Level-raising for automorphic forms on GL_n

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Abstract

We prove a level-raising result for regular algebraic conjugate self-dual automorphic representations of $GL_n(\mathbb{A}_E)$, where E is a CM number field, generalizing previous results of Thorne [T].

Keywords. Automorphic forms, Raising the level, Unitary Shimura varieties

1 Introduction

In [T], Thorne establishes instances of a conjectural 'Ihara lemma' of [CHT] for unitary groups of rank higher than 2, thereby simultaneously proving a level-raising result for regular algebraic, conjugate self-dual automorphic representations π of $GL_n(\mathbb{A}_E)$, where E is a CM number field. This generalizes classical level-raising results of Ribet [Ri] in the context of elliptic modular forms on SL_2. Thorne uses crucially the Shimura varieties arising from unitary similitude groups for establishing his result. Use of these varieties results in some additional hypotheses on the class of fields E appearing in that work. Inspired by his approach, our aim is to generalize his results for a larger class of cases by removing these hypotheses, by using certain locally symmetric spaces that arise from true unitary groups (as opposed to unitary similitude groups), and proving p-adic uniformization results for such models. Then we apply these results to the problem of raising the level for automorphic forms.

We briefly describe the main ideas of our work. Fix an integer $n > 1$. Let E be an imaginary CM number field. By definition, E is an imaginary quadratic extension of a totally real subfield, say F with $[F : \mathbb{Q}] = d$. Let π be a regular algebraic conjugate self-dual cuspidal representation of $GL_n(\mathbb{A}_E)$. If I is a definite unitary group in n variables

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associated to the extension E/F, then it is known that π can be 'transferred' to an automorphic representation Π of $I(\mathbb{A}_F)$. Such a representation of $I(\mathbb{A}_F)$ can be described in terms of spaces of algebraic modular forms, which admit integral structures. In [T], this representation is shown to further correspond to a representation $\tilde{\Pi}$ on $\tilde{I}$, which is defined as a group of unitary similitudes related to I. Let $\tilde{G}$ be an inner form of $\tilde{I}$ which differs from $\tilde{I}$ only at a fixed prime p and ∞. $\tilde{G}$ has type $U(n-1,1) \times U(n)^{d-1}$ at infinity, and it looks locally like the unit group of a division algebra at p. We note that inner forms obtained by such interchange of invariants appear in the case of p-adic uniformizations of Shimura curves, studied first by Cerednik [Ce]. Drinfeld [Dr] later found a natural proof of Cerednik's result using moduli interpretations. He was also able to prove existence of a semistable integral model for the case when base field is $\mathbb{Q}_p$. Rapoport and Zink [RZ] generalized these results for Shimura varieties wherever such a moduli interpretation was possible. In particular, their results apply to the groups of unitary similitudes $\tilde{I}$ and $\tilde{G}$. These results were used by Thorne to relate the space of algebraic automorphic forms on $\tilde{I}$ to the cohomology of Shimura variety arising from $\tilde{G}$. Unfortunately, passage from GL_n to $\tilde{I}$ forces some extra hypotheses on the number field E at the beginning. For instance, E is required to have the form $E = E_0.F$, where E_0 is an imaginary quadratic field. Since not every CM field E arises this way, removal of this hypothesis is worth pursuing. Similarly, p is required to be inert in F, but it seems plausible that a level-raising theorem can be proved when p is unramified in F.

In this work, we remove these hypotheses. The first ingredient in this work is to prove p-adic uniformization results for some locally symmetric spaces arising from the "true" unitary subgroup $G \subset \tilde{G}$, which is defined as the kernel of the similitude character on $\tilde{G}$. (Such groups in principle correspond to the group I.) We note that there is no moduli problem associated to these groups, hence we cannot follow the path taken by Rapoport and Zink. Instead we rely on results of Varshavsky [Va1], [Va2], who proved cases of p-adic uniformizations for unitary Shimura varieties by studying their complex uniformization using differential geometry. We first show existence of a canonical model $S(G,U)$ for the locally symmetric spaces over a field in which p is unramified. This is necessary for the existence of p-adic uniformization. We then prove a p-adic uniformization result for the varieties $S(G,U)$ in the spirit of Varshavsky's work.

Then we turn our attention to studying integral structures appearing on spaces of algebraic automorphic forms on these unitary groups. We define certain automorphic local systems $V_{\boldsymbol{\mu},\mathscr{O}}$ on the space $S(G,U)$. The link between the cohomology of these local systems and the spaces of algebraic automorphic forms $\mathcal{A}(U, W_{\boldsymbol{\mu},k})$ is given by the weight spectral sequence, and is explored by Thorne in his paper [T]. The E_1 page of this spectral sequence can be written in terms of algebraic automorphic forms $\mathcal{A}(U, W_{\boldsymbol{\mu},k})$ on the definite unitary group I. Let ℓ be the prime modulo which we seek a level-raising congruence at a place ν of F split in E such that $I(F_\nu) \cong GL_n(F_\nu)$. Then the E_2 page of the weight spectral sequence is related to the cohomology groups of the $O_F[1/\nu]$-arithmetic subgroups of $I(F_\nu)$. Furthermore, this spectral sequence degenerates when ℓ is a banal characteristic for $GL_n(F_\nu)$. (See [Vi] for information on banal characteristic.) This follows in particular from the Galois equivariance of the differentials in the spectral sequence. This allows us to prove, combined with ℓ-torsion-vanishing results of [LS] (or [Sh] with some further local hypotheses), that the cohomology of the arithmetic subgroups are ℓ-torsion-free, which in turn suffices to prove level-raising results for algebraic automorphic forms on I using the level-raising formalism of [T]. This in turn allows us to transfer to GL_n and prove a level-raising result for $GL_n(\mathbb{A}_E)$. We note that this in particular removes the hypothesis of p being inert in F from theorem 1.1 of [T].

We now describe the structure of this paper. In section 3, we define the unitary similitude group $\tilde{G}$ over $\mathbb{Q}$ and its unitary subgroup G and recall the result of Rapoport-Zink by way of motivation. In section 4, we define the locally symmetric spaces associated to G and use descent from the Shimura varieties for $\tilde{G}$ to describe a model for the locally symmetric spaces over a number field that is unramified at p. We then describe p-adic uniformizations of these models. In section 5, we define some ℓ-adic automorphic local systems on these spaces. In section 6, we define spaces of automorphic forms $\mathcal{A}(U, M)$ and describe the Hecke algebra action upon these spaces and the cohomology of local systems. In section 7, we recall the definition of the weight spectral sequence. In section 8, we describe the degeneration of this sequence in characteristic ℓ under the banal hypothesis to prove a level-raising result for the group I. In section 9, we apply this result to deduce level-raising for GL_n.

2 Unitary similitude Shimura varieties and p-adic uniformization

The material in this section is largely taken from [RZ]. In this section, we define the unitary and unitary similitude groups $(G, \tilde{G})$ precisely and state a p-adic uniformization result of Rapoport and Zink for the groups $\tilde{G}$. This will serve as motivation for the next section, although we emphasize that we do not use Rapoport-Zink's result or method of proof in the sections following this.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $n \in \mathbb{Z}_{>1}$. Let E be a CM imaginary field with totally real subfield F, $[F : \mathbb{Q}] = d$. We assume d is even. Let D be a central division algebra of dimension n^2 over E. Let $*$ be a positive involution of the second kind on D so that the invariants of $*$ on E is the subfield F. Let V be a left D-module and ψ an alternating pairing $\psi : V \times V \to \mathbb{Q}$ satisfying

$$\psi(dv, w) = \psi(v, d^* w) \tag{2.1}$$

for all $d \in D, v, w \in V$.

We define an algebraic group $\tilde{G}$ over $\mathbb{Q}$ by its functor of points :

$$\tilde{G}(R) := \{g \in GL_D(V \otimes R) \mid \psi(gv, gw) = c(g)\psi(v, w), c(g) \in R^\times\}. \tag{2.2}$$

where R is a $\mathbb{Q}$-algebra. In particular, $c = c(g)$ is the similitude character of $\tilde{G}$.

$\tilde{G}_{\mathbb{R}}$ can be embedded into a product of unitary similitude groups by making choice of a CM-type $\Phi \subset Hom(E, \mathbb{C})$ as follows. We can choose an isomorphism

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \Phi} D \otimes_{E,\tau} \mathbb{C} \cong \prod_{\tau \in \Phi} M_n(\mathbb{C})$$

such that $*$ corresponds to the operation $X \to \overline{X^t}$.

This decomposition induces an orthogonal decomposition with respect to ψ on

$$V \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \Phi} V \otimes_{E,\tau} \mathbb{C}.$$

There are isomorphisms $V \otimes_{E,\tau} \mathbb{C} \cong \mathbb{C}^n \otimes_{\mathbb{C}} W_\tau$ where $M_n(\mathbb{C})$ acts on the first factor. Then we can find a skew-hermitian form h_τ on W_τ so that

$$\psi_\tau(z_1 \otimes w_1 . z_2 \otimes w_2) = tr_{\mathbb{C}/\mathbb{R}}(\overline{z_1}^t z_2 h_\tau(w_1, w_2)). \tag{2.3}$$

We can find a basis of W_τ such that h_τ is represented by the matrix

$$diag(-i, -i, \cdots, -i, i, i, \cdots, i). \tag{2.4}$$

We denote the number of times $-i$ appears in this matrix by r_τ and the number of times i appears by r_{τ^c}. It can be seen that $r_\tau + r_{\tau^c} = (1/n) dim_E V$. Thus, the choices we have made so far imply that

$$\tilde{G}_{\mathbb{R}} \hookrightarrow \prod_{\tau \in \Phi} GU(r_\tau, r_{\tau^c}) \tag{2.5}$$

such that $\tilde{G}_{\mathbb{R}}$ is a normal subgroup with torus cokernel.

Under the above identifications, we write $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \tilde{G}_{\mathbb{R}}$ for the homomorphism

$$h : z \in \mathbb{C}^\times \to (diag(z, z, \cdots, z, \bar{z}, \bar{z}, \cdots, \bar{z}))_{\tau \in \Phi} \tag{2.6}$$

where again the number of times z appears is r_τ and the number of times $\bar{z}$ appears is r_{τ^c}. Let X denote the $\tilde{G}(\mathbb{R})$-conjugacy class of h inside the set of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \tilde{G}_{\mathbb{R}}$.

We define another algebraic group G over $\mathbb{Q}$ by its functor of points :

$$G(R) := \{g \in GL_D(V \otimes R) \mid \psi(gv, gw) = \psi(v, w)\}. \tag{2.7}$$

where R is a $\mathbb{Q}$-algebra. There is an exact sequence of algebraic groups

$$1 \to G \to \tilde{G} \xrightarrow{c} \mathbb{G}_m \to 1. \tag{2.8}$$

We let G' denote the derived subgroup of G. Note that it is also the derived subgroup of $\tilde{G}$. It is a subgroup of G consisting of matrices of determinant 1. It is a form of SL_n and in particular, simply connected. There is an exact sequence of algebraic groups

$$1 \to G' \to G \to (E^\times)^{N=1} \to 1 \tag{2.9}$$

where $(E^\times)^{N=1}$ is the torus of elements of $E^\times$ with norm 1 in F. We let $\tilde{T} := \tilde{G}/G'$ and $T' := G/G'$. Note that $\tilde{T}$ and T are the largest commutative quotients of the respective groups. We denote the natural map $\tilde{G} \to \tilde{T}$ by ν. Let $\tilde{Z}, Z$ denote the center of $\tilde{G}, G$ respectively.

All the definitions have been global so far. Hereon they will acquire a local flavor for the purpose of application to automorphic forms. We fix a rational prime p. Only in this section, we suppose that p is inert in F. Let ν be the unique prime of F over p. We assume that ν splits in E as $\omega\omega^c$. We fix embeddings ϕ_∞, ϕ_p of $\overline{\mathbb{Q}}$ into $\mathbb{C}, \overline{\mathbb{Q}}_p$ respectively. This choice induces a bijection of sets $Hom(E, \mathbb{C}) \leftrightarrow Hom(E, \overline{\mathbb{Q}}_p)$.

We assume that invariants of D at the places ω and ω^c are given respectively by $1/n$ and $-1/n$. At every other place of F, D is split. We take V to be a free module over D of rank 1.

We suppose that Φ corresponds to the set of embeddings inducing the p-adic place ω of E. Let $\tau_1, \cdots, \tau_d$ be the elements of Φ. We assume that $r_{\tau_1} = 1$ and $r_{\tau_i} = 0$ for $i = 2, \cdots, d$. Note that $r_{\tau_j} + r_{\tau_j^c} = n$ for all j by our hypothesis on V. We also assume that G is quasi-split at every finite place not dividing p. PEL data satisfying these assumptions exist, in particular if d is even, which we assume, cf. [HT].

We pause to note that under these assumptions

$$G_\mathbb{R} \cong \prod_{i=1}^{d} U(r_{\tau_i}, r_{\tau_i^c}) \cong U(1, n-1) \times U(0, n)^{d-1} \tag{2.10}$$

as shown in [HT]. It follows from the same reference that $\tilde{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times D_\omega^\times$ and thus, $G(\mathbb{Q}_p) \cong (D_\omega)^\times$.

With these assumptions, we have the following result, primarily due to Kottwitz [Ko2] following the theory of canonical models:

Proposition 2.1. *The pair $(\tilde{G}, X)$ is a Shimura datum. For $\tilde{U} \subset \tilde{G}(\mathbb{A}^\infty)$ a neat open compact subgroup, the Shimura varieties $S(\tilde{G}, \tilde{U})$ with $S(\tilde{G}, \tilde{U})(\mathbb{C}) := \tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A}^f) \times X/\tilde{U}$ are smooth projective algebraic varieties over $\mathbb{C}$ and admit canonical models over $\tau_1(F) \subset \mathbb{C}$.*

The varieties $S(\tilde{G}, \tilde{U})$ admit p-adic uniformizations. We describe them briefly. We refer the reader to [RZ] and [T] for more details.

Let K be a finite extension of $\mathbb{Q}_p$. We write Ω_{O_K} for the Drinfeld p-adic upper half plane over O_K. It is a p-adic formal scheme, formally locally of finite type over Spf O_K. There is a faithful action of $PGL_n(K)$ on Ω_{O_K}. Following Thorne [T], we define a p-adic formal scheme $\mathscr{M}^{\text{split}}$ by the formula

$$\mathscr{M}^{\text{split}} := \Omega_{O_K} \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times GL_n(K)/GL_n(K)^0 \tag{2.11}$$

where $GL_n(K)^0 \subset GL_n(K)$ is the open subgroup of matrices with determinant a p-adic unit and the sets on the right hand side are identified with the corresponding constant formal schemes over O_K. We define $\mathscr{M} := \mathscr{M}^{\text{split}}\hat{\otimes}_{O_K} O_{\mathbb{K}}$ where $\mathbb{K}$ denotes the completion of a maximal unramified extension of O_K.

Let $\nu = \phi_p\phi_\infty^{-1}$ be the embedding of $\tau_1(F)$ into $\overline{\mathbb{Q}}_p$. There exists an inner form $\tilde{I}$ of $\tilde{G}$ over $\mathbb{Q}$ such that

$$\tilde{G}(\mathbb{A}^{p,\infty}) \cong \tilde{I}(\mathbb{A}^{p,\infty}),$$
$$\tilde{I}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times GL_n(F_\nu).$$

Let $\mathbb{F}$ be the completion of the maximal unramified extension of F_ν. Then the group $\tilde{I}(\mathbb{Q})$ acts on $\Omega_{O_{F_\nu}}\hat{\otimes}O_{\mathbb{F}}$ through the map $\tilde{I}(\mathbb{Q}) \subset \tilde{I}(\mathbb{Q}_p) \to PGL_n(F_\nu)$ where the latter group acts on the Drinfeld p-adic upper half plane through its natural action. It also acts on $\tilde{G}(\mathbb{A}^\infty)/\tilde{U}_p$ where $\tilde{U}_p \subset \tilde{G}(\mathbb{Q}_p)$ is the unique maximal compact subgroup. We describe this action below.

We have

$$\tilde{G}(\mathbb{A}^\infty)/\tilde{U}_p \cong \tilde{G}(\mathbb{A}^{p,\infty}) \times \tilde{G}(\mathbb{Q}_p)/\tilde{U}_p \cong \tilde{I}(\mathbb{A}^{p,\infty}) \times \tilde{G}(\mathbb{Q}_p)/\tilde{U}_p.$$

$\tilde{I}(\mathbb{Q})$ acts diagonally on this product. It acts naturally on $\tilde{I}(\mathbb{A}^{p,\infty})$ and it acts on $\tilde{G}(\mathbb{Q}_p)/\tilde{U}_p$ through $\tilde{I}(\mathbb{Q}_p)$ as follows, cf. [RZ]. Let $(c, a) \in \tilde{I}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times GL_n(F_\nu)$ and $(c', a') \in \tilde{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times D_\omega^\times$. Let Π denote a uniformizer in $D_\omega^\times$. Then

$$(c, a).(c', a') \bmod \tilde{U}_p = (cc', \Pi^{\mathrm{val}_{F_\nu} \det a} a') \bmod \tilde{U}_p$$

where val_{F_ν} is normalized so that $\mathrm{val}_{F_\nu}(F_\nu^\times) = \mathbb{Z}$.

We say that an open compact subgroup $\tilde{U}^p$ of $\tilde{G}(\mathbb{A}^{p,\infty}) \cong \tilde{I}(\mathbb{A}^{p,\infty})$ is sufficiently small if there exists a prime $q \neq p$ such that the projection of $\tilde{U}^p$ to $\tilde{G}(\mathbb{Q}_q)$ contains no non-trivial elements of finite order.

With this notation, The following follows from Corollary 6.51 from [RZ].

Theorem 2.2. *(p-adic uniformization) For each sufficiently small open compact subgroup $\tilde{U}^p \subset \tilde{G}(\mathbb{A}^{p,\infty})$ there is an integral model of $S(\tilde{G}, \tilde{U}^p\tilde{U}_p) \otimes_{\tau_1(F),\nu} F_\nu$ over O_{F_ν} and a canonical isomorphism of formal schemes over Spf $O_{\mathbb{F}}$:*

$$\tilde{I}(\mathbb{Q})\backslash[\mathscr{M} \times \tilde{G}(\mathbb{A}^{p,\infty})/\tilde{U}^p \cong (S(\tilde{G}, \tilde{U}^p\tilde{U}_p) \otimes_{O_{F_\nu}} O_{\mathbb{F}})^\wedge.$$

This isomorphism is equivariant for the action of prime-to-p Hecke algebra which acts on both sides.

3 Canonical models for unitary Shimura varieties and their p-adic uniformization

3.1 Canonical models

In this section, we prove existence of a canonical model for locally symmetric spaces arising from G. From now on, we assume that p is unramified in F.

For $\tilde{U} = \tilde{U}^p \tilde{U}_p$ as above, we set $U := G(\mathbb{A}^f) \cap \tilde{U}$. Note that $U_p = O_{D_\omega}^\times$. We define

$$S(G,U)(\mathbb{C}) := G(\mathbb{Q}) \backslash G(\mathbb{A}^f) \times X/U. \tag{3.1}$$

This is a complex variety.

There is a natural map ϕ from $S(G,U)(\mathbb{C})$ to $S(\tilde{G},\tilde{U})(\mathbb{C})$ induced from the inclusion $i : G \hookrightarrow \tilde{G}$. We show in the following lemma that it is injective under a mild hypothesis inspired by a similar hypothesis in Carayol's work [Ca].

Lemma 3.1. *Let c be the similitude character as before. Assume that $\mathbb{G}_m(\mathbb{Q}) \cap c(\tilde{U}) = \{1\}$. Then*

$$\phi : S(G,U)(\mathbb{C}) \to S(\tilde{G},\tilde{U})(\mathbb{C})$$

sending a double coset representative $\overline{(g,x)}$ to $\overline{(g,x)}$ is injective.

Proof. Suppose $(g,x),(h,y) \in G(\mathbb{A}^f) \times X$ are such that their image in $S(\tilde{G},\tilde{U})(\mathbb{C})$ coincides. By definition, this implies that there exist $a \in \tilde{G}(\mathbb{Q}), u \in \tilde{U}$ such that $a.(g,x).u = (h,y)$. Hence, it follows that

$$agu = h \text{ and } ax = y.$$

If we prove that, in fact $a \in G(\mathbb{Q})$ and $u \in U$, then injectivity of ϕ follows. Recalling the exact sequence (2.8), we see that

$$c(agu) = c(au) = c(h) = 1.$$

Thus,

$$c(a) = c(u^{-1}).$$

By our assumption on the image of U under c, this forces

$$c(a) = c(u) = 1$$

which completes the proof. □

Since $\mathbb{G}_m(\mathbb{Q}) \cap c(\tilde{U})$ is a congruence subgroup of $\mathbb{Z}^\times$ by definition, we see that the hypothesis we introduce is not very strong. Indeed, a congruence subgroup of $\mathbb{Z}^\times$ can be trivial or the group $\{1, -1\}$. So that if $\tilde{U}$ is sufficiently small, then this intersection should be trivial.

We wish to identify $S(G, U)(\mathbb{C})$ with its image under ϕ. This is the content of our next proposition.

Proposition 3.2. $\phi(S(G, U)(\mathbb{C}))$ *is a union of connected components of the space* $S(\tilde{G}, \tilde{U})(\mathbb{C})$ *and* ϕ *is an isomorphism onto its image.*

Proof. This can be seen by writing each space of double cosets as a finite disjoint union of quotients of X by arithmetic subgroups indexed by the set of double coset representatives. (*cf.* Milne [M].)

Alternatively, from the definition of ϕ, it can be seen that it is a local homeomorphism and any bijective local homeomorphism is in fact a homeomorphism. $\square$

This identification of connected components can be further explored. There is a zero-dimensional Shimura variety associated with the data of $\tilde{T}$, the maximal commutative quotient of $\tilde{G}$, which can be identified with the set of connected components of $S(\tilde{G}, \tilde{U})$. We describe it briefly. Recall that Z is the center of G. There is a surjective homomorphism $Z \hookrightarrow \tilde{G} \xrightarrow{\nu} \tilde{T}$. We define

$$\tilde{T}(\mathbb{R})^\dagger := \mathrm{Im}(Z(\mathbb{R}) \to \tilde{T}(\mathbb{R})),$$

$$\tilde{T}(\mathbb{Q})^\dagger := \tilde{T}(\mathbb{Q}) \cap \tilde{T}(\mathbb{R})^\dagger.$$

Then, let

$$\tilde{Y} := \tilde{T}(\mathbb{R})/\tilde{T}(\mathbb{R})^\dagger.$$

$\tilde{Y}$ is a finite set. For any compact open set $\tilde{K} \subset \tilde{T}(\mathbb{A}^f)$, we can define a zero-dimensional Shimura variety

$$S(\tilde{T}, \tilde{K}) := \tilde{T}(\mathbb{Q})\backslash\tilde{T}(\mathbb{A}^f) \times \tilde{Y}/\tilde{K}.$$

It is known that

$$\pi_0(S(\tilde{G}, \tilde{U})) \cong S(\tilde{T}, \nu(\tilde{U})). \tag{3.2}$$

Similarly, we define

$$S(T, K) := T(\mathbb{Q})\backslash T(\mathbb{A}^f) \times Y/K. \tag{3.3}$$

where $T(\mathbb{Q})^\dagger$ and Y are defined in a similar fashion. We show that $S(T, \nu(U)$, like its (G, U)-counterpart, can be understood as a subgroup of $S(\tilde{T}, \nu(\tilde{U}))$.

Lemma 3.3. *Under the assumption $\mathbb{G}_m(\mathbb{Q}) \cap c(U) = \{1\}$, the natural map*

$$\theta : S(T, \nu(U)) \to S(\tilde{T}, \nu(\tilde{U}))$$

is injective.

Proof. Note that since G' is contained in the kernel of the similitude character c, c is well-defined at the level of T. The proof follows the same path we took for Lemma 1. □

Since $\tilde{T}(\mathbb{Q})$ is dense in $\tilde{T}(\mathbb{R})$, $Y \cong \tilde{T}(\mathbb{Q})/\tilde{T}(\mathbb{Q})^\dagger$. Hence, the space $\tilde{T}(\mathbb{Q})\backslash\tilde{T}(\mathbb{A}^f) \times \tilde{Y}/\tilde{K}$ can be identified with the set $\tilde{T}(\mathbb{Q})^\dagger\backslash\tilde{T}(\mathbb{A}^f)/\tilde{K}$. There is a natural map

$$\eta : \tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A}^f) \times X/\tilde{U} \to \tilde{T}(\mathbb{Q})^\dagger\backslash\tilde{T}(\mathbb{A}^f)/\nu(\tilde{U}) \tag{3.4}$$

given by $(g, x) \to \nu(g)$. There is a similar map η induced by ν from $S(G, U)(\mathbb{C})$ to $S(T, \nu(U))$. Since the maps ϕ and θ are injective, we can identify $S(G, U)(\mathbb{C})$ with a union of the connected components of $S(\tilde{G}, \tilde{U})(\mathbb{C})$ that map to $S(T, \nu(U))$ under η. We collect this information in the proposition below.

Proposition 3.4. *The diagram*

$$\begin{array}{ccc} S(G,U)(\mathbb{C}) & \hookrightarrow & S(\tilde{G},\tilde{U})(\mathbb{C}) \\ \downarrow & & \downarrow \\ S(T,\nu(U)) & \hookrightarrow & S(\tilde{T},\nu(\tilde{U})) \end{array} \tag{3.5}$$

is Cartesian.

Proof. Indeed this is the content of the preceding discussion. □

This observation about component groups is crucial. $S(\tilde{G}, \tilde{U})$ have canonical models defined over F, as can be seen for instance by the theory in [De]. We wish to show the existence of canonical models for $S(G, U)(\mathbb{C})$ over a field in which p is unramified. We do so below using the above proposition.

Theorem 3.5. *For each U satisfying our hypotheses, there exists a canonical model $S(G, U)$ of the complex variety $S(G, U)(\mathbb{C})$ over L where L is a finite extension of F unramified at p.*

Proof. We use the fact that $S(\tilde{G}, \tilde{U})$ already has a canonical model over F, *cf.* Theorem 1. Milne [M] describes the explicit action on $S(\tilde{T}, \nu(\tilde{U}))$ of $Aut(\mathbb{C}/E(\tilde{G}, X)$ where $E(G, X)$ is the reflex field. In this case, $E(G, X) = F$. Since $S(\tilde{T}, \nu(\tilde{U}))/S(T, \nu(U))$ is a finite abelian group, this action factors through a finite abelian quotient of $Aut(\mathbb{C}/F)$ which implies the existence of a finite abelian extension L of F over which there exists a model for $S(G, U)(\mathbb{C})$. In particular, since the level subgroup U is $O_{D_\omega}^\times$ at p, we conclude that p is unramified in L. □

3.2 p-adic uniformization

In this section, we describe p-adic uniformization of the canonical model $S(G,U)$ defined in Theorem 3.5. Following Varshavsky [Va1], [Va2], we set some notation. Since Varshavsky works with groups defined over F instead of those defined over $\mathbb{Q}$, the notation in this section is slightly different. He also works with a 'twist' of the Shimura datum that we use following Rapoport-Zink. It will be clear that the differences arise only as a difference of style as opposed to content. In particular, our locally symmetric spaces will be uniformized by the corresponding twist of the Drinfeld upper half plane.

Let $(D,*)$ be a central simple algebra over E of dimension n^2 with an involution of the second kind $*$ over F. Then one can define $\mathbf{G^{int}} := \mathbf{GU}(D,*)$, an algebraic group over F of unitary similitudes, as a functor of points -

$$\mathbf{G^{int}}(R) = \{d \in (D \otimes_F R)^\times | d.d^* \in R^\times\}$$

for every F-algebra R. Then $\tilde{G} := Res_{F/\mathbb{Q}}\mathbf{G^{int}}$, where $Res_{F/\mathbb{Q}}$ denotes the Weil restriction from F to $\mathbb{Q}$. $\tilde{G}$ is denoted by $\mathbf{H^{int}}$ in [Va2, Section 2]. The homomorphism $h : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbf{H^{int}}(\mathbb{R})$ is defined there by letting for each $z \in \mathbb{C}^\times$,

$$h(z) = (diag(1,\cdots,1,z/\bar{z})^{-1}; I_n; \cdots; I_n)$$

as, in our case, we have chosen the signature $(n-1,1)$ at only one embedding τ_1 of E in $\mathbb{C}$. Let M^{int} be the conjugacy class associated to h. It follows by comparing this definition with the definition of our conjugacy class X, that appears in the previous section that our uniformizations will differ by the corresponding twist, and that it suffices to prove a uniformization result for either of the conjugacy classes. (See remark 3.13 of *loc. cit.*)

Let $\tilde{\mathbf{X}}^{\mathbf{int}}$ be the canonical model associated to the Shimura variety associated to the Shimura datum $(\mathbf{H^{int}}, \mathbf{M^{int}})$ defined over the reflex field F. For the unitary group G as defined by us, we have an embedding of algebraic groups $G \hookrightarrow \tilde{G}$ and as seen in previous sections, there exists a canonical model for the locally symmetric spaces $S(G,U)$ defined over an extension of F, say L, in which p is unramified. Let ν be a prime of L lying above p such that its restriction to F, ν_1, splits in E as $\nu_1 = \omega\omega^c$ and ω, ω^c are the only places where the division algebra D is ramified with invariants $1/n$ and $-1/n$. Then we set

$$X^{int} := \tilde{\mathbf{X}}^{\mathbf{int}} \otimes F_{\nu_1},$$

and

$$S := (\varprojlim_U S(G,U)) \otimes_L L_\nu.$$

We show that S admits a p-adic uniformization.

For this purpose, we describe an inner form of $\tilde{G}$ obtained by switching primes. This is given using a central division algebra over E and an involution of second kind $(D', **)$

where the pair $(D, *)$ and $(D', **)$ are locally isomorphic at all finite places of F except at ν_1, where D' splits and $**$ is positive definite at all archimedian places of F. Then we define $\tilde{\mathbf{I}} := \mathbf{GU}(D', **)$. We also fix a central simple algebra $\tilde{D}_\omega$ over E_ω of invariant $1/n$. Then we define

$$\mathcal{I}' := F_{\nu_1}^\times \times \mathbf{I}(\mathbb{A}_F^{f;\nu_1})$$

and

$$\tilde{\mathcal{G}} := \tilde{D_\omega}^\times \times \mathcal{I}'.$$

Then we have isomorphisms

$$\tilde{\mathbf{I}}(\mathbb{A}_F^f) \cong GL_n(E_\omega) \times \mathcal{I}'$$

and

$$\mathbf{G}^{\mathbf{int}}(\mathbb{A}_F^f) \cong \tilde{\mathcal{G}}.$$

(Note that $\tilde{\mathcal{G}}$ is essentially the same as $\tilde{G}$ evaluated on finite adeles of $\mathbb{Q}$.)

For each level subgroup $T \subset \mathcal{I}'$, one considers the double quotient

$$\tilde{X}_T := T \backslash \Omega_{E_\omega} \times \mathcal{I}' / \tilde{\mathbf{I}}(F)$$

following [Va2], where Ω_{E_ω} is the rigid analytic Drinfeld p-adic upper half plane over E_ω. For every T as above, this has a structure of an analytic space and a projective scheme X_T. Set

$$X := \varprojlim_T X_T.$$

Then following essentially is the First Main Theorem of [Va1], [Va2] :

Theorem 3.6 (Varshavsky)**.** *There exists a $\tilde{\mathcal{G}}$-equivariant isomorphism*

$$\varphi : X \xrightarrow{\sim} X^{int}.$$

Remark 3.7. 1. Note that F_{ν_1} and E_ω are in fact the same fields. Hence both objects are defined over the same base.

2. Unraveling the definitions of above groups via Weil restriction, we see that proving p-adic uniformization results for spaces above (a priori defined over F so to speak) gives p-adic uniformization results for spaces over $\mathbb{Q}$.

We wish to use this result to prove p-adic uniformization for S using a corresponding inner form, which is a true unitary subgroup of $\tilde{\mathbf{I}}$. One can obtain a proof by suitably modifying techniques in *l*oc. cit., by proving that, in the notation that appears there, the stabilizer Δ of a connected component M is isomorphic to $PU(n-1, 1)$. We however prefer to give an alternate proof inspired by the example of Quaternionic Shimura varieties.

We define $\mathbf{I}$ to be the true unitary subgroup of $\tilde{\mathbf{I}} = \mathbf{GU}(D, **)$. Then we set

$$I' := \mathbf{I}(\mathbb{A}_F^{f;\nu_1})$$

and

$$\tilde{I} := \tilde{D}_{\nu_1}^{\times} \times I'$$

where $\tilde{D}$ is as before. There is a construction of an analytic space which is projective, say Y, completely analogous to the construction of X in the unitary similitude group case. It is given, for a level subgroup U', by :

$$(U' \backslash Y)^{an} \cong U' \backslash \Omega_{F_{\nu_1}} \times I'/\mathbf{I}(F).$$

We now come to the main theorem in this section.

Theorem 3.8. *There exists an $\tilde{I}$-equivariant isomorphism*

$$\varphi_{I,G} : Y \xrightarrow{\sim} S.$$

Proof. We have the diagonal embeddings of algebraic groups $\mathbf{I} \hookrightarrow \tilde{\mathbf{I}}$ and $\mathbf{G} \hookrightarrow \mathbf{G}^{\mathbf{int}}$, where $\mathbf{G}$ is the F-analogue of the unitary group G. These induce embeddings of groups

$$\tilde{I} = \tilde{D}_{\nu_1}^{\times} \times \mathbf{I}(\mathbb{A}_F^{f;\nu_1}) \hookrightarrow \tilde{D}_{\omega}^{\times} \times \tilde{\mathbf{I}}(\mathbb{A}_F^{f;\nu_1}) = \mathcal{G}_1 \backslash \tilde{\mathcal{G}}$$

where $\mathcal{G}_1$ is simply the group $\{1\} \times F_{\nu_1}^{\times} \times \{1\} \subset \tilde{G}$, which keeps track of similitude factors. Note that we are using the fact that ν_1 splits completely to identify $\tilde{D}_{\nu_1}^{\times}$ and $\tilde{D}_{\omega}^{\times}$ in this assertion.

Similarly, we have an embedding

$$\mathbf{G}(\mathbb{A}_F^f) \hookrightarrow \mathbf{G}^{\mathbf{int}}(\mathbb{A}_F^f) \to \mathcal{G}_1 \backslash \mathbf{G}^{\mathbf{int}}(\mathbb{A}_F^f)$$

commuting with the above identifications.

Then, we see that the natural embeddings

$$(\Omega_{F_{\nu_1}} \times \mathbf{I}(\mathbb{A}_F^{f;\nu_1}))/\mathbf{I}(F) \hookrightarrow (\Omega_{E_\omega} \times \tilde{\mathbf{I}}(\mathbb{A}_F^{f;\nu_1}))/\tilde{\mathbf{I}}(F)$$

(again, note that $F_{\nu_1} \cong E_\omega$) and

$$\mathbf{M}^{\mathbf{int}} \times \mathbf{G}(\mathbb{A}_F^f))/\mathbf{G}(F) \hookrightarrow \mathbf{M}^{\mathbf{int}} \times (\mathcal{G}_1 \backslash \mathbf{G}^{\mathbf{int}}(\mathbb{A}_F^f))/\mathbf{G}^{\mathbf{int}}(F)$$

where $\mathbf{M}^{\mathbf{int}}$ is the conjugacy class of homomorphisms arising from $\mathbf{H}^{\mathbf{int}}$. By GAGA, we see that these define embeddings $Y \hookrightarrow \mathcal{G}_1 \backslash X$ and $i : S_{\mathbb{C}} \hookrightarrow \mathcal{G}_1 \backslash \mathbf{X}_{\mathbb{C}}^{\mathbf{int}}$. We see that i is L_ν-rational by our earlier computation of the Galois action on the set of connected components. Alternatively, this also follows from proof of [Va2, Theorem 5.3].

We note now, that $\mathbf{P\tilde{I}} \xrightarrow{\sim} \mathbf{PI}$ and $\mathbf{PG} \xrightarrow{\sim} \mathbf{PG^{int}}$. This implies that modulo centers, above embeddings of schemes are actually isomorphisms :

$$Z(\tilde{I})\backslash Y \cong Z(\tilde{\mathcal{G}})\backslash X$$

and

$$Z(\mathbf{G}(\mathbb{A}_F^f))\backslash S \cong Z(\mathbf{G^{int}}(\mathbb{A}_F^f))\backslash X^{int}.$$

By Theorem 3.6, we have an isomorphism equivariant for $\tilde{\mathcal{G}}$-action

$$\varphi : X \xrightarrow{\sim} \mathbf{X^{int}}.$$

It induces an isomorphism, equivariant for $\mathcal{G}_1\backslash\tilde{\mathcal{G}}$-action

$$\tilde{\varphi} := \mathcal{G}_1\backslash X \xrightarrow{\sim} \mathcal{G}_1\backslash\mathbf{X^{int}}.$$

Let $x \in Y \subset \mathcal{G}_1\backslash X$. Then there exists $g \in Z(\tilde{\mathcal{G}})$ such that

$$g\tilde{\varphi}(x) \in S \subset \mathcal{G}_1\backslash X^{int}.$$

Thus, $g\tilde{\varphi}$ maps the $\tilde{I}$-orbit of x into S. By [Va1, Proposition 1.3.8 and 1.5.3], the $\tilde{I} = \mathbf{G}(\mathbb{A}_F^f)$-orbit is Zariski dense in both Y and S, we see that this gives us the required isomorphism $\varphi_{I,G} : Y \xrightarrow{\sim} S$.

□

We record the following corollary to this result, which is more suited for application to the problem of level-raising. This is analogous to the statement of Theorem 2.2. We now drop the subscript from the notation of the place ν_1 and refer to it as simply a fixed place ν. As in section 3, we define a variant of the Drinfeld upper half plane as follows :

$$\mathcal{M}^{\text{split}} := \Omega_{O_{F_\nu}} \times GL_n(F_\nu)/GL_n(F_\nu)^0 \tag{3.6}$$

where $GL_n(F_\nu)^0 \subset GL_n(F_\nu)$ is the open subgroup of matrices with determinant a p-adic unit and the sets on the right hand side are identified with the corresponding constant formal schemes over O_{F_ν}. We define $\mathcal{M} := \mathcal{M}^{\text{split}}\hat{\otimes}_{O_{F_\nu}} O_{\mathbb{F}}$ where $\mathbb{F}$ denotes the completion of a maximal unramified extension of F_ν. Then, we have, as a corollary -

Corollary 3.9. *We have the following isomorphism :*

$$I(\mathbb{Q})\backslash[\mathcal{M} \times G(\mathbb{A}^{p,\infty})/U^p \cong (S(G, U^pU_p) \otimes_{O_{F_\nu}} O_{\mathbb{F}})^\wedge$$

(where we have now dropped the boldface on I from Theorem 3.8 for the sake of brevity.)

Remark 3.10. We note that the map in the statement of the corollary no longer has a moduli interpretation, which is used in [RZ] to construct the isomorphism stated in Theorem 2.2. It is instead the "union of connected components" incarnation that we have mentioned earlier.

It is this corollary that will be used in future sections for applications to spaces of automorphic forms and automorphic local systems.

4 Automorphic local systems

From now on we will only consider sufficiently small open compact subgroups $U = U^pU_p$ with U_p being maximal compact, so that the models we have defined exist over a field in which p is unramified, and Corollary 3.9 holds true. We now describe some local systems on $S(G,U)$ that are related to algebraic representations of G and spaces of automorphic forms, analogous to the local systems appearing in [T].

Corresponding to the chosen CM type Φ, we get an isomorphism

$$G(\mathbb{C}) \cong \prod_{\tau\in\Phi} GL_n(\mathbb{C}).$$

Let $T \subset G \otimes_{\mathbb{Q}} \mathbb{C}$ be the product of diagonal maximal tori :

$$T(\mathbb{C}) \cong \prod_{\tau\in\Phi} \mathbb{C}^\times \times \ldots \times \mathbb{C}^\times.$$

Then we can write $X^*(T) \cong (\mathbb{Z}^n)^\Phi$ and we denote by $X^*(T)_+$ the subset of dominant weights $\boldsymbol{\mu} = (\mu_\tau)_{\tau\in\Phi}$, which are precisely the ones satisfying the condition

$$\mu_{\tau,1} \geq \mu_{\tau,2} \geq \ldots \geq \mu_{\tau,n}$$

for each embedding τ in Φ. If ℓ is a rational prime, we say $\boldsymbol{\mu}$ is ℓ-small if, for each $\tau \in \Phi$,

$$0 \leq \mu_{\tau,i} - \mu_{\tau,j} \leq \ell$$

for all i, j such that $0 \leq i < j \leq n$. For ℓ unramified in E and $\boldsymbol{\mu}$ ℓ-small, we can associate to $\boldsymbol{\mu}$ an ℓ-adic local system on $S(G,U)$ as follows. (See [HT], section $III.2$, [Ha], section 7.1.)

We fix an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$. Let K be a finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}_\ell}$ such that by assumption, the algebraic representation of $G \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ of highest weight $\iota^{-1}\boldsymbol{\mu}$, say $W_{\boldsymbol{\mu},K}$, is defined over K. Let $\mathscr{O}$ be the ring of integers of K with maximal ideal λ, and residue field k. Let U_ℓ be a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_\ell)$. Then, up to homothety, there is a unique U_ℓ-invariant $\mathscr{O}$-lattice of $W_{\boldsymbol{\mu},K}$. We choose one such and call it $W_{\boldsymbol{\mu},\mathscr{O}}$. Its uniqueness follows from the ℓ-small hypothesis by considering the reduction modulo ℓ.

Then, given an integer $m \geq 1$, we let $U(m) = U^p(m)U_p \subset U$ be an open compact subgroup acting trivially on $W_{\boldsymbol{\mu},\mathscr{O}/\lambda^m} = W_{\boldsymbol{\mu},\mathscr{O}} \otimes_{\mathscr{O}} \mathscr{O}/\lambda^m$. Then U acts on the constant sheaf defined by $W_{\boldsymbol{\mu},\mathscr{O}/\lambda^m}$ on $S(g, U(m))$, and the quotient defines an étale local system on $S(G, U)$, which we denote by $V_{\boldsymbol{\mu},\mathscr{O}/\lambda^m}$. We then take

$$V_{\boldsymbol{\mu},\mathscr{O}} := \varprojlim_m V_{\boldsymbol{\mu},\mathscr{O}/\lambda^m}$$

and

$$V_{\boldsymbol{\mu},K} := V_{\boldsymbol{\mu},\mathscr{O}} \otimes_{\mathscr{O}} K.$$

We note that sections of the local system $V_{\boldsymbol{\mu},\mathscr{O}/\lambda^m}$ over an étale open $T \to S(G, U)$ can be identified with the set of functions $f : \pi_0(S(G, U(m)) \times_{S(G,U)} T) \to W_{\boldsymbol{\mu}}$ such that for all $\sigma \in U, C \in \pi_0(S(G, U(m)) \times_{S(G,U)} T)$, we have $f(C\sigma) = \sigma^{-1} f(C)$.

5 Spaces of Automorphic forms

We recall the definition of spaces of automorphic forms with integral coefficients on the definite unitary group I, as defined in [T]. We recall the Hecke action on these spaces and the definition of cohomology of these spaces.

We take ℓ, K, $\mathscr{O}$, and k as before. Let $U_\ell \subset I(\mathbb{Q}_\ell)$ be an open compact subgroup, and suppose M is a finite $\mathscr{O}$-module on which U_ℓ acts continuously. Then the space of automorphic forms $\mathcal{A}(M)$ is defined to be the set of locally constant functions $f : I(\mathbb{A}^\infty) \to M$ such that for all $\gamma \in \mathbb{Q}, f(\gamma g) = f(g)$. This can be endowed with an action of the group $I(\mathbb{A}^\infty) \times U_\ell$ by the formula $(g.f)(h) := g_\ell f(hg)$ where g_ℓ is the projection to the ℓ-th component. If $U \subset I(\mathbb{A}^\infty) \times U_\ell$ is a subgroup, then we define $\mathcal{A}(U, M) := \mathcal{A}(M)^U$.

Then we record the following lemma (Lemma 2.2 from [T]) -

Lemma 5.1. *For $p \neq \ell$ prime, let U^p be an open compact subgroup of $I(\mathbb{A}^{p,\infty})$ whose projection to $I(\mathbb{Q}_\ell)$ is contained in U_ℓ. Then for any open compact subgroup $U_p \subset I(\mathbb{Q}_p)$, $\mathcal{A}(U^p, M)^{U_p}$ is a finite $\mathscr{O}$-module.*

Let $\boldsymbol{\mu}$ be a choice of ℓ-small dominant weight, and let $U = \prod_q U_q \subset I(\mathbb{A}^\infty)$ be an open compact subgroup. Then there is a finite free $\mathscr{O}$-module $W_{\boldsymbol{\mu},\mathscr{O}}$ and a corresponding space of automorphic forms $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}})$, which is a finite free $\mathscr{O}$-module. This space has an interpretation as an isotypic component of the space of automorphic forms $\mathcal{A}$ on I, a semisimple admissible representation of $I(\mathbb{A})$. Let $W_{\boldsymbol{\mu},\mathbb{C}}$ denote the representation of $I(\mathbb{R}) \subset I(\mathbb{C})$ which is the restriction of the algebraic representation with highest weight $\boldsymbol{\mu}$. Then we have an isomorphism:

$$\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}}) \otimes_{\mathscr{O},\iota} \mathbb{C} \cong Hom_{I(\mathbb{R})}(W_{\boldsymbol{\mu},\mathbb{C}}^\vee, \mathcal{A}).$$

5.1 Hecke action

If T is a finite set of primes containing ℓ, such that U_q is a hyperspecial maximal compact subgroup for all $q \notin T$, let $\mathbb{T}_T^{univ} = \mathscr{O}[\{T_1^\nu, \dots, T_n^\nu, (T_n^\nu)^{-1}\}]$ denote the universal Hecke algebra in infinitely many variables corresponding to the unramified Hecke operators at places ν of F which split in E and are away from T. This Hecke algebra acts on $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}})$ by $\mathscr{O}$-algebra endomorphisms, and on the cohomology of the local systems, $H^i(S(G,U)_{\overline{E}}, W_{\boldsymbol{\mu},\mathscr{O}})$ via the isomorphism $I(\mathbb{A}^{p,\infty}) \cong G(\mathbb{A}^{p,\infty})$.

If σ is an automorphic representation of $I(\mathbb{A})$ such that $(\sigma^\infty)^U \neq 0$ and $\sigma_\infty \cong W_{\boldsymbol{\mu},\mathbb{C}}^\vee$, we can associate to it a maximal ideal $m_\sigma \subset \mathbb{T}_T^{univ}$ by assigning to each Hecke operator the reduction modulo ℓ of its eigenvalue. If $\sigma^{'}$ is another automorphic representation of $I(\mathbb{A})$, we say that $\sigma^{'}$ *c*ontributes to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}})_{m_\sigma}$ if $((\sigma^{'})^\infty)^U \neq 0$ and $\sigma^{'}_\infty \cong W_{\boldsymbol{\mu},\mathbb{C}}^\vee$, and the intersection of $\iota^{-1}((\sigma^{'})^\infty)^U$ and $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}})_{m_\sigma}$ inside $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}}) \otimes_{\mathscr{O}} \overline{\mathbb{Q}_\ell}$ is nontrivial.

5.2 Cohomology

For $G = GL_n(F)$ where F is a finite extension of $\mathbb{Q}_p$ and $\mathscr{O}$ as before, cohomology groups $H^*(M)$ of a smooth $\mathscr{O}[G]$-module M are defined as follows.

Let $U_0 = GL_n(\mathscr{O}_F)$ be the standard maximal compact subgroup and $B \subset U_0$ be the standard Iwahori subgroup. Let ζ denote the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \varpi & 0 & \dots & 0 & 0 \end{pmatrix}$$

where ϖ is a fixed uniformizer for $\mathscr{O}_F$. For $i = 0, \dots, n-1$, let $U_i = \zeta^{-i} U_0 \zeta^i$. All of these contain B. For $A \subset \{0, 1, \dots, n-1\}$, we write $U_A := \cap_{i \in A} U_i$. Then we define a complex $C^\bullet(M)$ by the formula

$$C^i(M) = \bigoplus_{A \subset \{0,1,\cdots,n-1\}} M^{U_A},$$

where the direct sum is over subsets of cardinality $i+1$. The differential $d_i : C^i \to C^{i+1}$ is given by the sum of the restriction maps $r_{A,A'} : M^{U_A} \to M^{U_{A'}}$ for $A \subset A^{'}$, each multiplied by a sign $\epsilon(A, A^{'})$ which is given as follows. If $A^{'} = \{i_1, \cdots, \imath_r\}$ written in increasing order, and $A = A^{'} \setminus \{i_s\}$, then

$$\epsilon(A, A^{'}) = (-1)^s. \tag{5.1}$$

Then $H^*(M)$ is defined to be the hypercohomology of this complex.

The cohomology groups $H^i(\mathcal{A}(U^p), W_{\boldsymbol{\mu},k})$ will be key for proving the level-raising result. (See Theorem 4.6 from [T].)

6 Weight spectral sequence

We briefly recall the weight spectral sequence of Rapoport and Zink. A standard reference for this section is [Sa]. In particular, the fact that this sequence can be understood as the spectral sequence associated to a filtered object is crucial for its application in the later sections.

Let $S := Spec\mathscr{O}_F$ be the spectrum of a complete discrete valuation ring $\mathscr{O}_F$. As usual, s is the closed point of S and η is the generic point of S. Let $F := Frac\mathscr{O}_F$, and we fix an algebraic closure $\bar{F}$ of F. We write $\bar{s}, \bar{\eta}$ for the geometric points of S above s, η respectively. Suppose that $f : X \to S$ is a proper, strictly semistable morphism of relative dimension n. Then the special fiber X_s is a strict normal crossings divisor on X. We write $X_1, \ldots, X_h$ for its irreducible components. We suppose that each irreducible component X_i is smooth over $\kappa(s)$. For $A \subset \{1, \ldots, h\}$ we write X_A for the intersection $\bigcap_{i \in A} X_i$, and $X^{(m)} := \bigsqcup_{|A|=m+1} X_A$ for the disjoint union. Let K be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathscr{O}$, uniformizer λ, and residue field k, where ℓ is coprime to the residue characteristic of $\mathscr{O}_F$. Let $\Lambda = K, \mathscr{O}$, or k, and let V be a local system of flat Λ-modules on X. The weight spectral sequence is a spectral sequence

$$E_1^{p,q} = \bigoplus_{i \geq max(0,-p)} H^{q-2i}(X_{\bar{s}}^{(p+2i)}, V(-i)) \Rightarrow H^{p+q}(X_{\bar{K}}, V) \tag{6.1}$$

that computes the cohomology of the generic fiber in terms of the cohomology of the special fiber. This is equivariant for the action of G_F on both sides, and the differentials commute with this action. Note that the groups $E_1^{p,q}$ vanish for $q < 0$ and $q > 2n$. It can be seen to arise from the complex of nearby cycles $R\Psi V$, which is a filtered object.

We recall the construction of a simplicial complex $\mathcal{K}$ from [T] whose simplicial cohomology computes the first row on the E_1-page of the spectral sequence. We have

$$E_1^{p,0} = H^0(X_{\bar{s}}^{(p)}, V) \cong \bigoplus_{|A|=p+1} H^0(X_{A,\bar{s}}, V),$$

and unraveling the definition, the differential

$$d_1^{p,0} : E_1^{p,0} \to E_1^{p+1,0}$$

can be seen to be the sum of the canonical pullbacks $i_{A,A'} : H^0(X_{A,\bar{s}}, V) \to H^0(X_{A',\bar{s}}, V)$ each multiplied by precisely the sign $\epsilon(A, A^{'})$ defined in (5.1). The simplicial complex $\mathcal{K}$ is defined as follows. The vertices of $\mathcal{K}$ are in bijection with the X_i, and the set $\{X_{i_1}, \ldots, X_{i_r}\}$ corresponds to a simplex σ_A if and only if for $A = \{i_1, \ldots, i_r\}, X_A$ is nonempty. The coefficient system $\mathcal{V}$ corresponding to V is simply given by the global sections of the special fiber: $\sigma_A \to H^0(X_{A,\bar{s}}, V)$. We denote by $C^\bullet(\mathcal{K}, \mathcal{V})$ the complex computing the simplicial cohomology of $\mathcal{K}$ with coefficients in $\mathcal{V}$. Thus, by definition,

$$C^r(\mathcal{K},\mathcal{V}) = \bigoplus_{A\subset\{1,\ldots,h\}} H^0(X_{A,\bar{s}},V)$$

where the sum is over subsets A of cardinality $r+1$. The differential $d_r : C^r(\mathcal{K},\mathcal{V}) \to C^{r+1}(\mathcal{K},\mathcal{V})$ is given by the direct sum of restriction maps

$$res_{A,A'} : H^0(X_{A,\bar{s}},V) \to H^0(X_{A',\bar{s}},V)$$

each multiplied by the sign $\epsilon(A,A^{'})$ as before, and we have the following result -

Proposition 6.1. *There is a canonical isomorphism of complexes* $E_1^{\bullet,0} \cong C^\bullet(\mathcal{K},\mathcal{V})$.

7 Cohomology, degeneration, and level-raising

In this section, we use the description of irreducible components of the special fiber of the varieties $S(G,U)$ given by the p-adic uniformization result of Corollary 3.9, to compute first rows on E_1 and E_2 pages of the weight spectral sequence in terms of the spaces of automorphic forms $\mathcal{A}(U,W_{\boldsymbol{\mu},k})$. Then we prove a degeneration result following the use of a trick from [T] inspired by the use of weights in characteristic 0 to show that the spectral sequence degenerates at E_2 level. Finally, we put everything together to obtain a level-raising result for I and then GL_n.

7.1 Automorphic forms and weight spectral sequence

We have the p-adic uniformization result from Corollary 3.9 -

$$I(\mathbb{Q})\backslash[\mathcal{M}\times G(\mathbb{A}^{p,\infty})/U^p \cong (S(G,U^pU_p)\otimes_{O_{F_\nu}} O_{\mathbb{F}})^\wedge. \tag{7.1}$$

By the theory of Bruhat-Tits building BT of $PGL_n(F)$ and the Drinfeld upper half plane, the set of irreducible components in the special fiber of $\mathcal{M}$ is in bijection with the set $BT(0)\times\mathbb{Z}$, where $BT(i)$ for $i\in\mathbb{N}$ refers to the set of simplices of BT of dimension i. $BT(0)$ corresponds to homothety classes of $\mathscr{O}_F$-lattices $M\subset F^n$. We can in fact write down this set concretely in terms of the groups U_i defined in section 6.2. These are maximal compact subgroups that stabilize the distinct vertices, say $x_0,\ldots,x_{n-1}$, of the closure of the unique chamber of BT fixed by B, and their intersection is precisely B. We can define a coloring map $\kappa : BT(0)\times\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ by sending (M,b) to $\kappa(M,b) = log_q[M : \mathscr{O}_F^n] + b$, where q is the cardinality of the residue field. Then $\kappa(x_i,0)=i$ induces an isomorphism of $GL_n(F)$-sets

$$BT(0)\times\mathbb{Z}\cong\bigsqcup_{i=0}^{n-1} GL_n(F)/U_i.$$

Furthermore, for each $i = 0, \ldots, n-1$, there is a bijection between the set of nonempty $(i+1)$-fold intersections of irreducible components of the special fiber of $\mathcal{M}$ and the set

$$BT(i) \times \mathbb{Z} \cong \bigsqcup_{A \subset \{0,\ldots,n-1\}} GL_n(F)/U_A.$$

The disjoint union runs over subsets A of cardinality $i+1$. From this information, we see from (7.1), since U is sufficiently small, that the irreducible components of the special fiber are in bijection with the set

$$I(\mathbb{Q}) \setminus [BT(0) \times GL_n(F_\nu)/GL_n(F_\nu)^0 \times I(\mathbb{A}^{p,\infty})/U^p] \cong \bigsqcup_{i=0}^{n-1} I(\mathbb{Q}) \setminus I(\mathbb{A}^\infty)/U_i U^p. \quad (7.2)$$

Thus, for each $i = 0, \ldots, n-1$, there is a bijection

$$\pi_0(S(G,U)^{(i)}_{\bar{s}}) \cong \bigsqcup_{A \subset \{0,\ldots,n-1\}} I(\mathbb{Q}) \setminus I(\mathbb{A}^\infty)/U_A U^p,$$

the union running over subsets A of cardinality $i+1$.

To write down the weight spectral sequence for $S(G,U)$, we choose a partial ordering on the set of irreducible components of the special fiber, i.e. on the LHS of (7.2), given by the pullback under κ of the partial ordering $0 \leq 1 \leq \ldots \leq n-1$ defined on $\mathbb{Z}/n\mathbb{Z}$. Let $E_1^{p,q} \Rightarrow H^{p+q}(S(G,U)_{\bar{\eta}}, V_{\boldsymbol{\mu},k})$ be the weight spectral sequence.

Proposition 7.1. *1. For each $i = 0, \ldots, n-1$, there is a canonical isomorphism*

$$E_1^{i,0} \cong \bigoplus_{A \subset \{0,\ldots,n-1\}} \mathcal{A}(U^p U_A, W_{\boldsymbol{\mu},k})$$

where the direct sum runs over the set of all subsets A of cardinality $i+1$.

2. There is a canonical isomorphism of complexes

$$E_1^{\bullet,0} \cong C^\bullet(\mathcal{A}(U^p, W_{\boldsymbol{\mu},k}))$$

and therefore for each $i = 0, \ldots, n-1$,

$$E_2^{i,0} \cong H^i(\mathcal{A}(U^p, W_{\boldsymbol{\mu},k})).$$

Proof. We follow the proof of Proposition 6.4 from [T]. By definition, we have $E_1^{i,0} = H^0(S(G,U)^{(i)}_{\bar{s}}, V_{\boldsymbol{\mu},k})$. We have seen previously that this space of global sections can be identified with the space of functions $f : \pi_0(S(G,U(1))^{(i)}_{\bar{s}}) \to W_{\boldsymbol{\mu},k}$ satisfying the transformation relation $f(C\sigma) = \sigma^{-1} f(C)$ for all $C \in \pi_0(S(G,U(1))^{(i)}_{\bar{s}}), \sigma \in U$. The first isomorphism now follows from our identification of the set $\pi_0(S(G,U(1))^{(i)}_{\bar{s}})$ with the disjoint union $\bigsqcup_{A\subset\{0,\ldots,n-1\}} I(\mathbb{Q})\setminus I(\mathbb{A}^\infty)/U_A U(1)^p$ which is compatible with varying U, and the definition of the spaces $\mathcal{A}(U^p U_A, W_{\boldsymbol{\mu},k})$.

We note that under this isomorphism, the restriction maps of sections H^0 correspond to the natural inclusions of spaces of automorphic forms $\mathcal{A}(U^p U_A, W_{\boldsymbol{\mu},k}) \to \mathcal{A}(U^p U_{A'}, W_{\boldsymbol{\mu},k})$, and the signs $\epsilon(A, A^{'})$ agree by their definition. This implies that the differentials of the two complexes in the second part correspond under the isomorphism of the first part, giving us the desired isomorphisms of complexes. □

7.2 Degeneration

We now describe the degeneration result that is at the heart of this method. To this end, we first need to define another descent of the scheme $S(G,U)\otimes_{\mathscr{O}_{F_\nu}}\mathscr{O}_{\mathbb{F}}$ on which the action of Frobenius in characteristic ℓ is given by scalar weights. We define

$$S(G,U)^{split} := I(\mathbb{Q})\setminus[\mathcal{M}^{split}\times G(\mathbb{A}^{p,\infty})/U^p].$$

The local systems $V_{\boldsymbol{\mu},\Lambda}$ where $\Lambda = K, \mathscr{O}$, or $\mathscr{O}/\lambda^m$ admit descents to the scheme $S(G,U)^{split}$ by following the same definitions as before. $S(G,U)$ is not the same descent as $S(G,U)^{split}$ (in particular, Galois actions differ), but they become isomorphic after extension of scalars to $\mathbb{F}$. The following lemma and the proposition after that, then use essentially the fact that our schemes are a union of connected components of the similitude Shimura varieties.

Lemma 7.2. *1. The pullback of $V^{split}_{\boldsymbol{\mu},k}$ to any irreducible component of the special fiber of $S(G,U)^{split}$ is a constant sheaf.*

2. If $Y_1,\ldots,Y_s$ are irreducible components of the special fiber, then the action of the Frobenius element is by the scalar $q_\nu^{i/2}$ on the group $H^i((Y_1\cap\ldots\cap Y_s)_{\bar{s}}, V^{split}_{\boldsymbol{\mu},k})$.

Proof. Let $Y\subset S(G,U(1))^{split}$ be an irreducible component of the special fiber. If π : $S(G,U(1))^{split}S(G,U)^{split}$ is the natural projection, then the restriction $\pi|_Y$ induces an isomorphism from Y to its image in $S(G,U)^{split}$. Pullback of $V_{\boldsymbol{\mu},k}$ under the inverse of this isomorphism gives the first part. The second part then follows from the first and Proposition 3.1 from [T]. □

We recall that the group H^i appearing in the second part above is 0 if i is odd.

Proposition 7.3. *Let $r = 2s+1$. The differentials*

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$$

are all zero as long as $q_\nu^s \not\equiv 1$ mod ℓ.

Proof. The weight spectral sequence of a pair (X, V) where X is a strictly semistable scheme and V a local system on X only depends upon the extension of scalars $(X \otimes_{\mathscr{O}_{F_\nu}} \mathbb{F}, V)$ as a spectral sequence of abelian groups (though *not* as a sequence of Galois modules). This implies that it is sufficient to prove the assertion for the pair $(S(G,U)^{split}, V^{split}_{\boldsymbol{\mu},k})$. This follows from the fact that the differentials in the weight spectral sequence are Galois equivariant, and the previous lemma, combined together as in the proof of Proposition 6.5 in [T]. □

Corollary 7.4. *Suppose that ℓ is a banal characteristic for $GL_n(F_\nu)$. Then the weight spectral sequence for the pair $(S(G,U), V_{\boldsymbol{\mu}.k})$ degenerates at the E_2-level, and consequently there is an injection, equivariant for the prime-to-p Hecke algebra :*

$$H^i(\mathcal{A}(U^p, W_{\boldsymbol{\mu},k})) \hookrightarrow H^i(S(G, U_pU^p)_{\bar{F}_\nu}, V_{\boldsymbol{\mu},k}).$$

Proof. By the banal characteristic hypothesis, ℓ is coprime to the pro-order of $GL_n(F_\nu)$. This implies that none of the integers $q_\nu, q_\nu^2, \ldots, q_\nu^{n-1}$ are congruent to 1 modulo ℓ. Since the cohomology of $S(G,U)$ can only exist in the range $0, \ldots, 2(n-1)$, the groups $E_r^{p,q}$ can be nonzero only for that range. Then it follows from Proposition 7.3, that all the differentials for $r \geq 2$ are zero, giving the first assertion. In particular, as noted before, since the weight spectral sequence is associated to the filtered complex $R\Psi V$, we have $E_\infty^{q,0} \hookrightarrow H^q(S(G, U_pU^p)_{\bar{F}_\nu}, V_{\boldsymbol{\mu},k})$. The claim now follows from degeneration and Proposition 7.1. □

7.3 Level-raising for unitary groups

We now apply Corollary 7.4 to the problem of level-raising. For the rest of this subsection, we work in the notation from section 5.1. We fix an isomorphism $I(\mathbb{Q}_p) \cong GL_n(E_\omega)$. We assume that under this isomorphism, $U_p \cong B$, where B, as before, is the standard Iwahori subgroup of $GL_n(E_\omega)$. We then have the following analogue of level-raising theorem from [T] with relaxed hypotheses on F, E, and p, as mentioned earlier in the body of the paper:

Theorem 7.5 (Level-raising for I)**.** *Suppose that σ is as in section 6.1, and m_σ the associated maximal ideal of the Hecke algebra $\mathbb{T}_T^{univ}$. Suppose that the following hypotheses hold :*

1. *U^p is a sufficiently small open compact subgroup of $I(\mathbb{A}^{p,\infty})$, in particular satisfying the hypotheses of Lemma 3.1.*

2. *If σ' is another automorphic representation contributing to the space $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}})_{m_\sigma}$, then σ' considered as a representation of $GL_n(E_\omega)$ is a subquotient of a parabolic induction $n{-}Ind_Q^G St_a(\alpha) \otimes St_b(\beta)$ for some partition $a+b = n$ and Q the standard parabolic corresponding to this partition.*

3. *$\iota^{-1}\sigma_\omega$ satisfies the level-raising congruence* (4.2) *of [T].*

4. *$\boldsymbol{\mu}$ is ℓ-small and ℓ is a banal characteristic for $GL_n(E_\omega)$.*

5. *The groups $H^{n-2}(S(G, U_p U^p)_{\bar{F}_\nu}, V^\vee_{\boldsymbol{\mu},k})$ and $H^i(S(G, U_p^{'} U^p)_{\bar{F}_\nu}, V_{\boldsymbol{\mu},k})$ are zero.*

Then there exists another irreducible constituent $\sigma^{'}$ contributing to the space $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}})_{m_\sigma}$, such that $\sigma^{'}$ is an unramified twist of the Steinberg representation, i.e. $\sigma^{'}$ is obtained by raising the level of σ.

Remark 7.6. We again remind the reader that the fields F_ν and E_ω are the same.

Proof. By hypothesis 5 and Corollary 7.4, the groups, in characteristic ℓ, $H^{n-2}(\mathcal{A}(U^p, W_{\boldsymbol{\mu},k}))$ and $H^{n-2}(\mathcal{A}(U^p, W^\vee_{\boldsymbol{\mu},k}))$ are zero. On the other hand, in characteristic 0, we have a perfect $GL_n(E_\omega)$-equivariant pairing

$$\mathcal{A}(U, W_{\boldsymbol{\mu},\mathscr{O}}) \times \mathcal{A}(U, W^\vee_{\boldsymbol{\mu},\mathscr{O}}) \to \mathscr{O}$$

given by an analogous formula to the one appearing in Theorem 6.7 in [T]. The result then follows by a similar argument, appealing to the level-raising formalism developed there. □

8 Application to GL_n

In this section, we deduce our main theorem. It is similar to Theorem 7.1 in [T],with relaxed hypotheses on fields E and F. E is a CM fields as always with maximal totally real subfield F, $[F : \mathbb{Q}] = d$ is even. We no longer require that E be of the form $E_0.F$. The requirement that E/F be everywhere unramified is also relaxed. The level-raising prime p is assumed to be unramified in F, with a place ν above it of F assumed to be split in E as $\nu = \omega.\omega^c$. Let $n \geq 3$ be an integer, and $\ell \neq p$ a prime. We fix an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ as before.

Let n_1, n_2 be positive integers with $n = n_1 + n_2$. Suppose that π_1, π_2 are conjugate self-dual cuspidal automorphic representations of $GL_{n_1}(\mathbb{A}_E)$ and $GL_{n_2}(\mathbb{A}_E)$ respectively such that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. In Theorem 2.1 of [T], there is associated to π a continuous semisimple Galois representation $r_\iota(\pi) : G_E \to GL_n(\overline{\mathbb{Q}_\ell})$.

Theorem 8.1. *With π as above, suppose that $\iota^{-1}\pi_\omega$ satisfies the level-raising congruence* (4.2) *of [T]. Suppose further that the following hypotheses hold.*

1. *If t_ℓ is a generator of the ℓ-th part of the tame inertia group at ω, then $\overline{r_\iota(\pi)}(t_\ell)$ is a unipotent matrix with exactly two Jordan blocks.*

2. *ℓ is a banal characteristic for $GL_n(E_\omega)$.*

3. *The weight $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau:E\hookrightarrow\mathbb{C}}$ of π satisfy the following properties :*

   - *For each τ, and for each $0 \leq i < j \leq n$, we have $0 < \lambda_{\tau,i} - \lambda_{\tau,j} < \ell$.*
   - *There exists an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ such that the following inequalities hold :*

$$2n + \sum_{\tau:E\hookrightarrow\mathbb{C}} \sum_{j=1}^{n} (\lambda_{\tau,j} - 2\lfloor \lambda_{\tau,n}/2 \rfloor) \leq \ell,$$

$$2n + \sum_{\tau:E\hookrightarrow\mathbb{C}} \sum_{j=1}^{n} (2\lceil \lambda_{\tau,1}/2 \rceil - \lambda_{\tau,n+1-j}) \leq \ell.$$

4. *If π is ramified at a place γ of E, then γ is split over F.*

5. *π is unramified at the primes of E lying above ℓ, and ℓ is unramified in E.*

6. *$\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition (2.3) in [T], $n_1 \neq n_2$, and $n_1 n_2$ is even.*

Then there exists a regular algebraic conjugate self-dual cuspidal automorphic representation Π of $GL_n(\mathbb{A}_E)$ of weight $\boldsymbol{\lambda}$ such that there is an associated Galois representation $r_\iota(\Pi)$ to it satisfying $\overline{r_\iota(\pi)} \cong \overline{r_\iota(\Pi)}$ and Π_ω is an unramified twist of the Steinberg representation. If the places of F above ℓ are split in E, and π is ι-ordinary in the sense of [Ge, Definition 5.1.2], then we can even assume that Π is also ι-ordinary.

Proof. This follows from Theorem 7.5, once we make all the right identifications. By [CT, Proposition 2.9], and [La, Corollaire 5.3], we see that there exists an automorphic representation σ of I such that π is the base change of σ. (See Proposition 2.4 in [T] for more details.) Once we have σ in our hands, it is a matter of checking that various hypotheses of Theorem 7.5 are satisfied. Only the hypotheses (2) and (5) there need checking. Hypothesis (5) of that theorem follows from the torsion vanishing result of Lan and Suh [LS, Theorem 8.12] combined with the inequalities in hypothesis (3) of this theorem. One could also appeal to the torsion vanishing result of [Sh] after putting some extra hypotheses. The hypothesis (2) in that theorem is checked in a similar way as that of Theorem 7.1 of [T], from our hypothesis (1) on Jordan blocks. If σ' is the representation whose existence is guaranteed by Theorem 7.5 for σ as above, we use the reverse direction of the aforementioned base change results to obtain a representation Π having required properties. The last assertion about ordinary property follows after enlarging the Hecke algebra $\mathbb{T}_T^{univ}$ to contain the U_ℓ operators. □

Remark 8.2. The analogue of Theorem 1.1 from [T] with the hypothesis on p being 'p is unramified in F' as opposed to 'p is inert in F' follows from the above theorem in the same way as the proof there.

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