A SMOOTHING OF A RATIONAL SINGULARITY WITH NON FINITELY-GENERATED CANONICAL RING

An Honors Thesis Presented

By

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ABSTRACT

Let H be an isolated surface singularity with a one-parameter smoothing X. If the canonical algebra R of X is finitely generated (as an O_X-algebra) it can be used to define a small modification Y = Proj R of X such that K_Y is Q-Cartier. This is used in the construction of flips in the minimal model program.As discovered by Kollár--Shepherd-Barron in 1988, the finite generation holds when H is a quotient singularity. The strict transform of H is a so-called P-resolution, and all such can be determined combinatorially. They proved that Q-Gorenstein deformations of these P-resolutions describe the components of the versal deformation space of H.János Kollár conjectured that an analogous result holds when H is a rational singularity. In this paper, I will describe an approach, inspired by a paper of Cutkosky, to disprove the conjecture by constructing a smoothing with non-finitely generated canonical algebra, assuming a hypothesis we expect to be true.

A smoothing of a rational singularity with non finitely-generated canonical ring

Honors Thesis

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Contents

Abstract

Let X be an isolated surface singularity with a one-parameter smoothing $\mathscr{X}$. If the canonical algebra $\mathscr{R}$ of $\mathscr{X}$ is finitely generated (as an $\mathscr{O}_{\mathscr{X}}$-algebra) it can be used to define a small modification $\mathscr{Y} = \operatorname{Proj} \mathscr{R}$ of $\mathscr{X}$ such that $K_{\mathscr{Y}}$ is $\mathbb{Q}$-Cartier. This is used in the construction of flips in the minimal model program. As discovered by Kollár–Shepherd-Barron in 1988, the finite generation holds when X is a quotient singularity. The strict transform of X is a so-called P-resolution, and all such can be determined combinatorially. They proved that $\mathbb{Q}$-Gorenstein deformations of these P-resolutions describe the components of the versal deformation space of X. János Kollár conjectured that an analogous result holds when X is a rational singularity. In this paper, I will describe an approach, inspired by a paper of Cutkosky, to disprove the conjecture by constructing a smoothing with non-finitely generated canonical algebra, assuming a hypothesis we expect to be true.

Chapter 1

Introduction

Let $0 \in \mathscr{X}$ be an isolated singularity in a three dimensional complex space, or a variety over $\mathbb{C}$. In the minimal model program, an important question is the finite generation of the *canonical algebra*

$$\mathscr{R} := \bigoplus_{n \geq 0} \omega_{\mathscr{X}}^{[n]}$$

as an $\mathscr{O}_{\mathscr{X}}$-algebra. Here, $\omega_{\mathscr{X}}$ is the canonical sheaf on $\mathscr{X}$ and $\omega_{\mathscr{X}}^{[n]} := (\omega_{\mathscr{X}}^{**})^{\otimes n}$ is the nth-reflexive power. This sheaf is reflexive, and is isomorphic to the divisorial sheaf $\mathscr{O}(nK_{\mathscr{X}})$. The algebra $\mathscr{R}$, if finitely generated, induces a proper birational morphism $g : \mathscr{Y} := \underline{\mathrm{Proj}}_{\mathscr{X}}(\mathscr{R}) \to \mathscr{X}$. Here, $\mathscr{Y}$ is called the *canonical model* of $\mathscr{X}$, and is what's used to construct a flip/flop in the minimal model program. This model is $\mathbb{Q}$-Gorenstein with $K_{\mathscr{Y}}$ relatively ample, and hence is generally much better behaved than $\mathscr{X}$. Furthermore, $g : \mathscr{Y} \to \mathscr{X}$ is a *small modification*, i.e. g is an isomorphism outside a codimension 2 locus. Hence, this canonical model can be used to study several properties of $\mathscr{X}$.

In 1988, Kollár–Shepherd-Barron in [KSB88] considered a special case of this question. Let X be a hyperplane section through $0 \in \mathscr{X}$, and say $q \in X$ is the singular point. Can we say anything about the finite generation of $\mathscr{R}$ in terms of the surface singularity $q \in X$? They proved the following:

Theorem 1.0.1 ([KSB88], Thm 3.5)**.** *The canonical algebra is finitely generated if $q \in X$ is a* *quotient singularity.*

An alternate perspective is to think of $\mathscr{X}/\Delta$ as a one parameter smoothing of the germ of a rational surface singularity $q \in X$. Theorem 1.0.1 then states that the canonical algebra of the total space $\mathscr{X}$ is finitely generated. Let $\mathscr{Y}$ be the canonical model, and $Y \hookrightarrow \mathscr{Y}$ be the fiber over $0 \in \Delta := \operatorname{Spec} \mathbb{C}[[t]]$. We hence get the following diagram

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, dashed, "g_0"] & \mathscr{Y} \arrow[d, "g"] \\
X \arrow[r, hook] \arrow[d] & \mathscr{X} \arrow[d] \\
0 \arrow[r, hook] & \Delta
\end{tikzcd}$$

The small modification $\mathscr{Y} \to \mathscr{X}$ is an isomorphism over $(t \neq 0) \in \Delta$, and we thus get a resolution $Y \to X$ of the surface. This is an example of what's called a *P-resolution*, see Definition 1.0.3. Conversely, as quotient singularities are rational, a $\mathbb{Q}$-Gorenstein smoothing $\mathscr{Y}$ of a P-resolution $g_0 : Y \to X$ *blows down* to a one parameter smoothing of X given by $\mathscr{X} := \operatorname{Spec} H^0(\mathscr{O}_{\mathscr{Y}})$, see [Rie73, Thm. 2]. Hence we get a map $\operatorname{Def}^{QG} Y \to S \subset \operatorname{Def} X$, where S is an irreducible component of the versal base space of X. Additionally, one can show that there are enough P-resolutions so that this map gives a bijection of sets:

Theorem 1.0.2 ([KSB88], Thm 3.9)**.** *Let $q \in X$ be a quotient surface singularity. There is a bijective correspondence between P-resolutions of X and irreducible components of the versal base space of X.*

This generalizes the Artin component of the versal base space, which under this correspondence corresponds to the P-resolution obtained by contracting all the (-2) curves in the minimal resolution of q. Thus, we see that finite generation of $\mathscr{R}$ has an interesting consequence in terms of these P-resolutions, which is a more combinatorial encoding of this data. Formally,

Definition 1.0.3. A proper birational morphism $f : Y \to X$ of surfaces is a P-resolution of X if

(a) K_Y is f-ample;

(b) Y has at most T-singularities.

Here, a T-singularity is one which admits a $\mathbb{Q}$-Gorenstein one parameter smoothing.

These T-singularities can recursively be classified, see [KSB88, Prop. 3.11] for instance. Furthermore, these P-resolutions have a combinatorial structure, especially in the case of a cyclic quotient singularity:

Theorem 1.0.4 ([Chr91], [Ste91], [Alt98])**.** *Let $0 \in X$ be the cyclic quotient singularity $\frac{1}{n}(1, q)$, and let $\frac{n}{n-q} = [a_2, \dots, a_{e-1}]$ be the continued fraction of the dual. Then there is a bijection between P-resolutions of $0 \in X$ and continued fractions $[k_2, \dots, k_{e-1}]$ representing 0 with $k_i \leq a_i$ for all $2 \leq i \leq e-1$. Furthermore, there is a toric way of computing them.*

Since there is always the map $\operatorname{Def}^{QG} Y \to \operatorname{Def} X$ when $q \in X$ is a rational singularity, the best generalization of these results one can hope to have is when q is rational instead of quotient. This finite generation was conjectured by Kollár and first appeared in his survey:

Conjecture 1.0.5 ([Kol91], 6.2.1.)**.** *Let $0 \in X$ be a rational surface singularity, and $0 \in \mathscr{X}$ the total space of a one parameter smoothing. Then the canonical algebra*

$$\sum_{n=0}^{\infty} \mathscr{O}_{\mathscr{X}}(nK_{\mathscr{X}})$$

is a finitely generated $\mathscr{O}_{\mathscr{X}}$-algebra.

If this were true, then one would expect a generalization of Theorem 1.0.2, in terms of what are called P-modifications instead of P-resolutions. However, in this thesis we attempt to prove that this conjecture is false, under a result we expect to be true (see Question 1.2.2).

1.1 Background

We survey some interesting topics and prove relevant results that we use in this paper. We will mostly work in analytic setting, while most of our theory can be analogously developed for smooth varieties over $\mathbb{C}$. There is a subtle equivalence of results in these categories by the GAGA principle, however we won't worry too much about it.

1.1.1 Logarithmic sheaves

Let X be a complex manifold and D be a reduced normal crossing divisor on X, i.e. D is reduced and locally at intersections looks like the transversal intersection of hyperplanes $(x_1 x_2 \cdots x_k = 0)$. Let $i : U = X \backslash D \hookrightarrow X$ be the inclusion. Define the sheaf

$$\Omega^p(*D) := \varinjlim_{v} \Omega^p(vD) = i_* \Omega^p_U.$$

This is the sheaf of differential forms of poles of arbitrary order along D.

Definition 1.1.1. A p-form ω on X is said to be a logarithmic form if it has *logarithmic poles along* D, i.e. both ω and $d\omega$ have a pole of order at most one along D. Writing $\Omega^p(D) := \Omega^p \otimes \mathscr{O}_X(D)$, another way of writing this is

$$\{\omega \in \Omega^p(D) : d\omega \in \Omega^{p+1}(D)\}.$$

The data of these forms is the locally free sheaf denoted by $\Omega^p_X(\log D)$, which is naturally a subsheaf of $i_* \Omega^p_U$.

Outside D, this sheaf is isomorphic to Ω^p_U. Around D, this is has fairly simple local expression: consider an open set U where D is given by the local parameters $x_1 = \cdots = x_k = 0$ in $\mathbb{C}^n_{x_1,\ldots,x_n}$. Then $\Omega^1_X(\log D)(U)$ is the $\mathscr{O}_X(U)$-module generated by

$$\frac{dx_1}{x_1}, \ldots, \frac{dx_k}{x_k}, dx_{k+1}, \ldots, dx_n. \tag{1.1}$$

The sheaf $\Omega^p_X := \bigwedge^p \Omega^1_X$ is the sheaf of p-logarithmic forms. The terminology *logarithmic* comes from the fact that $d(\log z) = dz/z$, where the right side can be understood even in the algebraic context where $\log z$ cannot be defined. Observe that

$$\frac{d(fg)}{fg} = \frac{df}{f} + \frac{dg}{g},$$

and hence $\Omega^1_X(U)$ is generated by df/f for functions f supported on D. This is a coordinate-free description of the sections of Ω^1_X.

Definition 1.1.2. We define the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ as the dual of $\Omega^1_X(\log D)$. Concretely, if D is given by $(f = 0)$ over an open set U, this is the sheaf with sections

$$\{\partial \in \mathrm{Der}(\mathscr{O}_X(U)) : \partial f \in (f)\}.$$

This is roughly the sheaf of vector fields tangent along the smooth points of D.

Dualizing Equation 1.1, a local expression of $\mathscr{T}_X(-\log D)$ is

$$\left(x_1 \frac{\partial}{\partial x_1}\right) \mathscr{O}_X \oplus \cdots \oplus \left(x_k \frac{\partial}{\partial x_k}\right) \mathscr{O}_X \oplus \left(\frac{\partial}{\partial x_{k+1}}\right) \mathscr{O}_X \oplus \cdots \oplus \left(\frac{\partial}{\partial x_n}\right) \mathscr{O}_X$$

where $\partial/\partial x_i$ is the dual of dx_i, or equivalently the element $\{x_i \mapsto 1\}$ in $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}_{D_i}/\mathscr{I}^2_{D_i}, \mathscr{O}_X)$.

Lemma 1.1.3. *Let D be a reduced normal crossing divisor. We have the exact sequence of coherent sheaves*

$$0 \to \mathscr{T}_X(-\log D) \to \mathscr{T}_X \to \bigoplus\nolimits_j \mathscr{N}_{D_j/X} \to 0$$

where D_j are the irreducible components of D.

Proof. The first inclusion is clear, so it suffices to see that $\mathscr{N}_{D/X}$ is the cokernel. One way to see this is locally, where D is given by $x_1 = \cdots = x_k = 0$. The cokernel is then generated by

$$\left(x_i \frac{\partial}{\partial x_i}\right) \mathscr{O}_X / \left(\frac{\partial}{\partial x_i}\right) \mathscr{O}_X \cong \left(\frac{\partial}{\partial x_i}\right) \mathscr{O}_{D_i} \tag{1.2}$$

for $i \leq k$, where $D_j = (x_j = 0) \subset D$. These are exactly the generators $\mathscr{N}_{D_j/X} = \mathrm{coker}\{\mathscr{T}_{D_j} \to \mathscr{T}_X|_{D_j}\}$.

Another method to prove this is to first show

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \bigoplus_j \mathscr{O}_{D_j} \to 0. \tag{1.3}$$

The first map here is the inclusion, and the second map is the famous residue map. Given a local expression $(f = 0)$ of D_j, the map is $df/f \mapsto 1 \in \mathscr{O}_{D_j}$ extended linearly as a map of $\mathscr{O}_X$-modules. One only needs to check that a logarithmic form has no residue if and only if it is regular, i.e. in Ω^1_X. Indeed, it suffices to observe that $\alpha \in \mathscr{O}_X$ restricts to 0 on $\mathscr{O}_{D_j}$ if and only if $\alpha \in \mathscr{I}_{D_j}$, the ideal sheaf of D_j. Hence $(df/f) \wedge \alpha$ maps to 0 if and only if α/f is a regular section, proving the claim.

Then dualizing Equation 1.3 yields the exact sequence

$$0 \to \mathscr{T}_X(-\log D) \to \mathscr{T}_X \to \mathscr{E}xt^1(\mathscr{O}_{D_j}, \mathscr{O}_X) \to 0 \tag{1.4}$$

since $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_{D_j}, \mathscr{O}_X) \cong 0$ and $\mathscr{E}xt^1$ of $\Omega^1_X(\log D)$ vanishes since it is locally free. To compute $\mathscr{E}xt^1(\mathscr{O}_{D_j}, \mathscr{O}_X)$, dualize the exact sequence $0 \to \mathscr{I}_{D_j} \to \mathscr{O}_X \to \mathscr{O}_{D_j} \to 0$ to get

$$0 \cong \mathscr{O}^*_{D_j} \to \mathscr{O}^*_X \to \mathscr{I}^*_{D_j} \to \mathscr{E}xt^1(\mathscr{O}_{D_j}, \mathscr{O}_X) \to 0.$$

Here, the map $\mathscr{O}^*_X \to \mathscr{I}_{D_j}$ locally is $1 \mapsto (x_j \partial/\partial x_j)$. Thus, the cokernel of this map is generated by $(\partial/\partial x_j)\mathscr{O}_{D_j}$ like in Equation 1.2, and these generate the normal sheaf. Thus $\mathscr{E}xt^1(\mathscr{O}_{D_j}, \mathscr{O}_X) \cong \mathscr{N}_{D_j/X}$, and so we are done by Equation 1.4. □

In fact, we can generalize the proof of Lemma 1.1.3 to obtain the following more general exact sequence:

Lemma 1.1.4. *For any effective $D' \leq D$, we have the exact sequence*

$$0 \to \mathscr{T}_X(-\log D) \to \mathscr{T}_X(-\log(D - D')) \to \bigoplus_j \mathscr{N}_{D'_j/X} \to 0.$$

Observe that we have an obvious inclusion $\mathscr{T}_X(-D) \to \mathscr{T}_X(-\log D)$, since any vector field vanishing along D is tangent to it. This inclusion fits into an easy exact sequence:

Lemma 1.1.5 (cf. [Kaw78])**.** *We have the exact sequence*

$$0 \to \mathscr{T}_X(-D) \to \mathscr{T}_X(-\log D) \to \Theta_D \to 0$$

where Θ_D is the sheaf of derivations on D.

Proof. As before, there is a local proof, and a more global argument. Pick local parameters for D over $U \subset X$ so that it's given by $x_1 = \cdots = x_k = 0$ in $\mathbb{C}^n_{x_1,\ldots,x_n}$. The cokernel is then the $\mathscr{O}_X(U)$-module generated by

$$\left(\frac{\partial}{\partial x_i}\right)\mathscr{O}_X / \left(x_i\frac{\partial}{\partial x_i}\right)\mathscr{O}_X \cong \left(\frac{\partial}{\partial x_i}\right)\mathscr{O}_{D_i}$$

for $i > k$. These generate the tangent sheaf to D.

For a more global coordinate-free version, first tensor the standard exact sequence $0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$ by $\mathscr{T}_X$ to get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}_X(-D) \arrow[r] \arrow[d, "\alpha"] & \mathscr{T}_X \arrow[r] \arrow[d, "\beta"] & \mathscr{T}_X \otimes \mathscr{O}_D \arrow[r] \arrow[d, "\gamma"] & 0 \\
 & 0 \arrow[r] & \bigoplus_j \mathscr{N}_{D_j/X} \arrow[r] & \bigoplus_j \mathscr{N}_{D_j/X} \arrow[r] & 0
\end{tikzcd}$$

Clearly α is surjective, and so the snake lemma gives the exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to 0.$$

It suffices to observe that $\ker\alpha = \mathscr{T}_X(-D), \ker\beta = \mathscr{T}_X(-\log D)$ by Lemma 1.1.3, and lastly $\ker\gamma = \Theta_D$. $\square$

1.1.2 Rational Singularities

An important class of singularities that appear in the minimal model program are *rational singularities*. Consider a *rational curve* $E \subset Y$ on a smooth surface with negative self intersection. There is a contraction map $f : (Y, E) \to (X, p)$ with $p \in X$ possibly a singular point.

Since E was rational, we would like $p \in X$ to be an example of a rational singularity. For a more general *exceptional locus* (i.e. a simple normal crossing divisor with negative definite self intersection matrix), we want a notion of rationality of the locus.

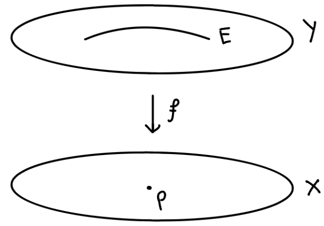

Figure 1.1: One rational exceptional curve

Lemma 1.1.6. *Let $E = \bigcup E_i$ be a simple normal crossing divisor with negative definite intersection matrix. Then there is a unique minimal positive divisor Z supported on E so that $Z \cdot E_i \leq 0$ for every E_i. Here, minimality means that any for any other Z' with this property, one finds $Z' \geq Z$.*

Proof. We first show existence. By negative definiteness, we can find a divisor Z with $Z \cdot E_i < 0$ for every E_i. Write $Z = A - B$ where A, B are effective with no common components. Then $A \cdot B \geq 0, B^2 \leq 0$ and $(A - B) \cdot B = Z \cdot B \leq 0$ show $B = 0$. Thus $Z = A$ is effective.

For any Z, W with this property, the divisor $\gcd(Z, W)$ also satisfies this. Indeed, write $Z = \sum z_i E_i$ and $W = \sum w_i E_i$. For some index j, assume without loss of generality that $z_j \leq w_j$. Then

$$\gcd(Z, W) \cdot E_j = z_j \cdot E_j^2 + \sum_{i \neq j} \min\{z_i, w_i\} E_i \cdot E_j \leq z_j \cdot E_j^2 + \sum_{i \neq j} z_i E_i \cdot E_j = Z \cdot E_j \leq 0.$$

Furthermore, $\gcd(Z, W) \geq \sum E_i$ since if $E_i \notin \mathrm{Supp}$, then $E_i \cdot \gcd(Z, W) > 0$. Hence the ordered set of such positive divisors has a minimal element. □

Definition 1.1.7. The cycle Z in Lemma 1.1.6 is called the fundamental cycle.

Similar to varieties, one can define the arithmetic genus of Z as $(-1)^{\dim Z}(\chi(\mathscr{O}_Z) - 1)$. When Z is a divisor on a surface Y, the exact sequence $0 \to \mathscr{O}_Y(-Z) \to \mathscr{O}_Y \to \mathscr{O}_Z \to 0$ and the Riemann-Roch theorem show

$$\chi(\mathscr{O}_Z) = \chi(\mathscr{O}_Y) - \chi(\mathscr{O}_Y(-Z)) = -\frac{1}{2}(-Z)(-Z - K)$$

and so $p_a(Z) = 1 + \frac{1}{2}(K \cdot Z + Z^2)$. One can prove that we always have $p_a(Z) \geq 0$ (cf. [Art66, Thm. 3]), and so this cycle Z turns out to be the right one to define rationality of a singularity.

Definition 1.1.8. A normal surface singularity $p \in X$ with minimal resolution $f : Y \to X$ is called rational if $p_a(Z) = 0$, where Z is the fundamental cycle of $f^{-1}(p)_{\mathrm{red}}$.

Example 1.1.9. Let $F \subset Y$ be a rational curve with three cyclic quotient singularities $\frac{1}{p}(1, a)$, $\frac{1}{q}(1, b)$ and $\frac{1}{r}(1, c)$, and say $F^2 < -1$. If there exists a contraction $(Y, F) \to (X, x)$ to a surface, we claim that $x \in X$ is a rational singularity.

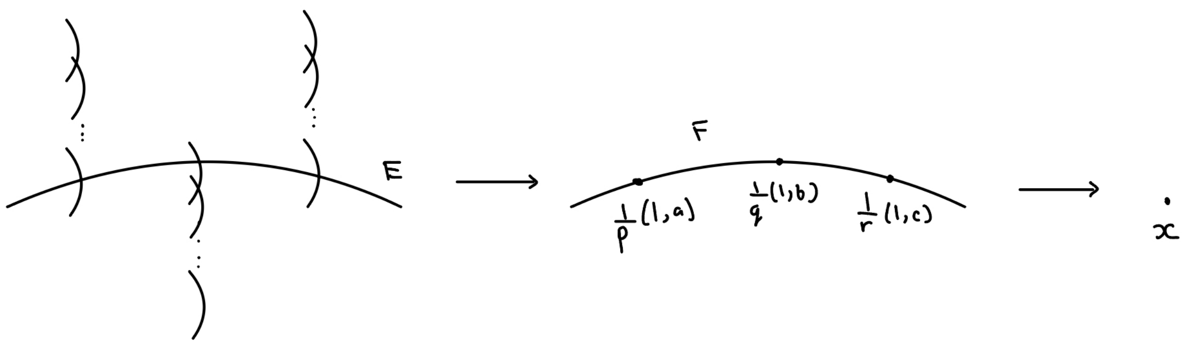

Figure 1.2: A rational fork with maps $\widetilde{Y} \to Y \to X$

Let $f : \widetilde{Y} \to Y$ be the minimal resolution of these three singularities, and set E to be the strict transform of F. Then $F^2 < -1$ implies $E^2 \le -2$. Let $D := f^{-1}(F)_{\mathrm{red}}$.

1. Suppose $E^2 < -2$. Then we claim the fundamental cycle is $Z = D$. Indeed, for any exceptional E_i, it is clear that $E_i^2 \le -2$ and it intersects at most two other curves, showing $Z \cdot E_i \le 0$. Since $E^2 \le -3$ and it intersects three exceptional curves, we find $Z \cdot E = 0$ as well. Write $Z = \sum E_i$. Let n be the number curves, so that there are $n - 1$ intersection points of curves in the locus. It is easy to then calculate that $Z^2 = \sum E_i^2 + 2(n-1)$ and $K \cdot Z = \sum(-E_i^2 - 2) = -\sum E_i^2 - 2n$. Hence $K \cdot Z + Z^2 = -2$, as desired.

2. Suppose $E^2 = -2$. Then we claim the fundamental cycle is $Z = D + E$. Indeed, $Z \cdot E_i \le 0$ for any exceptional E_i as argued above. Furthermore, $Z \cdot E = 2(-2) + 3 = -1 < 0$ too. Write $Z = 2E + \sum E_i$. Let n be the number curves E_i, so that there are $n - 3$ intersection points of curves in the locus $\{E_i\}$. It is easy to then calculate that $Z^2 = -8 + (\sum E_i^2 + 2(n-3)) + 4(3) = \sum E_i^2 + 2n - 2$ and $K \cdot Z = \sum(-E_i^2 - 2) = -\sum E_i^2 - 2n$. Thus $K \cdot Z + Z^2 = -2$, as desired.

We shall call the divisor D (or even the curve F, abusing notation) a rational fork.

We now consider an alternate definition which, although more technical, is more useful and easier to generalize.

Definition 1.1.10. An isolated singularity $p \in X$ is called a rational singularity if for one resolution (or equivalently, any resolution) $f : Y \to X$ satisfies $R^p f_* \mathscr{O}_X = 0$ for all $p > 0$.

Here, by a *resolution* of $p \in X$, we mean a proper birational morphism from a smooth variety Y to X. The first thing one needs to check is that the definition is independent of the choice of resolution. This follows from the following:

Lemma 1.1.11. *Let $f' : Y' \to X$ and $f : Y \to X$ be two resolutions of a normal variety X. Then*

$$R^p f'_* \mathscr{O}_{Y'} \simeq R^p f_* \mathscr{O}_Y \quad \forall\, p \geq 0.$$

Proof. Since we can pick a common resolution $Y'' \to X$ with maps $g' : Y'' \to Y'$ and $g : Y'' \to Y$ so that $g' \circ f' = g \circ f$, it suffices to prove the result in the case when there is a resolution map $\pi : Y' \to Y$.

$$\begin{tikzcd} Y' \arrow[d, "\pi"'] \arrow[dr, "f'"] & \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

The relative version of the Leray spectral sequence says $E_2^{p,q} = R^p f_*(R^q \pi_* \mathscr{O}_{Y'}) \Rightarrow R^{p+q} f'_* \mathscr{O}_{Y'}$. Since Y is smooth, $R^q \pi_* \mathscr{O}_{Y'} = 0$ for $q > 0$. By normality $R^p f_* \mathscr{O}_Y \simeq R^p f_*(\pi_* \mathscr{O}_{Y'}) \simeq R^p f'_* \mathscr{O}_{Y'}$. $\square$

The next thing we verify is that this coincides with our previous definition when X is a surface.

Lemma 1.1.12. *Let $f : Y \to X$ be the minimal resolution of a normal surface singularity $p \in X$, and say $X \backslash p$ is smooth. Let Z be the fundamental cycle of $f^{-1}(p)_{red}$. Then*

$$p_a(Z) = 0 \Longleftrightarrow R^i f_* \mathscr{O}_Y = 0, \quad i = 1, 2.$$

Proof. See [Art66, Prop. 1]. $\square$

It is well that a rational fork is log canonical iff $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\geq 1$. Clearly, hence, a general fork would not be log canonical, giving an infinite family of examples of rational but not log canonical singularities. In general, there are log canonical singularities which are not rational. A classical example is a cone over an elliptic curve. However, one can use a version of Fujita's vanishing theorem to show that log terminal singularities are always rational, cf. [KM98, Thm. 5.22]. For a normal Gorenstein variety, however, canonical is equivalent to rational. One proves this by showing the following important fact:

Lemma 1.1.13 (Lem. 5.12, [KM98]). *A normal singularity $p\in X$ is rational if and only if X is Cohen-Macaulay and $f_*\omega_Y\simeq\omega_X$ for any resolution $f:Y\to X$.*

Using this, if $K_Y=f^*K_X+E$ and X is Gorenstein (i.e. K_X is Cartier), then $f_*\mathscr{O}_Y(K_Y)=\mathscr{O}_X(K_X)$ shows E is effective. Hence $p\in X$ must be canonical, as claimed above.

1.1.3 Double Normal Crossing Singularities

When studying the moduli space $\overline{\mathcal{M}}_{K^2,\chi}$ of surfaces with fixed topological invariants, the boundary divisors, under some conditions, correspond to the so-called *orbifold double normal crossing* singularities. But first, we define double normal crossing surfaces.

Definition 1.1.14. A double normal crossing singularity is locally the intersection of two normal crossing hyperplanes $(x_1x_2=0)\subset\mathbb{C}^3$.

Let X be a double normal crossing surface made of two irreducible components $X_1\cup X_2$ intersecting transversally along the curve C. As usual, one defines the sheaf $\mathscr{T}^1_X:=\mathscr{E}xt^1(\Omega^1_X,\mathscr{O}_X)$. In this case, there is an explicit description of this in terms of bundles on X_1,X_2.

Lemma 1.1.15 ([Fri83], Prop. 2.3). *Let $X=X_1\cup X_2$ be a double normal crossing singularity. The sheaf $\mathscr{T}^1_X$ is supported on $C=X_1\cap X_2$, and is given by*

$$\mathscr{T}^1_X\simeq\mathscr{N}_{C/X_1}\otimes\mathscr{N}_{C/X_2}\simeq\mathscr{O}_C(C_1|_C+C_2|_C)$$

where $C_i:=C|_{X_i}$.

Consider the local picture over $U \subset X$ where X_i is given by $(x_i = 0)$. Locally, a section of $\mathscr{N}_{C/x_i}$ is generated by $\partial/\partial x_i$. Thus, $\partial/\partial x_1 \otimes \partial/\partial x_2$ generates $\mathscr{T}^1_X|_U$. Now recall that sections of $\mathscr{T}^1_X$ parameterize infinitesimal deformations. Explicitly, a section $s = f(x_3) \cdot (\partial/\partial x_1 \otimes \partial/\partial x_2)$ of $\mathscr{T}^1_X|_U$ corresponds to the first order deformation

$$(x_1 x_2 = f(x_3) \cdot t) \in \mathbb{C}^3 \times \operatorname{Spec}(\mathbb{C}[t]/(t^2)).$$

Definition 1.1.16. An orbifold double normal crossing singularity is locally the intersection of two normal crossing hyperplanes $(x_1 x_2 = 0) \subset \frac{1}{n}(1, -1, a)$ for some coprime n, a. On each surface, these singularities are cyclic quotient singularities.

We want to study deformations of these. Locally, there is a clear one, namely

$$(x_1 x_2 = t) \subset \tfrac{1}{n}(1, -1, a) \times \mathbb{C}_t.$$

Every such orbifold has a cyclic n-cover given by the quotient map

$$\mathbb{C}^3 \twoheadrightarrow \mathbb{C}^3 \Big/ \begin{pmatrix} \zeta_n & 0 & 0 \\ 0 & \zeta_n^{-1} & 0 \\ 0 & 0 & \zeta_n^a \end{pmatrix} =: \tfrac{1}{n}(1, -1, a).$$

This is an example of a so-called *index one cover*. We want our deformation to locally be induced by these. More formally,

Definition 1.1.17. A deformation of an orbifold germ $X \subset \frac{1}{n}(1, -1, a)$ is $\mathbb{Q}$-Gorenstein if it induced by an equivariant deformation of the index one cover.

To study global deformations and define analogues of the sheaves $\mathscr{T}^i$ and vector spaces $\mathbb{T}^i$, one must work on a sort of global cover that locally is just a double normal crossing surface. See [Hac12] for a detailed account on this. For our purposes, it is enough to know that the glued data of index one covers forms a Deligne-Mumford stack $\mathcal{X}$ with coarse moduli space X and induced map $q : \mathcal{X} \to X$. Locally, if $p : V \to U$ is an index one cover of some $U \subset X$ with the group $G = \mathbb{Z}/n\mathbb{Z}$ acting on V, then $q|_U$ is the map $[V/G] \to U$ from the quotient stack to its moduli space.

Definition 1.1.18. This stack $q : \mathcal{X} \to X$ is called a index one covering stack.

Given a coherent sheaf $\mathscr{F}$ on the étale site of $\mathcal{X}$, take a local index one cover $p : V \to U$ and define $\mathscr{F}_V$ to be the G-equivariant sheaf on V obtained by restricting $\mathscr{F}$ to V. Then we define the coherent sheaf $q_*\mathscr{F}$ on X by gluing the sheaves $(p_*\mathscr{F}|_V)^G$. On the stack $\mathcal{X}$, one can define the cotangent complex $\mathscr{L}_{\mathcal{X}/\mathbb{C}}$ on the étale site of $\mathcal{X}$ to study its deformations. Then, one can define the sheaves $\mathscr{T}^i_{QG,X} := q_* \mathscr{E}xt^i(L^i_{\mathcal{X}/\mathbb{C}}, \mathscr{O}_{\mathcal{X}})$ and the vector spaces $\mathbb{T}^i_{QG,X} := q_* \operatorname{Ext}^i(L^i_{\mathcal{X}/\mathbb{C}}, \mathscr{O}_{\mathcal{X}})$. These extend our previous theory to $\mathbb{Q}$-Gorenstein deformations of X, for instance

We can define the $\mathbb{Q}$-divisors $C_i := C_{X_i}|_C$ on each surface by moving C away from the singular points, of which there are a finite number. In this way, we can prove a result similar to Lemma 1.1.15, namely

$$\mathscr{T}^1_{QG,X} = \mathscr{O}_C(C_1|_C + C_2|_C). \tag{1.5}$$

Let $p : V \to U$ be a local index one cover of $\frac{1}{n}(1, -1, a)$. Noting that x_1x_2 is fixed under $G = \mathbb{Z}/n\mathbb{Z}$, a local section $s = f(x_3)\cdot(\partial/\partial x_1 \otimes \partial/\partial x_2) \in H^0(\mathscr{T}^1_V)$ is G-equivariant if $f = g^n$ for some $g \in \mathbb{C}[x_3]$. This induces a local section of $\mathscr{T}^1_{QG,U}$, which gives the first order deformation

$$(x_1x_2 = g(x_3^n)\cdot t) \in \tfrac{1}{n}(1, -1, a) \times \operatorname{Spec}(\mathbb{C}[t]/(t^2))$$

extending what we had for double normal crossing surfaces. Hence a necessary condition for the existence of a smoothing is $H^0(\mathscr{T}^1_{QG,X}) \neq 0$. We can say more than this.

Lemma 1.1.19. *If X admits a smoothing with reduced fiber X (i.e. a semi-stable degeneration), then $C_1^2 + C_2^2 = g(C)$. In this case we say X is d-semi-stable.*

Proof. We can define C_1, C_2 as $\mathbb{Q}$-divisors away from the singular points. Then from [Fri83, 2.4], we must have $\mathscr{O}_C(C_1|_C + C_2|_C) \simeq \mathscr{T}^1_{QG,X} \simeq \mathscr{O}_C$. Hence, Riemann-Roch shows $C_1^2 + C_2^2 = \deg \mathscr{T}^1_{QG,X} = g(C)$ as desired. $\square$

1.2 Sketch of approach

In section 2.2, we consider the rational elliptic surface Z with singular fibers I6, I3, I2 and I1, perform a series of blowups followed by blowdowns. This way, we get a rational singular surface S with $\rho(S) = 2$ and a rational curve C on it. In Proposition 2.2.1, we show that S is general type, that is K_S is ample. The curve C passes through the singular locus of S, which are three cyclic quotient singularities.

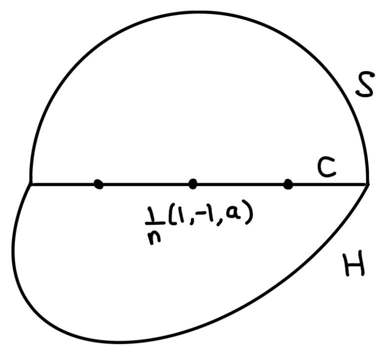

Figure 1.3: The surface $Y = S \cup H$

We glue S to an affine rational surface H along C so that $Y := S \cup H$ is an orbifold double normal crossing surface, and the self intersection number of the strict transform of $C|_H$ on the minimal resolution of H is -2. Thus $C|_H$ has three cyclic quotient singularities, dual to the ones on S, and H is smooth away from $C|_H$. The contraction of $C|_H$ gives a map $H \to X$ with a rational singularity $q \in X$ since $C|_H$ is a rational fork (cf. Section 1.1.2). This is our candidate to disprove the conjecture. We show that Y admits a smoothing $\mathscr{Y}/(0 \in \Delta)$ in Proposition 2.2.8, and that this blows down to a smoothing $\mathscr{X}/(0 \in \Delta)$ of X in Proposition 2.0.1, using methods inspired from the work of Lee&Park in [LP07]. In Section 2.1, we prove that the canonical algebra of $\mathscr{X}$ is not finitely generated using a proof similar to one used in [Cut88], assuming a result:

Conjecture 1.2.1. *Let the ray through C and $K_S + C$ in the nef cone of S intersect the extremal ray $(K_S + C) - \alpha C$. Then α is irrational.*

As $\rho(S) = 2$, the cone has two extremal rays. The image E of the curve J (see Figure 2.1) under the contraction $f : \widetilde{S} \to S$ has negative self intersection, and hence generates a ray of the nef cone.

However, equation 2.2 shows $(f^* K_S) \cdot J = -1 + \frac{9}{13} + \frac{5}{13} > 0$, and hence $(K_S + C) \cdot (f_* J) > C \cdot J$. Thus, the ray through C and $K_S + C$ intersects the other extremal ray.

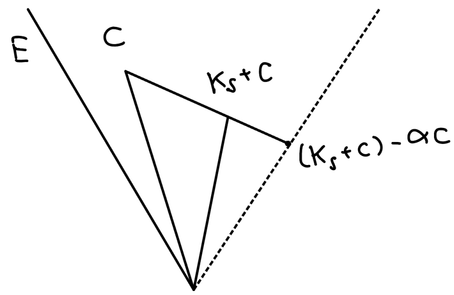

Figure 1.4: $\mathrm{nef}(S)$

Hence, Conjecture 1.2.1 is equivalent to the following question:

Question 1.2.2. *Does* $\mathrm{nef}(S)$ *have an irrational extremal ray?*

Since S is of general type, the cone theorem doesn't apply here. We have no concrete approach to prove this at the moment. It might be note-worthy to mention that this is related to an important and difficult line of questions in algebraic geometry, the most famous one being Nagata's conjecture.

1.3 Acknowledgments

I am extremely grateful to Professor Paul Hacking for being my advisor and mentoring me. His constant supply of ideas, motivation and support made this project possible. I had fruitful discussions with Professor Jenia Tevelev, to whom I am ever grateful for being my first mentor in the field of algebraic geometry. My advisor also had helpful discussions with János Kollár. I would also like to thank Professor Eyal Markman for being on the thesis committee. I would like to thank the organizers of the UMass 2024 Math REU, which is where this project started. Finally, I would like to thank my mother for being an perpetual source of love and support.

Chapter 2

Disproving Kollár's conjecture

Suppose we construct a rational surface singularity $q \in X$ with the following properties:

(a) There is a partial resolution $f_0 : Y \to X$ so that $Y = S \cup H$, is an *orbifold double normal crossing surface*, where $S := f_0^{-1}(q)$ is a surface with a rational curve $C := S|_H$. Here, H is the strict transform of X and $Y = S \cup H$ is reducible surface which is locally a simple normal crossing divisor along C.

(b) S is a (singular) surface so that

  (i) $H^1(\mathscr{O}_S) = H^2(\mathscr{O}_S) = 0$;

  (ii) S of general type, i.e. K_S is ample;

  (iii) a rational curve $C \subset S$ with three cyclic singularities of the form $\frac{1}{n}(1, a)$.

(c) Suppose the ray from C to $K_S + C$ intersects an irrational extremal ray of the nef cone $\mathrm{Nef}(S)$ at $(K_S + C) - \alpha C$ with $\alpha \notin \mathbb{Q}$.

(d) Y admits a smoothing $\mathscr{Y}$ over the disc $\Delta \cong \operatorname{Spec} \mathbb{C}[[t]]$.

Proposition 2.0.1. *The smoothing $\mathscr{Y}$ blows down to a smoothing $\mathscr{X}$ of X, i.e. there is a blowdown* *map f extending f_0 to a map over Δ so that the following diagram commutes*

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, dashed, "f_0"] \arrow[dd, bend right] & \mathscr{Y} \arrow[d, dashed, "f"] \arrow[dd, bend right] \\
X \arrow[r, hook] \arrow[d] & \mathscr{X} \arrow[d] \\
0 \arrow[r, hook] & \Delta
\end{tikzcd}$$

Proof. Consider the exact sequence $0 \to \mathscr{O}_Y \to \mathscr{O}_S \oplus \mathscr{O}_H \to \mathscr{O}_C \to 0$. Taking pushforwards,

$$0 \to \mathscr{O}_X \to \mathscr{O}_q \oplus \mathscr{O}_X \to \mathscr{O}_q \to R^1(f_0)_*\mathscr{O}_Y \to R^1(f_0)_*\mathscr{O}_S \oplus R^1(f_0)_*\mathscr{O}_H.$$

For $i > 0$, $R^i(f_0)_*\mathscr{O}_S = 0$ by the theorem of formal functions since $H^i(\mathscr{O}_{nS}) = 0$ (as S is rational) and $R^1(f_0)_*\mathscr{O}_H = 0$ since $H \to X$ is a resolution of the rational singularity q. Furthermore, $\mathscr{O}_q \oplus \mathscr{O}_X \to \mathscr{O}_q$ is surjective, and thus we conclude $R^1(f_0)_*\mathscr{O}_Y = 0$. We can thus construct the smoothing $\mathscr{X} := \operatorname{Spec} H^0(\mathscr{O}_{\mathscr{Y}})$ of X, see [Rie73, Thm. 2] $\square$

2.1 Non-finite generation of the canonical algebra of $\mathscr{X}$

We show that the canonical algebra

$$\mathscr{R} := \bigoplus_{n \geq 0} \mathscr{O}_{\mathscr{X}}(nK_{\mathscr{X}})$$

is not a finitely generated $\mathscr{O}_{\mathscr{X}}$-algebra. Assume not, and define W to be $\underline{\operatorname{Proj}}_{\mathscr{X}}(\mathscr{R})$ with the natural map $g : W \to \mathscr{X}$. As discussed in [Kol91, 6.2.2.], g is a small modification. Let $\psi : \mathscr{Y} \dashrightarrow W$ be the map $g \circ f^{-1}$ define on an open set of $\mathscr{Y}$, whose indeterminacy locus lies in S. Let $\tau : U \to \mathscr{Y}$ be a smooth resolution of the indeterminancies of ψ. Then there's an associated birational morphism $\lambda : U \to W$. Lastly, let $h = f \circ \tau : U \to \mathscr{X}$.

$$
\begin{tikzcd}
 & U \arrow[d, "\tau"'] \arrow[dr, "\lambda"] \arrow[dd, bend right=60, "h"'] & \\
 & (\mathscr{Y}, Y) \arrow[r, dashed, "\psi"] \arrow[d, "f"'] & W \arrow[dl, "g"] \\
 & (\mathscr{X}, X) &
\end{tikzcd}
$$

By a result of Elkik [Elk78][1], $q \in \mathscr{X}$ is also a rational singularity. We now work locally on the level of germs (at q). Rationality of q implies the inclusion $f_*\omega_{\mathscr{Y}} \hookrightarrow \omega_{\mathscr{X}}$ is an isomorphism, c Lemma 1.1.13. Let E' be an effective divisor on $\mathscr{X}$ so that $K_{\mathscr{X}} \sim -E'$ by working locally away from the poles. Let E be the strict transform of E' on $\mathscr{Y}$. Since $f_*K_{\mathscr{Y}} = -E'$, hence $K_{\mathscr{Y}} = -E - \mu S$ for some $\mu \in \mathbb{Q}$. As E, E' are effective, we claim that

$$f_*\mathscr{O}_{\mathscr{Y}}(-nE) = \mathscr{O}_{\mathscr{X}}(-nf_*E) = \mathscr{O}_{\mathscr{X}}(-nE') \quad \forall\, n \in \mathbb{Z}_{\geq 0}.$$

Indeed, recall [Kol13, Lem. 7.30]:

Lemma 2.1.1 (Fujita, 1985)**.** *Let $f\colon Y \to X$ be a proper, birational morphism. Let D be a $\mathbb{Z}$-divisor on X and assume that $D \sim_{\mathbb{Q}} f^*C + D_h + D_v$, C is some $\mathbb{Q}$-divisor, where D_v is f-exceptional, $\lfloor D_v \rfloor = 0$ and D_h is effective without exceptional components. Let B be any effective, f-exceptional divisor. Assume that*

(1) either X and Y are normal,

(2) or X and Y are S_2, f is an isomorphism outside a codimension 2 subscheme of X and Y is normal at the generic point of every exceptional divisor.

Then $\mathscr{O}_X(-f_*D) = f_*\mathscr{O}_Y(-D) = f_*\mathscr{O}_Y(B - D)$.

Using this, we find $f_*\mathscr{O}_{\mathscr{Y}}(n(K_{\mathscr{Y}} + \mu S)) = \mathscr{O}_{\mathscr{X}}(nK_{\mathscr{X}})$. Let $\mathscr{L} = \lambda^*\mathscr{O}_W(1)$, so that $\mathscr{O}_U(n\mathscr{L}) = \lambda^*\mathscr{O}_W(n)$ as τ is also a small modification. Then

$$h_*\mathscr{O}_U(n\mathscr{L}) = g_*\lambda_*(\lambda^*\mathscr{O}_W(n)) = g_*\mathscr{O}_W(n) = \mathscr{O}_{\mathscr{X}}(nK_{\mathscr{X}}) = f_*\mathscr{O}_{\mathscr{Y}}(n(K_{\mathscr{Y}} + \mu S)). \tag{2.1}$$

[1] also see [Eri14, Theorem 1.8]

by the projection formula and the fact that g is a small modification. Let $\tau_*\mathscr{L} = K_{\mathscr{Y}} + aS$ for some $a \in \mathbb{Q}$ (in particular, $a \neq \alpha$). As $S + H \sim 0$ on $\mathscr{Y}$, we find $(K_{\mathscr{Y}} + aS)|_S = (K_{\mathscr{Y}} + S + H)|_S - aH|_S = (K_S + C) - aC$. By construction, this divisor is nef on S if $a < \alpha$. Hence if $a < \alpha$, then $K_{\mathscr{Y}} + aS$ is f-nef as any exceptional curve of f lies on S. Consider now the following lemma:

Lemma 2.1.2. *Suppose Z is smooth with birational morphisms:*

$$\begin{tikzcd} (Z, \tau^*D) \arrow[dd, "h"'] \arrow[dr, "\tau"] & \\ & (Y, D) \arrow[dl, "f"] \\ X & \end{tikzcd}$$

Let $D \subset Y$ be an f-nef divisor, and F an irreducible codimension 1 subvariety of Z contracted under h. Then for any $b > 0$ rational, for $n \gg 0$ with $nb \in \mathbb{Z}$,

$$f_*\mathscr{O}_Y(n(D - b\tau_*F)) \subsetneq f_*\mathscr{O}_Y(nD).$$

Proof. From [Cut88, Lem. 2], there is an $r > 0$ such that $h_*\mathscr{O}_Z(n\tau^*D - rF) \subsetneq f_*\mathscr{O}_Y(nD)$. The result follows from the chain of inclusions

$$f_*\mathscr{O}_Y(n(D - b\tau_*F)) = f_*\mathscr{O}_Z(n(\tau_*(\tau^*D - bF))) = h_*\mathscr{O}_Z(n\tau^*D - nbF) \subseteq h_*\mathscr{O}_Z(n\tau^*D - rF) \subsetneq f_*\mathscr{O}_Y(nD)$$

for $n \gg 0$ with $nb \in \mathbb{Z}$ and $nb \geq r$. $\square$

Consider the case $a < \alpha$. If $a \leq \mu$, pick $a' \in (a, \min\{\alpha, \mu\})$ rational, possibly $a' = \mu$. Then $K_{\mathscr{Y}} + aS$ and $K_{\mathscr{Y}} + a'S$ are f-nef as explained above. Then Lemma 2.1.2 with $b = a' - a \in \mathbb{Q}_{>0}$ shows

$$h_*\mathscr{O}_U(n\mathscr{L}) = f_*\mathscr{O}_{\mathscr{Y}}(n(K_{\mathscr{Y}} + aS)) \subsetneq f_*\mathscr{O}_{\mathscr{Y}}(n(K_{\mathscr{Y}} + a'S)) \subseteq f_*\mathscr{O}_{\mathscr{Y}}(n(K_{\mathscr{Y}} + \mu S))$$

contradicting Equation 2.1. If $a > \mu$, in which case we pick $a' \in (\mu, a)$ and similarly argue $h_*\mathscr{O}_U(n\mathscr{L}) \supsetneq f_*\mathscr{O}_{\mathscr{Y}}(n(K_{\mathscr{Y}} + \mu S))$, again contradicting Equation 2.1.

Consider $a > \alpha$. As the ray of nef cone is extremal and irrational, it is also extremal in the effective cone. Hence $H^0(U, \tau_* \mathscr{L}|_S) = H^0(S, \mathscr{O}_S(K_S + C - aC)) = 0$ in this case. However $\tau^* \mathscr{O}_W(1)$ is clearly base-point free and has non-trivial sections, which in particular gives a non-vanishing section on $\mathscr{Y}$. Any such section doesn't vanish on S by base-point freeness, hence a contradiction.

2.2 Construction of Y

Consider a rational elliptic surface $\pi : Z = \mathrm{Bl}_9(\mathbb{P}^2) \to \mathbb{P}^1$ with singular fibers I6, I3, I2, I1. Observe that this has Mordell-Weil rank 8. Consider two sections that intersect neighboring divisors on the I_6 fiber, perform 4 blowups as shown, and contract every curve shown except the three (-1) curves and the section $\widetilde{C}$. This gives a surface S with a rational curve C on it, with the cyclic quotient singularities $\frac{1}{37}(1,17)$, $\frac{1}{26}(1,7)$ and $\frac{1}{3}(1,2)$. Recalling Example 1.1.9, C is a *rational fork*, and so contracting C gives a rational singularity which is not log-canonical since $\frac{1}{37} + \frac{1}{26} + \frac{1}{3} < 1$.

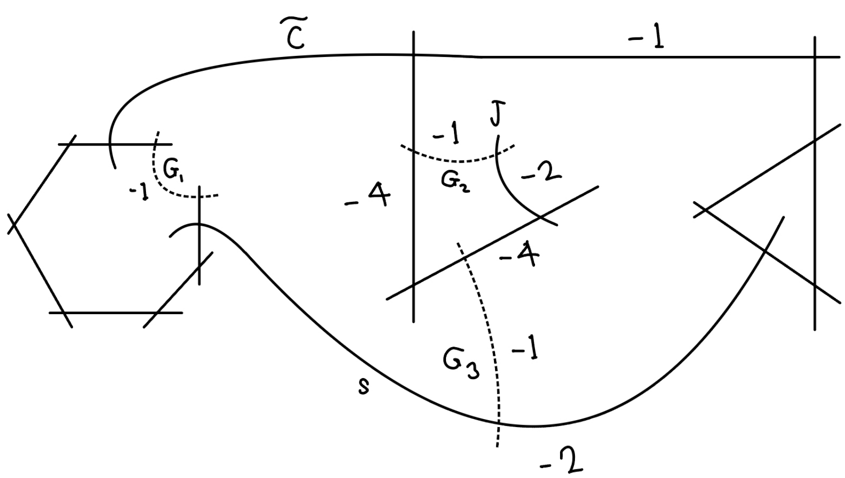

Figure 2.1: The surface $\widetilde{S}$

Proposition 2.2.1. *K_S is ample.*

Proof. We calculate

$$f^* K_S = K_{\widetilde{S}} + \left(\frac{19}{37} C_1^{(6)} + \cdots + \frac{24}{37} C_6^{(6)} + \frac{12}{37} S_1 \right) + \left(\frac{9}{13} C_1^{(2)} + \frac{10}{13} C_2^{(2)} + \frac{5}{13} C_3^{(2)} \right). \tag{2.2}$$

For a generic fiber F of $\pi : Z \to \mathbb{P}^1$, $f^*K_S \cdot F = \frac{12}{37} > 0$. Next, $f^*K_S \cdot C = -1 + \frac{19}{37} + \frac{9}{13} > 0$, $f^*K_S \cdot E_1 = -1 + \frac{19}{37} + \frac{24}{37} > 0$ $f^*K_S \cdot E_2 = -1 + \frac{9}{13} + \frac{5}{13} > 0$ and $f^*K_S \cdot E_3 = -1 + \frac{10}{13} + \frac{12}{37} > 0$. It is clearly that f^*K_S has positive degree on any other curve with self-intersection ≥ -2 since $K_{\widetilde{S}} \cdot G \geq 0$ for any such G. Thus, K_S is ample. $\square$

Next, we want to show $Y = S \cup H$ has a smoothing. Around any singular point, the germ of Y is analytically locally the orbifold $(xy = 0) \subset \frac{1}{n}(1, -1, a)$. This has a one-parameter smoothing given by $(xy = t) \subset \frac{1}{n}(1, -1, a) \times \mathbb{A}^1_t$. We want to show these glue together to give a smooth deformation of Y, for which it suffices to prove $H^2(\mathscr{T}_Y) = 0$. We do this in a series of results.

Lemma 2.2.2. *$f_*\mathscr{N}_{\widetilde{C}/\widetilde{S}} = \mathscr{N}_{C/S}$, where $\mathscr{N}_{C/S}$ is defined by the index one covering stack of S.*

Proof. S has *orbifold double normal crossing* singularities, and hence is semi-log canonical. Furthemore, it is smooth outside these orbifolds on C, and K_S is ample by Proposition 2.2.1. Thus S has an index one covering stack $q : S' \to S$, with $C' := f^{-1}(C)$ (cf. [Hac12, Sec. 3]). The map is an isomorphism outside the three singular points on C. Over the germ of each singular point $p \in C$, q restricts to a map q' which is a cyclic cover, étale over $S \backslash p$. For any coherent sheaf $\mathscr{F}$ on S', one can define the sheaf $q_*\mathscr{F}$ on S locally as $(q'_*\mathscr{F})^{\mathbb{Z}/n\mathbb{Z}}$ around each singular point. In this way, we define $\mathscr{N}_{C/S} := q_*\mathscr{N}_{C'/S'}$. On S', we have the exact sequence

$$0 \to \mathscr{O}_{S'} \to \mathscr{O}_{S'}(C') \to \mathscr{N}_{C'/S'} \to 0. \tag{2.3}$$

As we are working over $\mathbb{C}$, the functor $\mathscr{F} \mapsto (q'_*\mathscr{F})^{\mathbb{Z}/n\mathbb{Z}}$ is exact. Now, $q_*\mathscr{O}_{S'} = \mathscr{O}_S$ and $q_*\mathscr{O}_{S'}(C') = \mathscr{O}_S(C)$ by Hartog's property. Hence, Equation 2.3 implies $0 \to \mathscr{O}_S \to \mathscr{O}_S(C) \to \mathscr{N}_{C/S} \to 0$, giving an alternate description of $\mathscr{N}_{C/S}$ as $\mathrm{coker}\{\mathscr{O}_S \to \mathscr{O}_S(C)\}$.

Consider the exact sequence on $0 \to \mathscr{O}_{\widetilde{S}} \to \mathscr{O}_{\widetilde{S}}(\widetilde{C}) \to \mathscr{N}_{\widetilde{C}/\widetilde{S}} \to 0$ on $\widetilde{S}$. Since S has rational singularities, $R^1f_*\mathscr{O}_{\widetilde{S}} = 0$, and so we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & f_*\mathscr{O}_{\widetilde{S}} \arrow[r] \arrow[d, "\wr"] & f_*\mathscr{O}_{\widetilde{S}}(\widetilde{C}) \arrow[r] \arrow[d, "\wr"] & f_*\mathscr{N}_{\widetilde{C}/\widetilde{S}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_S \arrow[r] & \mathscr{O}_S(C) \arrow[r] & \mathscr{N}_{C/S} \arrow[r] & 0 \end{tikzcd}$$

The isomorphism $f_*\mathscr{O}_{\widetilde{S}} = \mathscr{O}_S$ follows from normality of S. To prove our lemma, it hence suffices to show $f_*\mathscr{O}_{\widetilde{S}}(\widetilde{C}) = \mathscr{O}_S(C)$. Since $\widetilde{S} \to S$ is the minimal resolution, $f^*C = \widetilde{C} + C_e = \widetilde{C} + \sum_i \alpha_i E_i$ with $\alpha_i \in [0,1)$, where E_i are f-exceptional curves. In particular, $\lfloor f^*C \rfloor = \widetilde{C}$, and so

$$f_*\mathscr{O}_{\widetilde{S}}(\widetilde{C}) = f_*\mathscr{O}_{\widetilde{S}}(\lfloor f^*C \rfloor) = \mathscr{O}_S(C)$$

follows from a more general result; in our case we can also argue by first noting the inclusion $f_*\mathscr{O}_{\widetilde{S}}(\widetilde{C}) \hookrightarrow \mathscr{O}_S(C)$, and then arguing that $g \in \mathscr{O}_S(C)$ pulls back to $f^*g \in \mathscr{O}_{\widetilde{S}}(\widetilde{C})$. As $(f^*g) + \widetilde{C} + C_e = (f^*g) + f^*C \geq 0$, it follows that $(f^*g) + \widetilde{C} \geq 0$ since f^*g is regular and $\lfloor C_e \rfloor = 0$. □

Lemma 2.2.3. $f_*\mathscr{T}_{\widetilde{S}}(-\log \widetilde{C}) \cong \mathscr{T}_S(-\log C)$.

Proof. We firstly claim there is an exact sequence $0 \to \mathscr{T}_S(-\log C) \to \mathscr{T}_S \to \mathscr{N}_{C/S}$. Indeed, again taking the index one covering stack $q : S' \to S$, the corresponding exact sequence is $0 \to \mathscr{T}_{S'}(-\log C') \to \mathscr{T}_{S'} \to \mathscr{N}_{C'/S'} \to 0$. Now $q_*\mathscr{T}_{S'} = \mathscr{T}_S$ and $q_*\mathscr{T}_{S'}(-\log C') = \mathscr{T}_S(-\log C)$ Then consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathscr{T}_{\widetilde{S}}(-\log \widetilde{C}) \arrow[r] \arrow[d] & f_*\mathscr{T}_{\widetilde{S}} \arrow[r] \arrow[d, "\wr"] & f_*\mathscr{N}_{\widetilde{C}/\widetilde{S}} \arrow[d, "\wr"] \\
0 \arrow[r] & \mathscr{T}_S(-\log C) \arrow[r] & \mathscr{T}_S \arrow[r] & \mathscr{N}_{C/S}
\end{tikzcd}$$

The isomorphism $f_*\mathscr{T}_{\widetilde{S}} \xrightarrow{\sim} \mathscr{T}_S$ follows from [WJ74, Prop. 1.2], and the second isomorphism follows from Lemma 2.2.2. The desired isomorphism then follows. □

Lemma 2.2.4. *Let E be the reduced exceptional divisor of $\widetilde{S} \to S$. Then $R^i f_*\mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C}+E)) = 0$ for $i = 1, 2$.*

Proof. As f is an isomorphism outside E, a one dimensional locus, $R^2 f_*\mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C}+E))$ is clearly 0. Now we know $R^1 f_*\mathscr{T}_{\widetilde{S}}(-\log E) = 0$ by [LP07, Lem. 1]. As E is the exceptional locus over a singular point, $f_*\mathscr{N}_{E/\widetilde{S}} = 0$. Using $f_*\mathscr{T}_{\widetilde{S}} \xrightarrow{\sim} \mathscr{T}_S$ (cf. [WJ74, Prop. 1.2]) and $f_*\mathscr{N}_{\widetilde{C}/\widetilde{S}} = \mathscr{N}_{C/S}$, (cf. Lem. 2.2.2), we obtain the following commutative diagram.

$$\begin{tikzcd}
f_*\mathscr{T}_{\widetilde{S}}(-\log E) \arrow[r] \arrow[d, hook] & f_*\mathscr{N}_{\widetilde{C}/S} \arrow[r] \arrow[d, "\wr"] & R^1f_*\mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C}+E)) \arrow[r] & R^1f_*\mathscr{T}_{\widetilde{S}}(-\log E)=0 \\
\mathscr{T}_S \arrow[r] \arrow[d] & \mathscr{N}_{\widetilde{C}/\widetilde{S}} \arrow[r] & 0 \\
f_*\mathscr{N}_{E/S}=0
\end{tikzcd}$$

Hence we get the vanishing of $R^1f_*\mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C}+E))$, as desired. □

Lemma 2.2.5. $H^2(\widetilde{S},\mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C}+E))=0$.

Proof. Consider first the following general lemma:

Lemma 2.2.6. *Let X be a smooth surface with with a divisor D and a (-1) curve $C\subset X$.*

1. $H^2(\mathscr{T}_X(-\log(D+C)))\cong H^2(\mathscr{T}_X(-\log D))$.
2. *If D is effective and reduced, then* $H^2(\mathscr{T}_X(-\log D))\cong H^2(\mathscr{T}_X(-D))$.

Proof. The exact sequence $0\to\mathscr{T}_X(-\log(D+C))\to\mathscr{T}_X(-\log D)\to\mathscr{N}_{C/X}\to 0$ (cf. Lem. 1.1.4) proves the result since $H^1(X,\mathscr{N}_{C/X})=H^1(C,\mathscr{N}_{C/X})=H^0(C,\mathscr{N}^*_{C/X}\otimes\omega_C)=0$ since this bundle has negative degree and $C\simeq\mathbb{P}^1$. For the second equality, the exact sequence $0\to\mathscr{T}_X(-D)\to\mathscr{T}_X(-\log D)\to\Theta_D\to 0$ (cf. Lem. 1.1.5) shows the result since for any $C\subset\operatorname{Supp} D$, $H^1(C,\Theta_D)=H^1(C,\Theta_C)=H^0(C,\omega_C^{\otimes 2})=0$. □

We abuse notation and let I_k denote the divisor of the I_k-fiber for $k=2,3,6$. Let D be the irreducible curve in I_3 which intersects the section s. As $I_k\sim F\sim -K_Z$, with F a general fiber of $\pi: Z\to\mathbb{P}^1$, we conclude

$$H^2(\mathscr{T}_Z(-I_6-I_2-I_3+D))=H^0(\Omega_Z(K_Z+I_6+I_2+I_3-D))=H^0(\Omega_Z(2F-D))\subseteq H^0(\Omega_Z(2F)).$$

By [LP07, Lem. 2], every such section comes from $H^0(\pi^*\Omega^1_{\mathbb{P}^1}(2))=H^0(\pi^*\mathscr{O}_{\mathbb{P}^1})=H^0(\mathscr{O}_Z)\cong\mathbb{C}$. If such a section vanishes on D, it must be the zero section, showing the cohomology groups are all

0. Lemma 2.2.6 then shows

$$H^2(\mathscr{T}_Z(-I_6 - I_2 - I_3 + D - s)) = 0$$

where s is the other section. Call the three (-1) curves G_i for $i = 1, 2, 3$, and let $J \neq G_2$ be the other (-1) curve in the inverse image of I_2. Then [LP07, Prop. 6] on the composite of blowups $\widetilde{S} \to Z$ shows

$$H^2(\mathscr{T}_{\widetilde{S}}(-E - G_1 - G_2 - G_3)) = H^2(\mathscr{T}_{\widetilde{S}}(-\widetilde{I}_6 - \widetilde{I}_3 - \widetilde{I}_2 + D - G_1 - G_2 - G_3 - J - s)) = 0,$$

where $\widetilde{I}_k$ is the strict transform of I_k. Lemma 2.2.6 then shows $H^2(\mathscr{T}_{\widetilde{S}}(-\log E)) = 0$, as desired. □

Lemma 2.2.7. $H^2(S, \mathscr{T}_S(-\log C)) = 0$.

Proof. Lemmas 2.2.5 and 2.2.6 show $H^2(\widetilde{S}, \mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C} + E))) = H^2(\widetilde{S}, \mathscr{T}_{\widetilde{S}}(-\log E)) = 0$. Taking the pushforwards of $0 \to \mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C} + E)) \to \mathscr{T}_{\widetilde{S}}(-\log \widetilde{C}) \to \mathscr{N}_{E/\widetilde{S}} \to 0$, Lemma 2.2.3 and $f_* \mathscr{N}_{E/\widetilde{S}} = 0$ show

$$f_* \mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C} + E)) \xrightarrow{\sim} \mathscr{T}_S(-\log C).$$

Using the Leray Spectral sequence, Lemma 2.2.4 and the vanishing of $H^2(\mathscr{T}_{\widetilde{S}}(\log(\widetilde{C} + E)))$, we find $H^2(S, f_* \mathscr{T}_{\widetilde{S}}(-\log(\widetilde{C} + E))) = 0$. Hence $H^2(\mathscr{T}_S(-\log C)) = 0$. □

Proposition 2.2.8. *$Y = S \cup H$ admits a smoothing $\mathscr{Y}$.*

Proof. Unless otherwise stated, we will work on the stack with coarse moduli space Y and closed substack with coarse moduli C, which we will also denote by Y and C, and similarly for the components S and H of Y. Then Y has normal crossing singularities and S and H are smooth (as stacks). We know Y locally is smoothable, so to show the obstruction space $\mathbb{T}^2$ vanishes, we must show $H^2(\mathscr{T}_Y) = 0, H^1(\mathscr{T}^1_{QG,Y}) = 0$.

Firstly, $\mathscr{T}^1_{\mathrm{QG},Y} := \mathscr{E}xt^1(\Omega^1_Y, \mathscr{O}_Y) \simeq \mathscr{O}_C(C|_S + C|_H) \simeq \mathscr{O}_C$ (cf. Eq. 1.5) since $C \simeq \mathbb{P}^1$ and one can compute $C|_S^2 + C|_H^2 = 0$. Hence, this is d-semi-stable (see Lemma 1.1.19), and $H^1(\mathscr{T}^1_{QG,Y})$ vanishes. We also find that there is an intrinsically defined sheaf $\Lambda^1_Y \subset a_* \Omega^1_{Y^\nu}(\log Y_{\mathrm{sing}})$, where $a : Y^\nu \to Y$

is the normalization, which is given by $\Omega^1_{\mathscr{Y}/\Delta}(\log Y)|_Y$ for any degeneration $\mathscr{Y}$ over $\Delta = \mathbb{C}[[t]]$. We have the exact sequence $0 \to \Omega_Y/\tau_Y \to \Lambda^1_Y \to \mathscr{O}_C \to 0$, where $\tau_Y \subset \Omega_Y$ is the torsion subsheaf, dualizing which gives

$$0 \to \mathscr{T}_Y(-\log) \to \mathscr{T}_Y \to \mathscr{E}xt^1(\mathscr{O}_C, \mathscr{O}_Y) \to 0 \tag{2.4}$$

as $\mathscr{H}om_{\mathscr{O}_Y}(\Omega_Y/\tau_Y, \mathscr{O}_Y) = \mathscr{H}om_{\mathscr{O}_Y}(\Omega_Y, \mathscr{O}_Y) = \mathscr{T}_Y, \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_C, \mathscr{O}_Y) = 0$ and $\mathscr{E}xt^1(\Lambda^1_Y, \mathscr{O}_Y) = 0$ since Λ^1_Y is locally free. Next, tensoring $0 \to \mathscr{O}_Y \to \mathscr{O}_S \oplus \mathscr{O}_H \to \mathscr{O}_S \to 0$ with the (locally free) sheaf $\mathscr{T}_Y(-\log)$ gives

$$0 \to \mathscr{T}_Y(-\log) \to \mathscr{T}_S(-\log C) \oplus \mathscr{T}_H(-\log C) \to \mathscr{T}_Y(-\log C)|_C \to 0. \tag{2.5}$$

Since C has three singular points, applying the exact functor $\mathscr{F} \mapsto q_*\mathscr{F}$ to the exact sequence $0 \to \mathscr{O}_C \to \mathscr{T}_S(-\log C)|_C \to \mathscr{T}_C \to 0$ gives

$$0 \to \mathscr{O}_C \to \mathscr{T}_S(-\log C)|_C \to \mathscr{T}_C(-3) \to 0 \tag{2.6}$$

which is now an exact sequence of sheaves on the scheme C. Since C is rational, $\mathscr{O}_C$ and $\mathscr{T}_C(-3) \simeq \mathscr{O}_C(-1)$ have vanishing H^1, and hence Equation 2.6 shows $H^1(T_S(-\log C)|_C) = 0$. Then Equation 2.5 shows

$$H^2(\mathscr{T}_Y(-\log)) \cong H^2(\mathscr{T}_S(-\log C)) \oplus H^2(\mathscr{T}_H(-\log C)). \tag{2.7}$$

Here $H^2(\mathscr{T}_S(-\log C)) = 0$ by Lemma 2.2.7. Furthermore, if $g : H \to Z$ is the contraction of C, then Z is an affine neighborhood of the rational singularity p.. Hence $H^2(\mathscr{T}_H(-\log C)) = R^2 g_* \mathscr{T}_H(-\log C) = 0$ by the theorem on formal functions since the thickenings $\mathscr{T}_H(-\log C) \times_Z \operatorname{Spec} \mathscr{O}_p/\mathfrak{m}_p^n$ for all $n \geq 0$ are supported on the curve C, which is one-dimensional. Thus, Equations 2.4 and 2.7 show $H^2(\mathscr{T}_Y) = 0$, finishing the proof. $\square$

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