Remote-state estimation with packet drop

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Abstract: In the remote estimation system, a transmitter observes a discrete-time symmetric countable state Markov process and decides to either transmit the current state of the Markov process or not transmit. The transmitted packet gets dropped in the communication channel with a probability ε. An estimator estimates the Markov process based on the received observations. When each transmission is costly, we characterize the minimum achievable cost of communication plus estimation error. When there is a constraint on the average number of transmissions, we characterize the minimum achievable estimation error. Transmission and estimation strategies that achieve these fundamental limits are also identified.

Keywords: Remote-state estimation, decentralized control, distortion-transmission function.

1. INTRODUCTION

Remote-state estimation refers to a scenario in which a sensor observes a stochastic process and determines whether or not to transmit each observation to a remote receiver. In this paper, we consider a model where the communication takes place over a TCP-like protocol; so either the transmitted packet is delivered without any error to the receiver or the packet is dropped.

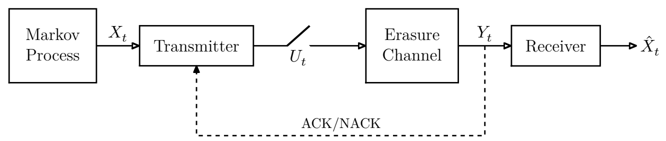

Fig. 1. The remote state estimation setup

Several variations of this setup has been considered in the literature. When the communication channel is ideal (i.e., there is no packet-drop), suboptimal and optimal transmission and estimation strategies are proposed in Imer and Basar (2005); Xu and Hespanha (2004); Lipsa and Martins (2011); Nayyar et al. (2013b); Molin and Hirche (2012); Chakravorty and Mahajan (2017). When there are packet drops, Li et al. (2013) consider the case when the transmitter can transmit only a fixed number of times, Xiaoqiang et al. (2016) consider the case when the probability of the packet-drop depends on the transmission power. Shi and Xie (2012) considers a similar setup with two energy levels and Dey et al. (2013) consider the case when the transmissions are noisy.

In this paper, we characterize the structure of optimal communication strategies as well as two fundamental trade-offs between communication and estimation: first when communication is costly and second when there is a constraint on the number of communications. In both cases, we identify communication strategies that achieve the optimal trade-offs.

2. PROBLEM FORMULATION

2.1 Remote estimation model

Consider the remote estimation setup shown in Fig. 1. A sensor observes a first-order time-homogeneous Markov process $\{X_t\}_{t\geq 0}$ with initial state $X_0 = 0$ and for $t \geq 0$,

$$X_{t+1} = aX_t + W_t, \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is an i.i.d. innovations process. For simplicity, in this paper we restrict attention to $a, X_t, W_t \in \mathbb{Z}$. The results extend naturally to the case when $a, W_t, X_t \in \mathbb{R}$. We assume that W_t is distributed according to a unimodal and symmetric probability mass function p, i.e., for all $e \in \mathbb{Z}_{\geq 0}$, $p_e = p_{-e}$ and $p_e \geq p_{e+1}$. To avoid the trivial case, we assume $p_0 < 1$.

After observing X_t, the sensor decides whether or not to transmit the current state. This decision is denoted by $U_t \in \{0, 1\}$, where $U_t = 0$ denotes *no transmission* and $U_t = 1$ denotes *transmission*.

If the transmitter decides to transmit (i.e., $U_t = 1$), X_t is transmitted over a wireless erasure channel and there is a probability $\varepsilon \in (0, 1)$ that the transmitted packet is dropped. Let $H_t \in \{0, 1\}$ denote the state of the channel at time t. $H_t = 0$ denotes that the channel is in the *OFF* state and a transmitted packet will be dropped; $H_t = 1$ denotes that channel is in the *ON* state and a transmitted packet will be received. We assume that $\{H_t\}_{t\geq 0}$ is an i.i.d. process with $\mathbb{P}(H_t = 0) = \varepsilon$. Moreover, $\{H_t\}_{t\geq 0}$ is independent of $\{X_t\}_{t\geq 0}$.

Transmission takes place using a TCP-like protocol, so there is an acknowledgment from the receiver to the transmitter when a packet is received successfully. This means that the transmitter observes $K_t = U_t H_t$, which indicates whether the packet was successfully received by the receiver ($K_t = 1$) or not ($K_t = 0$).

The received symbol, which is denoted by Y_t, is given by

$$Y_t = \begin{cases} X_t, & \text{if } K_t = 1 \\ \mathfrak{E}, & \text{if } K_t = 0, \end{cases} \tag{2}$$

where $Y_t = \mathfrak{E}$ denotes that no packet was received. Note that by observing K_t, the transmitter can compute Y_t. The transmitter uses this information to decide whether or not to transmit. In particular,

$$U_t = f_t(X_{0:t}, Y_{0:t-1}), \tag{3}$$

where $X_{0:t}$ and $Y_{0:t-1}$ are short-hand notations for $(X_0, \dots, X_t)$ and $(Y_0, \dots, Y_{t-1})$. The collection $f := \{f_t\}_{t\ge 0}$ of decision rules is called the *transmission strategy.*

After observing Y_t, the receiver generates an estimate $\{\hat{X}_t\}_{t\ge 0}$, $\hat{X}_t \in \mathbb{Z}$, using an *estimation strategy* $g := \{g_t\}_{t\ge 0}$, i.e.,

$$\hat{X}_t = g_t(Y_{0:t}). \tag{4}$$

The fidelity of estimation is measured by a per-step distortion $d(X_t - \hat{X}_t)$. We assume that:

- $d(0) = 0$ and for $e \neq 0$, $d(e) \neq 0$
- $d(\cdot)$ is even, i.e., $d(e) = d(-e)$
- $d(e)$ is increasing for $e \in \mathbb{Z}_{\ge 0}$.

2.2 The optimization problems

We are interested in two performance measures: expected total distortion and expected total number of transmission. Given any finite horizon strategy (f, g) for horizon T, the expected distortion is defined as

$$D_T(f, g) := \mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{T} d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

and the expected number of transmissions is defined as

$$N_T(f, g) := \mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{T} U_t \,\Big|\, X_0 = 0\Big].$$

Given any infinite horizon strategy (f, g) for discount factor β, $\beta \in (0, 1)$, the expected distortion is defined as

$$D_\beta(f, g) := (1-\beta)\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

and the expected number of transmissions is defined as

$$N_\beta(f, g) := (1-\beta)\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t \,\Big|\, X_0 = 0\Big].$$

We are interested in the following three optimization problems:

Problem 1 *(Costly communication, finite-horizon) In the model of Sec. 2.1, given a communication cost $\lambda \in \mathbb{R}_{>0}$ and a horizon T, find a transmission and estimation strategy (f^*, g^*) such that*

$$C_T^*(\lambda) := C_T(f^*, g^*; \lambda) = \inf_{(f,g)} C_T(f, g; \lambda), \tag{5}$$

where $C_T(f, g; \lambda) := D_T(f, g) + \lambda N_T(f, g)$ is the total communication cost and the infimum in (5) is taken over all history-dependent strategies of the form (3) and (4).

Problem 2 *(Costly communication, infinite-horizon) In the model of Sec. 2.1, given a discount factor $\beta \in (0, 1)$ and a communication cost $\lambda \in \mathbb{R}_{>0}$, find a transmission and estimation strategy (f^*, g^*) such that*

$$C_\beta^*(\lambda) := C_\beta(f^*, g^*; \lambda) = \inf_{(f,g)} C_\beta(f, g; \lambda), \tag{6}$$

where $C_\beta(f, g; \lambda) := D_\beta(f, g) + \lambda N_\beta(f, g)$ is the total communication cost and the infimum in (6) is taken over all history-dependent strategies of the form (3) and (4).

Problem 3 *(Constrained communication) In the model of Sec. 2.1, given a discount factor $\beta \in (0, 1)$ and a constraint $\alpha \in (0, 1)$, find a transmission and estimation strategy (f^*, g^*) such that*

$$D_\beta^*(\alpha) := D_\beta(f^*, g^*) = \inf_{(f,g): N_\beta(f,g) \le \alpha} D_\beta(f, g), \tag{7}$$

where the infimum is taken over all history-dependent strategies of the form (3) and (4).

Problems 1–3 are decentralized control problems. The system has two controllers or *agents*—the transmitter and the receiver—who have access to different information. In particular, the transmitter at time t has access to $(X_{0:t}, Y_{0:t-1})$ while the receiver at time t has access to $Y_{0:t}$. These two agents need to cooperate to minimize a common cost function given by (5), (6), or (7). Such decentralized control problems are investigated using *team theory* Mahajan et al. (2012).

In this paper we use the *person-by-person* approach in tandem with the *common information* approach to identify information states for both agents and obtain a dynamic programming decomposition. Then we use a partial order based on majorization to identify the structure of optimal transmission strategy. In particular, we show that optimal estimation strategy is similar to Kalman filtering and optimal transmission strategy is threshold-based. For Problems 2 and 3, we use ideas from renewal theory and constrained optimization to identify the optimal thresholds.

3. STRUCTURE OF OPTIMAL STRATEGIES

3.1 Person-by-person approach to remove irrelevant information at the transmitter

Proposition 1 *In Problem 1, there is no loss of optimality to restrict attention to transmission strategies of the form:*

$$U_t = f_t(X_t, Y_{0:t-1}).$$

PROOF Arbitrarily fix the estimation strategy g and consider the *best response* strategy at the transmitter. Similar to the argument given in Witsenhausen (1979); Teneketzis (2006), it can be shown that $(X_t, Y_{0:t-1})$ is an information state at the transmitter and therefore the result of proposition follows from Markov decision theory. ∎

3.2 Common-information based sufficient statistic for the transmitter and the receiver

Following Nayyar et al. (2013a), we split the information at the transmitter and the receiver into two parts: *common information* (which is the data that that is known to all future decision makers) and *local information* (which is the total data minus the common information). In particular, at the transmitter the common information is $Y_{0:t-1}$ and the local information is X_t while at the receiver the common information is $Y_{0:t}$ and the local information is empty. Now, consider the following centralized stochastic control problem, which we call the *coordinated system.* At time t, a virtual coordinator observes $Y_{0:t-1}$ (the common information at the transmitter at time t) and chooses a prescription $\phi_t : \mathbb{Z} \to \{0, 1\}$ according to a coordination strategy ψ_t, i.e.,

$$\phi_t = \psi_t(Y_{0:t-1}).$$

The decision U_t to transmit is generated according to $U_t = \phi_t(X_t)$. The received symbol Y_t is still given by (2). After observing Y_t, the coordinator generates an estimate $\hat{X}_t$ according to (4). The communication cost and the distortion function are the same as in Sec. 2.1.

As shown in Nayyar et al. (2013a) that the above centralized coordinated system is equivalent to the decentralized system considered in Problem 1. Since the coordinated system is centralized, an optimal coordinated strategy may be identified from an appropriate dynamic program. For that matter, define the following beliefs.

Definition 1 For any coordination strategy $\psi = (\psi_1, \dots, \psi_t)$, define $\Pi_{t|t-1}, \Pi_{t|t} \in \Delta(\mathbb{Z})$ as follows: for any $x \in \mathbb{Z}$,

$$\Pi_{t|t-1}(x) := \mathbb{P}^{\psi}(X_t = x \,|\, Y_{0:t-1})$$
$$\Pi_{t|t}(x) := \mathbb{P}^{\psi}(X_t = x \,|\, Y_{0:t}),$$

where $\Delta(\mathbb{Z})$ denotes the space of distributions on $\mathbb{Z}$. □

Note that when we condition on a particular realization of $Y_{0:t-1}$, then the realizations $\pi_{t|t-1}$ and $\pi_{t|t}$ of $\Pi_{t|t-1}$ and $\Pi_{t|t}$ are conditional distributions on X_t given $Y_{0:t-1}$. When we condition on the random variables $Y_{0:t-1}$, then $\Pi_{t|t-1}$ and $\Pi_{t|t}$ are distribution-valued random variables.

Based on Bayes rule, the update of $\pi_{t|t-1}$ and $\pi_{t|t}$ are given as follows.

Lemma 1 *There exists functions Q_t^1 and Q_t^2 such that for any coordination strategy ψ and any realization $y_{0:t}$ of $Y_{0:t}$,*

$$\pi_{t|t} = Q_t^1(\pi_{t|t-1}, \phi_t, y_t) \quad \text{and} \quad \pi_{t+1|t} = Q_t^2(\pi_{t|t}),$$

where

$$Q_t^1(\pi_{t|t-1}, \phi_t, y_t)(x) = \begin{cases} \delta(y_t), & \text{if } y_t \neq \mathfrak{E} \\ \frac{\pi_{t|t-1}(x)[\varepsilon\phi_t(x) + (1-\phi_t(x))]}{\sum_{x' \in \mathbb{Z}} \pi_{t|t-1}(x')[\varepsilon\phi_t(x') + (1-\phi_t(x'))]}, & \text{if } y_t = \mathfrak{E}, \end{cases} \tag{8}$$

and

$$Q_t^2(\pi_{t|t})(x) = \sum_{w \in \mathbb{Z}} p_w \pi_{t|t}(ax + w), \tag{9}$$

where $\delta(x')$ is the delta-distribution with unit mass at x'. □

The next proposition follows from (Nayyar et al., 2013b, Theorem 1).

Proposition 2 *Define recursively the following functions for a finite horizon T:*

$$V_{T+1|T}(\pi_{t|t-1}) := 0,$$

and for $t \in \{T, \cdots, 1\}$,

$$V_{t|t}(\pi_{t|t}) := \min_{\hat{x}_t \in \mathbb{Z}} \mathbb{E}\big[d(X_t - \hat{x}_t) \big| \Pi_{t|t} = \pi_{t|t}\big] + V_{t+1|t}(\pi_{t+1|t}), \tag{10}$$

where $\Pi_{t+1|t} = Q_t^2(\pi_{t|t})$ as given in (8), and

$$V_{t|t-1}(\pi_{t|t-1}) := \min_{\phi_t \in \mathcal{G}} \mathbb{E}\big[\lambda \mathbb{1}_{\{U_t = 1\}} + V_{t|t}(\Pi_{t|t}) \big| \Pi_{t|t-1} = \pi_{t|t-1}, \phi_t\big], \tag{11}$$

where $\Pi_{t|t} = Q_t^1(\pi_{t|t-1}, \phi_t, Y_t)$ is given in (8) and $\mathcal{G}$ is the set of all functions $\tilde{\phi}_t$ from $\mathbb{Z}$ to $\{0, 1\}$.

Then, for each realization of the post-transmission belief $\pi_{t|t}$ at time t, the minimizer in (10) exists and gives the optimal estimate at time t; for each realization of the pre-transmission belief $\pi_{t|t-1}$, the minimizer in (11) exists and gives the optimal prescription ϕ_t at time t. □

3.3 Structure of optimal strategies

To establish the structure of optimal transmission and estimation strategies, we state the following definitions from Nayyar et al. (2013b).

Definition 2 (ASU and even) A probability distribution ν on $\mathbb{Z}$ is said to be almost symmetric and unimodal (ASU) about a point $a \in \mathbb{Z}$, if for any $k \in \mathbb{Z}_{\geq 0}$,

$$\nu(a + k) \geq \nu(a - k) \geq \nu(a + k + 1).$$

If a distribution ν is ASU about 0 and $\nu(x) = \nu(-x)$, for all $x \in \mathbb{Z}$, then ν is said to be ASU and even. □

Definition 3 (ASU Rearrangement) The ASU rearrangement of a probability distribution ν, denoted by ν^+, is a permutation of ν such that for every $n \in \mathbb{Z}_{\geq 0}$,

$$\nu_n^+ \geq \nu_{-n}^+ \geq \nu_{n+1}^+.$$

Definition 4 (Majorization) Given two probability distributions ν_1 and ν_2 defined over $\mathbb{Z}$, ν_1 is said to majorize ν_2, which is denoted by $\nu_1 \succeq_m \nu_2$, if for all $n \in \mathbb{Z}_{\geq 0}$,

$$\sum_{-n}^{n} \nu_1^+(x) \geq \sum_{-n}^{n} \nu_2^+(x), \quad \sum_{-n}^{n+1} \nu_1^+(x) \geq \sum_{-n}^{n+1} \nu_2^+(x).$$

Definition 5 (Relation *R*) Given two probability distributions μ and $\tilde{\mu}$ defined over $\mathbb{Z}$, we say that a relation $\mathbf{R}$ exists between them, which is denoted by $\tilde{\mu}\mathbf{R}\mu$, if $\tilde{\mu} \succeq_m \mu$ and $\tilde{\mu}$ is ASU about some point $b \in \mathbb{Z}$. □

Definition 6 (ASU Schur-concavity) Let $H : \Delta(\mathbb{Z}) \to \mathbb{R}$ be a function that maps distributions on $\mathbb{Z}$ to real numbers. Then, H is said to be ASU Schur-concave if for any two distributions $\tilde{\mu}$ and μ, $\tilde{\mu}\mathbf{R}\mu$ implies $H(\tilde{\mu}) \leq H(\mu)$. □

The following three lemmas are important in establishing the structural result. The proofs are given in Appendix A.

The value functions defined in Proposition 2 satisfy the following:

Lemma 2 *$V_{t|t-1}$ and $V_{t|t}$ are ASU Schur-concave.* □

Using the above property, we can show the following:

Lemma 3 *If $\pi_{t|t}$ is ASU around θ, then θ is an arg min of the right hand side of (10).* □

Lemma 4 *If $\pi_{t|t-1}$ is ASU around θ, then the arg min of the right hand side of (11) is given by*

$$\phi_t(x) = \begin{cases} 1, & \text{if } |x - a\theta| > k_t(\pi_{t|t-1}) \\ 0, & \text{if } |x - a\theta| < k_t(\pi_{t|t-1}) \end{cases} \tag{12}$$

where k_t is a threshold that depends on $\pi_{t|t-1}$. When $|x - a\theta| = k_t(\pi_{t|t-1})$, either 0 or 1 may be chosen with an appropriate randomization probability. □

Definition 7 Define a processes $\{Z_t\}_{t\geq 0}$ and $\{E_t\}_{t\geq 0}$ as follows: $Z_0 = 0$,

$$Z_t = \begin{cases} X_t, & \text{if } Y_t \neq \mathfrak{E}, \\ aZ_{t-1}, & \text{if } Y_t = \mathfrak{E}, \end{cases}$$

and $E_t = X_t - aZ_{t-1}$. □

Theorem 1 (Structure of optimal strategies) *The optimal strategies have the following structure:*

(1) Structure of optimal estimation strategy*: For Problems 1–3, the optimal estimation strategy is time-homogeneous and Kalman-like, as given below:* $\hat{X}_0 = Z_0 = 0$ *and*

$$\hat{X}_t = g_t^*(Z_t) = Z_t = \begin{cases} X_t, & \text{if } Y_t \neq \mathfrak{E}, \\ a\hat{X}_{t-1}, & \text{if } Y_t = \mathfrak{E}; \end{cases} \tag{13}$$

(2) Structure of optimal transmission strategy*: For Problem 1, the optimal transmission strategy is a threshold-based strategy given by*

$$U_t = f_t^*(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k_t; \\ 0, & \text{if } |E_t| < k_t, \end{cases}$$

where $k_t \in \mathbb{Z}_{\geq 0}$ *are the time-varying thresholds that do not depend on* E_t.

For Problem 2, the optimal transmission strategy is threshold-based strategy given by

$$U_t = f_t^*(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k; \\ 0, & \text{if } |E_t| < k, \end{cases}$$

where $k \in \mathbb{Z}_{\geq 0}$ *is a time-homogeneous threshold.*

For Problem 3, the optimal strategy is a threshold-based randomized strategy given by

$$U_t = f_t^*(E_t) = \begin{cases} 1, & \text{if } |E_t| > k; \\ 1, & \text{w.p. } \theta^* \text{ if } |E_t| = k; \\ 0, & \text{w.p. } 1-\theta^* \text{ if } |E_t| = k; \\ 0, & \text{if } |E_t| < k, \end{cases}$$

where $k \in \mathbb{Z}_{\geq 0}$ *is a time-homogeneous threshold and* $\theta^* \in [0,1]$ *is an appropriate constant that depends on* α □

PROOF The results for Problem 1 follow from a forward induction argument. It can be recursively shown that $\pi_{t|t-1}$ is ASU about Z_{t-1} and $\pi_{t|t}$ is ASU about Z_t. The structure of the optimal transmission and estimation strategy follows from Lemmas 3 and 4. For the details of the proof, see Appendix B.

The results for Problem 2 can be shown by establishing appropriate regularity conditions under which the solution of infinite horizon Markov decision processes is time homogeneous (Puterman, 1994, Lemma 4.7.2).

For the proof for Problem 3, see Theorem 3.

The implication of Theorem 1 is the following. In general, in remote-state estimation problems, the structure of optimal estimation strategy depends on that of the optimal transmission strategy. However, as is shown in Theorem 1, the optimal estimation strategy can be characterized in closed form, independent of that of optimal transmission strategy. Thus, we can fix an estimation strategy of the form (13) and consider the optimization problem of finding the *best* transmission strategy corresponding the fixed estimation strategy. Since there is only one decision-maker (the transmitter), this optimization problem is centralized in nature. Since the optimal estimation strategy given by (13) is time-homogeneous, it can be shown that the optimal transmission strategy for infinite horizon is time-homogeneous and is given by the following dynamic program:

$$V_\beta(e) = \min\{V_\beta^1(e), V_\beta^0(e)\} \tag{14}$$

where

$$\begin{aligned} V_\beta^1(e) = (1-\varepsilon)\big(\lambda + \beta\mathbb{E}[V_\beta(E_{t+1}) \mid E_t = e, U_t = 1, C_t = 1] \\ + \varepsilon(\lambda + d(e) + \beta\mathbb{E}[V_\beta(E_{t+1}) \mid E_t = e, U_t = 1, C_t = 0]\big) \end{aligned}$$

and

$$V_\beta^0(e) = d(e) + \beta\mathbb{E}[V_\beta(E_{t+1}) \mid E_t = e, U_t = 0].$$

Let D_β^1 denote the performance of a strategy in which we transmit all the time. We assume that D_β^1 is uniformly bounded [1], say by $\bar{D}_\beta^1 < \infty$.

The above dynamic program has a unique solution due to the following reasons. When the per-step distortion $d(\cdot)$ is bounded, the existence of a unique and bounded solution follows from (Sennott, 1999, Proposition 4.7.1, Theorem 4.6.3). When $d(\cdot)$ is unbounded, then for any communication cost λ, we first define $e_0 \in \mathbb{Z}_{\geq 0} < \infty$ as:

$$e_0 := \min\Big\{e \,:\, d(e) \geq \frac{\bar{D}_\beta^1}{1-\beta}\Big\}.$$

Now, for any state e, $|e| > e_0$, the per-step cost $(1-\beta)d(e)$ of not transmitting is greater then the cost of transmitting at each step in the future, which is given by $\bar{D}_\beta^1$. Thus, the optimal action is to transmit, i.e., $f^*(e) = 1$.

Let $\mathcal{E}^* := \{e \,:\, |e| \geq e_0\}$. Then the countable-state state-process is equivalent to a finite-state Markov chain with state space $\{-e_0+1, \cdots, e_0-1\} \cup e^*$ (where e^* is a generic state for all states in the set $\mathcal{E}^*$). Since the state space is now finite, the dynamic program (14) has a unique and bounded time-homogeneous solution by the argument given for bounded $d(\cdot)$.

In the remainder of this paper, we show how to find an explicit solution of (14). For that matter, we first evaluate the performance of an arbitrary threshold-based strategy.

4. COMPUTING THE PERFORMANCE OF AN ARBITRARY THRESHOLD BASED STRATEGY

Let $f^{(k)}$ denote the threshold-based transmission strategy:

$$f^{(k)}(E_t) := \begin{cases} 1, & \text{if } |E_t| \geq k \\ 0, & \text{if } |E_t| < k. \end{cases}$$

For $\beta \in (0,1)$ and $e \in \mathbb{Z}$, define the following for a system that starts in state e and follows strategy $f^{(k)}$:

- $L_\beta^{(k)}(e)$: the expected distortion until the first transmission
- $M_\beta^{(k)}(e)$: the expected time until the first transmission
- $D_\beta^{(k)}(e)$: the expected distortion
- $N_\beta^{(k)}(e)$: the expected number of transmissions
- $C_\beta^{(k)}(e;\lambda)$: the expected total cost, i.e.,

$$C_\beta^{(k)}(e;\lambda) = D_\beta^{(k)}(e) + \lambda N_\beta^{(k)}(e), \quad \lambda \geq 0.$$

Note that under $f^{(k)}$, $\{E_t\}_{t\geq 0}$ is a Markov chain. From the balance equations, we get: for all $a, e \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$

[1] If D_β^1 is not uniformly bounded, then the performance of every strategy is infinite and seeking an optimal strategy is meaningless.

$$
L_\beta^{(k)}(e) = \begin{cases} \varepsilon\Big[d(e) + \beta \sum_{n\in\mathbb{Z}} p_{n-ae} L_\beta^{(k)}(n)\Big], & \text{if } |e| \ge k \\ d(e) + \beta \sum_{n\in\mathbb{Z}} p_{n-ae} L_\beta^{(k)}(n), & \text{if } |e| < k. \end{cases} \tag{15}
$$

and

$$
M_\beta^{(k)}(e) = \begin{cases} \varepsilon\Big[1 + \beta \sum_{n\in\mathbb{Z}} p_{n-ae} M_\beta^{(k)}(n)\Big], & \text{if } |e| \ge k \\ 1 + \beta \sum_{n\in\mathbb{Z}} p_{n-ae} M_\beta^{(k)}(n), & \text{if } |e| < k. \end{cases} \tag{16}
$$

Following the proof technique adopted in Chakravorty and Mahajan (2017), one can show the following (see Appendix C for the detailed proof).

Lemma 5 *Equations* (15) *and* (16) *have unique solutions* $L_\beta^{(k)}$ *and* $M_\beta^{(k)}$ *that are strictly increasing in* k. □

These solutions $L_\beta^{(k)}$ and $M_\beta^{(k)}$ can be computed using the techniques for finding fixed points of Bellman operators in countable state Markov decision processes; see Sennott (1999); White (1982); Cavazos-Cadena (1986).

Using ideas from renewal theory, we can evaluate the performance of $f^{(k)}$) (i.e., compute $D_\beta^{(k)}$, $N_\beta^{(k)}$, and $C_\beta^{(k)}$) from $L_\beta^{(k)}$ and $M_\beta^{(k)}$.

Proposition 3 *For any* $\beta \in (0,1)$*, the performance of strategy* $f^{(k)}$ *for costly communication is given as follows: For* $k \in \mathbb{Z}_{>0}$*,*

$$
D_\beta^{(k)}(0) \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)},
$$

$$
N_\beta^{(k)}(0) \coloneqq N_\beta(f^{(k)}, g^*) = \frac{1}{M_\beta^{(k)}(0)} - (1-\beta),
$$

and

$$
C_\beta^{(k)}(0;\lambda) \coloneqq C_\beta(f^{(k)}, g^*; \lambda) = \frac{L_\beta^{(k)}(0) + \lambda}{M_\beta^{(k)}(0)} - \lambda(1-\beta).
$$

See Appendix D for the proof.

Using Lemma 5 and Proposition 3, we can show the following:

Lemma 6 *For any* $\beta \in (0,1)$*,* $D_\beta^{(k)}(0)$ *is increasing in* k *and* $N_\beta^{(k)}(0)$ *is strictly decreasing in* k. □

See Appendix E for the proof.

5. OPTIMAL THRESHOLDS FOR COSTLY AND CONSTRAINED COMMUNICATION

Finally, we characterize the optimal strategies and optimal performances for Problems 2, and 3.

Definition 8 Given two (non–randomized) time–homogeneous strategies f_1 and f_2 and a randomization parameter $\theta \in (0,1)$, the *Bernoulli randomized strategy* (f_1, f_2, θ) is a strategy that randomizes between f_1 and f_2 at each stage; choosing f_1 with probability θ and f_2 with probability $(1-\theta)$. Such a strategy is called a Bernoulli randomized *simple* strategy if f_1 and f_2 differ on exactly one state i.e. there exists a state e_0 such that for all $e \neq e_0$, $f_1(e) = f_2(e)$. □

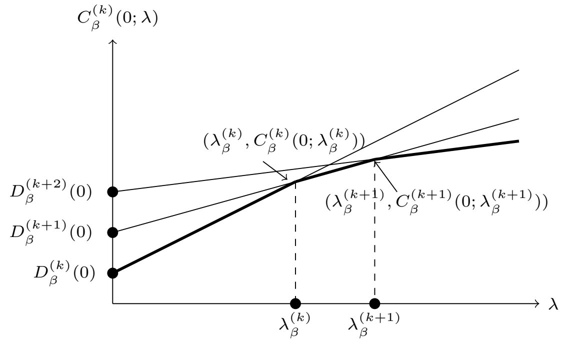

Fig. 2. The optimal costly performance as a function of λ.

The next two theorems characterize the performances for costly and constrained communication for infinite-horizon setup under the optimal communication strategies as given by Theorem 1.

The proofs are omitted due to space constraints; the proof idea is similar to that in Chakravorty and Mahajan (2017) (which may be considered as a special case with $\varepsilon = 0$).

Theorem 2 *(Characterization of optimal costly performance) For* $\beta \in (0,1]$*, let* $\mathbb{K}$ *denote* $\{k \in \mathbb{Z}_{\ge 0} : D_\beta^{(k+1)}(0) > D_\beta^{(k)}(0)\}$*. For* $k_n \in \mathbb{K}$*, define:*

$$
\lambda_\beta^{(k_n)} \coloneqq \frac{D_\beta^{(k_{n+1})}(0) - D_\beta^{(k_n)}(0)}{N_\beta^{(k_n)}(0) - N_\beta^{(k_{n+1})}(0)}. \tag{17}
$$

Then, we have the following.

(1) For any $k_n \in \mathbb{K}$ *and any* $\lambda \in (\lambda_\beta^{(k_{n-1})}, \lambda_\beta^{(k_n)}]$*, the strategy* $f^{(k_n)}$ *is optimal for Problem 2 with communication cost* λ*.*

(2) The optimal performance $C_\beta^*(\lambda)$ *is continuous, concave, increasing and piecewise linear in* λ*. The corner points of* $C_\beta^*(\lambda)$ *are given by* $\{(\lambda_\beta^{(k_n)}, D_\beta^{(k_n)}(0) + \lambda_\beta^{(k_n)} N_\beta^{(k_n)}(0))\}_{k_n \in \mathbb{K}}$ *(see Fig. 2).* □

See Appendix F for the proof.

Theorem 3 *(Characterization of optimal constrained performance) For any* $\beta \in (0,1)$ *and* $\alpha \in (0,1)$*, define*

$$
k_\beta^*(\alpha) = \sup\Big\{k \in \mathbb{Z}_{\ge 0} : M_\beta^{(k)} \le \frac{1}{1+\alpha-\beta}\Big\} \tag{18}
$$

$$
\theta_\beta^*(\alpha) = \frac{M_\beta^{(k^*+1)} - \frac{1}{1+\alpha-\beta}}{M_\beta^{(k^*+1)} - M_\beta^{(k^*)}}. \tag{19}
$$

For ease of notation, we use $k^* = k_\beta^*(\alpha)$ *and* $\theta^* = \theta_\beta^*(\alpha)$*.*

Let f^* *be the Bernoulli randomized simple strategy* $(f^{(k^*)}, f^{(k^*+1)}, \theta^*)$*, i.e.,*

$$
f^*(e) = \begin{cases} 0, & \textit{if } |e| < k^*; \\ 0, & \textit{w.p. } 1-\theta^*, \textit{ if } |e| = k^*; \\ 1, & \textit{w.p. } \theta^*, \textit{ if } |e| = k^*; \\ 1, & \textit{if } |e| > k^*. \end{cases}
$$

Then,

(1) (f^*, g^*) *is optimal for Problem 3 with constraint* α*.*

(2) Let $\alpha^{(k)} = N_\beta(f^{(k)}, g^*)$*. Then, for* $\alpha \in (\alpha^{(k+1)}, \alpha^{(k)})$*,* $k^* = k$ *and* $\theta^* = (\alpha - \alpha^{(k+1)})/(\alpha^{(k)} - \alpha^{(k+1)})$*, and*

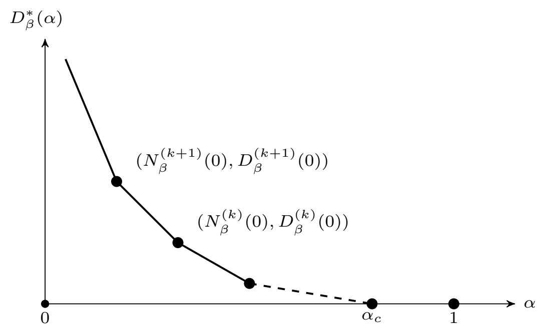

Fig. 3. $D^*_\beta(\alpha)$ as a function of α.

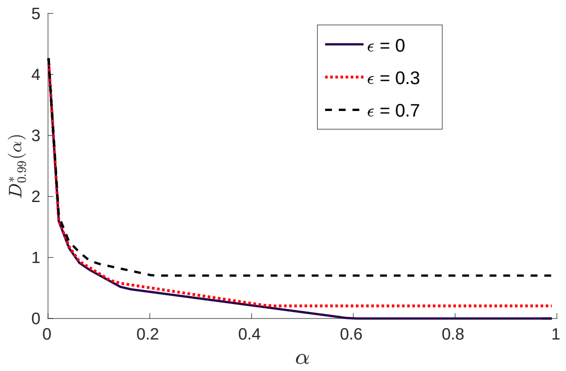

Fig. 4. Plots of $D^*_\beta(\alpha)$ versus α, for $\beta = 0.99$ and $\varepsilon \in \{0, 0.3, 0.7\}$.

the distortion-transmission function is given by

$$D^*_\beta(\alpha) = \theta^* D_\beta^{(k)} + (1-\theta^*) D_\beta^{(k+1)}. \tag{20}$$

Moreover, the distortion-transmission function is continuous, convex, decreasing and piecewise linear in α. Thus, the corner points of $D^*_\beta(\alpha)$ are given by $\{(N_\beta^{(k)}(0), D_\beta^{(k)}(0))\}_{k=1}^{\infty}$ (see Fig. 3). □

See Appendix F for the proof.

6. AN EXAMPLE: SYMMETRIC BIRTH-DEATH MARKOV CHAIN

In this section, we verify with a numerical example the main results for Problem 3 and analyze the variation of the distortion-transmission function with the packet-drop probability, ε. Consider an aperiodic, symmetric, birth-death Markov chain defined over $\mathbb{Z}$ with the transition probability matrix as given by:

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases}$$

where we assume that $p \in (0, \frac{1}{2})$. Let the distortion function be $d(e) = |e|$. The model satisfies (1) with $a = 1$. We verify the main results for $p = 0.3$, $\beta = 0.99$. Fig. 4 shows the distortion-transmission function as a function of α for $\varepsilon \in \{0, 0.3, 0.7\}$. We see from the plots that the optimal distortion increases with increase in the value of ε, which is in consistent with the intuition.

7. CONCLUSION

In this paper, we study the remote-state estimation problem for costly and constrained communication setup for erasure channel, where a transmitted packet is dropped with a known probability ε. We analyze the decentralized control problem with two decision makers—the transmitter and the receiver—in the light of person-by-person and common information approach to establish the structure of optimal communication strategies. Also, we provide the closed-form expressions for optimal thresholds and characterize the optimal performance.

For simplicity, we assumed that the observations are integer valued. But the results extend to real-valued observations in a manner similar to Chakravorty and Mahajan (2017). We also assumed that the sensor has perfect observations of the state. The results are also applicable when the sensor observes the state with noise. In that case, the sensor generates a local estimate of the state and whenever it transmits, it sends the local estimate.

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Appendix A. PROOFS OF LEMMAS 2–4

The proof relies on the properties given by Lemmas 7 and 8.

Lemma 7 *Given any two distributions $\pi_{t|t-1}$ and $\tilde{\pi}_{t|t-1}$ and any prescription ϕ_t, such that $\tilde{\pi}_{t|t-1}\mathbf{R}\pi_{t|t-1}$, there exists a prescription $\tilde{\phi}_t$ of the form (12) such that*

$$\sum_{n\in\mathbb{Z}} \pi_{t|t-1}(n)\phi_t(n) = \sum_{n\in\mathbb{Z}} \tilde{\pi}_{t|t-1}(n)\tilde{\phi}_t(n),$$

and $\tilde{\pi}_{t|t}\mathbf{R}\pi_{t|t}$, where $\tilde{\pi}^1_{t|t} := Q^1_t(\tilde{\pi}_{t|t-1}, \tilde{\phi}_t, \mathfrak{E})$ and $\pi^1_{t|t} := Q^1_t(\pi_{t|t-1}, \phi_t, \mathfrak{E})$ □

The proof of the lemma is same as in Nayyar et al. (2013b). Using Lemma 7 and Hardy-Littlewood inequality (Hardy et al. (1934)), one can show that

$$\min_{\hat{x}_t\in\mathbb{Z}} \mathbb{E}[d(X_t-\hat{x}_t) \,|\, \Pi_{t|t} = \pi_{t|t}] \ge \min_{\hat{x}_t\in\mathbb{Z}} \mathbb{E}[d(X_t-\hat{x}_t) \,|\, \Pi_{t|t} = \tilde{\pi}_{t|t}]. \tag{A.1}$$

Next, note that

$$\pi_{t+1|t} = \pi_{t|t} \star p,$$

where p is the probability mass function of the process noise W_t, which is ASU and even, and $\star$ denotes convolution.

Hence, one can show the following using algebraic calculation similar to (Hajek et al., 2008, Lemma 6.3), (Nayyar et al., 2013b, Lemma 12).

Lemma 8 *Given any two distributions $\pi_{t|t}$ and $\tilde{\pi}_{t|t}$ such that $\tilde{\pi}_{t|t}\mathbf{R}\pi_{t|t}$ and $\pi_{t+1|t} = Q^2_t(\pi_{t|t})$, we have $\tilde{\pi}_{t+1|t}\mathbf{R}\pi_{t+1|t}$, where $Q^2_t(\cdot)$ is given by (9).* □

A.1 Proof of Lemma 2

We prove the result by backward induction. $V_{T+1|T}$ and $V_{T+1|T+1}$ are both equal to 0 and hence are trivially ASU Schur-concave. This is the basis of the induction. Now, let us assume that $V_{t+1|t}$ is ASU Schur-concave. Consider any $\pi_{t|t}$ and $\tilde{\pi}_{t|t}$ such that $\tilde{\pi}_{t|t}\mathbf{R}\pi_{t|t}$. Then, by (10), (A.1), Lemma 8 and the induction hypothesis, we have

$$V_{t|t}(\pi_{t|t}) \ge V_{t|t}(\tilde{\pi}_{t|t}).$$

Hence, $V_{t|t}$ is ASU Schur-concave.

Claim 1 *For any $x, x' \in \mathbb{Z}$, a degenerate distribution δ, and an ASU Schur-concave function $V_{t|t}$, $V_{t|t}(\delta_x) = V_{t|t}(\delta_{x'})$, which we denote by $\Delta_{t|t}$.*

Proof We have that $\delta_x\mathbf{R}\delta_{x'}$, which implies $V_{t|t}(\delta_x) \le V_{t|t}(\delta_{x'})$. Similarly, we have that $\delta_{x'}\mathbf{R}\delta_x$, which implies $V_{t|t}(\delta_x) \ge V_{t|t}(\delta_{x'})$. The claim is proved by combining the two inequalities. ■

Next, consider any $\pi_{t|t-1}$ and $\tilde{\pi}_{t|t-1}$ such that $\tilde{\pi}_{t|t-1}\mathbf{R}\pi_{t|t-1}$. Let ϕ_t be the optimal prescription at $\pi_{t|t-1}$ and define $\tilde{\phi}_t$ as in Lemma 7. Note that by Lemma 7,

$$\mathbb{P}(U_t = 1 \,|\, \pi_{t|t-1}, \phi_t) = \mathbb{P}(U_t = 1 \,|\, \tilde{\pi}_{t|t-1}, \tilde{\phi}_t). \tag{A.2}$$

Furthermore,

$$\mathbb{P}(Y_t = \mathfrak{E} \,|\, \pi_{t|t-1}, \phi_t) = \mathbb{P}(Y_t = \mathfrak{E} \,|\, \tilde{\pi}_{t|t-1}, \tilde{\phi}_t), \tag{A.3}$$

and for any $x \in \mathbb{Z}$,

$$\mathbb{P}(Y_t \ne \mathfrak{E} \,|\, \pi_{t|t-1}, \phi_t) = \mathbb{P}(Y_t \ne \mathfrak{E} \,|\, \tilde{\pi}_{t|t-1}, \tilde{\phi}_t). \tag{A.4}$$

Observe that from (10) we have

$$\begin{aligned} &V_{t|t-1}(\pi_{t|t-1}) \\ &= \lambda\mathbb{P}(U_t = 1 \,|\, \pi_{t|t-1}, \phi_t) \\ &\qquad + \mathbb{P}(Y_t = \mathfrak{E} \,|\, \pi_{t|t-1}, \phi_t)V_{t|t}(\pi_{t|t}) \\ &\qquad\qquad + \mathbb{P}(Y_t \ne \mathfrak{E} \,|\, \pi_{t|t-1}, \phi_t)\Delta_{t|t} \\ &\overset{(a)}{\ge} \lambda\mathbb{P}(U_t = 1 \,|\, \tilde{\pi}_{t|t-1}, \tilde{\phi}_t) \\ &\qquad + \mathbb{P}(Y_t = \mathfrak{E} \,|\, \tilde{\pi}_{t|t-1}, \tilde{\phi}_t)V_{t|t}(\tilde{\pi}_{t|t}) \\ &\qquad\qquad + \mathbb{P}(Y_t \ne \mathfrak{E} \,|\, \tilde{\pi}_{t|t-1}, \tilde{\phi}_t)\Delta_{t|t} \\ &= V_{t|t-1}(\tilde{\pi}_{t|t-1}), \end{aligned} \tag{A.5}$$

where the inequality (a) uses (A.2), (A.3), (A.4), the fact that $V_{t|t}$ is ASU Schur-concave and Lemma 7. Thus, $V_{t|t-1}$ is ASU Schur-concave. This completes the induction step.

A.2 Proof of Lemma 3

Consider $\mathbb{E}[d(X_t-\theta)\,|\,\Pi_{t|t}=\pi_{t|t}]$. Since $d(X_t-\theta)$ is even and increasing about θ and $\pi_{t|t}$ is ASU around θ, we have by algebraic calculation that for any $\hat{x}_t \in \mathbb{Z}$,

$$\sum_{x\in\mathbb{Z}} d(x-\theta)\pi_{t|t}(x) = \sum_{x\in\mathbb{Z}} d^{\uparrow}(x-\theta)\pi^{+}_{t|t}(x) \leq \sum_{x\in\mathbb{Z}} d(x-\hat{x}_t)\pi_{t|t}(x),$$

where $d^{\uparrow}(\cdot)$ is the increasing rearrangement of the elements of $d(\cdot)$ and $\pi^{+}_{t|t}$ is ASU rearrangement of $\pi_{t|t}$. Hence, $\arg\min_{\hat{x}_t\in\mathbb{Z}} \mathbb{E}[d(X_t-\hat{x}_t)\,|\,\Pi_{t|t}=\pi_{t|t}]=\theta$.

A.3 Proof of Lemma 4

First, note that $\pi_{t|t-1}\mathbf{R}\pi_{t|t-1}$. Let ϕ_t be the optimal prescription at $\pi_{t|t-1}$. If ϕ_t is of the form (12), then we are done. If, however, ϕ_t is not of the form (12), construct $\tilde{\phi}_t$ as is given in Lemma 7. Then, by (A.5), the performance of $\tilde{\phi}_t$ is at least as good as that of ϕ_t. Thus, $\tilde{\phi}_t$ is also optimal.

Appendix B. PROOF OF THEOREM 1 FOR PROBLEM 1

To prove the theorem, let us first show the following.

Claim 2 *$\pi_{t|t-1}$ is ASU around Z_{t-1} and $\pi_{t|t}$ is ASU around Z_t.*

Proof We prove the claim by forward induction. The statement is trivially true for $\pi_{1|0}$ since $\pi_{1|0}$ being the the probability mass function of W_0, is ASU and even and $Z_0=0$. If $Y_1\neq\mathfrak{E}$ then $\pi_{1|1}=\delta_{X_1}$, which is ASU around $Z_1=X_1$. If $Y_1=\mathfrak{E}$, $\pi_{1|1}$ is updated according to (8), and so $\pi_{1|1}$ is ASU around Z_1. This is the basis of induction. Let us assume that the statement of the claim is true at t. Then, by Lemma 8, $\pi_{t+1|t}$ is ASU around Z_t. Now, since $\pi_{t+1|t}$ is ASU, by Lemma 4, ϕ_{t+1} is of the form (12). If $Y_{t+1}\neq\mathfrak{E}$, then $\pi_{t+1|t+1}$ is $\delta_{X_{t+1}}$, which is ASU around Z_{t+1}. If $Y_{t+1}=\mathfrak{E}$, $\pi_{t+1|t+1}$ updates according to (8) and hence, $\pi_{t+1|t+1}$ is ASU around Z_{t+1}. This completes the induction step. ∎

Hence, the optimal estimator is of the form (13) due to Claim 2 and Lemma 3 and by Lemma 4, the optimal transmitter is of the form (12).

Note that (12) is a randomized strategy, where the threshold depends on $\pi_{t|t-1}$. To argue that the optimal strategy is deterministic where the threshold does not depend on $\pi_{t|t-1}$, fix the estimator to be of the form (13) and consider the problem of finding the best transmitter. This is a centralized Markov decision problem and the optimality of deterministic threshold strategies (where the threshold does not depend on the state) can be proved following the argument of (Nayyar et al., 2013b, Theorem 4).

Appendix C. PROOF OF LEMMA 5

We prove the result for $L^{(k)}_\beta$. Similar argument holds for $M^{(k)}_\beta$.

Definition 9 Let us define the Hadamard product $\odot$ for vectors and matrices as follows:

- For two vectors of same dimension, v, w, $v\odot w$ denotes the elementwise product, i.e., the i-th element of the vector $v\odot w$ is given by $(v\odot w)_i = v_i w_i$,
- For a vector v and a matrix W with the same number of rows, $v\odot W$ is a matrix given by
$$[v\odot W]_{i,j} = v(i)W_{i,j}.$$

Note that $\odot$ is associative in the following sense. For vectors u, v and matrix W with compatible dimensions, $v\odot(Wu)=(v\odot W)u$. □

Define the vector $h^{(k)}$ as follows:

$$h^{(k)}_e := \begin{cases}\varepsilon, & \text{if } |e|\geq k,\\ 1, & \text{if } |e|<k.\end{cases}$$

Define matrix P on $\mathbb{Z}\times\mathbb{Z}$ as follows:

$$P_{en}=p_{|n-ae|}, \quad \forall e,n\in\mathbb{Z}.$$

Define the operator $\mathcal{B}^{(k)}$ as given by the following

$$\mathcal{B}^{(k)}v := h^{(k)}\odot(Pv) = (h^{(k)}\odot P)v.$$

Note that (15) can be written as

$$L^{(k)}_\beta = h^{(k)}\odot d + \beta\mathcal{B}^{(k)}L^{(k)}_\beta. \tag{C.1}$$

Lemma 9 *For $\beta,\varepsilon\in(0,1)$ and $k\in\mathbb{Z}_{\geq 0}$, $k<\infty$, the operator $\beta\mathcal{B}^{(k)}$ is a contraction.* □

Proof

$$\|\beta\mathcal{B}^{(k)}v\|_\infty = \beta\sup_{e\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} h^{(k)}_e P_{en}v_n \overset{(a)}{<} \beta\sup_{e\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} P_{en}v_w \leq \beta\|v\|_\infty\Big(\sup_{e\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} h^{(k)}_e P_{en}\Big) \overset{(b)}{<} \beta\|v\|_\infty,$$

where (a) and (b) hold as $\varepsilon<1$ and $k<\infty$. ∎

Similar to the argument for the value function, we can argue that $L^{(k)}_\beta$ is bounded. $\beta\mathcal{B}^{(k)}$ is a contraction operator from the space of bounded continuous functions on integers to itself. Thus, by Banach fixed point theorem, (C.1) has a unique fixed point solution given by

$$L^{(k)}_\beta = \Big[I-\beta\mathcal{B}^{(k)}\Big]^{-1}(h^{(k)}\odot d).$$

This proves the existence of the unique solution, which is the first part of the lemma.

To prove the monotonicity of the solutions, consider k, $l\in\mathbb{Z}_{\geq 0}$ such that $k<l$. A sample path starting from $e\in S^{(k)}$ must escape $S^{(k)}$ before it escapes $S^{(l)}$. Thus $L^{(l)}_\beta(e)\geq L^{(k)}_\beta(e)$. In addition, the above inequality is strict because W_t has a unimodal distribution. This completes the proof.

Appendix D. PROOF OF PROPOSITION 3

We start by noticing that the error process $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process. Therefore, the functions $D^{(k)}_\beta$ and $N^{(k)}_\beta$ may be thought as value functions when strategy $f^{(k)}$ is used. Thus, they satisfy the following fixed point equations: for $\beta\in(0,1)$,

$$D_\beta^{(k)}(e) = \begin{cases} \varepsilon\big((1-\beta)d(e) + \beta[\mathcal{B}D_\beta^{(k)}](e)\big), & \text{if } |e| \geq k \\ (1-\beta)d(e) + \beta[\mathcal{B}D_\beta^{(k)}](e), & \text{if } |e| < k, \end{cases}$$

$$N_\beta^{(k)}(e) = \begin{cases} \varepsilon\big((1-\beta) + \beta[\mathcal{B}N_\beta^{(k)}](e)\big), & \text{if } |e| \geq k \\ (1-\beta) + \beta[\mathcal{B}N_\beta^{(k)}](e), & \text{if } |e| < k \end{cases}$$

For $k > 0$, let $\tau^{(k)}$ denote the stopping time of first successful reception when the Markov process starting at state 0 at time $t = 0$ follows strategy $f^{(k)}$.

Then,

$$L_\beta^{(k)}(0) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_0) \;\Big|\; E_0 = 0\Big] \tag{D.1}$$

$$M_\beta^{(k)}(0) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \;\Big|\; E_0 = 0\Big] = \frac{1 - \mathbb{E}[\beta^{\tau^{(k)}} \mid E_0 = 0]}{1-\beta} \tag{D.2}$$

$$D_\beta^{(k)}(0) = \mathbb{E}\Big[(1-\beta)\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_0) + \beta^{\tau^{(k)}} D_\beta^{(k)}(0) \;\Big|\; E_t = 0\Big] \tag{D.3}$$

$$N_\beta^{(k)}(0) = \mathbb{E}\Big[(1-\beta)\sum_{t=0}^{\tau^{(k)}-1} \beta^t + \beta^{\tau^{(k)}} N_\beta^{(k)}(0) \;\Big|\; E_t = 0\Big]. \tag{D.4}$$

Substituting (D.1) and (D.2) in (D.3) we get

$$D_\beta^{(k)}(0) = (1-\beta)L_\beta^{(k)}(0) + [1 - (1-\beta)M_\beta^{(k)}(0)]D_\beta^{(k)}(0).$$

Rearranging, we get that

$$D_\beta^{(k)}(0) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)}.$$

Similarly, substituting (D.1) and (D.2) in (D.4) we get

$$N_\beta^{(k)}(0) = [1 - (1-\beta)M_\beta^{(k)}(0)][(1-\beta) + N_\beta^{(k)}(0)].$$

Rearranging, we get that

$$N_\beta^{(k)}(0) = \frac{1}{M_\beta^{(k)}(0)} - (1-\beta).$$

The expression for $C_\beta^{(k)}(0;\lambda)$ follows from the definition.

Appendix E. PROOF OF LEMMA 6

By Lemma 5 and Proposition 3, we have that $N_\beta^{(k)}(0)$ is strictly decreasing in k.

To prove the monotonicity of $D_\beta^{(k)}(0)$ in k, we restrict our attention to $a \in \mathbb{Z}_{>0}$. The result holds for any $a \in \mathbb{Z}$ due to (Chakravorty and Mahajan, 2017, Lemma 11).

For any $v : \mathbb{Z} \to \mathbb{R}$, define operator $\mathcal{B}$ as

$$[\mathcal{B}v](e) \coloneqq \sum_{w=-\infty}^{\infty} p_w v(ae + w), \quad \forall e \in \mathbb{Z}.$$

Or, equivalently,

$$[\mathcal{B}v](e) \coloneqq \sum_{n=-\infty}^{\infty} p_{n-ae} v(n), \quad \forall e \in \mathbb{Z}.$$

For any $\beta \in (0,1)$, $\varepsilon \in (0,1)$ and $k \in \mathbb{Z}_{\geq 0}$, define the operator $\mathcal{T}^{(k)} : (\mathbb{Z} \to \mathbb{R}) \to (\mathbb{Z} \to \mathbb{R})$ as follows. For any $D : \mathbb{Z} \to \mathbb{R}$,

$$[\mathcal{T}^{(k)}D](e) = \begin{cases} \varepsilon\big((1-\beta)d(e) + \beta[\mathcal{B}D](e)\big), & \text{if } |e| \geq k \\ (1-\beta)d(e) + \beta[\mathcal{B}D](e) & \text{if } |e| < k. \end{cases}$$

This operator is the Bellman operator for evaluating strategy $f^{(k)}$. Hence, it is a contraction and D is the unique fixed point of $\mathcal{T}^{(k)}$.

Define $D_\beta^{(k,0)} = D_\beta^{(k)}$, and for $m \in \mathbb{Z}_{>0}$, $D_\beta^{(k,m)} = \mathcal{T}^{(k+1)} D_\beta^{(k,m-1)}$.

For $|e| = k$, $D_\beta^{(k,1)}(e) = (1-\beta)d(e) + \beta[\mathcal{B}D_\beta^{(k)}](e)$ and $D_\beta^{(k)}(e) = \varepsilon((1-\beta)d(e) + \beta[\mathcal{B}D_\beta^{(k)}](e))$; hence, $D_\beta^{(k,1)}(e) > D_\beta^{(k)}(e)$, since $\varepsilon \in (0,1)$. For $|e| \neq k$, $D_\beta^{(k,1)}(e) = D_\beta^{(k)}(e)$ because both terms have the same expression. Hence, for all $e \in \mathbb{Z}$,

$$D_\beta^{(k,1)}(e) \geq D_\beta^{(k)}(e), \quad \text{or} \quad D_\beta^{(k,1)} \geq D_\beta^{(k)}.$$

If we apply the operator $\mathcal{T}^{(k+1)}$ to both sides, the monotonicity of $\mathcal{T}^{(k+1)}$ implies that $D_\beta^{(k,2)} \geq D_\beta^{(k,1)} \geq D_\beta^{(k)}$. Proceeding this way, we get that for any $m > 0$,

$$D_\beta^{(k+m)} \geq D_\beta^{(k)}. \tag{E.1}$$

Note that $\lim_{m\to\infty} D_\beta^{(k+m)} = D_\beta^{(k+1)}$, because $D_\beta^{(k+1)}$ is the unique fixed point of the operator $\mathcal{T}^{(k+1)}$. Thus, taking limit $m \to \infty$ in (E.1), we get that $D_\beta^{(k+1)} \geq D_\beta^{(k)}$.

Appendix F. PROOFS OF THEOREMS 2 AND 3

F.1 Proof of Theorem 2

We first characterize the structure of $C_\beta^{(k)}(0;\lambda)$.

Proposition 4 *For the model given in Section 2.1,*

(1) $C_\beta^{(k)}(0;\lambda)$ is submodular in (k,λ), i.e., for $l > k$, $C_\beta^{(l)}(0;\lambda) - C_\beta^{(k)}(0;\lambda)$ is decreasing in λ.

(2) Let $k_\beta^*(\lambda) = \arg\inf_{k\geq 0} C_\beta^{(k)}(0;\lambda)$ be the optimal k for a fixed λ. Then $k_\beta^*(\lambda)$ is increasing in λ. □

PROOF (1) $C_\beta^{(l)}(0;\lambda) - C_\beta^{(k)}(0;\lambda) = (D_\beta^{(l)}(0) - D_\beta^{(k)}(0)) - \lambda(N_\beta^{(k)}(0) - N_\beta^{(l)}(0))$. By Lemma 6, $N_\beta^{(k)}(0) - N_\beta^{(l)}(0)$ is positive, hence $C_\beta^{(l)}(0;\lambda) - C_\beta^{(k)}(0;\lambda)$ is decreasing in λ. Hence $C_\beta^{(k)}(0;\lambda)$ is submodular.

(2) Note that $k_\beta^*(\lambda) = \arg\inf_{k\geq 0} C_\beta^{(k)}(0;\lambda)$ can take a value ∞ (which corresponds to the strategy 'never communicate'). Thus, the domain of k is $\mathbb{X}_{\geq 0} \cup \{\infty\}$, which is compact. Hence, by (Topkis, 1998, Theorem 2.8.2), k_β^* is increasing in λ. ■

By Proposition 4, $k_\beta^*(\lambda) = \arg\inf_{k\geq 0} C_\beta^{(k)}(0;\lambda)$ is increasing in λ. Let $\mathbb{K}$ denote the set of all possible values of $k_\beta^*(\lambda)$. Since k is integer-valued, the plot of k_β^* vs λ must be a staircase function. In particular, there exists an increasing sequence $\{\lambda_\beta^{(k_n)}\}_{k_n \in \mathbb{K}}$ such that for $\lambda \in (\lambda_\beta^{(k_{n-1})}, \lambda_\beta^{(k_n)}]$, $k_\beta^*(\lambda) = k_n$. We will show that for any k_n,

$$C_\beta^{(k_n)}(0;\lambda_\beta^{(k_n)}) = C_\beta^{(k_n+1)}(0;\lambda_\beta^{(k_n)}),$$

or, equivalently,

$$\frac{L_\beta^{(k_n)}(0) + \lambda_\beta^{(k_n)}}{M_\beta^{(k_n)}(0)} = \frac{L_\beta^{(k_{n+1})}(0) + \lambda_\beta^{(k_n)}}{M_\beta^{(k_{n+1})}(0)}, \tag{F.1}$$

from which we get that $\lambda_\beta^{(k_n)}$ is given by (17) by using the relations given in Proposition 3.

Proof of (F.1) For any $\lambda \in (\lambda_\beta^{(k_{n-1})}, \lambda_\beta^{(k_n)}]$, $C_\beta^{(k_n)}(0;\lambda) \le C_\beta^{(k_{n+1})}(0;\lambda)$. In particular, for $\lambda = \lambda_\beta^{(k_n)}$,

$$C_\beta^{(k_n)}(0;\lambda_\beta^{(k_n)}) \le C_\beta^{(k_{n+1})}(0;\lambda_\beta^{(k_n)}). \tag{F.2}$$

Similarly, for any $\lambda \in (\lambda_\beta^{(k_n)}, \lambda_\beta^{(k_{n+1})}]$, $C_\beta^{(k_{n+1})}(0;\lambda) \le C_\beta^{(k_n)}(0;\lambda)$. Since both terms are continuous in λ, taking limit as $\lambda \downarrow \lambda_\beta^{(k_n)}$, we get

$$C_\beta^{(k_{n+1})}(0;\lambda_\beta^{(k_n)}) \le C_\beta^{(k_n)}(0;\lambda_\beta^{(k_n)}). \tag{F.3}$$

Eq. (F.1) follows from combining (F.2) and (F.3).

Proof of Part 1) By definition of $\lambda_\beta^{(k_n)}$, the strategy $f^{(k_n)}$ is optimal for $\lambda \in (\lambda_\beta^{(k_{n-1})}, \lambda_\beta^{(k_n)}]$.

Proof of Part 2) Recall $C_\beta^*(\lambda) = \inf_{k \ge 0} C_\beta^{(k)}(0;\lambda)$. By definition, for $\lambda \ge 0$, $C^{(k)}(0;\lambda)$, is increasing and affine in λ. Therefore, its pointwise minimum (over k) is increasing and concave in λ.

As shown in part 1), for $\lambda \in (\lambda_\beta^{(k_n)}, \lambda_\beta^{(k_{n+1})}]$, $C_\beta^*(\lambda) = C_\beta^{(k_{n+1})}(0;\lambda)$, which is linear (and continuous) in λ; hence, $C_\beta^*(\lambda)$ is piecewise linear. Finally, by (F.1), $C_\beta^{(k_n)}(0;\lambda^{(k_n)}) = C_\beta^{(k_{n+1})}(0;\lambda^{(k_n)})$. Therefore, at the corner points, $\lim_{\lambda \uparrow \lambda_\beta^{(k_{n+1})}} C_\beta^*(\lambda) = \lim_{\lambda \downarrow \lambda_\beta^{(k_{n+1})}} C_\beta^*(\lambda)$. Hence, $C_\beta^*(\lambda)$ is continuous in λ.

F.2 Proof of Theorem 3

Note that by definition, $\theta^* \in [0,1]$ and

$$\theta^* N_\beta(f^{(k^*)}, g^*) + (1-\theta^*) N_\beta(f^{(k^*+1)}, g^*) = \alpha. \tag{F.4}$$

Proof of Part 1) The proof relies on the following characterization of the optimal strategy stated in (Sennott, 2001, Proposition 1.2). The characterization was stated for the long-term average setup but a similar result can be shown for the discounted case as well, for example, by using the approach of Borkar (1988). Also, see (Luenberger, 1968, Theorem 8.4.1) for a similar sufficient condition for general constrained optimization problem.

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0,1]$ if the following conditions hold:

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$,
(C2) There exists a $\lambda^\circ \ge 0$ such that (f°, g°) is optimal for $C_\beta(f, g; \lambda^\circ)$.

We will show that the strategies (f^*, g^*) satisfy (C1) and (C2) with $\lambda^\circ = \lambda_\beta^{(k^*)}$.

(f^*, g^*) satisfy (C1) due to (F.4). For $\lambda = \lambda_\beta^{(k^*)}$, both $f^{(k^*)}$ and $f^{(k^*+1)}$ are optimal for $C_\beta(f, g; \lambda)$. Hence, any strategy randomizing between them, in particular f^*, is also optimal for $C_\beta(f, g; \lambda)$. Hence (f^*, g^*) satisfies (C2). Therefore, by (Sennott, 2001, Proposition 1.2), (f^*, g^*) is optimal for Problem 3.

Proof of Part 2) The expression of k^* and θ^* follow directly from (18) and (19). The form of $D_\beta^*(\alpha)$ given in (20) follows immediately from the fact that (f^*, g^*) is a Bernoulli randomized simple strategy.

$D_\beta^*(\alpha)$ is the solution to a constrained optimization problem with the constraint set $\{(f,g) : N_\beta(f,g) \le \alpha\}$. Therefore, it is decreasing and convex in the constraint α. The optimality of (f^*, g^*) implies (20). Piecewise linearity of $D_\beta^*(\alpha)$ follows from (20). Finally, by definition of $\alpha^{(k)}$ and θ, $\lim_{a \uparrow \alpha^{(k)}} D_\beta^*(\alpha) = D_\beta^{(k)}(0) = \lim_{a \downarrow \alpha^{(k)}} D_\beta^*(\alpha)$. Hence, $D_\beta^*(\alpha)$ is continuous in α.