Problem Set 6 (answers)

S352

1. The MLE can be obtained either by doing `1 / mean(expo)` or using `optim()`. $\hat{\lambda} = 0.338$.

   (a) To estimate the variance/standard error of $\hat{\lambda}$, we could use the Fisher information, $I(\hat{\lambda}) = l''(\hat{\lambda})$ and $\hat{V}(\hat{\lambda}) = -\frac{1}{I(\hat{\lambda})}$. From Chapter 2, we had $l''(\lambda) = -\frac{n}{\lambda^2}$. So

   $$\hat{V}(\hat{\lambda}) = -\frac{1}{I(\hat{\lambda})} = \frac{\hat{\lambda}^2}{n} = 0.00057.$$

   (b) For Poisson, $V(\hat{\lambda}) = V(\bar{X}) = \frac{\lambda}{n}$, so $\hat{V}(\hat{\lambda}) = \frac{\hat{\lambda}}{n}$. This formula is different from the one for λ in Exponential distribution in (a). (Note: Even though we are using the same notation for the rate parameter λ in Poisson distribution and Exponential distribution, the MLE for λ and the variance of the MLE is different!)

   (c) To compute the variance/standard error of $\hat{\lambda}$ with R, we could stuff the data and negative log-likelihood function into `optim()`.

   ```
   expo.nll <- function(lambda, data){
       lik <- dexp(data, lambda)
       return(-sum(log(lik)))
   }
   expo.mle <- optim(1, expo.nll, data = expo, method = "Brent",
                     lower = 0, upper = 50, hessian = TRUE)
   ```

   The Hessian is 1753, so the estimated variance is $1/1753 \approx 0.00057$.
   The estimated standard error is $1/\sqrt{1753} \approx 0.0239$.

   (d) 95% Wald confidence interval for λ is:

   $$\hat{\lambda} \pm 1.96 \cdot \hat{se}(\hat{\lambda}) = 0.338 \pm 1.96 \times 0.0239 = (0.337, 0.339).$$

   We are 95% confident that the true value of the parameter λ lies within the interval (0.337, 0.339). Since this interval doesn't include 0.5, we conclude from the sample that the true value of λ is not equal to 0.5, with 95% confidence.

   (e)

   $$H_0 : \lambda = 0.5$$

   It seems like the MLE is quite far from 0.5. We can get the Wald test statistic and then a P-value as follows:

```
# using theoretical results (and hypothesized lambda value in the denominator of W)
lambda.hat <- 1/mean(expo)
n <- length(expo)
w1 <- (lambda.hat - 0.5)^2 / (0.5^2/n)
1 - pchisq(w1, df = 1)

# using computational approach
w2 <- (expo.mle$par - 0.5)^2 / (1 / expo.mle$hessian)
1 - pchisq(w2, df = 1)
```

Since there was no specific instruction, using either theory approach or computational approach is fine. See code above. Both give a tiny P-value (around 4.5×10^{-6} and 1.1×10^{-11} respectively), so no, we have strong evidence to reject the null hypothesis and conclude that the true value of λ isn't 0.5.

2.

(a) The MLEs are the usual sample mean and plug-in estimate of variance: $\bar{x} = \frac{1}{n}\sum x_i = 162.06$ cm and $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x})^2 = 52.90$ cm^2.

```
> mean(womensheights)
[1] 162.0562
> mean(womensheights^2) - (mean(womensheights))^2
[1] 52.90342
```

(b)

$$\bar{Y} \sim N(\mu, \sigma^2/n)$$

A 99% Wald confidence interval for μ is

$$\bar{y} \pm \texttt{qnorm(0.995)} \cdot \hat{\sigma}/\sqrt{n} = (161.75, 162.36).$$

(c)

$$\hat{\sigma}^2 \approx N(\sigma^2, 2\sigma^4/n)$$

A 99% confidence interval for σ^2 is

$$\hat{\sigma}^2 \pm \texttt{qnorm(0.995)} \cdot \sqrt{2}\hat{\sigma}^2/\sqrt{n} = (49.76, 56.04).$$

(d)

$$H_0 : \mu = 161.3$$
$$H_1 : \mu \neq 161.3$$

We can get the Wald test statistic and then a P-value as follows:

```
w <- (162.0562 - 161.3)^2 / (52.90/3766)
1 - pchisq(w, df = 1)
```

The Wald test P-value is about 1.76×10^{-10}. The evidence is extremely strong for us to reject the null hypothesis that the true mean is 161.3 cm.

(e)
```
> t.test(womensheights, mu = 161.3, conf.level = 0.99)

One Sample t-test

data:  womensheights
t = 6.3795, df = 3765, p-value = 1.993e-10
alternative hypothesis: true mean is not equal to 161.3
99 percent confidence interval:
161.7507 162.3617
sample estimates:
mean of x
162.0562
```
The t-test P-value is about 1.99×10^{-10}, so we have strong evidence to reject the null that the true mean is 161.3 cm: looks like it's higher than that. A 99% confidence interval for the average of women's heights is 161.75 cm to 162.36 cm.

(f)

$$H_0 : \sigma = 5.3 \text{ equivallently } H_0 : \sigma^2 = 5.3^2 = 28.09$$

So under the null $\sigma_0^2 = 5.3^2$

$$\mathrm{Var}(\hat{\sigma}^2) \approx 2\sigma_0^4/n = 2 \times 5.3^4/3766 = 0.419.$$

Then the Wald test is

```
v <- 2 * 5.3^4 / 3766
w <- (52.90 - 5.3^2)^2 / v
1 - pchisq(w, df = 1)
```

giving a P-value of basically zero. The SD of American women's heights isn't nearly as small as 5.3 cm, perhaps because the U.S. has a much more heterogenous population than Belgium.