Mixing Properties of Stable Random Fields Indexed by Amenable Groups

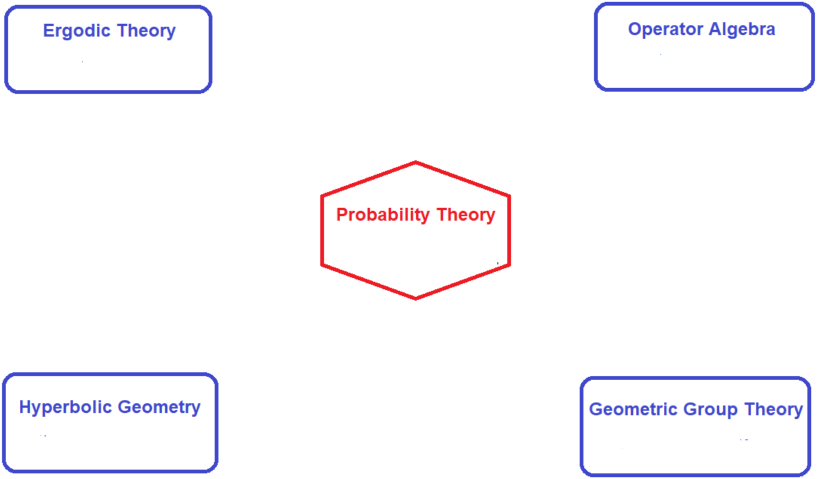
Parthanil Roy, Indian Statistical Institute

Joint work with **Mahan Mj (TIFR Mumbai)** and **Sourav Sarkar (University of Cambridge)**

arXiv:2205.15849

Motivation and Connections

What is this work about?



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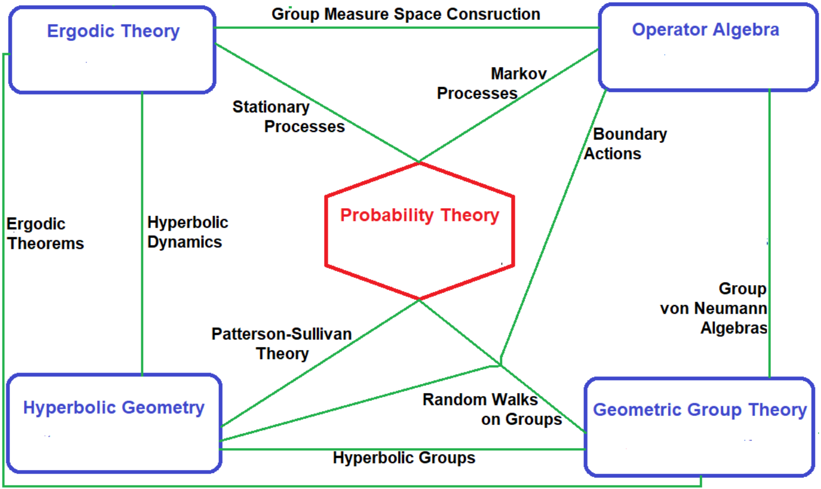

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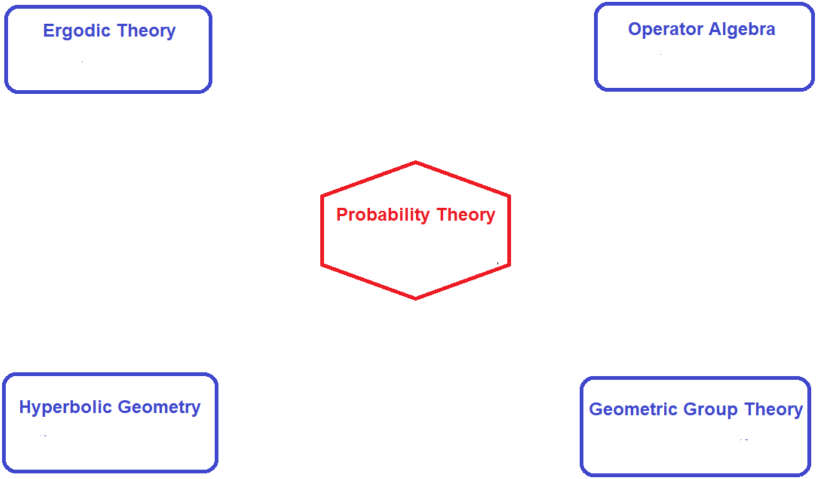

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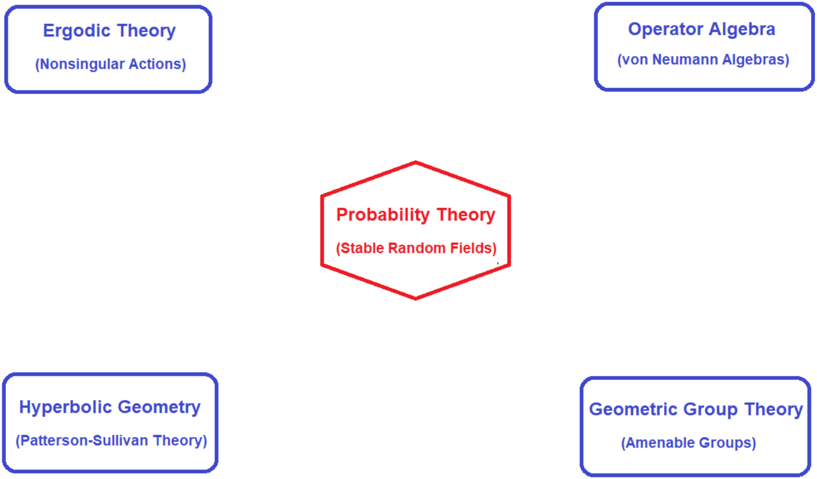

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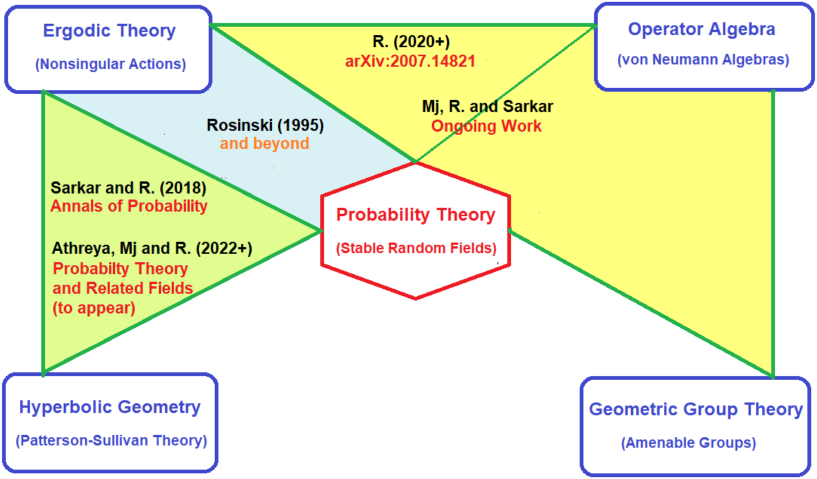

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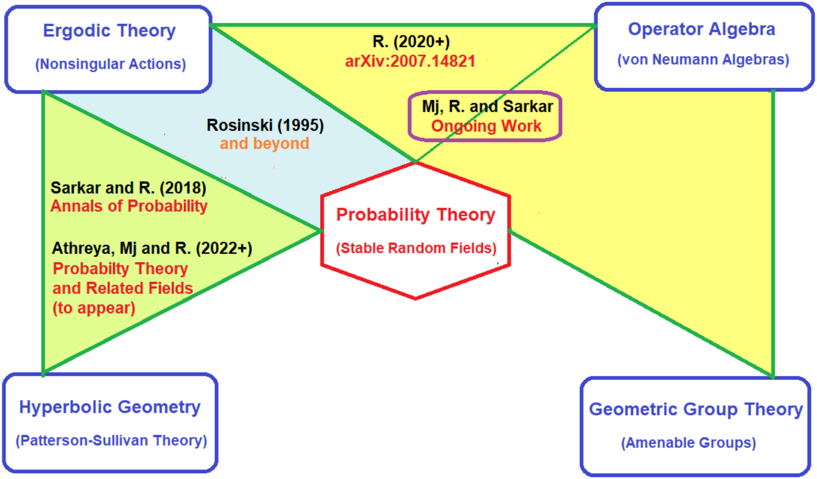

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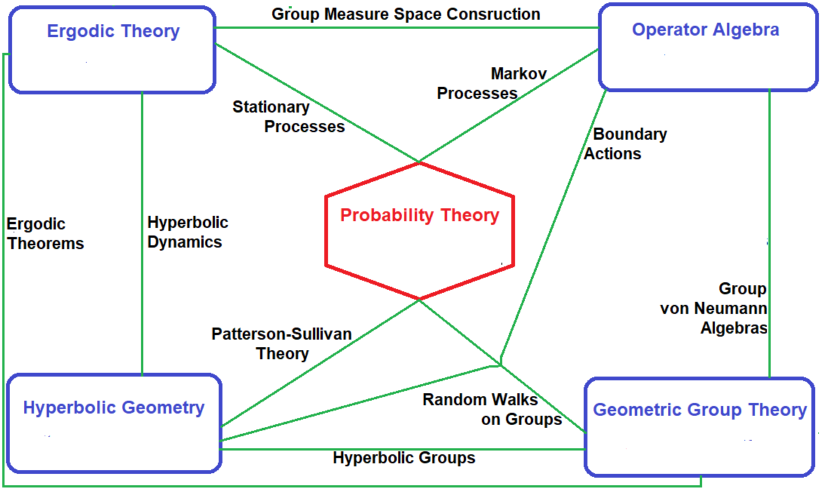

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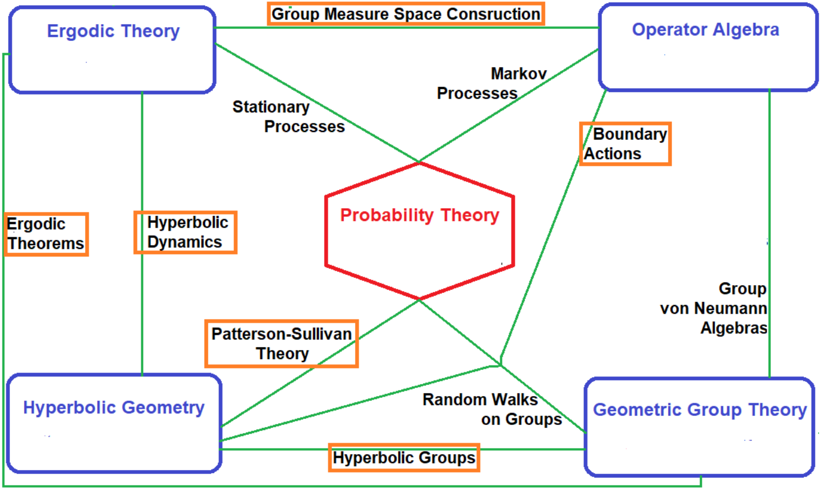

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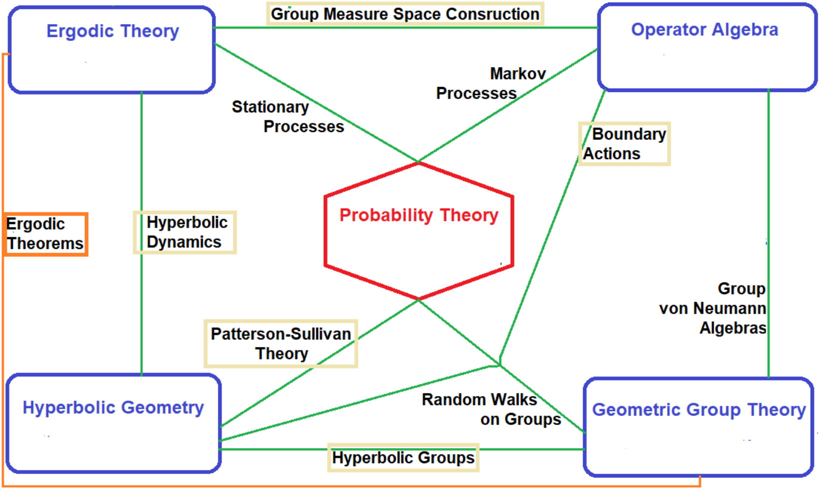

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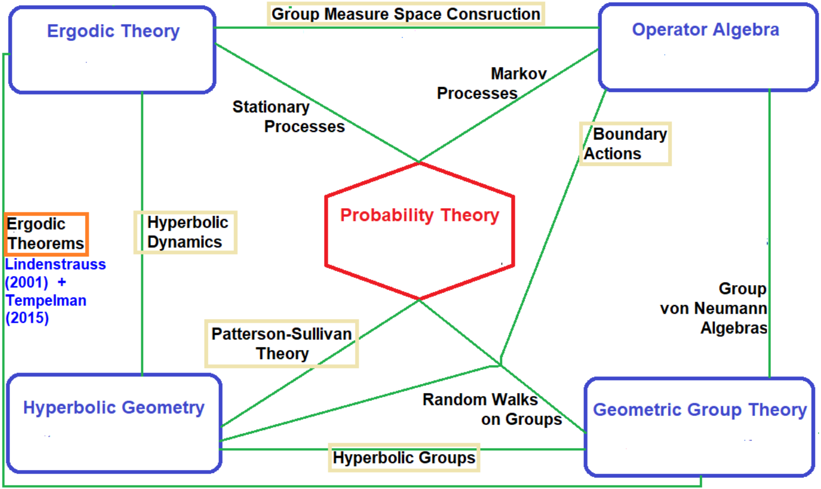

A Crash Course on Stable Random Fields

Symmetric α-stable distribution

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$\{f_g\}_{g \in G} :=$ a (spectral) representation of $\{X_g\}_{g \in G}$.

Questions or Comments or Concerns?

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Clearly, *invariant* $\Longrightarrow$ *quasi-invariant* but the converse is not true.

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The converse also holds, i.e., for any quasi-invariant G-action $\{\phi_g\}_{g \in G}$ on a σ-finite standard measure space $(\Lambda, \mathcal{A}, \mu)$ and any $f_e \in L^\alpha(\Lambda, \mu)$, if we define $\{f_g\}_{g \in G}$ by (2) $\left(\Longrightarrow \{f_g\}_{g \in G} \subset L^\alpha(\Lambda, \mu)\right)$, then there exists a stationary $S\alpha S$ random field $\{X_g\}_{g \in G}$ with representation $\{f_g\}_{g \in G}$, i.e., (1) holds.

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- Large deviations issues: Mikosch and Samorodnitsky (2000), Fasen and R. (2016)[SPA]
- Growth of maxima: Samorodnitsky (2004), R. and Samorodnitsky (2008)[JTP], Chakrabarty and R. (2013)[JTP], Owada and Samorodnitsky (2015), Athreya, Mj and R. (2022+)[PTRF]
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Broad Goal: Carry this link forward to the realms of *Geometric Group Theory* (amenable groups, hyperbolic groups, etc.) and *Operator Algebra* (von Neumann algebras).

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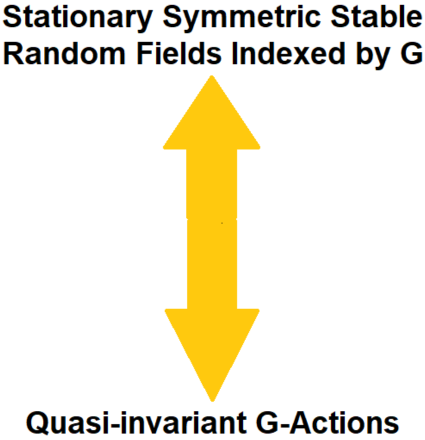

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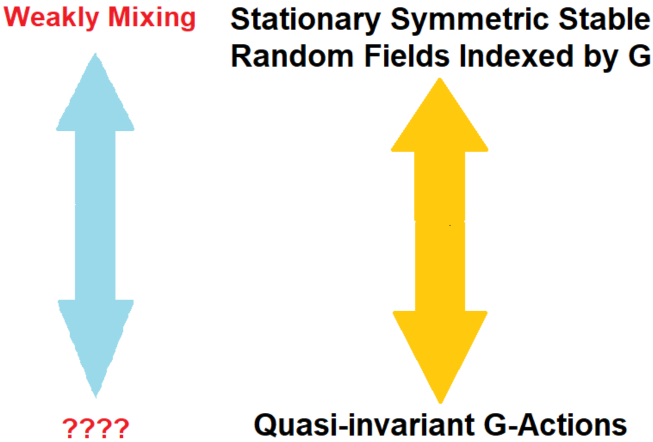
Weakly Mixing

Stationary Symmetric Stable Random Fields Indexed by G

Quasi-invariant G-Actions

Weak mixing and Rosinski representation



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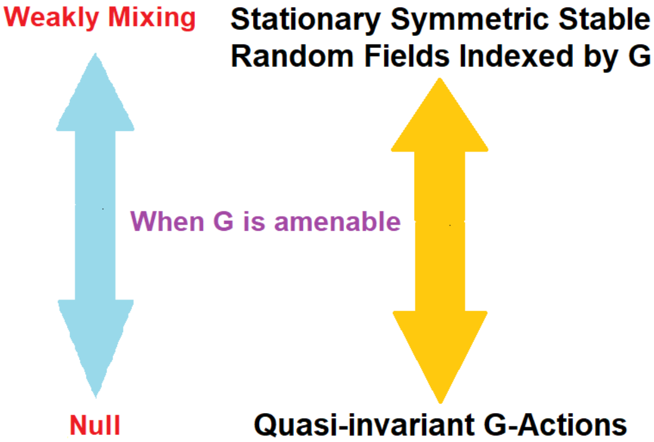

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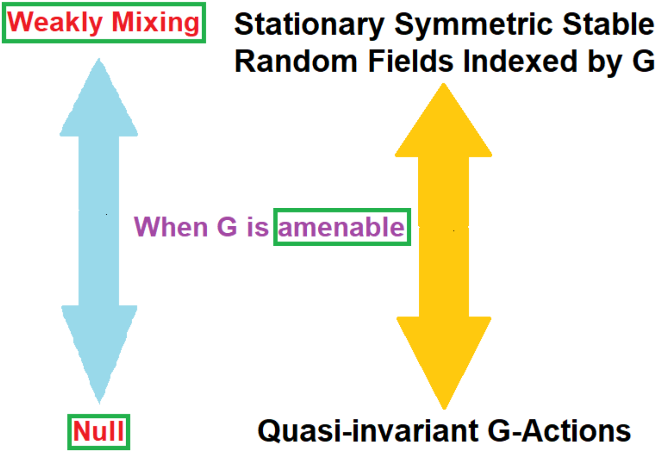

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Weak Mixing for Random Fields

The induced probability measure on $\mathbb{R}^G$

Recall that any (left) stationary SαS random field $\mathbf{X} = \{X_g\}_{g \in G}$ induces a measure-preserving (left) shift action (of G) on $\left(\mathbb{R}^G, \mathbb{P}_{\mathbf{X}}\right)$, where

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- When the underlying random field is weakly mixing, then we can apply ergodic theorem to investigate asymptotic properties of estimators/algorithms, large deviation issues, long run behaviour of solutions to SDEs, etc.

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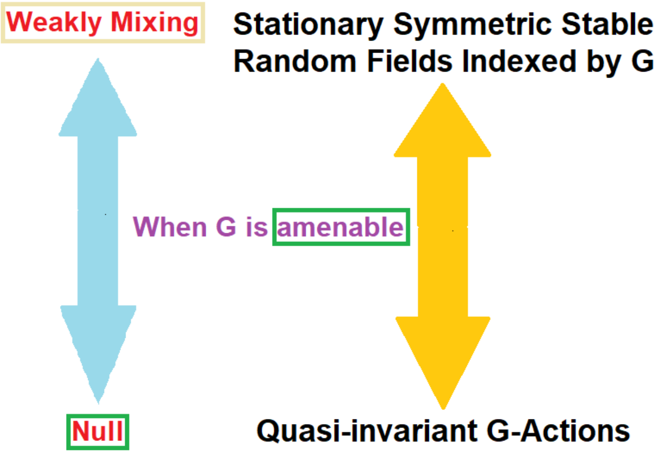
- $G = \mathbb{Z}^d$: Wang, R. and Stoev (2013) generalized the above result (for $d = 1$) to any $d \in \mathbb{N}$ using the work of Takahashi (1971).

Stable Random Fields Indexed by Amenable Groups

Weak mixing and Rosinski representation



Countable amenable groups

A countable amenable group is a countable group that admits an increasing Følner sequence $F_n \uparrow G$, i.e., an increasing sequence of exhausting finite subsets $F_n \subset G$ such that for all $g \in G$,

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2. Along any tempered Følner sequence, pointwise ergodic theorem holds for any finite measure preserving G-action.

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   - Avraham Re'em (2022+) [arXived a few days ago!]: used the machinery of absolutely continuous invariant probability measure + a Krengel-type criterion + Lindenstrauss (2001).

Questions or Comments or Concerns?

Recall: Rosinski (1995, 2000) representation

Given a stationary $S\alpha S$ $(0 < \alpha < 2)$ random field $\{X_g\}_{g \in G}$, there exist

(i) a σ-finite standard measure space $(\Lambda, \mathcal{A}, \mu)$,

(ii) a function $f_e : \Lambda \to \mathbb{R}$ such that $\|f_e\|_\alpha := \left(\int |f_e|^\alpha d\mu \right)^{1/\alpha} < \infty$, and

(iii) a quasi-invariant G-action $\{\phi_g\}_{g \in G}$ on $(\Lambda, \mathcal{A}, \mu)$

such that each real linear combination

$$\sum_{i=1}^{k} c_i X_{g_i} \sim S\alpha S \left(\left\| \sum_{i=1}^{k} c_i f_{g_i} \right\|_\alpha \right), \tag{3}$$

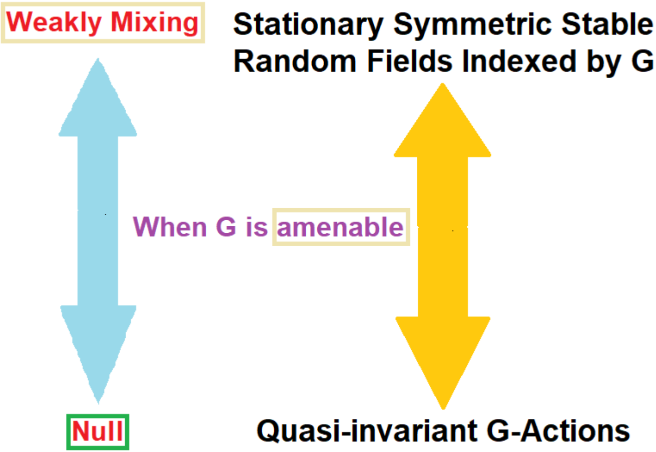
where

$$f_g = \pm \left(\frac{d\mu \circ \phi_g}{d\mu} \right)^{1/\alpha} f_e \circ \phi_g \,, \quad g \in G.$$

Converse also holds: given (i), (ii) and (iii), there exists a stationary $S\alpha S$ random field $\{X_g\}_{g \in G}$ satisfying (3).

$\{f_g\}_{g \in G} =$ **a** Rosinski representation of $\{X_g\}_{g \in G}$.

Weak mixing and Rosinski representation



Null quasi-invariant actions

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This *weaker dependence / asymptotic independence* may manifest itself in the form of weak mixing.

We have been able to prove this formally only when G is amenable. We believe that our result may be true for a much bigger class of groups and/or actions.

Weakly mixing random fields indexed by amenable groups

Let G be a countably infinite amenable group and $\{F_n\}_{n\geq 1}$ is any increasing Følner sequence.

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Dye (1965) [Cor 1, pg 129] or Bergelson and Gorodnik (2004) [Thm 1.6]: A random field $\{X_g\}_{g\in G}$ is weakly mixing if and only if for all $k \geq 1$, for all $g_1, g_2, \ldots, g_k, h \in G$ and for all Borel $A, B \subset \mathbb{R}^k$,

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Choose the Følner sequence to be increasing and tempered according to Lindenstrauss (2001).

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$$\frac{1}{|F_n|}\sum_{h\in F_n}\Big|P\big[(X_{hg_1}, X_{hg_2}, \ldots, X_{hg_k}) \in A,\ (X_{g_1}, X_{g_2}, \ldots, X_{g_k}) \in B\big] - P\big[(X_{g_1}, X_{g_2}, \ldots, X_{g_k}) \in A\big]\, P\big[(X_{g_1}, X_{g_2}, \ldots, X_{g_k}) \in B\big]\Big| \to 0$$

as $n \to \infty$.

Choose the Følner sequence to be increasing and tempered according to Lindenstrauss (2001). These will give us analytic and ergodic theoretic advantages, respectively.

One of our main results

Theorem (Mj, R. and Sarkar (2022+))

Supoose G is a countably infinite amenable group and $\mathbf{X} := \{X_g\}_{g \in G}$ is a left stationary symmetric α-stable $(0 < \alpha < 2)$ random field generated by a quasi-invariant G-action $\{\phi_g\}_{g \in G}$ in its Rosinski representation. Then $\mathbf{X}$ is weakly mixing if and only if $\{\phi_g\}_{g \in G}$ is a null action.

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- There are many solvable groups (e.g., suitable discrete subgroups of Lie groups) that admit many natural null actions.

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- We apply this result + a truncation argument to establish our result for any amenable group G provided the underlying action is (σ-finite) measure preserving.
- Generalization from the measure preserving to the quasi-invariant case is carried out using Maharam (1964) skew-product.

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$$G \curvearrowright \big(\Lambda \times (0,\infty), \mu \otimes Leb\big) \quad \text{by} \quad g^{-1}.(x,y) = \left(\phi_g(x), \frac{y}{\omega_g(x)}\right).$$

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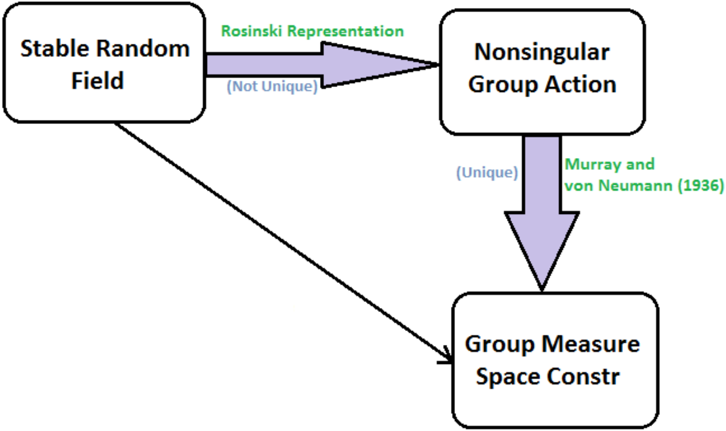
  - How about groups having Kazhdan's property (T)?

Thank You Very Much

Stable fields and von Neumann Algebras



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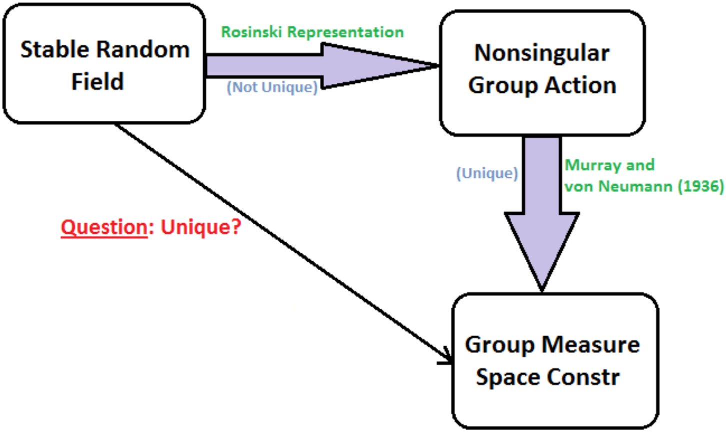

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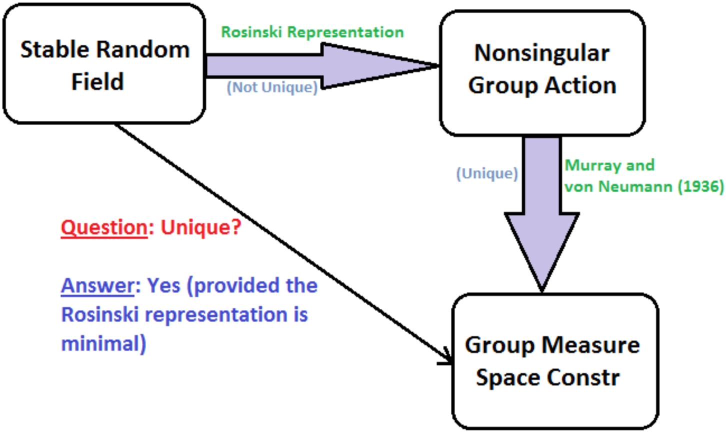

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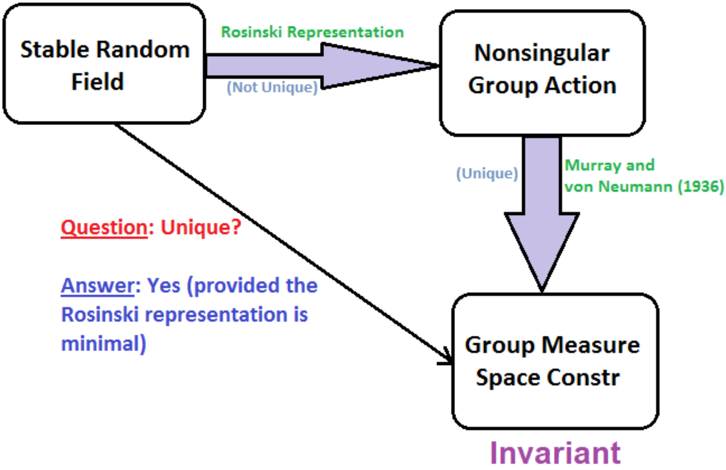

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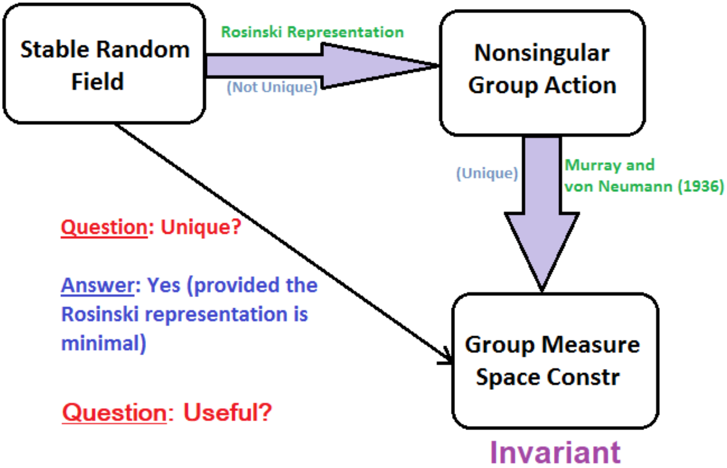

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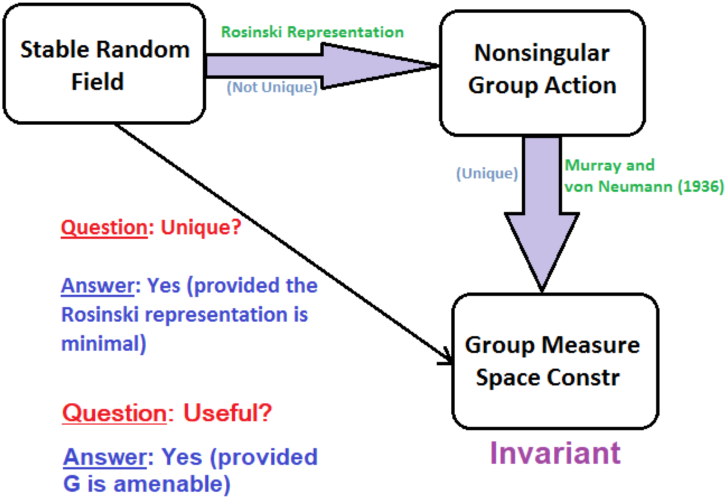

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Theorem (R. (2020+) for $G = \mathbb{Z}^d$, Mj, R. and Sarkar (2022+))

Weak mixing is a W^*-rigid property for stationary $S\alpha S$ random fields indexed by amenable groups.

Connection to orbit equivalence

Corollary (R. (2020+) for $G = \mathbb{Z}^d$, Mj, R. and Sarkar (2022+))

If two stationary $S\alpha S$ random fields indexed by (possibly two different) amenable groups are generated by orbit equivalent free quasi-invariant actions, then one is weak mixing if and only if the other one is so.

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Connetions to hyperbolic geometry

Suppose a discrete non-elementary hyperbolic group (after fixing the basepoint to be e)

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$$F_d \curvearrowright \big(\Lambda = \partial F_d \times \partial F_d \setminus Diagonal,\ \mu = \text{ a suitable (σ-finite) measure}\big).$$

Group Measure Space Construction

Koopman representation

G-action $\{\phi_t\}$ lifts to the space of all real-valued measurable functions on S by

$$\sigma_t g = g \circ \phi_t, \;\; t \in G.$$

This lifted action preserves the $\mathcal{L}^\infty$-norm but not other $\mathcal{L}^p$-norms.

However, for each $t \in G$, $\pi_t : \mathcal{L}^2(S, \mu) \to \mathcal{L}^2(S, \mu)$ given by

$$(\pi_t g)(s) = g \circ \phi_t(s) \left(\frac{d\mu \circ \phi_t}{d\mu}(s) \right)^{1/2}, \; s \in S$$

defines an isometry. The unitary representation $\{\pi_t\}_{t \in G}$ of G inside $\mathcal{L}^2(S, \mu)$ is called the Koopman representation.

The crossed product relation

Using the cocycle relationship

$$\frac{d\mu \circ \phi_{uv}}{d\mu} = \frac{d\mu \circ \phi_u}{d\mu} \, \sigma_u \left(\frac{d\mu \circ \phi_v}{d\mu} \right), \quad u, v \in G,$$

one gets that for all $a \in \mathcal{L}^\infty(S, \mu)$ (thought of as acting on $\mathcal{L}^2(S, \mu)$ by multiplication), for all $t \in G$ and for all $g \in \mathcal{L}^2(S, \mu)$,

$$(\pi_t \, a \, \pi_{t^{-1}} g)(s) = ((\sigma_t a) g)(s), \quad s \in S. \tag{5}$$

In other words, the Koopman representation "normalizes" $\mathcal{L}^\infty(S, \mu)$ inside $\mathcal{B}(\mathcal{L}^2(S, \mu))$. The group measure space construction is a space, where the crossed product relation (5) is internalized.

Consider the von Neumann algebra

$$\mathcal{B}(l^2(G) \otimes \mathcal{L}^2(S,\mu)) = \overline{\mathcal{B}(l^2(G)) \otimes \mathcal{B}(\mathcal{L}^2(S,\mu))}$$

(with the closure being taken with respect to the weak/strong operator topology). Define a representation of G by $t \mapsto u_t := \lambda_t \otimes \pi_t$, where $\{\lambda_t\}$ is the left regular representation and $\{\pi_t\}$ is the Koopman representation. We also represent $\mathcal{L}^\infty(S,\mu)$ by $a \mapsto 1 \otimes \mathcal{M}_a$, where $\mathcal{M}_a$ is the multiplication (by a) operator on $\mathcal{L}^2(S,\mu)$. It can be checked that the following "internal" crossed product relation holds:

$$u_t(1 \otimes \mathcal{M}_a)u_{t^{-1}} = 1 \otimes \mathcal{M}_{\sigma_t a}\,.$$

Define the *group measure space construction* (also known as *crossed product construction*) as

$$\mathcal{L}^\infty(S,\mu) \rtimes G := \{u_t, 1 \otimes \mathcal{M}_a : t \in G,\ a \in \mathcal{L}^\infty(S,\mu)\}''.$$

Connections to ergodic theory

It can be shown that the internal crossed product relation implies that any $x \in \mathcal{L}^\infty(S,\mu) \rtimes G$ can be uniquely written as $x = \sum_{t\in G} a_t u_t$ with $\{a_t : t \in G\} \subseteq \mathcal{L}^\infty(S,\mu)$. Thus, we can view x as a $|G| \times |G|$ matrix with entries coming from $\mathcal{L}^\infty(S,\mu)$ that are the same along each left group-diagonal; see, e.g, Jones (2009).

Theorem (see, e.g, Peterson (2013))

The following results hold for a nonsingular G-action $\{\phi_t\}$ and the corresponding group measure space construction defined above.

1. *If the action $\{\phi_t\}_{t\in G}$ is free and ergodic, then $\mathcal{L}^\infty(S,\mu) \rtimes G$ is a factor.*
2. *If $\mathcal{L}^\infty(S,\mu) \rtimes G$ is a factor, then $\{\phi_t\}_{t\in G}$ is ergodic.*
3. *If $\{\phi_t\}_{t\in G}$ is free and ergodic, then the factor $\mathcal{L}^\infty(S,\mu) \rtimes G$ is of type II_1 if and only if $\{\phi_t\}_{t\in G}$ is a positive action.*

Furthermore, if the two nonsingular actions (not necessarily of the same group) are orbit-equivalent, then the corresponding group measure space constructions are isomorphic as von Neumann algebras

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