Experiment 4: Equipotential Surfaces Lab Report

Name: Aditya Ramdasi
Course Code: PHY-2010-1
TAs: Kamal Nayan, Kartik Tiwari

Instructor: Prof. Susmita Saha
Lab Partner: Aaryan Nagpal
TF: Rohit Kumar Vishwakarma

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1 Introduction

Electrostatics is the branch of physics that studies the effect of stationary charged particles on other test charges in their vicinity. This effect manifests itself in the form of electric fields, which can be thought of as an electric property of every point in space, near a charged particle.
Coulomb's Law gives the expression for the electric field at a distance r from a single point charge q as follows -

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{1}$$

where $\hat{r}$ points in the **radially outward** direction, and q is substituted with its sign to determine the direction of $\mathbf{E}$. For an extended charged object, we can use the **Principle of Superposition** to find the net electric field at any point in space. Theoretically, this would be equivalent to the vector sum of the electric fields generated by the constituent point charges.
However, this is very difficult to do practically, due to the vector nature of the field. The concept of an **electric potential** V simplifies the calculation of a static electric field. **Electric Potential** (V) is a **scalar quantity**. Thus, it can be represented fully just by a single number at every point in space. Whereas, to describe the electric field at every point without the use of V, we need to know 3 numbers (components in all three directions). The relation between **electric field** (E) and **electric potential** (V) is given by -

$$\mathbf{E} = -\nabla\mathbf{V}$$

In this experiment, we try to know more about the nature of electric fields between some specific geometries by analyzing their electric potential at various points.

2 Aim

- **Part A:** To find and justify the equipotential surfaces between two "infinite" parallel bars as electrodes.
- **Part B:** To find and justify the equipotential surfaces between two concentric cylinders as electrodes.
- **Part C:** To explore the equipotential surfaces for any other chosen configuration of electrodes.

3 Apparatus

1. An acrylic tray (to setup the electrodes in)
2. Water as an electrolytic medium (inside the acrylic tray)
3. 2 metallic bar electrodes (to generate electric field for Part A)
4. 2 metallic cylindrical electrodes (to generate electric field for Part B)
5. A pointed probe attached to an $x-y$ stage (to move around and measure the potential at specific locations inside the electric field)
6. A mount for the pointed probe (if necessary)
7. Hand Gloves (to protect from electrical accidents while taking observations)
8. Graph paper (to keep track of the probe's positions inside the electric field)
9. 1 AC Power supply (with voltage stabilizer)
10. 1 digital multimeter (to measure output voltage at various points)
11. Wires (to make the connections in the circuit)

4 Theory

4.1 Relating electric field E to electric potential V -

Maxwell's equations describe static electric fields using its divergence and curl, as follows -

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2}$$

$$\nabla \times \mathbf{E} = 0 \tag{3}$$

Equation (3) tells us that a static electric field is **conservative** in nature, since it has no curl. Every conservative vector field can be associated with a corresponding scalar field (without any loss of information). Since the **line integrals** of the vector field are independent of the path and only dependent on the initial and final points, a scalar field is sufficient to contain all the information of that vector field. However, this scalar field must be such that it incurs the **maximum change possible** along the direction of our vector field. An operator that has exactly this property is the **gradient** (∇). Thus, our electric field **E** can be expressed as a **gradient of a scalar field** f -

$$E = \nabla f$$

Let us define this scalar field f to be our **electric potential**, denoted by V. Since V represents the amount of work done in bringing a unit charge from infinity to a finite distance $\mathbf{r}$, it acquires a negative sign (since $V(\infty) = 0$). Thus, we get our relation -

$$\mathbf{E} = -\nabla \mathbf{V} \tag{4}$$

The same result can be arrived at using the method of **Helmholtz decomposition** on the electric field. The Helmholtz theorem states that any vector field $F(r)$ can be *uniquely* decomposed into the sum of a longitudinal constituent (whose curl is zero) and a transverse constituent (whose divergence is zero). Here, we know that the curl of our electric field is zero (see Equation (3)). Thus, we get the same Equation (4) using this **Helmholtz decomposition**.

4.2 Poisson and Laplace equation -

On taking the divergence of both sides of Equation (4), we get -

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \nabla \cdot (-\nabla \mathbf{V}) \\ &= -\nabla^2 V \end{aligned}$$

Using Equation (2) on the above result, we get that -

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{5}$$

This result is known as **Poisson's Equation**. However, we know that the *charge density* ρ inside a conductor is zero. Thus, in the context of conductors, this Poisson Equation takes the form -

$$\nabla^2 V = 0 \tag{6}$$

This result is known as **Laplace's Equation**. The condition for applying this equation is that the charge density in the bulk of the material must be zero. since the net electric field inside a conductor is always zero, we can apply and solve the Laplace's Equation to obtain an expression for V for certain configurations of conductors.

4.3 Parallel Plate Capacitor

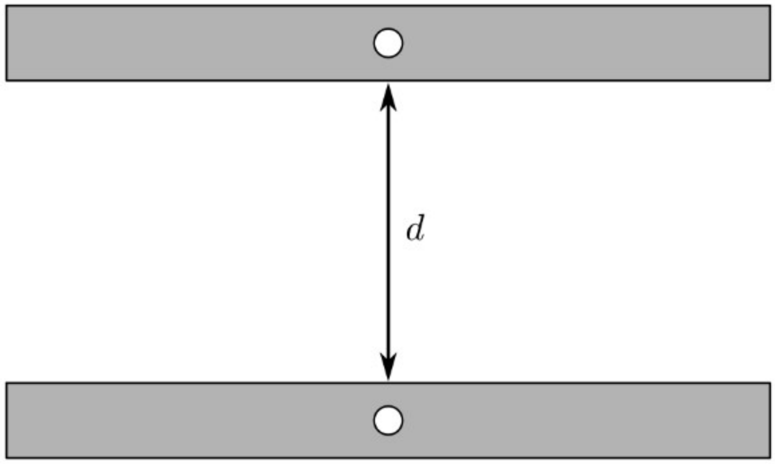

Figure 1: 'Infinite' parallel bars

The first configuration of conducting electrodes we study in this experiment is as shown above, in Figure 1. Let us assume that the bottom electrode is at a potential V_0, while the top electrode is **grounded** i.e. $V = 0$. Since this is a 2D configuration, we apply **Laplace's Equation** to this geometry to get -

$$\nabla^2 V = 0$$
$$\implies \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Since the electrodes are infinite in the horizontal (x) direction, V should be independent of x. Using this physical insight, we eliminate a variable x from our previous equation and convert the **partial differential** to a **complete differential** and get -

$$\frac{d^2 V}{dy^2} = 0$$

On integrating this once with respect to y, we get -

$$\frac{dV}{dy} = C_1$$

where C_1 is some constant value. To get an expression for V, we need to integrate this expression one more time. Since the potential is V_0 at the top electrode and 0 at the bottom, we measure the distance (y) from the bottom electrode. We then substitute the appropriate limits on the integrals, and integrate both sides to get -

$$\begin{aligned} dV &= C_1 dy \\ \int_0^{V(y)} dV &= \int_0^y C_1 dy \\ \implies V(y) - 0 &= C_1(y - 0) \\ \implies V(y) &= C_1 y \end{aligned}$$

To find the value of C_1, we use the fact that $V(d) = V_0$ and get $C_1 = V_0/d$. Thus, we have our final (theoretical) expression for **potential** V in this 'infinitely' long bar electrode configuration -

$$\boxed{V(y) = V_0\left(\frac{y}{d}\right)} \tag{7}$$

4.4 Concentric Cylindrical Capacitor -

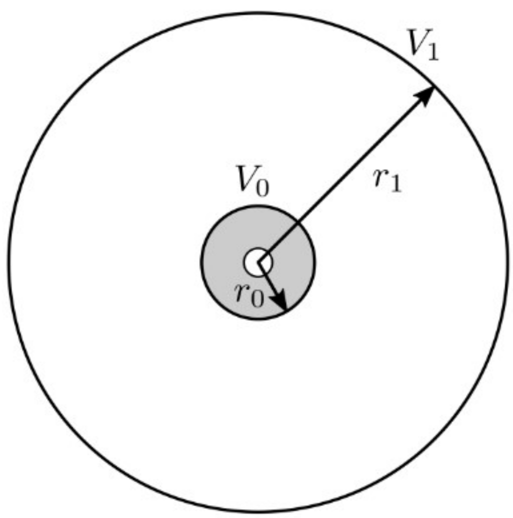

Figure 2: Concentric cylindrical electrodes

We follow a similar approach towards solving for an expression for V in this configuration, as we did in the previous configuration. Since this is a spherically symmetric 2D system, we write the Laplacian in **2D polar coordinates**. We use the following coordinate transformations -

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

We then differentiate the above relations twice and compare with the Cartesian expression for the Laplacian to get the equivalent expression for the Laplacian in 2D polar coordinates, which is as follows -

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2}\right) = 0$$

Since our system is symmetric for all angles θ (see Figure 2), we can say that -

$$\frac{\partial^2 V}{\partial \theta^2} = 0$$

Thus, the final form of our Laplace equation becomes -

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) &= 0 \\ \implies \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) &= 0 \end{aligned}$$

We now solve this differential equation by separating variables and assuming constants as follows (partial differential converts to complete differential since there is only one variable, r) -

$$\begin{aligned} \frac{d}{dr}\left(r\frac{dV}{dr}\right) &= 0 \\ \implies r\frac{dV}{dr} &= C_1 \\ \implies dV &= \frac{C_1}{r}dr \\ \implies \int_{V_0}^{V(r)} dV &= C_1 \int_{r_0}^{r} \frac{dr}{r} \\ \implies V(r) - V_0 &= C_1 \ln\left(\frac{r}{r_0}\right) \\ \implies V(r) &= C_1 \ln\left(\frac{r}{r_0}\right) + V_0 \end{aligned}$$

Thus, $V(r)$ follows the form as expressed in the lab handout. Here, note that we are considering $r_0 < r < r_1$. Now, to find the value of C_1, we use the boundary condition $V(r_1) = V_1$ and get -

$$\begin{aligned} V_1 &= C_1 \ln\left(\frac{r_1}{r_0}\right) + V_0 \\ \implies C_1 &= \frac{V_1 - V_0}{\ln\left(\frac{r_1}{r_0}\right)} \end{aligned}$$

Now, we substitute this value of C_1 back into our expression for $V(r)$ to get the final result for $V(r)$ as follows -

$$\boxed{V(r) = \frac{(V_1 - V_0)\ln\left(\frac{r}{r_0}\right)}{\ln\left(\frac{r_1}{r_0}\right)} + V_0} \tag{8}$$

We verify the predictions of Equations (7) and (8) through our experimental results.

5 Part A

5.1 Procedure

In this part we find the equipotential curves between two parallel bar electrodes.

1. Draw the standard $x-y$ coordinate system on a graph paper and fix it on a flat surface.
2. Place the (transparent) acrylic tray over this graph paper, and use the coordinate system as a reference to mark out positions.
3. Place the two long electrodes parallel to the x - axis symmetrically about the origin.
4. Fill the acrylic tray with water to approximately 75% of its height. This will act as an electrolyte.
5. Connect the electrodes to the AC power supply such that one electrode is at a potential V_0 and the other is grounded ($V = 0$).
6. Using the multimeter, test the potential difference between the electrodes, in volts. If it differs substantially from the voltage supplied by the power unit, then fix the circuit to minimize this difference.
7. Connect the positive wire of the multimeter to the probe and measure the approximate change in potential from one end of the tray to another.
8. Fix a few (well distributed) values of potentials between V_0 and 0 , corresponding to certain locations in the tray.
9. Find and record the coordinates of points where the voltages are equal to these values of voltages. Use the $x - y$ stage to move the probe to different locations.
10. Plot these points on a graph in Python, with points at the same voltage having the same colour. Plotting enough of these points will give us the equipotential curves.
11. Plot a $V(y)$ vs y/d graph to observe and experimentally verify the relation between these two quantities.

5.2 Observations

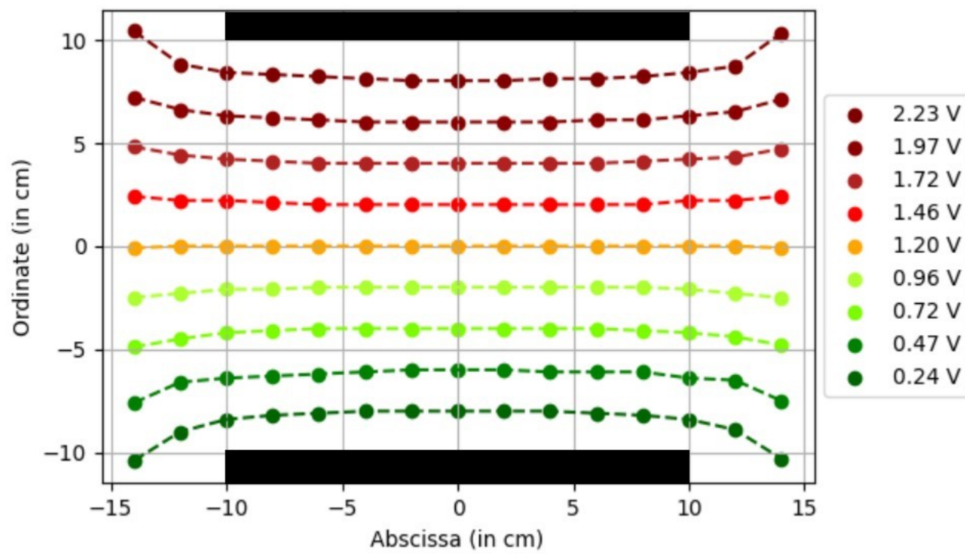

Figure 3: Equipotential surfaces for parallel bars

The **black bands** in the above figure represent the approximate position and size of the two **parallel electrodes**. The contour plots represent the recorded equipotential points, with a **heat-mapping** colour scheme used to signify the gradual movement from high to low potential. As seen in the above plot, the distance d between the two electrodes is **20 cm**. The input voltage was $\mathbf{2.62} \pm \mathbf{0.02}$ volts, noted from the voltmeter input readings.

From the **theoretically derived expression** for equipotential curves of this configuration (Equation 7), we expect them to be **parallel** to the electrodes. We can indeed observe and verify from the plotted data above that it follows the expected trend, at least in the middle portions. It seems to **curve** a little towards the edges. An explanation for this can be found in the discussion section of the report.

To verify Equation (7) directly, we can plot a graph of $V(y)$ vs y/d and observe the trend. Note that since measurements of y are taken from the origin at the center of the acrylic tray, we need to convert the negative values to positive before plotting them. Here, we also divide these y values by d to normalize them, so that the slope of this graph directly gives us the **input voltage** V_0. The table below shows us the **non-normalized** data -

Potential Difference V(y) in volts	Distance y from grounded electrode (cm)
0.24	2.00
0.47	4.00
0.72	6.00
0.96	8.00
1.20	10.00
1.46	12.00
1.72	14.00
1.97	16.00
2.23	18.00

Table 1: $V(y)$ data for parallel electrodes

On plotting the above data, we get the following graph -

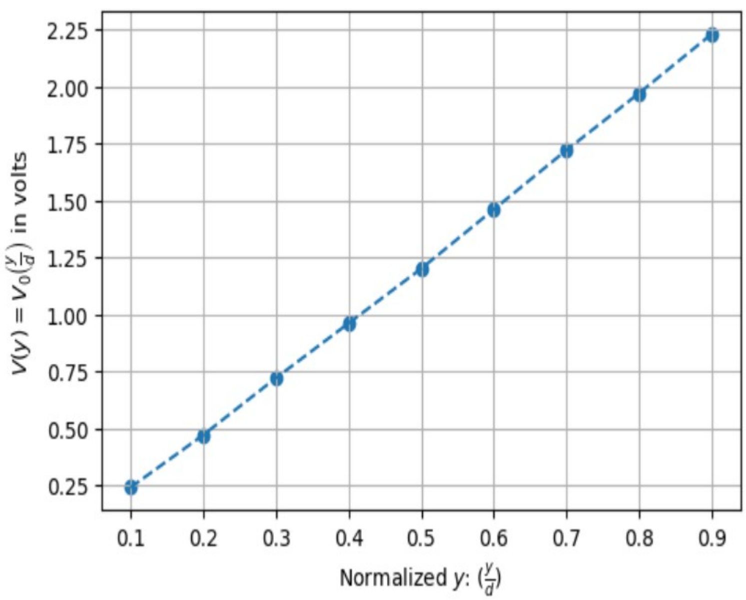

Figure 4: Linear trend of electric potential $V(y)$

The slope of this graph gives us the input voltage of the circuit, V_0. Using Python's inbuilt `np.polyfit()` function, the slope of the above graph was found to be **2.493 volts**. As mentioned below in the error analysis section, the **uncertainty** in measurement of $V(y)$ is equal to $\pm$**0.02 volts**. The uncertainty in measurement of y is approximately **5mm** (see **refraction uncertainty in discussion section**. The error bars, are thus too small to be seen clearly.

5.3 Error analysis

- Input voltage observed through multimeter $= 2.62 \pm 0.02$ volts

- Input voltage obtained from the slope of the graph = 2.49 volts

Thus, we can use the following expression to get the % error in the value of V_0 -

$$\begin{aligned}\% \text{ error} &= \frac{(2.62 \pm 0.02) - 2.49}{2.62 \pm 0.02} \times 100 \\ &= 4.19\% \text{ to } 5.68\%\end{aligned}$$

5.4 Results

Thus, we can see the linear variation of the potential with increasing distance from the grounded electrode.

6 Part B

In this part we try to find the equipotential curves between two concentric cylindrical electrodes.

6.1 Procedure

1. Replace the bar electrodes with two concentric cylindrical metal electrodes in the setup for Part A. Ensure that both of them are centered at the origin.

2. Set one of the cylinder at a higher potential V_0 and ground the other cylinder.

3. Choose increments of equal length in the radii of concentric circles while recording points of equal potential.

4. Follow the same procedure as done in Part A to plot the equipotential points for this configuration.

5. Plot a $V(r)$ vs $\ln(r/r_0)$ graph to observe and experimentally verify the relation between these two quantities.

6.2 Observations

The two cylindrical electrodes used in this part had their radii equal to **2cm** and **9cm** respectively. As we will see in the discussion however, they were not perfectly circular and slightly elongated along the x axis. Thus, we took the voltages at 5 equiradial points from the common center of the electrodes. The graph below shows the **schematic equipotential curve** (adjusted limits of x, y) along with the approximate positions and sizes of the cylindrical electrodes -

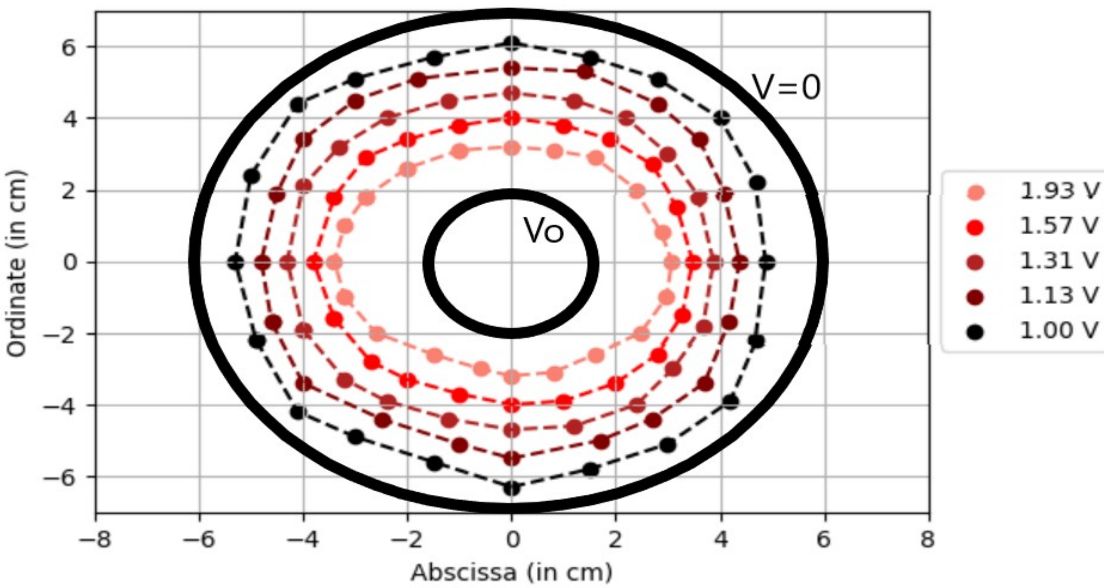

Figure 5: Schematic equipotential curves

The **actual distribution** of equipotential points are represented by the graph below -

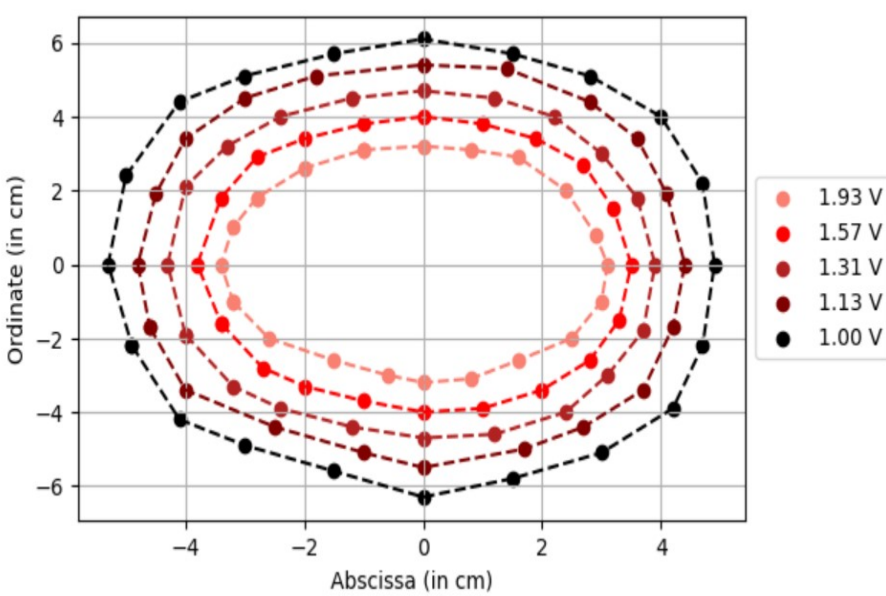

Figure 6: Actual equipotential curves

The slightly elliptic nature of the equipotential curves is a matter of contention. A possible justification for this observation is given in the discussion section.

The table below shows us the voltages obtained at various incremental radial distances from the center -

Potential difference V(y) in volts	Radial distance from origin (cm)
1.93	3.50
1.57	4.50
1.31	5.50
1.13	6.50
1.00	7.50

Table 2: $V(y)$ data for concentric cylindrical electrodes

On plotting the above data, the graph below shows us the **decreasing** nature of the potential difference for **constant increments in radial distance** -

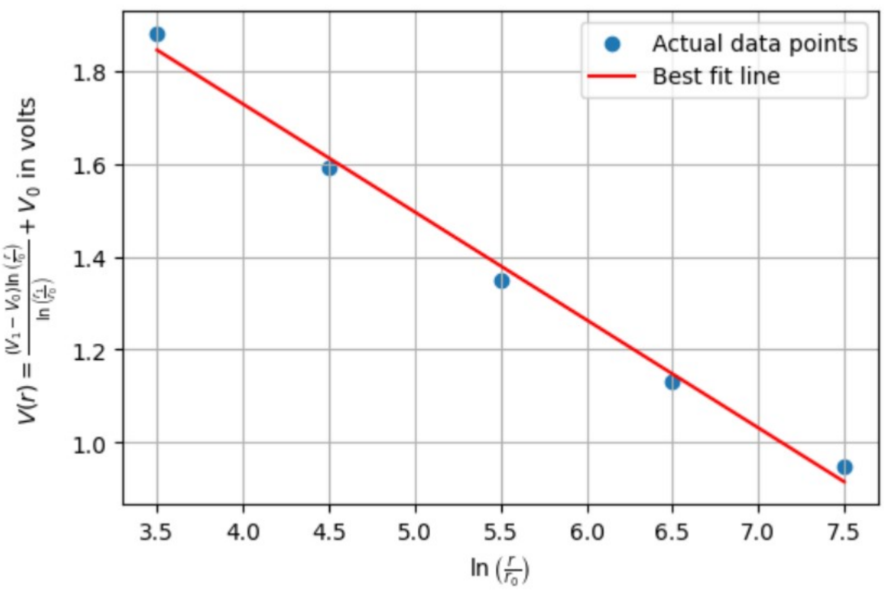

Figure 7: $V(r)$ vs $\ln \frac{r}{r_0}$

The **negative slope** of this graph shows that the magnitude of the drop in potential decreases as we go further away from the center, in equal increments (of 1 cm). We can get the **input voltage** V_0 from the y-intercept of this graph (not the slope), which comes out to be **2.653V** using the `np.polyfit()` function. The slight deviation from the linear trend can be explained by the irregularities in the shape of the electrodes, as elaborated in the discussion.

6.3 Error Analysis

- Input voltage observed through multimeter $= 2.62 \pm 0.02$ volts
- Input voltage obtained from the intercept of the graph $= 2.653$ volts

Thus, we can use the following expression to get the % error in the value of V_0 -

$$\begin{aligned}\% \text{ error} &= \frac{|(2.62 \pm 0.02) - 2.653|}{2.62 \pm 0.02} \times 100 \\ &\approx 1.14\%\end{aligned}$$

6.4 Results

Thus, we can see that for equal increments of radial distances in this configuration, the voltage drops as expected, and the magnitude of drop decreases as we go farther away from the center.

7 Part C

In this part we choose any configuration of electrodes and try to find the equipotential curves for that configuration.

7.1 Procedure

1. Replace the concentric cylindrical electrodes with two small cylindrical electrodes of equal radii, kept symmetrically about the origin and y axis.
2. Set one electrode at a higher potential V_0 and the other one at $V = 0$
3. Follow the same procedure as in Parts A and B to get the equipotential curves for this configuration.

7.2 Observations

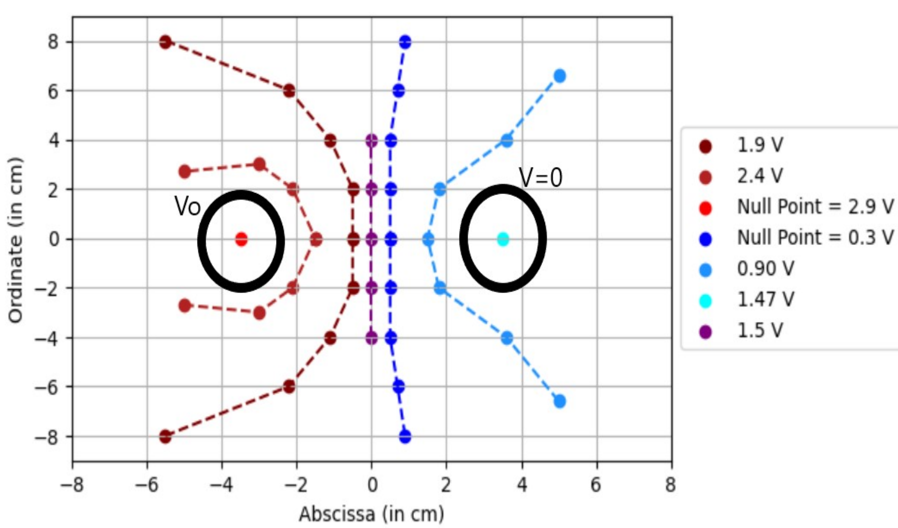

Figure 8: Equipotential surfaces for dipole configuration

The **black circles** on the above graph represent the approximate size and location of the cylindrical electrodes fixed in the acrylic tray. The left electrode was kept at a higher potential V_0, while the right one was grounded.
We can clearly see the **equipotential curves** traced out by the points for the above configuration. There is an interesting variation in potential observed at the "null points" present at the center of both electrodes.

7.3 Results

Thus, these experimentally obtained equipotential curves verify the theoretically expected nature of **equipotential curves** for an electric dipole (albeit with slight variation at the null points).

8 Precautions

1. Arrange the electrodes properly and keep them fixed. Use some tape to fixate their location onto the tray.

2. Ensure that none of the wires dip in the electrolyte solution (water) when taking the readings, as it may short the circuit.

3. Ensure that gloves are worn on both hands at all times when the AC power supply is on, while performing the experiment.

4. Make sure the probe's tip just grazes the surface of the water. It is better if it does not touch the surface of the tray, as that hampers the readings (see discussion point 4).

9 Discussion

1. **'Infinite' parallel electrode:** We can see that the equipotential curves bend a little towards the edges in Part A. This is because the electrodes used are not actually infinitely long in the x direction. They are of a finite length.

2. **Refraction error:** An important source of error identified in the recording of equipotential point locations was the bending of the apparent image of the probe where it touched the graph surface. Due to this refraction bending, the actual coordinates of the probe were slightly skewed from the recorded coordinates for some data. To minimize this error we tried to look at the probe from exactly above its tip, and recorded its apparent position for all data. This resulted in the same shift for all data, reducing the overall effect of the error in the inferences.

3. **Connector weight:** For the configurations of electrodes in Part B and C, we experienced a unique logistical problem while setting up the circuit. Due to the weight of the connector, the electrode tilts and topples over quite frequently. It is very challenging

to record correct data when this keeps happening every once in a while. To fix this we used tape and glued the lighter electrodes onto their locations.

4. **Touching the tray's surface:** An interesting observation was noted in the difference of voltage values recorded when the probe just touched the water, versus when it touched the surface of the acrylic tray. These two values of voltage varied significantly for some readings, and in such cases the reading for only water contact were considered and recorded.

5. **Inhomogeneity in Part B:** A possible attempt to justify the slighty oval-like elongated shape of the equipotential curves obtained in Part B can be made by observing the composition of the electrodes themselves. Instead of a single circular electrode, the actual electrode used in the experiment was made by joining two semi-circular electrodes at the top and bottom edges.The image below shows this joint -

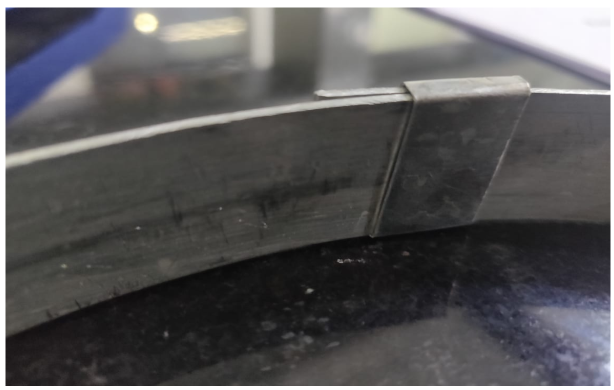

Figure 9: Inhomogeneity in cylindrical electrodes

This may be a reason why the electric fields and in turn the equipotential curves of the configuration got slightly squished in the vertical direction, giving them an elliptical character.

10 References

- Ashoka University, Undergraduate Physics Lab 2 Handout, Equipotential Curves.
- Introduction to Electrodynamics, David J. Griffiths.

11 Appendix

Please look at the attached code files to see all recorded data for graphs plotted in this experiment.