Assignment No. 1

Unit I : Linear Differencial Equations

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Q1.] $(D^2-3D+2)y = \frac{1}{e^{e^{-x}}} + \cos e^{-x}$.

→ We see e^{-x} many times, so, put
$e^{-x} = t$

∴ Differenciating,
$-e^{-x}\,dx = dt \Rightarrow -t\,dx = dt$

∴ $dx = \frac{dt}{-t}$ —①

Now,
$A.E. = D^2-3D+2 = 0$
$D^2-2D-D+2=0 \Rightarrow D(D-2)-1(D-2)=0$
∴ $D = 1, 2.$ → Real and distinct roots.
∴ $\boxed{y_c = c_1e^{x} + c_2e^{2x}}$ —②

Now,
$$y_p = \frac{1}{\phi(D)} f(x) = \frac{1}{D^2-3D+2}\left[e^{-e^{-x}} + \cos e^{-x}\right]$$
$$= \underbrace{\frac{1}{(D-2)(D-1)} e^{-e^{-x}}}_{y_{p1}} + \underbrace{\frac{1}{D^2-3D+2}\cos e^{-x}}_{y_{p2}}$$

∴ $y_{p1} = \frac{1}{(D-2)\underline{(D-1)}} e^{-e^{-x}}$

We know, $\frac{1}{D-m} X = e^{mx}\int e^{-mx} X\,dx$ —③

∴ $y_{p1} = \frac{1}{(D-2)} e^{x}\int e^{-x}\cdot e^{-e^{-x}}\,dx.$

Put $e^{-x} = t$, ∴ $dx = \frac{dt}{-t}$ ...(from ①).

$\therefore y_{p1} = \frac{1}{(D-2)} e^{x} \int t \cdot e^{-t} \cdot \frac{dt}{-t} = \frac{1}{(D-2)} e^{x} \int -e^{-t} dt$

$= \frac{1}{(D-2)} e^{x} (-1) e^{-t} = \frac{1}{(D-2)} e^{x} \cdot e^{-e^{-x}}$

Again using ③,

$y_{p1} = e^{2x} \int e^{-2x} \cdot e^{x} \cdot e^{-e^{-x}} dx$

$= e^{2x} \int e^{-x} \cdot e^{-e^{-x}} dx$

Put $e^{-x} = t$, $\therefore dx = dt/-t$ ...(from ①)

$\therefore y_{p1} = e^{2x} \int t \cdot e^{-t} \frac{dt}{-t} = e^{2x} \int -e^{-t} dt = e^{2x} [e^{-t}]$

$\therefore \boxed{y_{p1} = e^{2x} \cdot e^{-e^{-x}}}$

Now, $y_{p2} = \frac{1}{D^2 - 3D + 2} \cos e^{-x}$.

$y_c = c_1 e^{x} + c_2 e^{2x}$ —②

Put, $y_1 = e^{x}$, $y_2 = e^{2x}$.

$\therefore w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ ...(Wronskian) $\therefore f(x) = \cos e^{-x}$

$= \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$.

$\therefore w = e^{2x}$

$u = \int \frac{-y_2 \cdot f(x)}{w} dx$.

$= \int -\frac{e^{2x} \cdot \cos e^{-x}}{e^{3x}} dx = \int -e^{-x} \cos e^{-x} dx$.

Put $e^{-x} = t \Rightarrow dx = \frac{dt}{-t}$

$\therefore u = \int -t \cos t \frac{dt}{-t} = \int \cos t \, dt = \sin t = \sin e^{-x}$.

$u = \sin e^{-x}$.

Now, $v = \int \frac{y_1 \cdot f(x)}{\omega} dx$

$= \int \frac{e^{x} \cdot \cos e^{-x}}{e^{3x}} dx = \int e^{-2x} \cdot \cos e^{-x} \, dx$

Put $e^{-x} = t \Rightarrow dx = \frac{dt}{-t}$

$\therefore v = \int t^2 \cdot \cos t \frac{dt}{-t} = -\int t \cos t \, dt$

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$= -\left[t \sin t - \int 1 \cdot \sin t \, dt \right] = -[t \sin t - (-\cos t)]$

$\therefore v = -t \sin t - \cos t = -e^{-x} \sin e^{-x} - \cos e^{-x}.$

Now, $y_{p2} = u \cdot y_1 + v \cdot y_2$

$= [\sin e^{-x}] e^{x} + [-e^{-x} \sin e^{-x} - \cos e^{-x}] \cdot e^{2x}$

$= \cancel{e^{x} \sin e^{-x}} - \cancel{e^{x} \sin e^{-x}} - e^{2x} \cos e^{-x}.$

$\therefore \boxed{y_{p2} = -e^{2x} \cdot \cos e^{-x}}$

$\therefore y = y_c + y_p$ i.e. $y_c + y_{p1} + y_{p2}$

$\therefore y = c_1 e^{x} + c_2 e^{2x} + e^{2x} \cdot e^{-e^{-x}} + - e^{2x} \cdot \cos e^{-x}$

OR

$\boxed{y = c_1 e^{x} + e^{2x} [c_2 + e^{-e^{-x}} - \cos e^{-x}]}$

Q2] $(D^2 + 1) y = \tan x.$

$\rightarrow$ AE: $D^2 + 1 = 0$

$\therefore D = \pm i \rightarrow$ complex roots.

$\therefore \boxed{y_c = c_1 \cos x + c_2 \sin x}$

Now, $y_p = \frac{1}{\phi(D)} X = \frac{1}{D^2 + 1} \tan x.$

We know, $y_c = c_1 \cos x + c_2 \sin x$

Let, $y_1 = \cos x$, $y_2 = \sin x$.

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ ... (Wronskian) , $f(x) = \tan x$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$\therefore W = 1.$

Now, $u = \int \frac{-y_2 \cdot f(x)}{W} dx = \int - \frac{\sin x \cdot \tan x}{1} dx$

$= - \int \frac{\sin^2 x}{\cos x} dx = -\int \left(\frac{1 - \cos^2 x}{\cos x} \right) dx$

$= \int \cos x \, dx - \int \sec x \, dx = \sin x - \ln(\sec x + \tan x).$

$\therefore u = \sin x - \ln(\sec x + \tan x).$

Now, $v = \int \frac{y_1 \cdot f(x)}{W} dx = \int \frac{\cos x \cdot \tan x}{1} dx = \int \cos x \cdot \frac{\sin x}{\cos x} dx$

$\therefore v = -\cos x$

$\therefore y_p = u \cdot y_1 + v \cdot y_2$

$= [\sin x - \ln(\sec x + \tan x)] \cdot \cos x + [-\cos x] \sin x$

$= \sin x \cdot \cos x - \cos x \cdot \ln(\sec x + \tan x) - \sin x \cdot \cos x.$

$\therefore \boxed{y_p = -\cos x \cdot \ln(\sec x + \tan x)}$

$\therefore y = y_c + y_p$

$\boxed{\therefore y = c_1 \cos x + c_2 \sin x - \cos x \cdot \ln(\sec x + \tan x)}.$

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Q3.] $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x \cdot e^{2x} \cdot \sin 3x.$

→ i.e. $(D^2 - 4D + 4)y = x \cdot e^{2x} \cdot \sin 3x.$

$AE \Rightarrow (D-2)^2 = 0$

$\therefore D = 2, 2 \rightarrow$ Real repeated roots.

$\therefore \boxed{y_c = (c_1 + c_2 x) e^{2x}}$

Now, $y_p = \frac{1}{\phi(D)} X = \frac{1}{D^2 - 4D + 4} x \cdot e^{2x} \cdot \sin 3x.$

$= \frac{1}{D^2 - 4D + 4} e^{2x} \cdot (x \sin 3x).$

We know, $\frac{1}{\phi(D)} e^{ax} \cdot V = e^{ax} \frac{1}{\phi(D+a)} \cdot V$

$\therefore$ Replace $D \rightarrow D + a.$

$\therefore y_p = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} \cdot x \sin 3x = e^{2x} \frac{1}{D^2 + \cancel{4} + \cancel{4D} - \cancel{4D} \cancel{-8} + \cancel{4}} x \sin 3x.$

$\therefore y_p = e^{2x} \cdot \frac{1}{D^2} x \sin 3x$

We know, $\frac{1}{D}$ means to integrate.

$\therefore y_p = e^{2x} \frac{1}{D} \int \underset{D}{x} \cdot \underbrace{\sin 3x}_{I} \, dx$

$= e^{2x} \frac{1}{D} \left[-x \frac{\cos 3x}{3} + \int \frac{1 \cdot \cos 3x \, dx}{3} \right] = \frac{1}{D} \left[\frac{-x \cos 3x}{3} + \frac{\sin 3x}{9} \right]$

$= e^{2x} \left[\int -\overset{D}{x} \frac{\overset{I}{\cos 3x}}{3} dx + \int \frac{\sin 3x \, dx}{9} \right]$

$= e^{2x} \left\{ -\frac{1}{3} \left[\frac{x \sin 3x}{3} - \int \frac{1 \cdot \sin 3x}{3} \right] + \int \frac{\sin 3x}{9} \right\}$

$= e^{2x} \left\{ -\frac{1}{3} \left[\frac{x \sin 3x}{3} + \frac{\overset{\cos}{\cancel{\sin}} 3x}{9} \right] \cancel{+} \frac{-\cos 3x}{27} \right\}$

$\therefore \boxed{y_p = e^{2x} \left[\frac{-x \sin 3x}{9} - \frac{2 \cos 3x}{27} \right]}$

$\therefore y = y_c + y_p$

$$\therefore \boxed{y = (c_1 + c_2 x)e^{2x} - e^{2x}\left[\frac{x\sin 3x}{9} + \frac{2\cos 3x}{27}\right]}$$

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Q.4] $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$. — (1)

→ Let $(2x+3) = e^z$. — (2)

This eq^n of of Legendre's type. Put $\frac{d}{dz} \equiv D_1$, $\frac{d}{dy} \equiv D$.

∴ We know,

$(2x+3)^2 D^2 y = (2)^2 D_1 (D_1 - 1) y$.

and $(2x+3) D y = (2)^1 D_1 y$.

∴ Substituting in eq^n (1),

$4D_1(D_1-1)y - 2(2D_1)y - 12y = 6x$.

∴ $[4D_1^2 - 4D_1 - 4D_1 - 12] y = 6x$.

∴ $\boxed{[D_1^2 - 2D_1 - 3] y = \frac{3}{2} x. \text{ — (3) } = \frac{3}{2}\left(\frac{e^z - 3}{2}\right)}$ ...(from 2)

Solving eqⁿ (3):

AE: $D_1^2 - 2D_1 - 3 = 0 \Rightarrow (D_1+1)(D_1-3) = 0$

$\therefore D_1 = -1, 3 \rightarrow$ Real and distinct.

$$\therefore \boxed{y_c = c_1 e^{-z} + c_2 e^{3z}}$$

Now, $y_p = \frac{1}{\phi(D)} f(z) = \left[\frac{1}{D_1^2 - 2D_1 - 3}\right] \frac{3}{4}(e^{z} - 3)$.

$$= \frac{3}{4}\left[\frac{1}{D_1^2 - 3D_1 - 3} e^{z} - \frac{1}{D_1^2 - 2D_1 - 3} 3e^{0z}\right]$$

$$= \frac{3}{4}\left[\frac{e^{z}}{1-3-3} - \frac{1}{0-0-3} 3\right]$$

[Replacing $D_1 \rightarrow 1$, e^{1z}.] [Replacing $D_1 \rightarrow 0$, e^{0z}]

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$$y_p = \frac{3}{4}\left[\frac{e^{z}}{-4} + 1\right]$$

$$\therefore \boxed{y_p = \frac{3}{4}\left[\frac{4 - e^{z}}{4}\right]}$$

Finally, $y = y_c + y_p$

$$\therefore y = c_1 e^{-z} + c_2 e^{3z} + \frac{3}{4}\left[\frac{4 - e^{z}}{4}\right]$$

Resubstituting, $e^{z} = 2x + 3$.

$$\therefore y = \frac{c_1}{(2x+3)} + c_2(2x+3)^3 + \frac{3}{4}\left[\frac{4-(2x+3)}{4}\right]$$

$$\therefore \boxed{y = c_2(2x+3)^3 + c_1(2x+3)^{-1} + \frac{3}{16}[1-2x]}$$

Q5] $\dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dz}{4xy^2 - 2z}$.

→ This is of type symmetrical simultaneous D.E.

Solving by grouping:

Consider first and second ratio

$\dfrac{dx}{2x} = \dfrac{dy}{-y}$ → variable separable.

$\dfrac{dx}{2x} + \dfrac{dy}{y} = 0$

Integrating,

$\dfrac{\ln x}{2} + \ln y = c_1 \Rightarrow \ln x + 2\ln y = c_2 \Rightarrow \ln xy^2 = c_2$.

$\therefore$ $xy^2 = e^{c_2}$ i.e. $\boxed{xy^2 = c_3}$ → ①

Now, consider first and third ratio,

$\dfrac{dx}{2x} = \dfrac{dz}{4xy^2 - 2z} \Rightarrow \dfrac{dx}{2x} = \dfrac{dz}{4c_3 - 2z}$ ...(using ①).

$\therefore$ $\dfrac{dx}{2x} + \dfrac{dz}{2z - 4c_3} = 0$ → variable separable.

$\therefore \quad \frac{dx}{x} + \frac{dz}{z-2c_3} = 0$

Integrating,

$\ln x + \ln(z-2c_3) = c_4$

$\therefore \ln[(x)\cdot(z-2c_3)] = c_4.$

$\therefore \quad x(z-2c_3) = e^{c_4}$ or $= c_5.$

$\therefore \quad \boxed{x(z-2c_3) = c_5}$

$\therefore \quad xy^2 = c_3$ and $x(z-2c_3) = c_5$

i.e.

$$\boxed{xy^2 = c \text{ and } x(z-2c) = c_1 \text{ is the solution.}}$$

Q6] $\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0.$

→ We will have to solve these D.E. simultaneously.

In operator form $D \equiv \frac{d}{dt}$.

$\therefore \quad (D+5)x - 2y = t \quad \rightarrow (1)$

$\therefore \quad 2x + (D+1)y = 0 \quad \rightarrow (2)$

To solve for x, eliminate y,

i.e. $(D+1)\cdot(1) + 2\cdot(2)$ gives,

$(D+1)(D+5)x - \cancel{2(D+1)y} = (D+1)t$

$+ \quad 4x + \cancel{2(D+1)y} = 0$

$[(D+1)(D+5)+4]x = (D+1)t$

i.e. $(D^2+6D+9)x = 1+t$

↳ This is LDE with constant coefficients in x and t.

AE $\Rightarrow D^2+6D+9=0 \Rightarrow D = -3, -3 \rightarrow$ Real & repetitive.

$\therefore \boxed{x_c = (c_1 + c_2 t)e^{-3t}}$

Now,

$$x_p = \frac{1}{D^2+6D+9}(1+t) = \frac{1}{9}\left[\frac{1}{1+\left(\frac{D^2+6D}{9}\right)}\right](1+t).$$

$$= \frac{1}{9}\left[1-\left(\frac{D^2+6D}{9}\right)+\cdots\right](1+t).$$

$$= \frac{1}{9}\left[1-\frac{2D}{3}+\cdots\right](1+t).$$

$$= \frac{1}{9}\left[1+t-\frac{2}{3}\right] = \frac{t}{27}.$$

$$\therefore \boxed{x_p = \frac{1}{9}\left[t + \frac{1}{3}\right]}$$

To find y_p, $(D+5)x - 2y = t \rightarrow (1)$

$$\therefore y = \frac{1}{2}\left[\frac{dx}{dt} + 5x - t\right] \rightarrow (3)$$

But $x = x_c + x_p$.

$$x = (c_1 + c_2 t)e^{-3t} + \frac{1}{9}\left[t + \frac{1}{3}\right] \qquad \therefore d(uv) = uv' + vu'$$

$$\frac{dx}{dt} = -3c_1 e^{-3t} + c_2\left[e^{-3t} + t(-3)e^{-3t}\right] + \frac{1}{9}$$

$$\frac{dx}{dt} = e^{-3t}(-3c_1 + c_2 - 3t) + \frac{1}{9}$$

Replace in (3),

$$y = \frac{1}{2}\left[e^{-3t}(-3c_1 + c_2 - 3t) + \frac{1}{9} + (5c_1 + 5c_2 t)e^{-3t} + \frac{5t}{9} + \frac{5}{27} - t\right]$$

$$\therefore y = \frac{1}{2}\left[e^{-3t}(-3c_1 + c_2 - 3t + 5c_1 + 5c_2 t) + \left(\frac{5t}{9} - t\right) + \left(\frac{1}{9} + \frac{5}{27}\right)\right]$$

$$\therefore \boxed{y = \frac{1}{2}\left[2c_1 e^{-3t} + 2c_2 t e^{-3t} + c_2 e^{-3t} - \frac{4t}{9} + \frac{8}{27}\right]}$$

X