Lower bounds Using Formulas and Rectangles

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Overview

Literature

Khrapchenko proved a lower bound of n^2 for parity function $f(x_1, ..., x_n) = x_1 \oplus x_2 \oplus ... \oplus x_n$. Later, **Rychknov** observed that Khrapchenko's argument reduces the lower bound problems for DeMorgans formulas to a combinatorial problem about covering of Rectangles by monochromatic subrectangles.

Definitions

Rectangles

A $n-$dimensional *combinational rectangle* or *rectangle*, is a Cartesian product $R = A \times B$ of two disjoint subsets A and B of vectors in $\{0,1\}^n$. A *subrectangle* of R is a subset $S \subseteq R$ which itself forms a rectangle.

Separator f

A boolean function $f : \{0,1\}^n \to \{0,1\}$ separates a rectangle $A \times B$ if $f(A) = 1$ and $f(B) = 0$, that is $f(a) = 1 \forall a \in A$ and $f(b) = 0 \forall b \in B$.

Monochromatic Rectangles

Rectangles separated by a boolean variable x_i or by its negation $\neg x_i$ are called *monochromatic*. R is monochromatic
if $\exists i$ such that $a_i \neq b_i \forall (a,b) \in R$

Important Points

- If the monochromatic rectangles are separated by non-negated variables x_i, they are called *positively monochromatic*.
  For such cases, $a_i = 1$ and $b_i = 0$ $\forall$ edges $(a, b) \in R$
- Every $n-$dimensional rectangle can be bound covered by at most $2n$ monochromatic rectangles separated by variables $x_1, x_2, ..., x_n$ and their negations. These rectangles however overlap, but we are interested in coverings by non-overlapping rectangles.

Tiling Number

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The *tiling number* $\chi(R)$ of a rectangle R is the smallest number t such that R can be decomposed into t disjoint monochromatic rectangles. For a boolean function f, $\chi(f) = \chi(f^{-1}(1) \times f^{-1}(0))$.

- *Monotone tiling numbers*, $\chi_+(R), \chi_+(f)$, allow only *positively* monochromatic rectangles. In general they may not be defined : $f(\vec{0}) = 1, f(\vec{1}) = 0$.
- If f is *monotone boolean function* then $f(a) = 1, f(b) = 0 \Rightarrow \exists i, a_i = 1, b_i = 0$. $\chi_+(f)$ well defined.
- Any $n-$dimensional rectangle can be covered by $2n$ monochromatic rectangles, if disjointness is relaxed :
  $M_{i,a} = \{(x, y) \in R | x_i = a, y_i = 1 - a\} \qquad (a = 0, 1; i = 1, ..., n)$
  Called *canonical monochromatic rectangles*. Thus the disjointness constraint makes $\chi(R)$ nontrivial.

Rychkov's Lemma

Let $L(f)$ be the smallest leafsize of a DeMorgan formula computing f. A formula is monotone if it has no negated variables as input literals. If f is a monotone boolean function, then $L_+(f)$ denotes the smallest leafsize of a monotone DeMorgan formula computing f.

Lemma (Rychkov's Lemma)

For every boolean function f and for every monotone boolean function g, $L(f) \geq \chi(f)$ and $L_+(f) \geq \chi_+(f)$

Proof

- The proof goes by *Induction* on $L(f)$.
- Base Case : $L(f) = 1$ then f is a single variable x_i or its negation. R itself is monochromatic rectangle.
- Induction Step : Let $t = L(f)$, assume theorem holds for $\forall$ boolean functions g with $L(g) \leq t - 1$.
- Consider the minimal formula f, and assume that the last gate is an *And* gate (the case for Or gate is similar).
- Thus $f = f_0 \wedge f_1$ for some boolean functions f_0, f_1, such that $L(f_0) + L(f_1) = L(f)$

Proof

- Suppose f separates $R = A \times B, f(A) = 1, f(B) = 0$
- Consider $B_0 = \{b \in B | f_0(b) = 0\}$
- Then f_0 separates $R_0 = A \times B_0$ and f_1 separates $R_1 = A \times (B \setminus B_0)$.
- Using induction hypothesis, $\chi(R_i) \leq L(f_i)$ for both $i = 0, 1$. Hence , $\chi(R) \leq \chi(R_0) + \chi(R_1) \leq L(f_0) + L(f_1) = L(f)$
- Proof for monotone case is same with basis case replaced with $L_+(g) = 1$. Here R itself is positively monochromatic rectangle. Inductive step is same.

Implications

- It is not known whether some polynomial inverse of Rychkov's lemma holds.
- Only a 'quasi-polynomial' inverse is known $L(f) \leq \chi(f)^{2log(\chi(f))}$, lemma 3.9
- Since boolean functions with $L(f) = \Omega(2^n/log(n))$ exist (Theorem 1.12), the above inequality implies that boolean functions f of n variables such that $\chi(f) \geq 2^{(1-o(1))\sqrt{n}}$ exists.

References

Stasys Jukna (2011)
Boolean Function Complexity - Advances and Frontiers

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