Question about measurability of optimal control strategies in static teams

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Let

- $(\mathcal{X}_0, \mathfrak{F}_0)$, $(\mathcal{X}_1, \mathfrak{F}_1)$, $(\mathcal{X}_2, \mathfrak{F}_2)$, $(\mathcal{U}_1, \mathfrak{G}_1)$, $(\mathcal{U}_2, \mathfrak{G}_2)$, $(\mathcal{W}, \mathfrak{H})$ be Polish spaces.
- (X_0, X_1, X_2, W) be random variables defined on a common probability space $(\Omega, \mathfrak{F}, P)$, where $X_i \in \mathcal{X}_i$ and is $\mathfrak{F}/\mathfrak{F}_i$ measurable, for $i \in \{0, 1, 2\}$, and $W \in \mathcal{W}$ is $\mathfrak{F}/\mathfrak{H}$ measurable.
- For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be the class of measurable functions from $(\mathcal{X}_0 \times \mathcal{X}_i, \mathfrak{F}_0 \otimes \mathfrak{F}_i)$ to $(\mathcal{U}_i, \mathfrak{G}_i)$
- For $i \in \{1, 2\}$, let $\mathcal{D}_i$ be the class of measurable functions from $(\mathcal{X}_i, \mathfrak{F}_i)$ to $(\mathcal{U}_i, \mathfrak{G}_i)$.
- Let ℓ be a measurable function from $(\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{W}, \mathfrak{G}_1 \otimes \mathfrak{G}_2 \otimes \mathfrak{H})$ to $(\mathbb{R}, \mathcal{B})$. We refer to ℓ as the cost function.

Consider the following optimization problem:

$$\inf_{\substack{f_1 \in \mathcal{C}_1 \\ f_2 \in \mathcal{C}_2}} \mathbb{E}\big[\ell\big(f_1(X_0, X_1), f_2(X_0, X_2), W\big)\big] \tag{P1}$$

This problem can be simplified under the following assumption:

(A) The spaces $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{W}$ are finite sets and $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{H}$ are the corresponding power set sigma-algebras..

Under (A), we can ignore measurability constraints. Define:

- $\mathcal{A}_i$ to be the space of functions from $\mathcal{X}_0$ to $\mathcal{D}_i$, $i = 1, 2$.

For a function $h_1 \in \mathcal{A}_1$ and $x_0 \in \mathcal{X}_0$, $h_1(x_0)$ is an element in $\mathcal{D}_1$; hence $h_1(x_0)$ is a function from $\mathcal{X}_1$ to $\mathcal{D}_1$.

Consider the following optimization problem:

$$\min_{\substack{h_1 \in \mathcal{A}_1 \\ h_2 \in \mathcal{A}_2}} \mathbb{E}\big[\ell\big(h_1(X_0)(X_1), h_2(X_0)(X_2), W\big)\big], \tag{P2}$$

where $h_i(X_0)(X_i)$ denotes the value of the function $h_i(X_0)(\cdot)$ at X_i, $i = 1, 2$.

It is straightforward to show that the optimization problem P2 can be solved by solving the following family of optimization problems:

$$\forall x_0 \in \mathcal{X}_0, \quad \min_{\substack{g_1 \in \mathcal{D}_1 \\ g_2 \in \mathcal{D}_2}} \mathbb{E}\big[\ell\big(g_1(X_1), g_2(X_2), W\big) \,\big|\, X_0 = x_0\big] \tag{P3}$$

For each $x_0 \in \mathcal{X}_0$, let $(h_1^*(x_0), h_2^*(x_0)) \in \mathcal{D}_1 \times \mathcal{D}_2$ denote the arg min in P3 (assuming there is a fixed rule for breaking ties). Note that the function h_i^*, $i = 1, 2$, maps $\mathcal{X}_0$ to $\mathcal{D}_i$, that is, $h_i^* \in \mathcal{A}_i$.

The functions h_1^* and h_2^* can be used to find the minimizing f_1 and f_2 in Problem P1. For that purpose, we define the following notion of consistency.

Definition 1 We say $(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ and $(h_1, h_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ are *consistent* if for all $(x_0, x_1, x_2) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2$

$$f_1(x_0, x_1) = g_1(x_1) \text{ and } f_2(x_0, x_2) = g_2(x_2), \quad \text{where } (g_1, g_2) = (h_1(x_0), h_2(x_0)). \tag{1}$$

An immediate implication of this definition is that if (f_1, f_2) and (h_1, h_2) are consistent, then

$$\mathbb{E}\big[\ell\big(f_1(X_0, X_1), f_2(X_0, X_2), W\big)\big] = \mathbb{E}\big[\ell\big(h_1(X_0)(X_1), h_2(X_0)(X_2), W\big)\big]$$

where $h_i(X_0)(X_i)$ denotes the value of the function $h_i(X_0)(\cdot)$ at X_i.

Proposition 1 *Under (A),*

1. *For all $(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, there exists an $(h_1, h_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ that is consistent with (f_1, f_2), and hence achieves the same expected cost.*
2. *For all $(h_1, h_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, there exists $(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ that is consistent with h, and hence achieves the same expected cost.*

Therefore, an (f_1^*, f_2^*) that is consistent with the (h_1^*, h_2^*) defined by the family of optimization problems in P3 is optimal in Problem P1.

The proof is immediate. We can use the definition of consistency to construct the respective functions by simply ensuring that for all $x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2$:

$$f_i(x_0, x_i) = h_i(x_0)(x_i), \qquad i = 1, 2.$$

Parts 1 and 2 of the proposition imply that the optimal expected cost in Problems P1 and P2 are the same. Therefore, an (f_1^*, f_2^*) that is consistent with the (h_1^*, h_2^*) achieves the optimal expected cost in Problem P1.

Note that Problem (P3) is simpler to solve than Problem (P1). In particular, if $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$ are binary valued, a brute force solution of (P1) requires 2^8 computations (there are 4^2 possibilities for both f_1 and f_2) while a brute force solution of (P3) requires 2^5 computations (for each value of x_0, there are 2^2 possibilities for g_1 and g_2).

Question

Can a claim similar to Proposition 1 be made when Assumption (A) is not true? This would require defining an appropriate sigma algebra on $\mathcal{A}_1 \times \mathcal{A}_2$ and perhaps weakening the definition of consistency such that the equality in (1) hold almost everywhere.