Low-Dimensional Solutions for Optimal Control of Subsystems Coupled Over a Directed Network

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Background and Motivation

- Large network systems require scalable, low-complexity control.
- Structural properties like symmetry allow simplifications via spectral decomposition.

Applications of Large-Scale Network-Coupled Systems

- Large-scale networked systems
- Smart grids and power networks
- Distributed control of multi-agent systems
- Cyber-physical systems and IoT networks

Literature Review

Identification of conditions for low-complexity control synthesis and implementation:

- Simplified control structures (ensemble control)

Li, *"Ensemble control of finite-dimensional time-varying linear systems"*, 2011.

- Structure reduction and the use of matrix sparsity

Benner, *"Solving large-scale control problems"*, 2004.

- Low-dimensional spectral decomposition approximations using state aggregation

Aoki, *"Control of large-scale dynamic systems by aggregation"*, 1968.

Contributions of the Article

- **Goal**: extend spectral decomposition methods to directed networks, from the work on undirected networks of:

Gao and Mahajan, *"Optimal Control of Network-Coupled Subsystems: Spectral Decomposition and Low-Dimensional Solutions"*, 2022.

- We solve the LQR problem on a system made of networked subsystems, whose dynamics depend on local state/control and a weighted network field.
- Normal coupling matrices admit complex spectral decomposition enabling problem decoupling.
- The global LQR problem is decomposed into local LQR subproblems independent of network size, reducing computational cost and allowing localized implementation.

System Model

- Network $\mathcal{N}$ of n nodes with coupling matrix $M = [m_{ji}]$; local fields: $x_i^{\mathcal{G}}(t) = \sum_{j \in \mathcal{N}} x_j(t) m_{ji}$, $u_i^{\mathcal{G}}(t) = \sum_{j \in \mathcal{N}} u_j(t) m_{ji}$.

Discrete-time local dynamics with finite horizon

$$x_i(t+1) = Ax_i(t) + Bu_i(t) + D\, x_i^{\mathcal{G}}(t) + E\, u_i^{\mathcal{G}}(t) + \xi_i(t), \ \ t \leq T-1$$

- Matrix form with $x(t) = \text{cols}(x_1, \ldots, x_n)$, $u(t) = \text{cols}(u_1, \ldots, u_n)$, $x^{\mathcal{G}}(t) = x(t)M$, $u^{\mathcal{G}}(t) = u(t)M$:

$$x(t+1) = Ax(t) + Bu(t) + D\, x(t)M + E\, u(t)M + \xi(t).$$

- Noise: $\xi_i(t)$ i.i.d., zero mean, finite variance; system starts from $x(0)$.

System Performance and Control Objective

- Quadratic instantaneous cost:
  $c(x(t), u(t)) = \langle x(t), Qx(t) \rangle_{M_q} + \langle u(t), Ru(t) \rangle_{M_r}$, with
  $\langle x, y \rangle_P = \mathrm{Tr}(x^\dagger y P)$.

- Terminal cost at T:

  $$c_T(x(T)) = \langle x(T), Q_T x(T) \rangle_{M_q}.$$

- Control policy:
  state–feedback

  $$u(t) = g_t(x(t)).$$

Objective: minimize expected total cost

$$J(g) = \mathbb{E}\Big[\sum_{t=0}^{T-1} c(x(t), u(t)) + c_T(x(T)) \Big]$$

- **Problem**: Choose a control policy $g = (g_0, \dots, g_{T-1})$ solving the optimization problem under system dynamics.

Assumptions on the Coupled Dynamics

Assumption (A0)

M is normal, including the case where the network is undirected or circulant.

Assumption (A1)

Weight coupling matrices M_q and M_r commute with M (e.g., polynomials of M).

- **Consequence**: M, M_q, and M_r share eigenvectors $\{v^\ell\}_{\ell \in \mathcal{N}}$, allowing a common spectral decomposition.
- Eigenvalues are $M : \{\lambda^\ell\}_{\ell \in \mathcal{N}}$, $M_q : \{q^\ell\}_{\ell \in \mathcal{N}}$, $M_r : \{r^\ell\}_{\ell \in \mathcal{N}}$.

Spectral Decomposition of the Dynamics

- Local states/controls: $x^{\ell}(t) = x(t) v^{\ell} (v^{\ell})^{\dagger}, u^{\ell}(t) = u(t) v^{\ell} (v^{\ell})^{\dagger}$
- Local states/controls are decomposed per eigenmode as $x_i(t) = \sum_{\ell \in \mathcal{N}} x_i^{\ell}(t)$, $u_i(t) = \sum_{\ell \in \mathcal{N}} u_i^{\ell}(t)$.

Dynamics decoupled for each mode $\ell \in \mathcal{N}$

$$x_i^{\ell}(t+1) = (A + \lambda^{\ell} D) x_i^{\ell}(t) + (B + \lambda^{\ell} E) u_i^{\ell}(t) + \xi_i^{\ell}(t)$$

- Enables mode-wise analysis and scalable control, but non-symmetry means some dynamics are complex.

Assumptions on the Coupled Cost

(A2) Eigenvalue conditions:

$\mathrm{Re}(q^\ell) \geq 0$, $\mathrm{Re}(r^\ell) > 0$ for all $\ell \in \mathcal{N}$.

(A3) Weight matrices:

Q, Q_T symmetric positive semidefinite, R symmetric positive definite.

- Ensures per-mode cost is nonnegative and standard finite-horizon LQR assumptions hold.
- Together with (A1), these assumptions guarantee that the cost is well-defined and each mode can be analyzed independently.

Spectral Decomposition of the Cost

- Partition eigenvalues of M: $\mathcal{N}_r$ = real, $\mathcal{N}_c^+$ = complex with $\mathrm{Im}(\lambda) > 0$, $\mathcal{N}_c^-$ = complex with $\mathrm{Im}(\lambda) < 0$
- Instantaneous cost decomposes per eigenmode:

$$c(x(t), u(t)) = \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{N}} c^\ell(x_i^\ell, u_i^\ell), \quad c^\ell(x_i^\ell, u_i^\ell) = q^\ell x_i^{\ell\dagger} Q x_i^\ell + r^\ell u_i^{\ell\dagger} R u_i^\ell$$

- We combine contributions from complex-conjugate modes, ensuring real costs:

$$c(x(t), u(t)) = \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{N}} \bar{c}^\ell(x_i^\ell, u_i^\ell), \text{ where}$$

$$\bar{c}^\ell(x_i^\ell, u_i^\ell) = \mathrm{Re}(q^\ell) x_i^{\ell\dagger} Q x_i^\ell + \mathrm{Re}(r^\ell) u_i^{\ell\dagger} R u_i^\ell$$

If (ℓ^+, ℓ^-) form complex-conjugate pairs, then $\bar{c}^{\ell^+}$ and $\bar{c}^{\ell^-}$ are the same.

Main Results

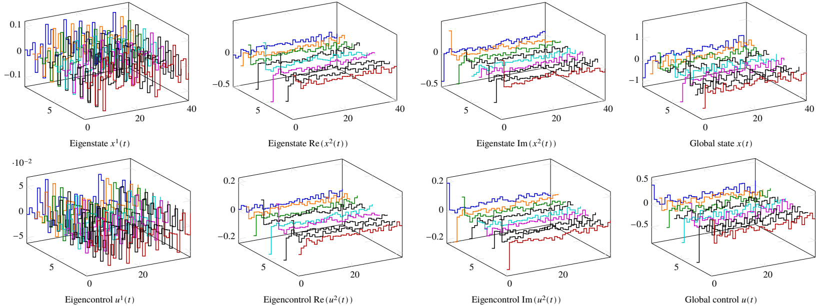
- **Optimal Control Law:** For each $i \in \mathcal{N}$, the solution of the LQR problem is:

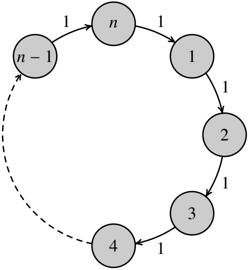
$$u_i(t) = \sum_{\ell \in \mathcal{N}_r} K^\ell(t) x_i^\ell(t) + 2 \sum_{\ell \in \mathcal{N}_c^+} \mathrm{Re}\left(K^\ell(t) x_i^\ell(t)\right),$$

where each $K^\ell(t)$ is computed from a Riccati equation, with size *independent of* n, corresponding to the eigenmode $\ell \in \mathcal{N}$.

- Dynamics and cost decouple per eigenmode ℓ.
- If (ℓ^+, ℓ^-) form complex-conjugate pairs, the Riccati equations are identical, meaning fewer equations to solve.
- Subsystem dimension d_x: centralized Riccati $nd_x \times nd_x$ vs. n decoupled $d_x \times d_x$ equations, giving large savings.

Numerical Simulation





- $n = 9$, scalar subsystems, $A = 1, B = 2, D = 1, E = 2$, $Q = 5, R = 1, Q_T = 10,\ T = 40$, Gaussian noise.
- $n_1 = 5$ separate Riccati equations instead of 9×9 centralized implies big computational savings.

Conclusion

- Dynamics and cost of networked subsystems decouple per eigenmode of M.
- Optimal control law computed mode-wise via n small Riccati equations.
- Complex-conjugate modes share Riccati equations which leads to fewer computations.
- Computational savings: from one large $nd_x \times nd_x$ Riccati to n small $d_x \times d_x$ Riccati equations.
- Scalable, mode-wise control applicable to large networks with normal coupling matrices.

Thank you!

References I

[1] J.-S. Li, "Ensemble control of finite-dimensional time-varying linear systems,", vol. 56, no. 2, pp. 345–357, 2011.

[2] P. Benner, "Solving large-scale control problems,", vol. 24, no. 1, pp. 44–59, 2004.

[3] M. Aoki, "Control of large-scale dynamic systems by aggregation,", vol. 13, no. 3, pp. 246–253, 1968.

[4] S. Gao and A. Mahajan, "Optimal control of network-coupled subsystems: Spectral decomposition and low-dimensional solutions,", vol. 9, no. 2, pp. 657–669, Jun. 2022, ISSN: 2372-2533. DOI: 10.1109/tcns.2021.3124259.