frac{1}{2n} = \underbrace{\left(1 + \frac{1}{3} + \frac{1}{5} + \dots \frac{1}{2n-1}\right)}_{A} - \underbrace{\left(\frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2n}\right)}_{B} = \underline{\underline{A - B}}$$

We can write : $A - B = A + B - 2B$ (add & subtract B)

$$\therefore \text{RHS} = (A + B) - (2B)$$

$$= \left[\left(1 + \frac{1}{3} + \frac{1}{5} + \dots \frac{1}{2n-1}\right) + \left(\frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2n}\right)\right] - 2\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \frac{1}{2n}\right]$$

$$= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \frac{1}{2n-1} + \frac{1}{2n}\right] - \left[1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}\right]$$

$$= H_{2n} - H_n$$

$$= \text{LHS}$$

$\therefore$ $\boxed{\text{RHS} = \text{LHS}}$ Thus Proved!!

Problem B.2

It is well known that squared brackets do not simply square the individual terms:

$$(1+2)^2 \neq 1^2 + 2^2$$

$$(1+2+3)^2 \neq 1^2 + 2^2 + 3^2$$

Instead, we add a correction term ψ to make the equations hold true:

$$(1+2)^2 = 1^2 + 2^2 + \psi_2$$

$$(1+2+3)^2 = 1^2 + 2^2 + 3^2 + \psi_3$$

$$\ldots$$

$$(1+2+3+\ldots+n)^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 + \psi_n$$

Show that the correction term ψ_n has the following form and determine the values of α and β:

$$\psi_n = \frac{n^4 - n^2}{\alpha} + \frac{n^3 - n}{\beta}$$

Solution :-

We use the following two results here ⟶

① $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = 1^2+2^2+\ldots n^2$
[sum of first 'n' natural number squares]

② $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = 1+2+3+\ldots n$
[sum of first 'n' natural numbers]

$\therefore\ (1+2+3+\ldots n)^2 = 1^2+2^2+3^2+\ldots n^2 + \psi_n$

$\Rightarrow\ \psi_n = \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{n(n+1)(2n+1)}{6}\right]$

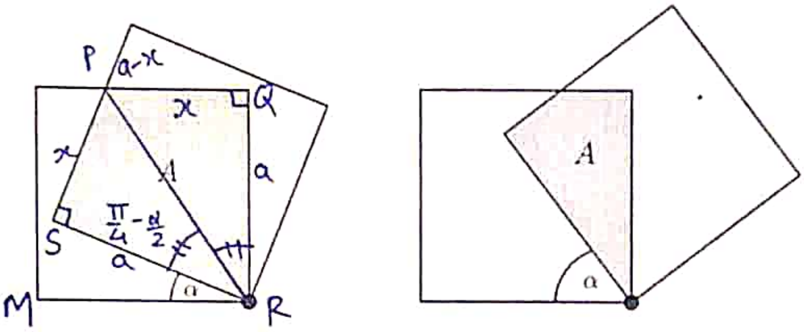
$= n(n+1)\left[\frac{n^2+n}{4} - \frac{2n+1}{6}\right] = \frac{3n^4-3n^2}{12} + \frac{2n^3-2n}{12} = \frac{n^4-n^2}{4} + \frac{n^3-n}{6}$

$\therefore\ \boxed{\psi_n = \frac{n^4-n^2}{4} + \frac{n^3-n}{6}}$

ψ_n is clearly of the given form and $\boxed{\alpha = 4,\ \beta = 6}$

Problem B.3

You are given two overlaying squares with side length a. One of the squares is fixed at the bottom right corner and rotated by an angle of α (see drawing). Find an expression for the enclosed area $A(\alpha)$ between the two squares with respect to the rotation angle α.



Solution :-

Refer to the above figure for this solution. It is representative of any 'α' value. $\alpha \in [0, \frac{\pi}{2}]$

Since both squares have side length 'a', $\overline{SR} = \overline{QR} = a$

$\therefore$ $\boxed{\triangle PSR \cong \triangle PQR}$ [by RHS criterion of congruence] — (1)

$\therefore m\angle PRS = m\angle PRQ$ (from (1)) $= \dfrac{90^\circ - \alpha}{2} = 45^\circ - \dfrac{\alpha}{2}$

[given that $m\angle QRM = 90^\circ$]

In $\triangle PSR$, let $l(PS) = x$. $\Rightarrow \tan(\angle PRS) = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \dfrac{PS}{SR} = \dfrac{x}{a}$

$\therefore$ $\boxed{x = a\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}$ - (2) $\quad$ $\boxed{PS = PQ = x}$

The shaded enclosed area 'A' $=$ sum of $ar(\triangle PSR) + ar(\triangle PQR)$

$= \frac{1}{2} \times PS \times SR + \frac{1}{2} \times PQ \times QR$

$= \dfrac{ax}{2} + \dfrac{ax}{2} = ax$

$\therefore A = ax = a \cdot a\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = a^2 \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

$\therefore$ $\boxed{A(\alpha) = a^2 \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = a^2\left[\dfrac{1 - \tan(\alpha/2)}{1 + \tan(\alpha/2)}\right]}$

$\left\{\tan(a-b) = \dfrac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}\right\}$

$[\tan(\pi/4) = 1]$

Problem C.1

For this problem, we define the fractional part of $x \in \mathbb{R}_{\geq 0}$ as

$$\{x\} = x - \lfloor x \rfloor$$

where $\lfloor x \rfloor$ is the integer part of x, i.e., the greatest integer less than or equal to x.

(a) Draw the function $\{x\}$ in a coordinate system for $0 \leq x \leq 3$.

(b) Find the area A_n under the graph of $\{x\}$ between 0 and $n \in \mathbb{N}$ as given by:

$$A_n = \int_0^n \{x\}\, dx$$

Remember the definition of H_n from problem B.1. H_n grows similar to $\log(n)$ and they define the well-known constant γ in mathematics:

$$\gamma = \lim_{n\to\infty} (H_n - \log(n)) = 0.5772...$$

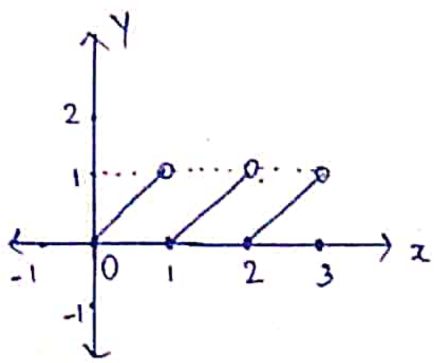
(c) Use this to prove the following identity:

$$\int_1^\infty \frac{\{x\}}{x^2}\, dx = 1 - \gamma$$

Hint: Split the integral into individual sums for each integer value.

Solution :—

(a) $f(x) = \{x\} \rightarrow$



(b) $A_n = \int_0^n \{x\}\, dx = ?$

Area under graph of $\{x\}$ between '0' and $n \in N$ is the sum of areas of 'n' right angled triangles, with base = 1 and height = 1.

$\therefore A_n = n\left[\frac{1}{2} \times b \times h\right]$

$= n\left[\frac{1}{2} \times 1 \times 1\right]$

$= \frac{n}{2}$

$\therefore \boxed{A_n = \int_0^n \{x\}\, dx = \frac{n}{2}}$

(C) To prove : $\boxed{\int_1^\infty \frac{\{x\}}{x^2}dx = 1-\gamma = 1-\lim_{n\to\infty}(H_n - \log(n))}$

Let us split the integral at every integer value of limits.

$$\therefore \int_1^\infty \frac{\{x\}}{x^2} = \int_1^2 \frac{\{x\}}{x^2}dx + \int_2^3 \frac{\{x\}}{x^2}dx + \dots \int_{n-1}^n \frac{\{x\}}{x^2}dx \quad [n\to\infty]$$

$$= \int_1^2 \frac{x-1}{x^2}dx + \int_2^3 \frac{x-2}{x^2}dx + \dots \int_{n-1}^n \frac{x-(n-1)}{x^2}dx.$$

$$= \int_1^2\left(\frac{1}{x}-\frac{1}{x^2}\right)dx + \int_2^3\left(\frac{1}{x}-\frac{2}{x^2}\right)dx + \dots \int_{n-1}^n\left(\frac{1}{x}-\frac{n-1}{x^2}\right)dx$$

$$= \int_1^2 \frac{1}{x}dx + \int_2^3 \frac{1}{x}dx \dots \int_{n-1}^n \frac{1}{x}dx + \left[\int_1^2 \frac{-1}{x^2}dx + \int_2^3 \frac{-2}{x^2}dx + \dots \int_{n-1}^n \frac{-(n-1)}{x^2}dx\right]$$

$$= \int_1^n \frac{1}{x}dx + \left[\frac{1}{x}\Big]_1^2 + \frac{2}{x}\Big]_2^3 + \frac{3}{x}\Big]_3^4 \dots \frac{n-1}{x}\Big]_{n-1}^n\right]$$

$$= \ln(x)\Big]_1^n + \left[\left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{2}{3}-\frac{2}{2}\right)+\left(\frac{3}{4}-\frac{3}{3}\right)+\dots\left(\frac{n-1}{n}-\frac{n-1}{n-1}\right)\right]$$

$$= \lim_{n\to\infty}\log(n) + \left[\left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}\dots\frac{n-1}{n}\right) - (1+1+1\dots (n-1) \text{ times})\right]$$

$$= \lim_{n\to\infty}\log(n) + \left[\sum_{n=1}^{\infty}\frac{n-1}{n} - (n-1)\right]$$

$$= \lim_{n\to\infty}\log(n) + \left[\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right) - n+1\right]$$

$$= \lim_{n\to\infty}\log(n) + \left[\cancel{n} - H_n - \cancel{n}+1\right] \qquad \left[\because \sum_{k=1}^{n}\frac{1}{k} = H_k\right]$$

$$= \lim_{n\to\infty}\left[\log(n) - H_n\right] + 1 = -\gamma + 1$$

$$\Rightarrow \boxed{\int_1^\infty \frac{\{x\}}{x^2} = -\gamma+1 = 1-\gamma}$$ Thus Proved!

Problem C.2

This problem requires you to read following scientific article:

Sum of Reciprocals of Germain Primes.
Wagstaff, Samuel S. *Journal of Integer Sequences*, 24 (2021).
Link: https://cs.uwaterloo.ca/journals/JIS/VOL24/Wagstaff/wag4.pdf

Use the content of the article to work on the problems (a-f) below:

(a) What is the difference between twin primes and Germain primes? Give examples for both.

- Twin prime 'p' is such that either 'p+2' or 'p-2' is also a prime.
  Eg. 3&5, 5&7, 11&13, etc
- Germain prime 'p' is such that '2p+1' is also a prime. Eg. 11, 29, 41 etc.

(b) Which numbers does the set $\mathcal{S}_{1,0}$ represent and what is the value of $S'_{1,2}(4 \cdot 10^{18})$?

$\bar{S}_{a,b} = \{p : ap+b \text{ is prime}\}$ $\therefore$ $\bar{S}_{1,0} = \{p : 1p+0 = p \text{ is a prime}\}$

$\therefore$ $\bar{S}_{1,0}$ represents the set of all prime numbers.

$S'_{1,2}(4 \cdot 10^{18}) \approx S'_{1,2}(x) \Big|_{\lim x \to \infty}$

= Brun's constant 'B' $\boxed{\approx 1.9021}$

(c) In the proof of Theorem 1, explain why $\sum_{p \le x, p \in \mathcal{S}_{a,b}} \frac{1}{p} = \sum_{t=1}^{x} \frac{\pi_{a,b}(t) - \pi_{a,b}(t-1)}{t}$?

(d) Explain the difference between Table 1 and Table 3.

Table 1 shows lower bounds for $S_{a,b}$ for a few values of x_0 (powers of 10)

Table 3 shows most probable values for $S_{a,b}$ $\left[S_{a,b}(x_0) + \frac{2c_2}{\log x_0}\right]$ $[4C_2 \text{ for } S'_{1,2}]$

(e) Use Theorem 3 to calculate an upper bound for $\pi_{1,16}(e^{100})$ in orders of magnitude. $\boxed{D = 8.37}$

$\pi_{1,16}(e^{100}) < \dfrac{16c_2^{\prime}(e^{100})}{\log(e^{100})\left[8.37 + \log(e^{100})\right]} \Rightarrow \pi_{1,16}(e^{100}) < \dfrac{16C_2 e^{100}}{(100)(108.37)} \approx \dfrac{10^{43}}{10^4} \approx \boxed{10^{39}}$

(f) Show in detail why the left- and right-hand side of equation (1) in Theorem 4 are equal.