Sufficient statistics for linear control strategies in decentralized systems with partial history sharing

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Abstract: In decentralized control systems with linear dynamics, quadratic cost, and Gaussian disturbance (also called decentralized LQG systems) linear control strategies are not always optimal. Nonetheless, linear control strategies are appealing due to their analytic and implementation simplicity. In this paper, we investigate decentralized LQG systems with partial history sharing information structure and identify finite-dimensional sufficient statistics for such systems. Unlike prior work on decentralized LQG systems, we do not assume partially nestedness or quadratic invariance. Our approach is based on the common information approach of Nayyar et al., 2013 and exploits the linearity of the system dynamics and control strategies. To illustrate our methodology, we identify sufficient statistics for linear strategies in decentralized systems where controllers communicate over a strongly connected graph with finite delays, and for decentralized systems consisting of coupled subsystems with control sharing or one-sided one step delay sharing information structures.

1 Introduction

With the increasing applications of networked control systems, the problem of finding the best linear control strategy for decentralized systems with linear dynamics, quadratic cost, and Gaussian disturbances (henceforth referred to as decentralized LQG systems) has received considerable attention in recent years [1] (and references therein).

In centralized LQG systems, linear control strategies are globally optimal, the best linear control strategies can be computed by backward Riccati recursions, the best linear control is a function of the controller's estimate of the state of the plant and this estimate is updated using Kalman filtering equations. In contrast, the problem of finding the best linear control strategies for decentralized LQG systems has the following salient features:

1. In general, linear control strategies are not globally optimal, i.e., there may exist non-linear control strategies that outperform linear strategies as is illustrated by the Witsenhausen counterexample [2] and memoryless control in Gaussian noise [3]. Linear strategies are globally optimal only when the control problem has specific information structures such as static [4], partially nested [5], or stochastically nested [6] information structures and their variations.

2. In general, the problem of finding the best linear control strategies is not convex. It may be converted to a convex model matching problem only when the sparsity pattern of the plant and the controller have specific structure such as funnel causality [7] or quadratic invariance [8] and their variations.

3. In general, the best linear control strategy may not have a finite-dimensional sufficient statistic, i.e., it may not be possible to represent the best linear controller by a finite set of estimates that are generated by recursions of finite order as is illustrated by the two controller completely decentralized system considered in [9]. The best linear strategies are known to have a finite-dimensional sufficient statistic only for specific examples [10–18]. Note that all of these examples have partially nested information structure and some of these examples have quadratic invariant sparsity structure. It is generally believed that the best linear control strategies in partially nested and quadratic invariant systems will have finite-dimensional sufficient statistic.

In this paper, we investigate the third aspect of decentralized LQG systems described above, viz., whether finite-dimensional sufficient statistics for linear control strategies can be identified for some subclass of decentralized LQG systems. In particular, we investigate decentralized LQG systems with partial history sharing information structure [19], which is a generalization of several well-known information structures of decentralized control. The partial history sharing model, in general, is not partially nested or quadratic invariant. Our main results for this model are presented in Section 3 and can be summarized as follows:

1. we identify finite-dimensional sufficient statistics for the best linear control strategy; and

2. we show that the update equation of these sufficient statistics is similar to Kalman filter updates.

In Section 4, we apply these results to decentralized control systems in which the controllers communicate along a strongly connected graph with finite delay between any pair of controllers. In Section 5, we show that these results can also be applied to problems that do not directly conform to the partial history sharing model but can be converted to problems with partial history sharing by using a person-by-person approach.

To the best of our knowledge, **_these are the first results on sufficient statistics for best linear strategies in decentralized LQG systems that are neither partially nested nor quadratic invariant._** Our results suggest that the form of the sufficient statistics is a consequence of linearity of system dynamics and control strategies rather than partial nestedness or quadratic invariance of the information structure.

Our solution methodology is based on the *common information approach* developed in [20] and used in [19] for decentralized control systems with partial history sharing. However, our results cannot be derived directly using the results of [19]. For a general decentralized system with partial history sharing, the results of [19] provide the structure of globally optimal control strategies *and* a dynamic programming decomposition. In this paper, we exploit linearity (of control strategies and of the underlying decentralized system) to address

only the problem of finding the structure of best linear strategy. We do not address the problem of computing the best linear strategy. This narrower focus allows us to get simpler results than in [19].

Even with finite-dimensional sufficient statistics, the problem of computing the best linear strategies is, in general, a non-convex optimization problem. Nonetheless, when the system is either partially nested or quadratic invariant, it may be possible to use finite-dimensional sufficient statistics to compute best linear or globally optimal strategies. For example, an approach similar to ours was used in [18] to identify sufficient statistics for best linear control strategies (that were also globally optimal) for a two player decentralized LQG team that is partially nested and quadratic invariant. The authors of [18] then exploited the partially nested nature of the system to identify explicit expressions for the best linear control strategies.

Notation

Uppercase letters denote random variables/vectors and lowercase letters denote their realization. Bold uppercase letters denote matrices. $\mathbb{P}(\cdot)$ denotes the probability of an event and $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. $\mathbb{R}$ denotes the set of real numbers.

For a sequence of (column) vectors X, Y, Z, ..., the notation $\text{vec}(X, Y, Z, \dots)$ denotes the vector $[X^\mathsf{T}, Y^\mathsf{T}, Z^\mathsf{T}, \dots]^\mathsf{T}$. The vector $\text{vec}(X_1, \dots, X_t)$ is also denoted by $X_{1:t}$.

The notation $\mathbf{A} = \text{diag}(\mathbf{B}, \mathbf{C}, \mathbf{D})$ denotes a block diagonal matrices with blocks $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ on the diagonal. $\mathbf{A}^\mathsf{T}$ denotes the transpose of a matrix and $\text{Tr}[\mathbf{A}]$ denotes the trace of a matrix.

The notation $\mathbf{0}_{n \times m}$ denotes a $n \times m$ all zeros matrix; $\mathbf{0}_n$ is a short-hand for $\mathbf{0}_{n \times n}$; $\mathbf{I}_n$ denotes a $n \times n$ identity matrix. We omit the subscripts when dimensions can be inferred from context.

For any two random vectors X and Y, we say that X is a sub-vector of Y, and denote it by $X \subset Y$, if the set of all components of X is a subset of the set of all components of Y. More formally, $X \subset Y$ if there exists a row-stochastic binary matrix $\mathbf{P}$ (i.e., all its elements are 0 or 1 and each row has a single 1) such that $X = \mathbf{P}Y$.

2   Problem formulation

2.1   Model

Consider a linear dynamic system with n controllers and a partial history sharing information structure [19]. We follow the same notation as [19] and, for completeness, restate the model below.

The system operates in discrete time for a horizon T. Let $X_t \in \mathbb{R}^{d_x}$ denote the state of the system at time t, $U_t^i \in \mathbb{R}^{d_u^i}$ denote the control action of controller i, $i = 1, \dots, n$ at time t, and U_t denote the vector $\text{vec}(U_t^1, \dots, U_t^n)$.

The initial state X_1 has a Gaussian probability distribution $\mathcal{N}(0, \Sigma_x)$ and evolves according to

$$X_{t+1} = \mathbf{A}_t X_t + \mathbf{B}_t U_t + W_t^0 \tag{1}$$

where $\mathbf{A}_t$ and $\mathbf{B}_t$ are matrices of appropriate dimensions and $\{W_t^0\}_{t=1}^T$ is a sequence of i.i.d. zero-mean Gaussian random variables with probability distribution $\mathcal{N}(0, \Sigma_{w^0})$.

As in [19], at any time t, each controller has access to three types of data: the current observation Y_t^i, the local memory M_t^i, and the shared memory $Z_{1:t-1}$. The details of the information structure will be described later. We use Y_t to denote $\text{vec}(Y_t^1, \dots, Y_t^n)$ and M_t to denote $\text{vec}(M_t^1, \dots, M_t^n)$. Let $d_y = d_y^1 + \cdots + d_y^n$ and $d_m = d_m^1 + \cdots + d_m^n$, so that $Y_t \in \mathbb{R}^{d_y}$ and $M_t \in \mathbb{R}^{d_m}$.

We restrict attention to linear control strategies and assume that controller i's strategy is of the form:

$$U_t^i = \mathbf{K}_t^i Z_{1:t-1} + \mathbf{G}_t^i Y_t^i + \mathbf{H}_t^i M_t^i \tag{2}$$

where $\mathbf{K}_t^i$, $\mathbf{G}_t^i$, and $\mathbf{H}_t^i$ are matrices of appropriate dimensions. The collection of $\{(\mathbf{K}_t^i, \mathbf{G}_t^i, \mathbf{H}_t^i)\}_{t=1}^T$ is referred to as the *control strategy* of controller i.

Combining (2) for all controllers, we get

$$U_t = \mathbf{K}_t Z_{1:t-1} + \mathbf{G}_t Y_t + \mathbf{H}_t M_t, \tag{3}$$

where $\mathbf{K}_t = [\mathbf{K}_t^{1\mathsf{T}} \mid \cdots \mid \mathbf{K}_t^{n\mathsf{T}}]^{\mathsf{T}}$, $\mathbf{G}_t = \mathrm{diag}(\mathbf{G}_t^1, \ldots, \mathbf{G}_t^n)$ and $\mathbf{H}_t = \mathrm{diag}(\mathbf{H}_t^1, \ldots, \mathbf{H}_t^n)$.

At time t, the system incurs a quadratic cost $\ell_t(X_t, U_t)$ given by

$$\ell_t(X_t, U_t) = X_t^{\mathsf{T}} \mathbf{Q}_t X_t + U_t^{\mathsf{T}} \mathbf{R}_t U_t \tag{4}$$

where $\mathbf{Q}_t$ is positive semi-definite and $\mathbf{R}_t$ is positive definite matrices of appropriate dimensions.

We are interested in choosing control strategies of all controllers to minimize

$$\mathbb{E}\Big[\sum_{t=1}^T \ell_t(X_t, U_t)\Big], \tag{5}$$

where the expectation is with respect to the joint probability measure on $(X_{1:T}, U_{1:T})$ induced by the choice of the control strategies.

2.2 Partial history sharing information structure

As described earlier, controller i has access to three types of data at time t: the current observation Y_t^i, the local memory M_t^i, and the shared memory $Z_{1:t-1}$. These variables are given as follows:

1. The current local observation $Y_t^i \in \mathbb{R}^{d_y^i}$ of controller i is given by

$$Y_t^i = \mathbf{C}_t^i X_t + W_t^i \tag{6}$$

where $\mathbf{C}_t^i$ is a matrix of appropriate dimensions and $\{W_t^i\}_{t=1}^T$ is a sequence of i.i.d. zero-mean Gaussian random variables with probability distribution $\mathcal{N}(0, \Sigma_{w^i})$. The random variables in the collection $\{X_1, W_t^j, t = 1, \ldots, T, j = 0, 1, \ldots, n\}$, called *primitive random variables*, are mutually independent. Combining (6) for all controllers, we get

$$Y_t = \mathbf{C}_t X_t + W_t^{1:n},$$

where $W_t^{1:n} = \mathrm{vec}(W_t^1, \ldots, W_t^n)$ and $\mathbf{C}_t = [\mathbf{C}_t^{1\mathsf{T}} \mid \cdots \mid \mathbf{C}_t^{n\mathsf{T}}]^{\mathsf{T}}$.

2. The local memory $M_t^i \in \mathbb{R}^{d_m^i}$ of controller i is a subset of the history of its local observations and actions:

$$M_t^i \subset \{Y_{1:t-1}^i, U_{1:t-1}^i\}. \tag{7}$$

At $t = 1$, the local memory is empty, which we will represent by the convention $M_1^i := 0$.

3. In addition, all controllers have access to a shared memory $Z_{1:t-1}$, where $Z_t = \mathrm{vec}(Z_t^1, \ldots, Z_t^n)$. The shared memory $Z_{1:t-1}$ is a subset of the history of observations and actions of all controllers:

$$Z_{1:t-1} \subset \{Y_{1:t-1}, U_{1:t-1}\}. \tag{8}$$

At $t = 1$, the shared memory is empty, $Z_0 := 0$; at each time $Z_t \in \mathbb{R}^{d_z}$. We will refer to the shared memory as the *common information* among the controllers.

The local and shared memories are updated as follows: after taking the control action at time t, controller i sends a subvector Z_t^i of its local information $\{M_t^i, Y_t^i, U_t^i\}$ to the shared memory. We assume that the protocol of choosing the subset Z_t^i is pre-specified. After sending data Z_t^i to the shared memory, controller i updates its local memory according to a pre-specified protocol such that $M_{t+1}^i \subset \{M_t^i, Y_t^i, U_t^i\} \setminus Z_t^i$, which ensures that the contents of the local and shared memories do not overlap.

The process of generating the new local memory M_{t+1}^i and Z_t^i described above can be written in terms of the following equations:

$$M_{t+1}^i = \mathbf{P}_{mm,t}^i M_t^i + \mathbf{P}_{my,t}^i Y_t^i + \mathbf{P}_{mu,t}^i U_t^i \tag{9}$$

and

$$Z_t^i = \mathbf{P}_{zm,t}^i M_t^i + \mathbf{P}_{zy,t}^i Y_t^i + \mathbf{P}_{zu,t}^i U_t^i, \tag{10}$$

where $\mathbf{P}_{**,t}^i$ are matrices that satisfy the following properties:

A1. Each entry of $\mathbf{P}_{**,t}^i$ is either 0 or 1.

A2. The matrix

$$\begin{bmatrix} \mathbf{P}_{mm,t}^i & \mathbf{P}_{my,t}^i & \mathbf{P}_{mu,t}^i \\ \mathbf{P}_{zm,t}^i & \mathbf{P}_{zy,t}^i & \mathbf{P}_{zu,t}^i \end{bmatrix}$$

is doubly stochastic (that is, each row and column sum is 1).

Note that the $\mathbf{P}_{**,t}^i$ matrices are specified a priori based on the memory update protocols of the system. Also note that properties A1 and A2 are a consequence of these memory update protocols. We refer the reader to [19] for several examples of partial history sharing information structures.

Combining (9) for all controllers, we get

$$M_{t+1} = \mathbf{P}_{mm,t} M_t + \mathbf{P}_{my,t} Y_t + \mathbf{P}_{mu,t} U_t \tag{11}$$

where $\mathbf{P}_{mm,t} = \mathrm{diag}(\mathbf{P}_{mm,t}^1, \ldots, \mathbf{P}_{mm,t}^n)$, $\mathbf{P}_{my,t} = \mathrm{diag}(\mathbf{P}_{my,t}^1, \ldots, \mathbf{P}_{my,t}^n)$, $\mathbf{P}_{mu,t} = \mathrm{diag}(\mathbf{P}_{mu,t}^1, \ldots, \mathbf{P}_{mu,t}^n)$.

Similarly, combining (10) for all controller gives

$$Z_t = \mathbf{P}_{zm,t} M_t + \mathbf{P}_{zy,t} Y_t + \mathbf{P}_{zu,t} U_t \tag{12}$$

where $\mathbf{P}_{zm,t} = \mathrm{diag}(\mathbf{P}_{zm,t}^1, \ldots, \mathbf{P}_{zm,t}^n)$, $\mathbf{P}_{zy,t} = \mathrm{diag}(\mathbf{P}_{zy,t}^1, \ldots, \mathbf{P}_{zy,t}^n)$, $\mathbf{P}_{zu,t} = \mathrm{diag}(\mathbf{P}_{zu,t}^1, \ldots, \mathbf{P}_{zu,t}^n)$.

An example of the above model is the delayed sharing information structure [21], in which the shared memory consists of k steps old observations and control actions all controllers, i.e., $Z_{1:t-1} = \mathrm{vec}(Y_{1:t-k}, U_{1:t-k})$ and the local memory consists of the observations and actions taken at $t-k+1$, $\ldots$, $t-1$, i.e., $M_t^i = \mathrm{vec}(Y_{t-k+1:t-1}^i, U_{t-k+1:t-1}^i)$. In particular, when the delay $k = 2$, then $M_t^i = \mathrm{vec}(Y_{t-1}^i, U_{t-1}^i)$, $Z_t^i = \mathrm{vec}(Y_{t-1}^i, U_{t-1}^i)$, and the equations for generating M_{t+1}^i and Z_t^i can be written as

$$M_{t+1}^i = \mathbf{0} M_t^i + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} Y_t^i + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} U_t^i$$

and

$$Z_t^i = \mathbf{I} M_t^i + \mathbf{0} Y_t^i + \mathbf{0} U_t^i.$$

2.3   Generalized partial history sharing information structure

We now describe the generalized version of the partial history sharing information structure. As in the original partial history sharing model, controller i has access to three types of data at time t: the current observation Y_t^i, a shared memory $Z_{1:t-1}$ that is available to all controllers, and a local memory M_t^i with $M_1^i := 0$ and $Z_0 := 0$. The difference between the original model and the generalized one lies in the memory update rules. In the partial history sharing model, the local and shared memories are updated according to (11) and (12), where $\mathbf{P}_{**,t}$ are block diagonal matrices and $\mathbf{P}_{**,t}^i$ satisfy properties A1 and A2. In *generalized partial history sharing information structure*, the local and shared memory update rules still satisfy (11) and (12), but we allow $\mathbf{P}_{**,t}$ to be arbitrary matrices. We will describe examples of this information structure in Section 4.

Remark 1 In some cases, the local memory M_t^i is always empty. In such systems, the update equations (9)-(12) can be replaced by

$$Z_t^i = \mathbf{P}_{zy,t}^i Y_t^i + \mathbf{P}_{zu,t}^i U_t^i, \tag{13}$$

$$Z_t = \mathbf{P}_{zy,t} Y_t + \mathbf{P}_{zu,t} U_t. \tag{14}$$

2.4   Problem formulation and main result

We are interested in the problem of finding the best linear control strategies. Specifically:

Problem (P1) *For the model described above, given horizon T, the matrices $\mathbf{A}_t$, $\mathbf{B}_t$, $\mathbf{C}_t^i$, $\mathbf{Q}_t$, $\mathbf{R}_t$, the covariance matrices Σ_x, Σ_{w^i}, and the protocols for updating the local and shared memory, find a control strategy of the form (2) that minimizes the expected total cost given by (5).*

One of the difficulties for Problem (P1) is that the shared memory $Z_{1:t-1}$ available to all controllers is increasing with time; consequently, the size of the gain matrices $\mathbf{K}_t$ in (3) is increasing as well. Our main result is to identify an appropriate sufficient statistic $\check{S}_t$ of $Z_{1:t-1}$ that has the same dimension as $\text{vec}(X_t, M_t)$.

Main Result *In Problem (P1), the best linear control strategies are of the form*

$$U_t = \tilde{\mathbf{K}}_t \tilde{\mathbf{L}}_t \check{S}_t + \mathbf{G}_t Y_t + \mathbf{H}_t M_t, \tag{15}$$

where $\check{S}_t = (\hat{X}_t, \hat{M}_t)$ is a common information based estimate given by

$$\check{S}_t = \mathbb{E}[\text{vec}(X_t, M_t) \mid Z_{1:t-1}].$$

$\check{S}_t$ evolves according to a Kalman filtering recursion given in Lemma 2. The matrices $\{\tilde{\mathbf{L}}_t\}_{t=1}^T$ are given by $\tilde{\mathbf{L}}_t = \begin{bmatrix} \mathbf{I}_{d_x} & \mathbf{0}_{d_x \times d_m} \\ \mathbf{C}_t & \mathbf{0}_{d_y \times d_m} \\ \mathbf{0}_{d_m \times d_x} & \mathbf{I}_{d_m} \end{bmatrix}$, and the matrices $\{\tilde{\mathbf{K}}_t\}_{t=1}^T$ are determined off-line by a backward Riccati recursion given in Theorem 1.

2.5   Salient features of the result

The above structural result shows that in the best linear strategy, the control action at each time depends on the current local observation, the current local memory, and a common information based estimate of the system state and the local memories of all controllers. Thus, **the sufficient statistic is finite dimensional**.

Unlike prior work on structural results for decentralized control problems, our result relies on the *linearity* of the decentralized system and of the control strategies; and not on partial nestedness or quadratic invariance.

The result basically follows from two simple observations: (i) under linear strategies, control actions can be viewed as superposition of two components—a local information based component and a common information based component; and (ii) once the matrices for calculating the local information based component have been fixed, the problem of choosing the common information based component reduces to a centralized LQG problem. The details are presented in the next section.

3   Proof of the main result

The main idea of the proof is as follows. Arbitrarily fix the matrices $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$ and consider the sub-problem of finding the best choice of matrices $\mathbf{K}_{1:T}$ to minimize the total expected cost given by (5). We argue that the resulting sub-problem is a centralized stochastic control problem for which $\check{S}_t$ is a sufficient statistic, thereby establishing the main result.

3.1   A sub-problem and the induced centralized system

Following [19], we introduce a new decision maker—the *coordinator*—that sequentially observes the *common information* process $\{Z_t\}_{t=1}^T$ and chooses actions $\tilde{U}_t = \text{vec}(\tilde{U}_t^1, \ldots, \tilde{U}_t^n)$ where

$$\tilde{U}_t^i = \mathbf{K}_t^i Z_{1:t-1}. \tag{16}$$

The controllers of the original system are passive agents that generate U_t^i according to

$$U_t^i = \tilde{U}_t^i + \mathbf{G}_t^i Y_t^i + \mathbf{H}_t^i M_t^i. \tag{17}$$

Combining (16) and (17) for all i, we get

$$\tilde{U}_t = \mathbf{K}_t Z_{1:t-1}, \tag{18}$$
$$U_t = \tilde{U}_t + \mathbf{G}_t Y_t + \mathbf{H}_t M_t; \tag{19}$$

where $\mathbf{G}_t$ and $\mathbf{H}_t$ are block diagonal matrices and $\mathbf{K}_t$ is a stacked matrix as defined earlier in (3).

As in [19], the optimization problem at the coordinator is equivalent to a partially observed centralized stochastic control problem, which we call the *coordinated system*. Define the state $\tilde{X}_t$ and the observation $\tilde{Y}_t$ of this coordinated system as:

$$\tilde{X}_t = \text{vec}(X_t, Y_t, M_t), \tag{20}$$
$$\tilde{Y}_t = Z_{t-1}. \tag{21}$$

Then the control action $\tilde{U}_t$ of this system is chosen according to (18) which is a linear function of the observation history.

The coordinated system is a centralized system LQG system with linear dynamics, linear observations, quadratic cost, and Gaussian disturbance. In particular:

1. The coordinated system has linear dynamics which may be written as

$$\tilde{X}_{t+1} = \tilde{\mathbf{A}}_t \tilde{X}_t + \tilde{\mathbf{B}}_t \tilde{U}_t + \tilde{\mathbf{F}}_t W_t \tag{22}$$

where $W_t = \text{vec}(W_t^0, W_{t+1}^1, \ldots, W_{t+1}^n)$, and $\tilde{\mathbf{A}}_t$, $\tilde{\mathbf{B}}_t$, and $\tilde{\mathbf{F}}_t$ are matrices of appropriate dimensions that are obtained by combining (1), (6), (11), (18), and (19) and are given by

$$\tilde{\mathbf{A}}_t = \begin{bmatrix} \mathbf{A}_t & \mathbf{B_t G_t} & \mathbf{B_t H_t} \\ \mathbf{C_{t+1} A_t} & \mathbf{C_{t+1} B_t G_t} & \mathbf{C_{t+1} B_t H_t} \\ \mathbf{0} & \mathbf{P}_{my,t} + \mathbf{P}_{mu,t}\mathbf{G}_t & \mathbf{P}_{mm,t} + \mathbf{P}_{mu,t}\mathbf{H}_t \end{bmatrix}, \tag{23}$$

$$\tilde{\mathbf{B}}_t = \begin{bmatrix} \mathbf{B}_t \\ \mathbf{C_{t+1} B_t} \\ \mathbf{P}_{mu,t} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{F}}_t = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \tag{24}$$

where the blocks in the first column of $\tilde{\mathbf{F}}_t$ have dimensions compatible with W_t^0 and the blocks in the second column have dimensions compatible with $\text{vec}(W_t^1, \ldots, W_t^n)$.

2. The observations are linear in the state and the control and may be written as

$$\tilde{Y}_1 = \mathbf{0}, \tag{25}$$
$$\tilde{Y}_t = \tilde{\mathbf{C}}_t \tilde{X}_{t-1} + \tilde{\mathbf{D}}_t \tilde{U}_{t-1}, \quad t > 1; \tag{26}$$

where $\tilde{\mathbf{C}}_t$ and $\tilde{\mathbf{D}}_t$ are matrices of appropriate dimensions given by

$$\tilde{\mathbf{C}}_t = \begin{bmatrix} \mathbf{0} & \mathbf{P}_{zy,t-1} + \mathbf{P}_{zu,t-1}\mathbf{G}_{t-1} & \mathbf{P}_{zm,t-1} + \mathbf{P}_{zu,t-1}\mathbf{H}_{t-1} \end{bmatrix}, \tag{27}$$
$$\tilde{\mathbf{D}}_t = \mathbf{P}_{zu,t-1}. \tag{28}$$

3. The per-step cost is quadratic in the state and control action and may be written as

$$\ell_t(X_t, U_t) = \tilde{\ell}_t(\tilde{X}_t, \tilde{U}_t) = \begin{bmatrix} \tilde{X}_t^\intercal & \tilde{U}_t^\intercal \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Q}}_t & \tilde{\mathbf{N}}_t \\ \tilde{\mathbf{N}}_t^\intercal & \tilde{\mathbf{R}}_t \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ \tilde{U}_t \end{bmatrix} \tag{29}$$

where $\tilde{\mathbf{Q}}_t$, $\tilde{\mathbf{N}}_t$, $\tilde{\mathbf{R}}_t$ are obtained by combining (4) and (19) and are given by

$$\tilde{\mathbf{Q}}_t = \begin{bmatrix} \mathbf{Q}_t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G_t^\intercal R_t G_t} & \mathbf{G_t^\intercal R_t H_t} \\ \mathbf{0} & \mathbf{H_t^\intercal R_t G_t} & \mathbf{H_t^\intercal R_t H_t} \end{bmatrix}, \tag{30}$$

$$\tilde{\mathbf{N}}_t = \begin{bmatrix} \mathbf{0} \\ \mathbf{G_t^\intercal R_t} \\ \mathbf{H_t^\intercal R_t} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{R}}_t = \mathbf{R}_t. \tag{31}$$

Recall that we assumed that $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$ are fixed. The auxiliary matrices $\tilde{\mathbf{A}}_t$, $\tilde{\mathbf{C}}_t$, $\tilde{\mathbf{Q}}_t$ and $\tilde{\mathbf{N}}_t$ defined above depend on $\mathbf{G}_t$ and $\mathbf{H}_t$.

3.2 Structure of the optimal controller

The coordinated system defined above is a centralized partially observed LQG system. Therefore, from standard results in linear stochastic control [22], the state estimate can be computed recursively as follows.

Lemma 1 *Define $\check{X}_t$ as the coordinator's estimate of the state $\tilde{X}_t$ given by*

$$\check{X}_t = \mathbb{E}[\tilde{X}_t \mid \tilde{Y}_{1:t}, \tilde{U}_{1:t-1}]$$

The initial value of the state estimate is given by $\check{X}_1 = 0$. For $t > 1$, the state estimate may be updated as follows

$$\check{X}_{t+1} = \tilde{\mathbf{A}}_t \check{X}_t + \tilde{\mathbf{B}}_t \tilde{U}_t + \tilde{\mathbf{A}}_t \tilde{\mathbf{P}}_t \tilde{\mathbf{C}}_{t+1}^\intercal \left[\tilde{\mathbf{C}}_{t+1} \tilde{\mathbf{P}}_t \tilde{\mathbf{C}}_{t+1}^\intercal \right]^{-1} \left(\tilde{Y}_{t+1} - \tilde{\mathbf{C}}_{t+1} \check{X}_t - \tilde{\mathbf{D}}_{t+1} \tilde{U}_t \right) \tag{32}$$

where

$$\tilde{\mathbf{P}}_t = \mathbb{E}[(\tilde{X}_t - \check{X}_t)^2 \mid \tilde{Y}_{1:t}, \tilde{U}_{1:t-1}],$$

which may be computed a priori by solving the following forward Riccati recursion:

$$\tilde{\mathbf{P}}_1 = \mathrm{diag}\left(\Sigma_x, \mathbf{0}_{d_y \times d_y}, \mathbf{0}_{d_m \times d_m}\right), \tag{33}$$

$$\tilde{\mathbf{P}}_{t+1} = \tilde{\mathbf{A}}_t \tilde{\mathbf{P}}_t \tilde{\mathbf{A}}_t^\intercal + \tilde{\Sigma}_W \tag{34}$$

$$- \tilde{\mathbf{A}}_t \tilde{\mathbf{P}}_t \tilde{\mathbf{C}}_{t+1}^\intercal \left[\tilde{\mathbf{C}}_{t+1} \tilde{\mathbf{P}}_t \tilde{\mathbf{C}}_{t+1}^\intercal \right]^{-1} \tilde{\mathbf{C}}_{t+1} \tilde{\mathbf{P}}_t \tilde{\mathbf{A}}_t^\intercal; \tag{35}$$

where $d_y = \sum_{i=1}^n d_y^i$, $d_m = \sum_{i=1}^n d_m^i$, and $\tilde{\Sigma}_W$ is the covariance of $\tilde{\mathbf{F}}_t W_t$ which is given by

$$\mathrm{diag}\left(\Sigma_{w^0}, \Sigma_{w^1}, \ldots, \Sigma_{w^n}, \mathbf{0}\right)$$

where $\mathbf{0}$ is a square matrix of dimension same as M_t.

It is possible to replace $\check{X}_t$ by a lower-dimensional estimate. Let

$$\mathbf{P}_{sx} = \begin{bmatrix} \mathbf{I}_{d_x} & \mathbf{0}_{d_x \times d_y} & \mathbf{0}_{d_x \times d_m} \\ \mathbf{0}_{d_m \times d_x} & \mathbf{0}_{d_m \times d_y} & \mathbf{I}_{d_m} \end{bmatrix}$$

and

$$\tilde{\mathbf{L}}_t = \begin{bmatrix} \mathbf{I}_{d_x} & \mathbf{0}_{d_x \times d_m} \\ \mathbf{C}_t & \mathbf{0}_{d_y \times d_m} \\ \mathbf{0}_{d_m \times d_x} & \mathbf{I}_{d_m} \end{bmatrix}.$$

Then, we have the following.

Lemma 2 *Define*

$$\breve{S}_t = \mathbb{E}[\text{vec}(X_t, M_t) \mid Z_{1:t-1}, \tilde{U}_{1:t-1}].$$

Then

$$\breve{S}_t = \mathbf{P}_{sx}\breve{X}_t \quad and \quad \breve{X}_t = \tilde{\mathbf{L}}_t\breve{S}_t. \tag{36}$$

Furthermore, the initial value $\breve{S}_1 = 0$. For $t > 1$, the estimate may be updated as follows:

$$\breve{S}_{t+1} = \mathbf{P}_{sx}\Big[\tilde{\mathbf{A}}_t\tilde{\mathbf{L}}_t\breve{S}_t + \tilde{\mathbf{B}}_t\tilde{U}_t + \tilde{\mathbf{A}}_t\tilde{\mathbf{P}}_t\tilde{\mathbf{C}}_{t+1}^{\mathsf{T}}[\tilde{\mathbf{C}}_{t+1}\tilde{\mathbf{P}}_t\tilde{\mathbf{C}}_{t+1}]^{-1}(\tilde{Y}_{t+1} - \tilde{\mathbf{C}}_{t+1}\tilde{\mathbf{L}}_t\breve{S}_t - \tilde{\mathbf{D}}_{t+1}\tilde{U}_t)\Big]. \tag{37}$$

Proof. Eq. (36) is an immediate consequence of the definition of $\breve{X}_t$ and $\breve{S}_t$. Combining (36) with (32) gives (37). □

Theorem 1 *In the coordinated system described in Section 3.1, the optimal action of the coordinator is given by*

$$\tilde{U}_t = \tilde{\mathbf{K}}_t\tilde{\mathbf{L}}_t\breve{S}_t \tag{38}$$

where the gain matrices $\{\tilde{\mathbf{K}}_t\}_{t=1}^T$ are given by

$$\tilde{\mathbf{K}}_t = -[\tilde{\mathbf{R}}_t + \tilde{\mathbf{B}}_t^{\mathsf{T}}\mathbf{S}_{t+1}\tilde{\mathbf{B}}_t]^{-1}\mathbf{\Lambda}_t$$

where

$$\mathbf{\Lambda}_t = \tilde{\mathbf{N}}_t + \tilde{\mathbf{B}}_t^{\mathsf{T}}\mathbf{S}_{t+1}\tilde{\mathbf{A}}_t$$

and the matrices $\{\mathbf{S}_t\}_{t=1}^T$ are given by backward Riccati recursion:

$$\mathbf{S}_T = \tilde{\mathbf{Q}}_T,$$
$$\mathbf{S}_t = \tilde{\mathbf{A}}_t^{\mathsf{T}}\mathbf{S}_{t+1}\tilde{\mathbf{A}}_t + \tilde{\mathbf{Q}}_t - \mathbf{\Lambda}_t^{\mathsf{T}}[\tilde{\mathbf{R}}_t + \tilde{\mathbf{B}}_t^{\mathsf{T}}\mathbf{S}_{t+1}\tilde{\mathbf{B}}_t]^{-1}\mathbf{\Lambda}_t.$$

The total cost of the above strategy is given by

$$J = \sum_{\tau=1}^T \text{Tr}[\tilde{\mathbf{P}}_t\tilde{\mathbf{Q}}_t + (\tilde{\Sigma}_W + \tilde{\mathbf{A}}_t\tilde{\mathbf{P}}_t\tilde{\mathbf{A}}_t^{\mathsf{T}} - \tilde{\mathbf{P}}_{t+1})\mathbf{S}_{t+1}]. \tag{39}$$

(Note that the matrices $(\tilde{\mathbf{K}}_{1:T}, \mathbf{S}_{1:T}, \tilde{\mathbf{P}}_{1:T})$ obtained in this section depend on the choice of the matrices $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$.)

Proof. Since the coordinated system is a centralized partially observed LQG system, the coordinator's state estimate $\breve{X}_t$ is a sufficient statistic. Hence, the optimal strategy for the coordinator is of the form $\tilde{U}_t = \tilde{\mathbf{K}}_t\breve{X}_t$, where $\tilde{\mathbf{K}}_t$ is computed as described above. The result of the Theorem then follows from (36). □

Since any linear strategy in the coordinated system can be implemented in the original system and vice versa, the above result gives the following structure of best linear strategies in the original system.

Theorem 2 *In Problem (P1), the best linear control strategies are of the form*

$$U_t = \tilde{U}_t + \mathbf{G}_t Y_t + \mathbf{H}_t M_t$$
$$= \tilde{\mathbf{K}}_t\tilde{\mathbf{L}}_t\breve{S}_t + \mathbf{G}_t Y_t + \mathbf{H}_t M_t. \tag{40}$$

where $\mathbf{G}_t = \text{diag}(\mathbf{G}_t^1, \ldots, \mathbf{G}_t^n)$, $\mathbf{H}_t = \text{diag}(\mathbf{H}_t^1, \ldots, \mathbf{H}_t^n)$,

$$\breve{S}_t = \text{vec}(\hat{X}_t, \hat{M}_t) = \mathbb{E}[\text{vec}(X_t, M_t) \mid Z_{1:t-1}, \tilde{U}_{1:t-1}],$$

and the evolution of $\breve{S}_t$, the gain matrices $\tilde{\mathbf{K}}_t$ and the system performance are the same as in Lemma 2 and Theorem 1.

Remark 2 Let $\tilde{\mathbf{K}}_t = [\tilde{\mathbf{K}}_t^{1\,\mathsf{T}} \mid \cdots \mid \tilde{\mathbf{K}}_t^{n\,\mathsf{T}}]^{\mathsf{T}}$. Then, the control action of each controller may be written as

$$U_t^i = \tilde{\mathbf{K}}_t^i \tilde{\mathbf{L}}_t \breve{S}_t + \mathbf{G}_t^i Y_t^i + \mathbf{H}_t^i M_t^i.$$

Note that each controller is using its local information (Y_t^i, M_t^i) and an estimate $\breve{S}_t$ based on the common information $Z_{1:t-1}$. Since this structure holds for any choice of $\mathbf{G}_{1:T}, \mathbf{H}_{1:T}$, we say that $\breve{S}_t$ serves as a sufficient statistic in place of the entire common information $Z_{1:t-1}$.

Remark 3 Note that for a given choice of $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$, Theorem 1 identifies the optimal $\tilde{\mathbf{K}}_t^i$ matrices and the associated cost. To emphasize the dependence of matrices $\tilde{\mathbf{K}}_{1:T}^i$ on the choice of $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$, we use the notation $\tilde{\mathbf{K}}_{1:T}^i(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$ for the optimal $\tilde{\mathbf{K}}_t^i$ identified in Theorem 1. We can also write the total cost achieved by the gain matrices identified in Theorem 1 as a function of $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$ and $\tilde{\mathbf{K}}_{1:T}^i(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$:

$$\text{Optimal cost for a given } (\mathbf{G}_{1:T}, \mathbf{H}_{1:T}) \; = J(\mathbf{G}_{1:T}, \mathbf{H}_{1:T}, \tilde{\mathbf{K}}_{1:T}^i(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})).$$

In order to find the best linear strategy, we need to optimize the the above function with respect to $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$ — which may be a non-convex optimization problem due to the complicated dependence of the matrices $\tilde{\mathbf{K}}_{1:T}^i$ on $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$.

3.3 Generalization to models with common observations

In some cases, in addition to the shared memory, controllers may also have a common observation Y_t^{com} about the state of the system given as

$$Y_t^{com} = \mathbf{C}_t^{com} X_t + W_t^{com},$$

where $W_t^{com}, t = 1, 2, \ldots, T$ is a sequence of i.i.d. Gaussian variables that are independent of the all the other primitive random variables. Each controller can select its action according to a *linear* control law of the form

$$U_t^i = g_t^i(Y_t^i, M_t^i, Z_{1:t-1}, Y_{1:t}^{com}).$$

The methodology of Theorem 1 can easily be adapted for this model by allowing the coordinator to choose action $\tilde{U}_t = \text{vec}(\tilde{U}_t^1, \ldots, \tilde{U}_t^n)$ based on the shared memory *and* the history of common observations. That is,

$$\tilde{U}_t^i = \mathbf{K}_t^i \,\text{vec}(Z_{1:t-1}, Y_{1:t}^{com}). \tag{41}$$

Following the same arguments as before, the coordinator's problem once again becomes a classical LQG problem, thus establishing the result of Theorem 1 for this case with $\breve{S}_t$ now defined as

$$\breve{S}_t = \mathbb{E}[(X_t, M_t) \mid Z_{1:t-1}, Y_{1:t}^{com}, \tilde{U}_{1:t-1}]$$

3.4 Comparison with [19]

For decentralized control system with partial history sharing information structure, it is shown in [19] that the sufficient statistic of the shared memory $Z_{1:t-1}$ is given by the posterior probability distribution on (X_t, M_t). In contrast, the result of Theorem 1 shows that when attention is restricted to linear strategies, the sufficient statistic is given by the conditional mean $\breve{S}_t$ of (X_t, M_t). Our result is consistent with that of [19] because under linear strategies, the posterior probability distribution is Gaussian and is completely characterized by the conditional mean and covariance. The conditional covariance is data independent and is computed off-line using (33) and (35).

Although the methodology used in proving Theorem 1 and the solution methodology of [19] are similar, it is not possible to derive the result of Theorem 1 by directly using the results of [19]. In [19], the coordinator solves a global optimization problem to determine how controllers should use their local information. On the other hand, to prove the result of Theorem 1, we arbitrarily fix the components of the control laws that use the local information and then find the structure of the best response strategies at the coordinator.

In contrast to the approach of [19] which gives the structure of globally optimal control laws and a dynamic programming decomposition, our approach only gives the *structure* of best linear control laws. It is not possible, in general, to use our approach to *compute* the best linear control laws. The question whether the approach proposed in this paper simplifies for partially nested teams warrants further investigation.

4 Delayed sharing information structure

In this section, we illustrate our results using the specific example of delayed sharing information structures. We consider two cases: (i) one with symmetric delays where the observations and actions of any controller are available to all other controllers after a delay of k time steps and (ii) the asymmetric delay case where the communication delay from controller j to controller i is $k_{ij} < \infty$.

4.1 Symmetric delays

In delayed sharing information structure, each controller's observations and control actions are shared with all other controllers after a delay of $k \geq 1$ time steps [21]. The system dynamics, local observations, and cost function are the same as in Section 2.1.

In the language of partial history sharing model, the shared memory in this case consists of all observations and control actions that are at least k time-steps old, that is, $Z_{1:t-1} = \text{vec}(Y^1_{1:t-k}, U^1_{1:t-k}, \ldots, Y^n_{1:t-k}, U^n_{1:t-k})$; and the local memory consists of the observations and actions taken at $t-k+1$, ..., $t-1$, that is, $M^i_t = \text{vec}(Y^i_{t-1}, U^i_{t-1}, \ldots, Y^i_{t-k+1}, U^i_{t-k+1})$.

Therefore, the result of Theorem 1 applies to this model with

- the $\mathbf{P}^i_{**}$ matrices in the memory update equations (9) and (10) are given by

$$\mathbf{P}^i_{mm,t} = \begin{bmatrix} \mathbf{0}_{d^i_y \times d^i_m} \\ \mathbf{0}_{d^i_u \times d^i_m} \\ \mathbf{I}_{(k-2)(d^i_y+d^i_u)} \ \mathbf{0}_{(k-2)(d^i_y+d^i_u)\times(d^i_y+d^i_u)} \end{bmatrix},$$

$$\mathbf{P}^i_{my,t} = \begin{bmatrix} \mathbf{I}_{d^i_y} \\ \mathbf{0}_{d^i_u \times d^i_y} \\ \mathbf{0}_{(k-2)(d^i_y+d^i_u)\times d^i_y} \end{bmatrix},$$

$$\mathbf{P}^i_{mu,t} = \begin{bmatrix} \mathbf{0}_{d^i_y \times d^i_u} \\ \mathbf{I}_{d^i_u} \\ \mathbf{0}_{(k-2)(d^i_y+d^i_u)\times d^i_u} \end{bmatrix},$$

$$\mathbf{P}^i_{zm,t} = \begin{bmatrix} \mathbf{0}_{(d^i_y+d^i_u)\times(k-2)(d^i_y+d^i_u)} \ \mathbf{I}_{(d^i_y+d^i_u)} \end{bmatrix},$$

and $\mathbf{P}^i_{zy,t} = \mathbf{0}$ and $\mathbf{P}^i_{zu,t} = \mathbf{0}$.
- and the estimate of Theorem 1 as

$$\breve{S}_t = \mathbb{E}[\text{vec}(X_t, M_t) \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-1}].$$

Recall that the evolution of the sufficient statistic $\breve{S}_t$ depends on the choice of matrices $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$ in the control strategy. Such a dependence is also present in the sufficient statistic for optimal control laws for the general delayed sharing model [21]. Hence, restricting attention to linear control strategies does not lead to a two-way separation of estimation and control in delayed sharing information structures.

An alternative sufficient statistic for delayed-sharing model is given as follows:

Corollary 1 *For the symmetric delay sharing model, $\tilde{U}_t$ in Theorems 1 and 2 can be written as a linear function of*

$$S_t = \text{vec}(\hat{X}_{t-k+1|t-k}, \tilde{U}_{t-k+1:t-1}, Y_{t-2k+2:t-k}, U_{t-2k+2:t-k}), \tag{42}$$

where

$$\hat{X}_{t-k+1|t-k} = \mathbb{E}[X_{t-k+1} \mid Y_{1:t-k}, U_{1:t-k}].$$

Furthermore, $\hat{X}_{t+1|t}$ is updated according to:

$$\hat{X}_{1|0} = 0, \tag{43}$$

$$\hat{X}_{t+1|t} = \mathbf{A}_t \hat{X}_{t|t-1} + \mathbf{B}_t U_t + \mathbf{A}_t \mathbf{P}_t \mathbf{C}_t^\mathsf{T} [\mathbf{C}_t \mathbf{P}_t \mathbf{C}_t^\mathsf{T} + \Sigma_w]^{-1} (Y_t - \mathbf{C}_t \hat{X}_{t|t-1}); \tag{44}$$

where $\Sigma_w = \mathrm{diag}(\Sigma_{w^1}, \ldots, \Sigma_{w^n})$ and $\mathbf{P}_t = \mathbb{E}[(X_t - \hat{X}_{t|t-1})^2 \mid Y_{1:t-1}, U_{1:t-1}]$, which can be precomputed as follows:

$$\mathbf{P}_1 = \Sigma_x;$$

$$\mathbf{P}_{t+1} = \mathbf{A}_t \mathbf{P}_t \mathbf{A}_t^\mathsf{T} + \Sigma_{w^0} - \mathbf{A}_t \mathbf{P}_t \mathbf{C}_t^\mathsf{T} [\mathbf{C}_t \mathbf{P}_t \mathbf{C}_t^\mathsf{T} + \Sigma_w]^{-1} \mathbf{C}_t \mathbf{P}_t \mathbf{A}_t^\mathsf{T}.$$

See Appendix A for a proof. Corollary 1 shows that S_t is a sufficient statistic for $(Y_{1:t-k}, U_{1:t-k})$. This sufficient statistic consists of three parts:

1. A strategy-independent $k-1$ step window $(Y_{t-2k+2:t-k}, U_{t-2k+2:t-k})$ of the history of observations and actions that are available to all controllers.
2. A strategy-independent estimate of the k-step delayed state X_{t-k+1} based on the history of common information. Note that the update of $\hat{X}_{t-k+1|t-k}$ does not depend on the matrices $(\mathbf{G}_{1:T}, \mathbf{H}_{1:T})$.
3. A *strategy-dependent* $k-1$ step window of the history of coordinated control actions $\tilde{U}_{t-k+1:t-1}$.

This structure is similar to the optimal controller derived in [21, second structural result].

For the special case of one-step delay sharing (delay $k=1$), the result of Corollary 1 simplifies as follows.

Corollary 2 *For the one-step delay sharing model (sharing delay $k=1$), the sufficient statistic S_t of Corollary 1 is given by $S_t = \hat{X}_{t|t-1}$, where $\hat{X}_{t|t-1}$ and its evolution is defined in Corollary 1. Thus, $\tilde{U}_t$ in Theorem 1 and 2 can be written as a linear function of $\hat{X}_{t|t-1}$; hence, the control action U_t^i of controller i can be written as a linear function of $\hat{X}_{t|t-1}$ and Y_t^i.*

Corollary 2 is equivalent to the result obtained in [10, 11].

Note that a key conceptual simplification for the one-step delay sharing is that the sufficient statistic S_t is strategy independent. A similar simplification was shown for the general delayed sharing model in [21]

4.2 Asymmetric delays

In this model, controller i observes the observations and control actions of controller j with a delay of $k_{ij} < \infty$. The information available to controller i at time t consists of

$$I_t^i = \{Y_{1:t}^i, U_{1:t-1}^i\} \cup \bigcup_{j \neq i} \{Y_{1:t-k_{ij}}^j, U_{1:t-k_{ij}}^j\}.$$

All delays are finite. For convenience, define $k_{ii} := 1$. Then, the information available to controller i at time t can be written as

$$I_t^i = \{Y_t^i\} \cup \bigcup_{j=1}^n \{Y_{1:t-k_{ij}}^j, U_{1:t-k_{ij}}^j\}.$$

This information structure arises when controllers communicate along a *strongly connected graph with finite delay between any pair of controllers*. The system dynamics, local observations, and cost function are the same as in Section 2.1. Similar models have been considered in [23–28]. Note that unlike these models, we do not assume any sparsity structure on the matrices $\mathbf{A}_t, \mathbf{B}_t$ and $\mathbf{C}_t$ in the system model.

Such a model has the generalized partial history sharing information structure. As an illustration, consider the 3 controller system shown in Figure 1. Controllers 1 and 2 share information with 1-step delay, controllers 2 and 3 share information with 1-step delay but controllers 1 and 3 share information with 2-step delay, that is,

$$k_{12} = k_{21} = 1, \quad k_{23} = k_{32} = 1, \quad k_{13} = k_{31} = 2.$$
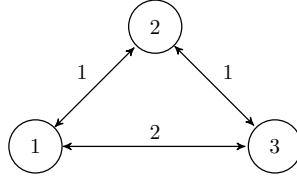
Figure 1: An example of a system with asymmetric delayed sharing. The number on the arrows denote the delay in flow of information.

The shared memory at time t is given by

$$Z_{1:t-1} = \text{vec}(Y^1_{1:t-2}, U^1_{1:t-2}, Y^2_{1:t-1}, U^2_{1:t-1}, Y^3_{1:t-2}, U^3_{1:t-2});$$

the local memories are

$$M^1_t = \text{vec}(Y^1_{t-1}, U^1_{t-1}),$$
$$M^2_t = \text{vec}(Y^1_{t-1}, U^1_{t-1}, Y^3_{t-1}, U^3_{t-1}),$$
$$M^3_t = \text{vec}(Y^3_{t-1}, U^3_{t-1});$$

and the increment in shared memory at time t is

$$Z_t = \text{vec}(Y^1_{t-1}, U^1_{t-1}, Y^2_t, U^2_t, Y^3_{t-1}, U^3_{t-1}).$$

The update of the local and shared memories may be written as (11) and (12) with

$$\mathbf{P}_{mm,t} = \mathbf{0}$$

$$\mathbf{P}_{my,t} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_{mu,t} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{P}_{zm,t} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{P}_{zy,t} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_{zu,t} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Similar to the above example, the general model with asymmetric delays may be considered as a special case of the generalized partial history sharing model. For that matter, define $k^*_j := \max_i k_{ij}$. Thus, k^*_j is the delay after which controller j's current information is available to all other controllers. In the above example, $k^*_1 = k^*_3 = 2$ and $k^*_2 = 1$.

Then, the common information available to all controllers at time t is

$$Z_{1:t-1} = \text{vec}(Y^1_{1:t-k^*_1}, U^1_{1:t-k^*_1} \ldots, Y^n_{1:t-k^*_n}, U^n_{1:t-k^*_n}),$$

and the local memory of controller i is

$$M_t^i = I_t^i \setminus \{Y_t^i, Z_{1:t-1}\}$$
$$= \text{vec}(Y_{t-k_{i1}}^1, U_{t-k_{i1}}^1, \ldots, Y_{t-k_i^*+1}^1, U_{t-k_i^*+1}^1,$$
$$Y_{t-k_{i2}}^2, U_{t-k_{i2}}^2, \ldots, Y_{t-k_i^*+1}^2, U_{t-k_i^*+1}^2,$$
$$\ldots\ldots$$
$$Y_{t-k_{in}}^n, U_{t-k_{in}}^n, \ldots, Y_{t-k_i^*+1}^n, U_{t-k_i^*+1}^n).$$

To facilitate writing the memory update equations of the form in (9) and (10) for the general asymmetric delay model, it is helpful to define the following vectors:

$$L_t^i = \text{vec}(Y_{t-1}^i, U_{t-1}^i, \ldots, Y_{t-k_i^*+1}^i, U_{t-k_i^*+1}^i). \tag{45}$$

L_t^i denotes the observations and control actions of controller i that have not yet been shared with all controllers. L_t^i takes values in $\mathbb{R}^{d_l^i}$. L_t^i is always a sub-vector of M_t^i. Note that L_t^i may be distinct from M_t^i in general (see the example above). More explicitly, the relation between L_t^i and M_t^i can be written as

$$L_t^i = \begin{bmatrix} \mathbf{0}_{\sum_{j<i}(k_j^*-k_{ij})(d_y^j+d_u^j)\times(k_i^*-1)(d_y^i+d_u^i)} \\ \mathbf{I}_{(k_i^*-1)(d_y^i+d_u^i)} \\ \mathbf{0}_{\sum_{j>i}(k_j^*-k_{ij})(d_y^j+d_u^j)\times(k_i^*-1)(d_y^i+d_u^i)} \end{bmatrix}^\intercal M_t^i \tag{46}$$

Define $L_t = \text{vec}(L_t^1, \ldots, L_t^n)$. Note that M_t^i is a sub-vector of L_t. The explicit relation between M_t^i and L_t can be written as

$$M_t^i = \text{diag}(J_{i1}, \ldots, J_{in})L_t, \tag{47}$$

where

$$J_{ij} = [\mathbf{0}_{(k_j^*-k_{ij})(d_y^j+d_u^j)\times(k_{ij}-1)(d_y^j+d_u^j)}\ \mathbf{I}_{(k_j^*-k_{ij})(d_y^j+d_u^j)}]. \tag{48}$$

Furthermore, L_t^i has an update equation similar to (10):

$$L_{t+1}^i = \tilde{\mathbf{P}}_{\ell\ell}^i L_t^i + \tilde{\mathbf{P}}_{\ell y}^i Y_t^i + \tilde{\mathbf{P}}_{\ell u}^i U_t^i \tag{49}$$

where

$$\tilde{\mathbf{P}}_{\ell m}^i = \begin{bmatrix} & \mathbf{0}_{d_y^i \times d_l^i} \\ & \mathbf{0}_{d_u^i \times d_l^i} \\ \mathbf{I}_{(k_i^*-2)(d_y^i+d_u^i)} & \mathbf{0}_{(k_i^*-2)(d_y^i+d_u^i)\times(d_y^i+d_u^i)} \end{bmatrix},$$

$$\tilde{\mathbf{P}}_{\ell y}^i = \begin{bmatrix} \mathbf{I}_{d_y^i} \\ \mathbf{0}_{d_u^i \times d_y^i} \\ \mathbf{0}_{(k_i^*-2)(d_y^i+d_u^i)\times d_y^i} \end{bmatrix}, \quad \tilde{\mathbf{P}}_{\ell u}^i = \begin{bmatrix} \mathbf{0}_{d_y^i \times d_u^i} \\ \mathbf{I}_{d_u^i} \\ \mathbf{0}_{(k_i^*-2)(d_y^i+d_u^i)\times d_u^i} \end{bmatrix}.$$

The increment in shared memory can be written in terms of L_t^i as

$$Z_t^i = [\mathbf{0}_{(d_y^i+d_u^i)\times(k_i^*-2)(d_y^i+d_u^i)}\ \mathbf{I}_{(d_y^i+d_u^i)}] L_t^i. \tag{50}$$

Therefore, the result of Theorem 1 applies to this model with

- The analogue of (11) obtained by combining (46), (49) and (48).
- The analogue of (12) obtained by combining (46) and (50).
- and the estimate of Theorem 1 as

$$\check{S}_t = \mathbb{E}[\text{vec}(X_t, M_t) \mid \text{vec}(Y_{1:t-k_1^*}^1, U_{1:t-k_1^*}^1 \ldots, Y_{1:t-k_n^*}^n, U_{1:t-k_n^*}^n), \tilde{U}_{1:t-1}].$$

Analogous to Corollary 1, we also have the following result in this model.

Corollary 3 *Define $k^* = \max_{i,j} k_{ij}$. For the asymmetric delay sharing model, $\tilde{U}_t$ of Theorems 1 and 2 may be written as a linear function of*

$$S_t = \text{vec}(\hat{X}_{t-k^*+1|t-k^*}, \tilde{U}_{t-k^*+1:t-1}, Y_{t-2k^*+2:t-k^*}, U_{t-2k^*+2:t-k^*})$$

where

$$\hat{X}_{t-k^*+1|t-k^*} = \mathbb{E}[X_{t-k^*+1} \mid Y_{1:t-k^*}, U_{1:t-k^*}].$$

The proof is similar to the proof of Corollary 1 in Appendix A.

A 3 controller system with asymmetric delays (in particular, $k_{21} = k_{32} = k_{13} = 1$ and $k_{12} = k_{23} = k_{31} = 2$) and a partially nested information structure is considerd in [26]. The authors of [26] identify optimal control strategies whose structural form is similar to our result above. Note that our results hold for *any* strongly connected communication graph with finite delays.

5 Models that reduce to partial history sharing

The approach presented in this paper is also applicable to models that are not partial history sharing as such but can be reduced to one by using a *person-by-person approach* [1]. We illustrate this by means of two examples presented below.

5.1 Coupled subsystems with control sharing

In the control sharing model considered in [29][1] the system consists on n-subsystems; each subsystem has a co-located control station. Let X_t^i denote the state of subsystem i and U_t^i the control action of controller i. Let $X_t = \text{vec}(X_t^1, \ldots, X_t^n)$ and $U_t = \text{vec}(U_t^1, \ldots, U_t^n)$. The system dynamics are given by

$$X_{t+1}^i = \mathbf{A}_t^i X_t^i + \mathbf{B}_t^i U_t + W_t^i$$

where $\mathbf{A}_t^i$ and $\mathbf{B}_t^i$ are matrices of appropriate dimensions. Note that the next state of subsystem i depends on the current state of subsystem i and the control actions of all controllers. The noise processes $\{W_t^i\}_{t=1}^{\infty}$ are mutually independent and independent across time. The cost is quadratic and given by (4).

Control station i observes the state of control station i and the one-step delayed control actions of all controllers. Each controller has *perfect recall*. Therefore, action U_t^i must be chosen based on the data $(X_{1:t}^i, U_{1:t-1})$. It is shown in [29, Proposition 3] using a person-by-person approach that there is no loss of optimality in shedding $X_{1:t-1}^i$ and choosing U_t^i based on the data $(X_t^i, U_{1:t-1})$. We restrict attention to controllers that are linear functions of this data, i.e., controllers for the form

$$U_t^i = \mathbf{K}_t^i U_{1:t-1} + \mathbf{G}_t^i X_t^i$$

This model fits the general partial history sharing model described in Section 2.1 with

- the local memory M_t^i is empty;
- the local observation Y_t^i is X_t^i;
- the shared memory $Z_{1:t-1}$ is $U_{1:t-1}$
- the update of the shared memory given by (13) where $\mathbf{P}_{zy}^i = \mathbf{0}$, $\mathbf{P}_{zu}^i = \mathbf{I}$ and $\mathbf{P}_{**} = \text{diag}(\mathbf{P}_{**}^1, \ldots, \mathbf{P}_{**}^n)$.

The results of Theorem 1 apply to this model with

$$\check{S}_t = \mathbb{E}[X_t \mid U_{1:t-1}].$$

For this model, it is known that linear strategies are not globally optimal. The optimal non-linear control strategy is given by the embedding of the observations in the control actions [30].

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[1]The model presented here is simpler than the model described in [29]. The results also extend to the generalized models considered in [29], but we restrict attention to the more simpler model for ease of exposition.

5.2   One-sided one-step delayed sharing

Consider two coupled subsystems with one-sided one-step delayed sharing. Let X_t^i denote the state of subsystem i and U_t^i denote the control action of subsystem i. Let $X_t = \mathrm{vec}(X_t^1, X_t^2)$ and $U_t = \mathrm{vec}(U_t^1, U_t^2)$. The dynamics are arbitrary and given by (1). At each time, controller 1 observes $\mathrm{vec}(X_t^1, X_{t-1}^2)$: the current state of subsystem 1 and the one-step delayed state of subsystem 2; controller 2 observes X_t^2: the current state of subsystem 2. Thus, controller 1 chooses its control actions based on the data $(X_{1:t}^1, U_{1:t-1}^1, X_{1:t-1}^2, U_{1:t-1}^2)$ and controller 2 based on $(X_{1:t}^2, U_{1:t-1}^2)$. The cost is quadratic and given by (4).

When $\mathbf{A}$ and $\mathbf{B}$ are lower block triangular, the model is partially nested [5]. Such a model was considered in [16, 17]. A minor variation of this model (which was also partially nested) was also considered in [12, 15]. The sparsity assumptions on $\mathbf{A}$ and $\mathbf{B}$ are needed to prove global optimality of linear strategies; but, as we show below, not to identify the sufficient statistics for linear strategies.

The structure of controller 1 can be simplified by using a person-by-person approach. For any arbitrary choice of control strategy for controller 2, the subproblem of finding the *best response* strategy at controller 1 is a centralized stochastic control problem. It can be shown that $(X_t^1, X_{1:t-1}^2, U_{1:t-1}^2)$ is an information state of this subproblem. Therefore, there is no loss of optimality in choosing U_t^1 based on the data $(X_t^1, X_{1:t-1}^2, U_{1:t-1}^2)$. We restrict attention to controllers that are linear functions of the available data, i.e., controllers of the form

$$U_t^i = \mathbf{K}_t^i \, \mathrm{vec}(X_{1:t-1}^2, U_{1:t-1}^2) + G_t^i X_t^i;$$

This model fits the general partial history sharing model described in Section 2.1 with

- the local memory M_t^i is empty;
- the local observation Y_t^i is X_t^i;
- the shared memory $Z_{1:t-1}$ is $\mathrm{vec}(X_{1:t-1}^2, U_{1:t-1}^2)$;
- the update of the shared memory given by (13) where $\mathbf{P}_{zy}^1 = \mathbf{0}$, $\mathbf{P}_{zu}^2 = \mathbf{0}$, $\mathbf{P}_{zy}^2 = \mathbf{I}$, $\mathbf{P}_{zu}^2 = \mathbf{I}$, and $\mathbf{P}_{**} = \mathrm{diag}(\mathbf{P}_{**}^1, \ldots, \mathbf{P}_{**}^n)$.

The results of Theorem 1 apply to this model with

$$\breve{S}_t = \mathbb{E}[X_t \mid X_{1:t-1}^2, U_{1:t-1}^2, \tilde{U}_{1:t-1}].$$

The above structural result is similar to the result obtained in [16, 17]. However, unlike [16, 17], our model does not have a partially nested information structure. This suggests that the structure of the best linear control law is a consequence of the linearity of control strategies rather than the partially nested information structure.

6   Conclusion

Linear control strategies for LQG systems are appealing due to their analytical and implementation simplicity. However, to fully leverage the advantages of linear strategies, we need to identify finite-dimensional sufficient statistics for best linear strategies that can be easily updated. We identified such a result in Theorem 1 for decentralized systems with partial history sharing information structures. The result relied on the linearity of the decentralized system and is applicable to models that are neither partially nested nor quadratically invariant.

We focused on the partial history sharing model in this paper because it provides a common model for decentralized systems where controllers' local information remains finite dimensional but the common information increases with time.

We showed that our results provide sufficient statistics for different variations of delayed sharing information structures, including those with asymmetric delays that arise when controllers communicate along a strongly connected graph.

We also showed that our approach is applicable to some decentralized systems where local information is also increasing with time, provided one can first employ a person by person optimality approach to find a preliminary sufficient statistic which ensures that local information is finite dimensional.

We have focused only on finding the structure of best linear control strategies in this paper. It is not possible, in general, to use our approach to compute the best linear control strategies. Even in the absence of a complete methodology to find the best linear strategies, the structural results of Theorem 1 are useful because they restrict the solution space to search for best linear strategies. Furthermore, as is the case with the sufficient statistics in centralized stochastic control, the sufficient statistics of Theorem 1 allow us to formulate the problem of finding and implementing the best linear control strategies over an infinite horizon.

A   Proof of Corollary 1

To prove the result, we will argue that $\check{S}_t = \text{vec}(\hat{X}_t, \hat{M}_t)$ is a linear function of S_t for the symmetric delay sharing model. Therefore, the control law of Theorem 1 can be written in the form specified in Corollary 1.

Observe that according to the coordinated system dynamics in (22), (X_t, M_t) is a linear function of $\tilde{X}_{t-k+1} = \text{vec}(X_{t-k+1}, Y_{t-k+1}, M_{t-k+1})$, $\tilde{U}_{t-k+1:t-1}$ and $W^0_{t-k+1:t-1}, W^{1:n}_{t-k+1:t-1}$. Therefore, by linearity of conditional expectation, $(\hat{X}_t, \hat{M}_t)$ is a linear function of the following three terms

1. $\mathbb{E}[\text{vec}(X_{t-k+1}, Y_{t-k+1}, M_{t-k+1}) \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-k}]$.
2. $\mathbb{E}[\tilde{U}_{t-k+1:t-1} \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-k}]$.
3. $\mathbb{E}[W_{t-k+1:t-1} \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-k}]$.

Consider each of these terms separately. Recall that in delayed sharing information structure $M_{t-k+1} = \text{vec}(Y_{t-2k+2:t-k}, U_{t-2k+2:t-k})$ which are included in the right hand side of conditioning in the first term. Therefore,

$$\mathbb{E}[M_{t-k+1} \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-1}] = \text{vec}(Y_{t-2k+2:t-k}, U_{t-2k+2:t-k}). \tag{51}$$

Furthermore, using (6)

$$\begin{aligned} \mathbb{E}[Y^i_{t-k+1} &\mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-1}] \\ &= \mathbf{C}^i \mathbb{E}[X_{t-k+1} \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-1}] \\ &= \mathbf{C}^i \mathbb{E}[X_{t-k+1} \mid Y_{1:t-k}, U_{1:t-k}] \end{aligned} \tag{52}$$

where we removed $\tilde{U}^i_{1:t-1}$ from the right hand side of conditioning because it is a function of $(Y^i_{1:t-k}, U^i_{1:t-k})$ which are included in the right hand side of conditioning. Combining (51) and (52), we get that $\mathbb{E}[\text{vec}(X_{t-k+1}, Y_{t-k+1}, M_{t-k+1}) \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-k}]$ is a linear function of $(\hat{X}_{t-k+1|t-k}, Y_{t-2k+2:t-k}, U_{t-2k+2:t-k})$, which is a sub-vector of S_t.

The second term $\mathbb{E}[\tilde{U}_{t-k+1:t-1} \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-k}]$ is simply $\tilde{U}_{t-k+1:t-1}$ which is also a sub-vector of S_t.

Since the primitive random variables are independent, the third term $\mathbb{E}[W_{t-k+1:t-1} \mid Y_{1:t-k}, U_{1:t-k}, \tilde{U}_{1:t-k}]$ is 0.

Therefore, $\check{S}_t = \text{vec}(\hat{X}_t, \hat{M}_t)$ is a linear function of S_t, which implies the result of the corollary.

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