Experiment 2: Kater's Pendulum
Lab Report

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1 Introduction

The variation in the value of gravitational acceleration 'g' was first noted by **Jean Richer**, a French scientist, in 1671[1]. He recognized a time delay of 2.5 minutes between the clocks of **French Guiana** and **Paris**. This was later proved by **Isaac Newton** in **1687** by taking into account the **oblateness** of the Earth's shape.

The simplest way to measure the value of g in contemporary experimental physics is by using the **Simple Pendulum.** The expression for the **time period '*T*'** a simple pendulum (for small oscillations) gives us the value of g, and is -

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{1}$$

However, this equation relies heavily on the **accuracy** of our **length measurement**, which is a source of **substantial error** considering the practical limitations of measuring length to a high degree of accuracy.

The **compound pendulum** is a **rigid, extended body** with mass distributed throughout its length. It swings in the vertical plane, about any axis horizontal to the body. Thus, the resultant force acts through the pendulum's **centre**

of mass. The compound pendulum gives us a slightly better chance at measuring the value of g accurately, since its expression for **Time Period '*T*'** uses both, the **length** measurement and a theoretically calculated quantity called **radius of gyration**, k. This reduces the influence of the error in measuring l on the overall value of g. The expression of **time period '*T*'** for small oscillations of a **compound pendulum** is given by -

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}} \tag{2}$$

where l is the length between the pendulum's **center of mass** and **point of suspension**.

In 1817, a **British physicist** named **Henry Kater** suggested an even more refined way to measure the value of g experimentally. **Huygens' principle** states that the **point of suspension** and the **center of mass** of a pendulum can be changed, by keeping the time period **constant**. Using this principle, Kater developed his technique of measuring g experimentally to even better degrees of accuracy. The **final expression** for g using Kater's Pendulum is as follows -

$$\frac{8\pi^2}{g} = \frac{T_A^2 + T_B^2}{l_A + l_B} + \frac{T_A^2 - T_B^2}{l_A - l_B} \tag{3}$$

where l_A, l_B are the distances between the center of mass and the **knife edges** and T_A, T_B are the time periods in the **respective configurations**, of the Kater's pendulum.

2 Aim

- **Part A:** To find four points on the Kater's pendulum with the same time period
- **Part B:** To find the local value of **gravitational acceleration '*g*'** using the **Kater's pendulum.**
- **Part C:** To perform a detailed error analysis for the experimental data collected, and report the final g values along with **the appropriate error margins**.

3 Apparatus

1. Kater's Pendulum (a thick rod of mass M, with two knife-edges)
2. Two unequal masses M_1, M_2 made of wood and metal.
3. Stopwatch (to measure time period of oscillations)
4. Meter tape (to measure distances on the pendulum)
5. Markers (to fix positions of knife-edges and weights on the thick rod)

4 Theory

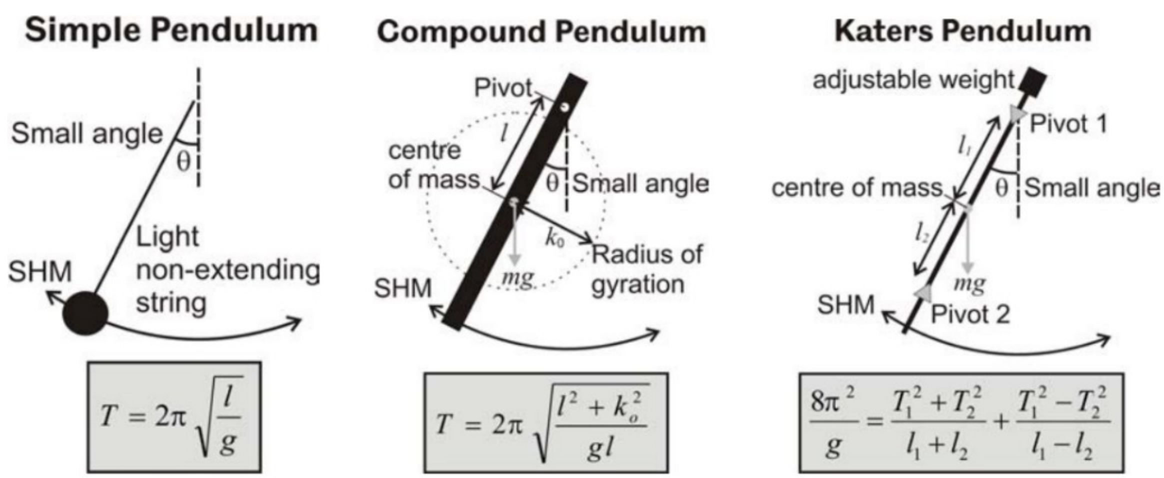

Figure 1: 3 ways to measure 'g'

Figure 1 summarizes the 3 pendulums and their expressions for measuring 'g' experimentally, as explained in the Introduction Section.

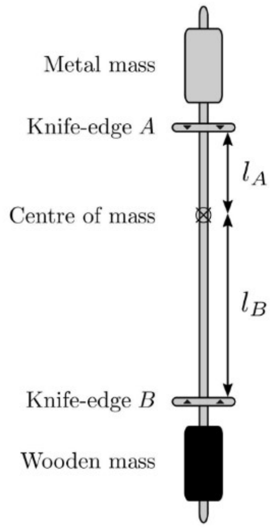

Figure 2: Kater's Pendulum

Consider small amplitude oscillations of the **Kater's Pendulum** given in Figure 2. Then, we have **two possible orientations** for suspending the same pendulum and measuring time periods, which are as follows -

1. T_A = time period when pendulum is suspended about Knife Edge A, at a distance l_A from the center of mass.

2. T_B = time period when pendulum is suspended about Knife Edge B, at a distance l_B from the center of mass.

In each of these orientations, we use Equation 2 to get the following expressions-

$$T_A = 2\pi\sqrt{\frac{k^2 + l_A^2}{g l_A}}$$

$$T_B = 2\pi\sqrt{\frac{k^2 + l_B^2}{g l_B}}$$

To eliminate the **radius of gyration, *k***, we equate the expressions for k from both equations to get -

$$\frac{T_A^2 l_A g - 4\pi^2 l_A^2}{4\pi^2} = \frac{T_B^2 l_B g - 4\pi^2 l_B^2}{4\pi^2}$$

$$\implies g(l_A T_A^2 - l_B T_B^2) = 4\pi^2(l_A^2 - l_B^2)$$

$$\implies \frac{l_A T_A^2 - l_B T_B^2}{l_A^2 - l_B^2} = \frac{4\pi^2}{g}$$

Now, to get an expression of the form as given in Figure 1, we equate the LHS of the above equation to the sum of two fractions, as given below -

$$\begin{aligned}
\frac{l_A T_A^2 - l_B T_B^2}{l_A^2 - l_B^2} &= \frac{P}{l_A + l_B} + \frac{Q}{l_A - l_B} \\
\implies \frac{l_A T_A^2 - l_B T_B^2}{l_A^2 - l_B^2} &= \frac{P(l_A - l_B) + Q(l_A + l_B)}{l_A^2 - l_B^2} \\
\implies l_A T_A^2 - l_B T_B^2 &= l_A(P + Q) - l_B(P - Q)
\end{aligned}$$

On comparing the coefficients of l_A and l_B, we get the following -

$$\begin{aligned}
P + Q &= T_A^2 \\
P - Q &= T_B^2
\end{aligned}$$

Adding both equations and solving for P, Q we get -

$$\begin{aligned}
P &= \frac{T_A^2 + T_B^2}{2} \\
Q &= \frac{T_A^2 - T_B^2}{2}
\end{aligned}$$

Thus, on substituting the values of P, Q in the original equation, we get the expression -

$$\begin{aligned}
\frac{4\pi^2}{g} &= \frac{T_A^2 + T_B^2}{2(l_A + l_B)} + \frac{T_A^2 - T_B^2}{2(l_A - l_B)} \\
\implies \frac{1}{g} &= \frac{1}{4\pi^2}\left(\frac{T_A^2 + T_B^2}{2(l_A + l_B)} + \frac{T_A^2 - T_B^2}{2(l_A - l_B)}\right)
\end{aligned}$$

There is still a problem in this expression. We can neither calculate nor measure the length $(l_A - l_B)$ **accurately** in our experiment, since the **center of mass** of the compound pendulum is very hard to find. Thus, we try to eliminate that entire term, by finding a configuration of the pendulum, such that $T_A = T_B$. On substuting this in the above equation, we finally get our desired expression for the value of g -

$$g = 4\pi^2 \left(\frac{l_A + l_B}{T^2}\right) \tag{4}$$

5 Part A

In this part, we aim to demonstrate that there are 4 possible points of suspension on the Kater's Pendulum for which it oscillates with the same time period.

5.1 Procedure

1. Measure the position of the center of mass of the pendulum's thick rod by balancing it on a heavy edge. Mark this position with a marker.

2. Mark out 9 roughly equidistant points on each side of the rod, between the center of mass and both ends.

3. Place the knife edge at one of the end positions, and measure the time period for 10 oscillations at this position.

4. Repeat this for every position of the knife-edge, and fill these values into 2 NumPy arrays; one of knife-edge positions (l) and another of time period of 10 oscillations (T). Remember to flip the rod when the position of the knife-edge crosses the center of mass.

5. After measuring T for all marked locations, plot the T vs l graph and also plot a trend-line (dashed).

6. Verify that for a certain T value, there are 4 possible values of l on the rod by cutting the curve with a horizontal line parallel to the x axis, at an appropriate height.

5.2 Observations

- Total length of cylindrical rod of Kater's pendulum = 115 cm
- Position of center of mass of rod = 57.5 cm (from either ends)

The following Table shows us the time needed for 10 oscillations of the rod for different positions of the knife-edge (point of suspension) on both sides of its **center of mass** -

Position of knife-edge from center of rod (cm)	Time for 10 oscillations (sec)	Position of knife-edge from center of rod (cm)	Time for 10 oscillations (sec)
56.5	17.41	-8.5	28.71
50.5	17.11	-12.0	22.81
43.5	16.74	-15.5	20.33
36.5	16.56	-22.5	17.64
29.5	16.72	-29.5	16.74
22.5	17.83	-36.5	16.73
15.5	20.14	-43.5	16.72
12.0	22.76	-50.5	17.18
8.5	28.72	-56.5	17.68

Table 1: Positions of knife edge with time for 10 oscillations

If we plot the data-points in Table 1, (T vs l) we get the following graph -

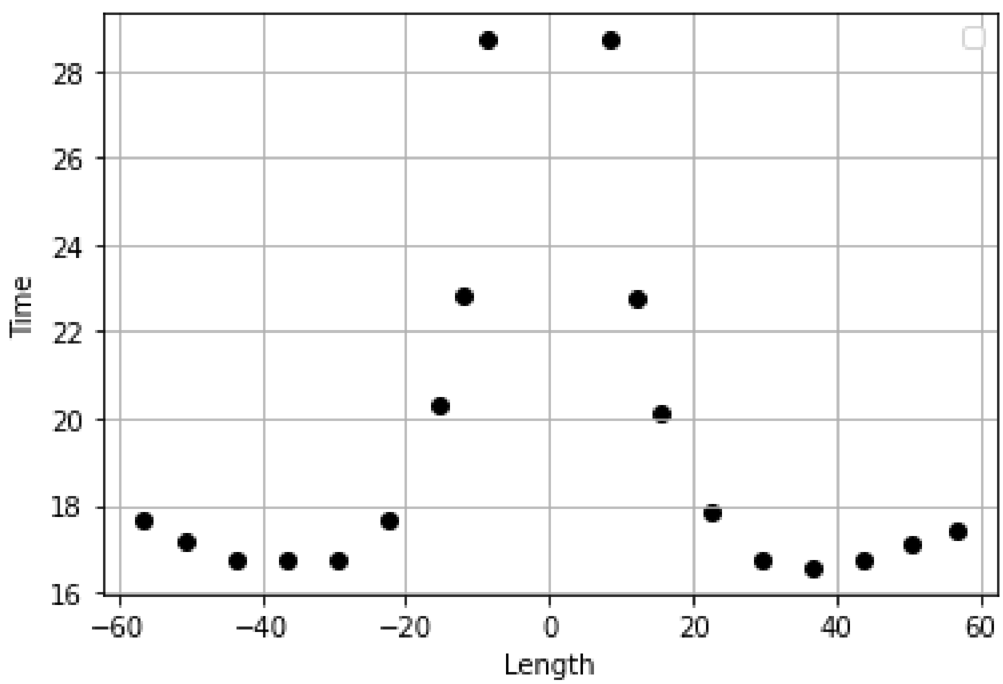

Figure 3: Scatter plot of data in Table 1

On plotting the trend-line, we can see the pattern that the graph exhibits. We then draw a single horizontal line at some appropriate height ($T = 17.3$ sec), to see that it cuts the plot at 4 distinct positions, as seen below -

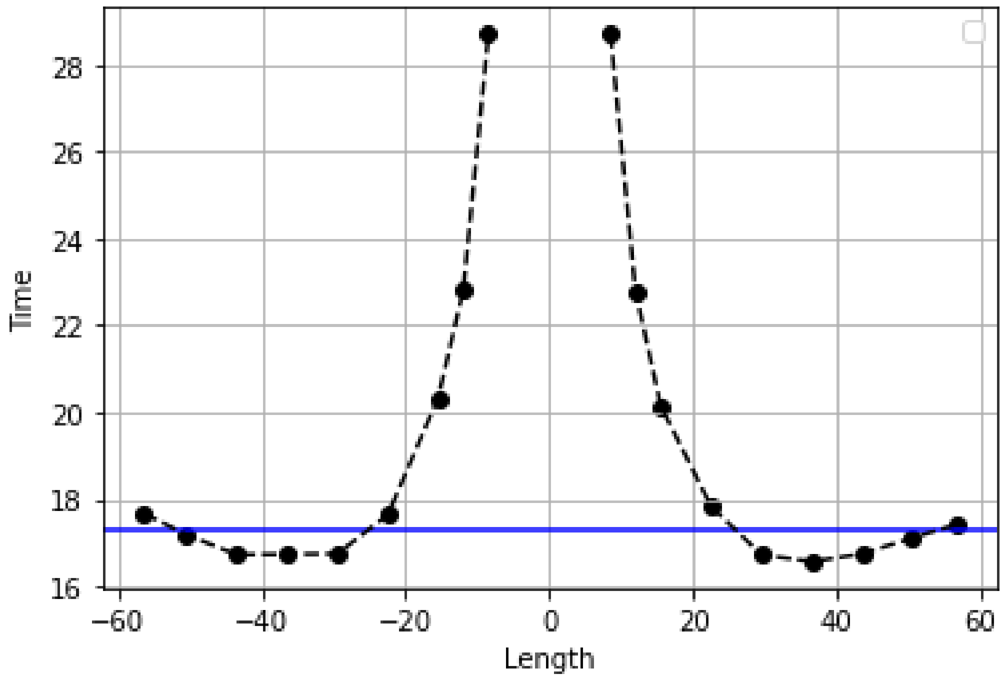

Figure 4: Trend-line and verification line

5.3 Result

Thus, we can clearly see from Figure 4 that for some value of time T (=17.3 sec), there are **4 distinct points** on the pendulum which have the **same time period**.

6 Part B

In this part, we use the result obtained in Part A to find the local value of gravitational acceleration g using a special configuration of the Kater's Pendulum.

6.1 Procedure

1. The first aim of this procedure is to systematically find such a configuration of positions for the two knife-edges and two masses, on the central rod, such that $T_A = T_B$.

2. There are 4 distance-variables in this configuration of the Kater's pendulum; the distances between the center of the rod and Knife-Edge A, Knife-Edge

B, Wooden Mass, and Metal Mass (see Figure 2). To get a sense of how T_A, T_B vary, we fix three of these 4 distance-parameters and vary the fourth one, to see its effect on the time periods.

3. Fix the position of the wooden mass and knife-edge B (sticking to each other) at one end of the rod. This will remain constant throughout the experiment.

4. Mark the center of mass of the compound pendulum (rod with masses and knife-edges) before taking every observation for T_A, T_B. Use this to calculate distances to masses and knife-edges for that trial.

5. Fix knife-edge A at some arbitrary distance from the center of the rod. Now, measure values of T_A, T_B for two different positions of the metal mass in this configuration; one when the metal mass sticks to the knife-edge A, while the other when it is at one end of the rod.

6. If the sign of $T_A - T_B$ changes for these two positions, fix knife-edge A to its current position. If not, change knife-edge A's position and repeat the above procedure until the sign of $T_A - T_B$ changes in between both positions. Once knife-edge A is fixed, measure the value of $l_A + l_B$ (see Figure 2)

7. Once the position of knife-edge A is fixed, we just have one variable parameter - the position of the metal mass. This *has to* be in a certain range, somewhere between knife-edge A and the end of the metal rod. Vary the position of the metal mass slightly and measure T_A, T_B.

8. Repeat the center of mass measurements and $T_A - T_B$ measurements for slightly different positions of the metal mass and note down the variation in a table. Do this until $T_A - T_B$ becomes very close to 0.

9. Once $T_A = T_B$ is reached, fix that configuration and note down all length measurements. This is our final configuration.

10. Now, measure T_A, T_B for 100 oscillations of the pendulum in this final configuration. Ensure that the oscillations are counted correctly. Substitute the values of T_A, T_B obtained in Equation 4 to get the experimental value of g.

6.2 Observations

After fixing wooden mass and knife-edge B to one end of the rod, the first trial arbitrary position chosen for knife-edge A was at a distance of of **81.6 cm** from knife-edge B. On measuring $T_A - T_B$ values for both positions of the metal mass in this configuration, the sign of $T_A - T_B$ was found to change. We finalize the position of knife-edge A at that position. Thus, our constant value of $l_A + l_B =$ **81.6 cm**.

With three parameters fixed, we now vary the 4^{th} parameter (position of metal mass) within our fixed range. We get values with alternating signs for $T_A - T_B$, which finally converges to almost zero (+0.01). The following table shows the data recorded -

Trial No.	T_A (sec)	T_B (sec)	$T_A - T_B$ (sec)	l_A (cm)	l_B (cm)	$l_A + l_B$ (cm)	$l_A - l_B$ (cm)
1	17.31	17.53	-0.22	28.5	53.1	81.6	-24.6
2	19.89	18.44	+1.45	24.6	57.6	81.6	-32.4
3	17.96	17.99	-0.03	27.5	54.1	81.6	-26.6
4	17.87	17.70	+0.17	28	53.6	81.6	-25.6
5	17.89	17.96	-0.07	27.6	54	81.6	-26.4
6	18.02	18.01	+0.01	27.4	54.2	81.6	-26.8

Table 2: Table for finding configuration for $T_A = T_B$

We can clearly see the magnitude of $(T_A - T_B)$ reducing and the sign alternating in Table 2. It finally converges to **0.01** at the end. **We now have our final configuration of the Kater's pendulum.**

We now measure the time T_A, T_B for **100 oscillations** of the pendulum in this configuration. The values come out to be as follows -

$$\mathbf{T_A} \text{ for } \mathbf{100\ oscillations = 181.09\ sec}$$
$$\mathbf{T_B} \text{ for } \mathbf{100\ oscillations = 180.39\ sec}$$

6.3 Result

On dividing these values of time by 100 and substituting them in Equation 4, we get the following values for g -

$$\mathbf{g} \text{ from } \mathbf{T_A} = \mathbf{9.823\ m/s^2}$$
$$\mathbf{g} \text{ from } \mathbf{T_B} = \mathbf{9.899\ m/s^2}$$

7 Part C

7.1 Sources and magnitude of errors

The two main sources of error in this experiment are the **instruments** used to measure the two quantities - **length** and **time**.

- **Length:** The standard meter-tape was used to measure all the lengths in this experiment. The **least count** of the meter-tape was **1 mm**, or 0.001 m. $\boxed{\therefore \delta l = 0.001 \text{ m}}$
- **Time:** The **least count** of the stopwatch used in this experiment was **0.01 sec**. However, this is irrelevant since the major source of error in time measurement was **human reaction time**, which propagates a much larger error than the least count of the stopwatch. After 5 attempts at starting and stopping the stopwatch as quickly as possible and finding the arithmetic mean of those 5 values, my reaction time was approximately **0.19 seconds**. $\boxed{\therefore \delta T = 0.19 \text{ sec}}$

7.2 Expression for relative error

Equation 4 gives us the final relation between g and T which we use to calculate the value of g from T. We rewrite it here for convenience again -

$$g = 4\pi^2 \left(\frac{l_A + l_B}{T^2} \right)$$

Thus, to get relative error for g, we differentiate the equation and divide it by the same **original quantity** to get the **relative error** as follows -

$$\frac{\delta g}{g} = \frac{\delta(l_A + l_B)}{l_A + l_B} + 2\left(\frac{\delta T}{T}\right) \tag{5}$$

We substitute the values specified above into this equation, individually for both T_A, T_B to get -

$$\begin{aligned}
\left(\frac{\delta g}{g}\right)_{T_A} &= \frac{0.001}{0.816} + 2\left(\frac{0.19}{181.09}\right) \\
&\approx 0.0033 \\
\left(\frac{\delta g}{g}\right)_{T_B} &= \frac{0.001}{0.816} + 2\left(\frac{0.19}{180.39}\right) \\
&\approx 0.0033
\end{aligned}$$

7.3 Percentage error

The **percentage error** is given by $100 \times \frac{\delta g}{g}$. Thus, we get the **approximate value** of **theoretical** percentage error for both $T_A, T_B = \mathbf{0.33\%}$

To calculate the **practical percentage error**, we can use the following expression -

$$\%\ \text{experimental error} = \frac{|g_{\text{true}} - g_{\text{obtained}}|}{g_{\text{obtained}}} \times 100$$

If we consider $g_{\text{true}} = 9.806\ \text{m}/s^2$, then we get following experimental percentage errors for T_A, T_B -

$$\begin{aligned}
(\%\ \text{experimental error})_{T_A} &= \frac{|9.806 - 9.823|}{9.823} \times 100 \\
&\approx 0.17\% \\
(\%\ \text{experimental error})_{T_B} &= \frac{|9.806 - 9.899|}{9.899} \times 100 \\
&\approx 0.93\%
\end{aligned}$$

Thus, we understand that the experimental error in $\mathbf{T_A}$ is within an acceptable range, while the experimental error in $\mathbf{T_B}$ is slightly higher than the theoretically obtained range.
To get the error range in terms of absolute values, we multiply the respective **percentage error** with the **actual value** of g obtained, as follows -

$$\begin{aligned}
9.823 \times 0.17\% = 0.017 \\
9.899 \times 0.93\% = 0.092
\end{aligned}$$

7.4 Result

Thus, we finally get the following values of g from this experiment -

$$(g)_{T_A} = 9.823 \pm 0.017\,\mathrm{m/s^2}$$
$$(g)_{T_B} = 9.899 \pm 0.092\,\mathrm{m/s^2}$$

8 Precautions

1. The rod of the Kater's Pendulum (combined with other unequal masses) is quite heavy. Ensure a firm grip while holding it. Do not keep your legs directly beneath the rod while suspending it on the pivot to record the time periods for oscillations.

2. Ensure that the knife-edges are tightened with a plier (especially knife-edge A, near the metal mass) to prevent them from sliding and changing their positions while taking the measurements.

3. Ensure that the rod of the pendulum is flipped as we cross the center of mass of the rod, while taking measurements in Part A. If the center of mass goes above the knife-edge, the rod will topple over and may hurt someone or break something in its vicinity.

9 Discussion

1. If we rewrite Equation (2) as a quadratic in terms of length l, then the solutions of the that quadratic will correspond to those points on the rod which have the same Time Period of oscillation.

2. There will be 2 unique solutions to this quadratic equation - $l, k^2/l$. But since these are just on side of the rod with respect to its center, each solution will also have a corresponding point on the negative (other) side of the rod i.e $-l, -k^2/l$.

3. Thus, we can imagine the 4 points of equal T as being the points of intersection of 2 concentric circles of radius l and k^2/l, with the central rod.

4. While adjusting the configuration of masses and knife-edges in order to make $T_A = T_B$, we know that $l_A - l_B \neq 0$, since the wooden and metal mass are unequal. Thus, the distances between center of mass (of compound pendulum) and both knife-edges should be unequal.

5. Experimentally, $l_A + l_B$ is not the sum of two separate lengths on the rod. It is collectively just equivalent to the distance between the two knife-edges. For this reason, in our experiment, once we fix both knife-edges (and vary position of metal mass), the value of $l_A + l_B$ remains constant.

6. The **radius of gyration** k is a theoretically calculated quantity which proves very useful for calculations involving the **M.O.I** of non-trivially shaped objects. The **parallel M.O.I** (axis along length of cylinder) of our cylindrical rod of radius R is given by -

$$I_{||} = \frac{1}{2}MR^2$$

   Using the definition of **radius of gyration** k, we equate this expression to get -

$$\frac{1}{2}MR^2 = Mk^2$$
$$\implies k = \frac{R}{\sqrt{2}}$$

7. While finding the 4 points of equal time period on the pendulum in Part A, we can **ignore** the M.O.I of the knife edge, since we're using the knife-edge as the point of suspension for the pendulum. Any (uniform) mass at the origin of our coordinate system can be ignored for M.O.I calculations.

8. While calculating **lengths between masses**, it was assumed that both of them (wooden and metal) have uniform mass distribution. Thus, the lengths calculated were taken from the **center** of the length of the masses. However, this may not be a very good assumption, and thus could potentially **induce more error** in the length readings.

9. On discussing with peers, another potentially significant source of error in this experiment could be due the use of the **wooden mass**. Due

to its extreme less weight, it may experience significantly more **air resistance** than the metal mass, hampering the value of T_B. We tried to **minimize this** by fixing the **wooden mass** and its corresponding knife-edge at one end of the rod.

10. **Kater's Pendulum** gives a **higher accuracy** for the value of g compared to the Simple Pendulum, since there's a significant constraint on length measurement in the simple pendulum. We don't know the **exact position** of the center of mass of the bob (since mass distribution could be non-uniform), due to which l can't be measured accurately. Also, the string is assumed to be **ideal** (mass-less), but it will have some finite mass. All these assumptions **decrease the accuracy** of g value from Simple Pendulum, thus making Kater's pendulum more accurate.

10 References

1. University of Sheffield, 2nd year laboratory script, Kater's Pendulum (image credit - Figure 1)

2. Kater's Pendulum: A brief history

3. Ashoka University, Physics Lab 2 Handout, Kater's Pendulum (image credit - Figure 2)