Problem Set 8 (answers)

S352

1. (a) The MLE is $\hat{p} = 0.6$ and its standard error is about $\sqrt{0.6 \times 0.4/1000} \approx 0.0155$. A 95% Wald interval would be $0.6 \pm 1.96 \times 0.0155$, or about 57.0% to 63.0%.

   (b)
```
> library(binom)
> binom.profile(600, 1000)
method    x    n mean     lower     upper
1 profile 600 1000  0.6 0.5694091 0.6300791
```
   The binomial profile likelihood interval for p goes from 0.569 to 0.630, very slightly different from the Wald interval. These two CIs are close due to the large sample size.

   (c)
```
p <- seq(0.5, 0.7, 0.001)
swift.loglik <- log(dbinom(600, 1000, p))
plot(p, swift.loglik, type = "l",
     main = "Log-likelihood for Taylor Swift favorability",
     xlab = "p = proportion favorable to Swift", ylab = "l(p)")
abline(h = log(dbinom(600, 1000, 0.6)), col = "red")
abline(h = log(dbinom(600, 1000, 0.5694091)), col = "blue")
```

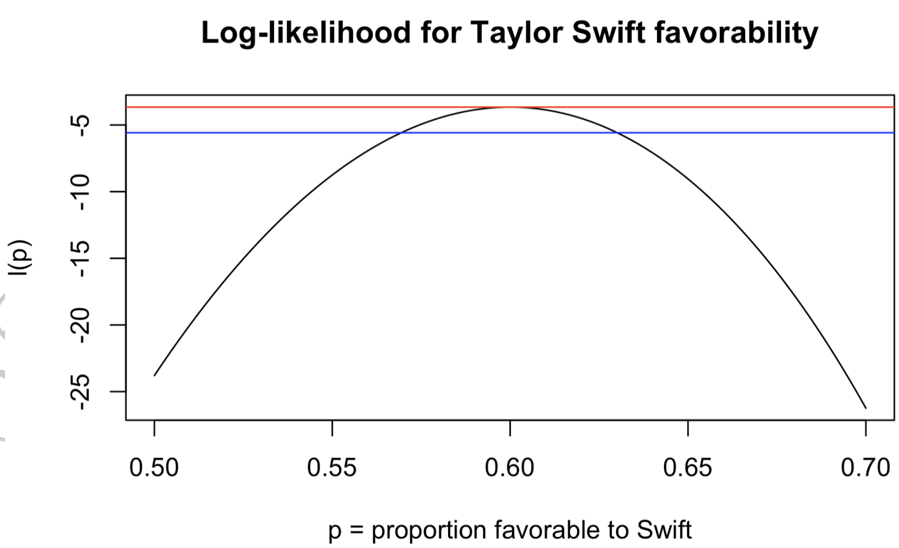

Figure 1: Log-likelihood for Q1(b).

1.92 is the vertical distance between the two horizontal lines.

(d) By the invariance principle, the MLE of logit(p) is logit($\hat{p}$).

$$\text{logit}(0.6) = \log \frac{0.6}{1-0.6} = \log 1.5 \approx 0.405.$$

(e) Applying the logit transformation to the confidence bounds from (b):

```
log(binom.profile(600, 1000)$lower / (1 - binom.profile(600, 1000)$lower))
log(binom.profile(600, 1000)$upper / (1 - binom.profile(600, 1000)$upper))
```

gives a 95% confidence interval of $[0.279, 0.533]$ for logit(p).

2. The fully-specified null hypothesis is that the probability of winning is 0.125 from each of the eight gates. Compare observed and expected counts:

```
observed <- c(29, 19, 18, 25, 17, 10, 15, 11)
n <- sum(observed)
expected <- rep(n/8, 8)
G <- 2 * sum(observed * log(observed / expected))
1 - pchisq(G, df = 8 - 1)
```

The G-statistic is 16.1. Comparing this to a chi-squared distribution with 7 degrees of freedom gives a P-value of 0.024. There's some evidence against the null, and hence some evidence that a horse's starting position *does* affect its chance of winning. In particular, it looks like it's easier to win when starting from position 1.

3.
```
observed <- c(926, 288, 293, 104)
n <- sum(observed)
expected <- n * c(9/16, 3/16, 3/16, 1/16)
G <- 2 * sum(observed * log(observed / expected))
1 - pchisq(G, df = 4 - 1)
```

The G-statistic is 1.478. Comparing this to a chi-squared distribution with 3 degrees of freedom gives a P-value of 0.69. The observations are consistent with the expected proportions.

4. (a)
```
laliga <- scan("laliga.txt")
x.bar <- mean(laliga)
```
The sample mean is the ML estimate of λ. Here, it's 2.503.

(b) The variance of the Poisson distribution is λ. The variance of the sample mean, or MLE of λ, is $\text{Var}(\hat{\lambda}) = \text{Var}(\bar{X}) = \lambda/n$, so the estimated variance of $\hat{\lambda}$ is:
`x.bar/length(laliga)` ≈ 0.0066.

(c)
```
# Wald
se <- sqrt(x.bar / 380)
wald.ci <- x.bar + qnorm(c(.01, .99)) * se
# (2.313841, 2.691423)

# Profile
laliga.nll <- function(lambda){
    lik <- dpois(laliga, lambda)
```

```
    return(-sum(log(lik)))
}
library(Bhat)
control.list <- list(label = "lambda", est = x.bar, low = 0, upp = 99)
profile.ci <- plkhci(control.list, laliga.nll, "lambda", prob = 0.98)
# (2.318555, 2.696199)

# t
t.ci <- t.test(laliga, conf.level = 0.98)$conf.int
t.ci
# (2.296151, 2.709112)
```

All of these methods give a lower bound of 2.29–2.32 and an upper bound of 2.69–2.71.

(d) Likelihood ratio test:

```
# Unrestricted loglik
loglik.max <- sum(log(dpois(laliga, mean(laliga))))
# Restricted loglik
loglik.null <- sum(log(dpois(laliga, 2.5)))
# LR stat
X <- 2 * (loglik.max - loglik.null)
# P-value
1 - pchisq(X, df = 1)
```

This test gives a test statistic of 0.001 and a P-value of about 0.974. We cannot reject the null hypothesis.

Wald test:

```
W <- (x.bar - 2.5)^2 / (2.5 / 380)
1 - pchisq(W, df = 1)
```

This test gives a Wald statistic of 0.001 and a P-value of about 0.974. We cannot reject the null hypothesis.

t-test:

```
t.test(laliga, mu = 2.5)
```

This test gives a t-statistic of 0.03 and a P-value of about 0.976. We cannot reject the null hypothesis.

Regardless of the test, we don't have any evidence to reject the hypothesis that the expected number of goals per game is 2.5.