When to communicate information in two-player teams

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The remote-state estimation setup

The remote-state estimation setup

$$U_t = f_t(X_{1:t}, U_{1:t-1}),\ Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \mathfrak{E}, & \text{if } U_t = 0 \end{cases},\ \hat{X}_t = g_t(Y_{1:t})$$

Communication cost: λ
Per-step distortion: $d(X_t - \hat{X}_t)$

The remote-state estimation setup

Discounted setup: $\beta \in (0,1)$

- $D_\beta(f,g) := (1-\beta)\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$
- $N_\beta(f,g) := (1-\beta)\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t \,\Big|\, X_0 = 0\Big]$

The remote-state estimation setup

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Communication cost: λ
Per-step distortion: $d(X_t - \hat{X}_t)$

$$C^*_\beta(\lambda) := C_\beta(f^*, g^*) := \inf_{(f,g)} D_\beta(f, g) + \lambda N_\beta(f, g), \; \beta \in (0, 1]$$

The remote-state estimation setup

- Two decision-makers
- Common objective function
- Non-classical information structure - **Difficulty**!

Team problem

Team vs Game

- Strategic game - players are willing to cooperate to minimize identical utility functions
- Cooperative game - not asking how the total utility is split among the players

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We seek global optimum !

Team problem

Team: Multiple decision makers to achieve a common goal

A brief literature overview: Economics

- Marschak - "Elements for theory of teams", Management Science, 1955
- Radner - "Team decision problems", Ann Math Stats, 1962
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Static teams ! Similar to Bayesian games.

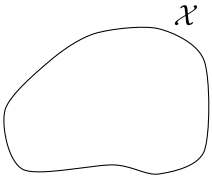
Team problem

A brief literature overview: Systems and control

- Witsenhausen - “Separation of estimation and control”, IEEE Proc 1971; “On information structures, feedback and causality”, SICON 1971,
- Ho, Chu - “Team decision theory and information structures”, IEEE TAC 1972
- ...

Dynamic teams ! Action of one player affects others' decisions. Similar to Bayesian dynamic games.

An illustration of why the problem is difficult



Transmitter : transmit / not transmit
$(\color{blue}{U=1})$ / $(\color{red}{U=0})$

Receiver : estimate of x

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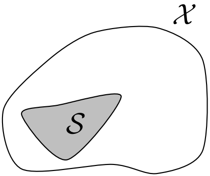

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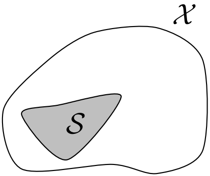

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- Transmitter: $Y = \begin{cases} x, & \text{if } x \notin \mathcal{S} \\ \mathfrak{E}, & \text{if } x \in \mathcal{S}. \end{cases}$
- Receiver: $\min_{\hat{x}} d(\mathcal{S}, \hat{x})$, where $d(\mathcal{S}, \hat{x}) = \mathbb{E}[d(X - \hat{x}) \,|\, X \in \mathcal{S}]$.

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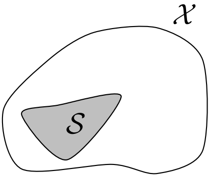

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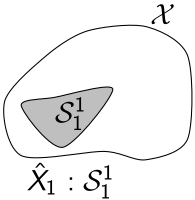
Combinatorial optimization

$$\min_{\mathcal{S}, \hat{x}} \lambda \mathbb{P}(x \notin \mathcal{S}) + d(\mathcal{S}, \hat{x}) \mathbb{P}(x \in \mathcal{S})$$
$$\implies \min_{\mathcal{S}} \Big[\lambda \mathbb{P}(x \notin \mathcal{S}) + \min_{\hat{x}} d(\mathcal{S}, \hat{x}) \mathbb{P}(x \in \mathcal{S}) \Big]$$

Dynamic version



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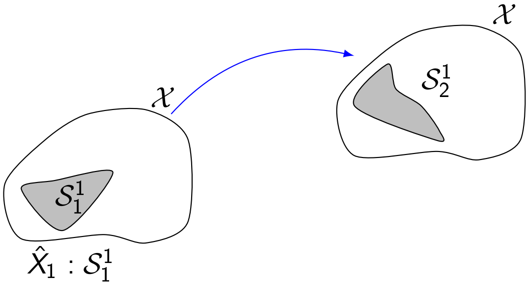
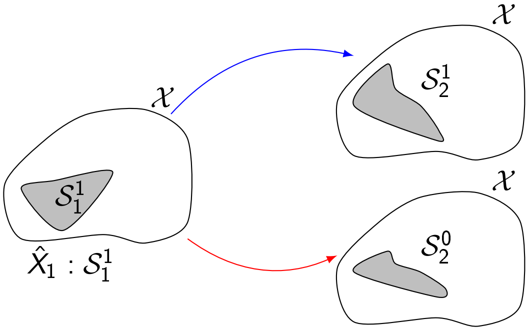


$\mathcal{S}_2^1(X_1, U_1 = 1)$

$\hat{X}_2 : \mathcal{S}_2^1, X_1$

Dynamic version



$\mathcal{S}_2^1(X_1, U_1 = 1)$

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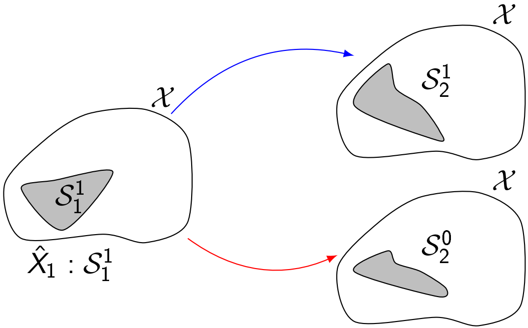

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- We want to find the sufficient statistic of all past decisions and realizations

The remote state estimation

- Classical information - instead of
  f(history of observation), use f(information state),
  Dynamic program:
  $V(\text{Information state}) = \min_{\text{action}} \mathcal{B}_{\text{action}} V(\text{Information state})$

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Common information approach for non-classical information - Nayyar, Mahajan, Teneketzis, TAC 2013

Dynamic team: common information approach - centralized stochastic problem

Original system

$$f_t \ \boxed{X_t, Y_{1:t-1}} \ U_t$$

$$g_t \ \boxed{Y_t, Y_{1:t-1}} \ \hat{X}_t$$

Dynamic team: common information approach - centralized stochastic problem

Original system	Coordinated system
f_t $\boxed{X_t, Y_{1:t-1}}$ U_t	ϕ_t $\boxed{X_t}$ U_t
g_t $\boxed{Y_t, Y_{1:t-1}}$ $\hat{X}_t$	γ_t $\boxed{Y_t}$ $\hat{X}_t$

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- $U_t = f_t(x_t, y_{1:t-1}) = \boldsymbol{h_t^1}(\boldsymbol{y_{1:t-1}})(\boldsymbol{x_t})$
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Decentralized control: Common information approach [Nayyar, Mahajan, Teneketzis, TAC 2013]

- Structure of optimal strategy - instead of f(history of observation), use

  f(local information, common information based state)

- Optimal strategy: solution of DP

$$V(\text{Information state}) = \min_{\tilde{\phi}:\text{local info}\to\text{action}} \mathcal{B}_{\tilde{\phi}} V(\text{Information state})$$

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Belief state: information state

$$\Pi_t := \mathbb{P}(X_t \mid Y_{1:t-1})$$

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Dynamic program

$$V_t(\pi) = \min_{\phi,\gamma} \mathbb{E}[\lambda(\phi_t(X_t)) + d(X_t, \gamma_t(\phi_t(X_t))) + V_{t+1}(\Pi_{t+1}) \mid \Pi_t = \pi, \phi_t = \phi, \gamma_t = \gamma]$$

The remote state estimation process

Source process $X_{t+1} = aX_t + W_t$, W_t i.i.d.

- $a, X_t, W_t \in \mathbb{Z}$
- $W_t \sim$ unimodal and symmetric distribution
  $p : \forall e \in \mathbb{Z}_{\geq 0},\ p_e = p_{-e},\ p_e \geq p_{e+1};\ p_1 > 0$
- Distortion: for all $e \in \mathbb{Z}$, $d(e)$,
  $d(0) = 0$, for $e \neq 0$, $d(e) \neq 0$, $d(\cdot)$ is even and increasing on $\mathbb{Z}_{\geq 0}$

Analysis - proof outline

- Identify structure of optimal strategies.
- Find the best strategy with that structure.

Structure of optimal strategies [1]

Optimal estimation $\hat{X}_t = \begin{cases} x, & \text{if } Y_t = x \\ a\hat{X}_{t-1}, & \text{if } Y_t = \mathfrak{E} \end{cases}$

Time homogeneous!

Optimal transmission strategy Let $E_t = X_t - a\hat{X}_{t-1}$ be the error process and f_t be the threshold based strategy such that

$$f_t(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k_t \\ 0, & \text{if } |E_t| < k_t. \end{cases}$$

[1] [Lipsa-Martins 2011] and [Nayyar-Basar-Teneketzis-Veeravalli 2013] - finite horizon

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Tools

Majorization, Schur-concavity, convolution

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One can show that the results generalize to infinite horizon setup; the optimal thresholds are time - homogeneous.

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Performance of threshold based strategies, $\beta \in (0, 1]$

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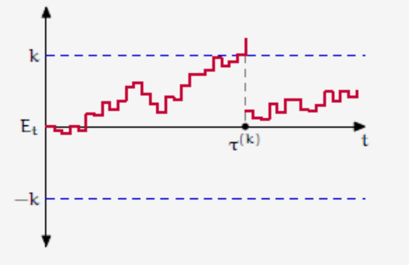

$\tau^{(k)}$: stopping time when the Markov process starting at state 0 at time $t = 0$ enters the set $\{e \in \mathbb{Z} : |e| \geq k\}$

Performance of threshold based strategies, $\beta \in (0, 1]$

Fix a threshold based startegy $f^{(k)}$. Define

- $L_\beta^{(k)}(e)$: the expected distortion until the first transmission, starting from state e.
- $M_\beta^{(k)}(e)$: the expected time until the first transmission, starting from state e.

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- Closed form expressions for $L_\beta^{(k)}(0)$ and $M_\beta^{(k)}(0)$ using matrix inversion.

$$L_\beta^{(k)}(0) = \{[I_{2k-1} - \beta P^{(k)}]^{-1} d^{(k)}\}_0,$$
$$M_\beta^{(k)}(0) = \{[I_{2k-1} - \beta P^{(k)}]^{-1} \mathbf{1}_{2k-1}\}_0.$$

Performance of threshold based strategies, $\beta \in (0, 1]$

- $D_\beta^{(k)}$: the expected distortion.
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- $N_\beta^{(k)}$: the expected number of transmissions.

Renewal relationship, $\beta \in (0, 1]$

$$D_\beta^{(k)} = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)}, \qquad N_\beta^{(k)} = \frac{1}{M_\beta^{(k)}(0)} - (1 - \beta)$$

Main result

Characterize the optimal threshold for a given communication cost - Tricky !

Instead, we characterize the optimal communication cost for a given threshold

Main result

- $k^*_\beta(\lambda) \in \mathbb{Z}_{\geq 0} := \arg\min_{k \in \mathbb{Z}_{\geq 0}} C^{(k)}_\beta(\lambda)$.
- $C^{(k)}_\beta(\lambda)$ is submodular function - $k^*_\beta(\lambda)$ is increasing in λ

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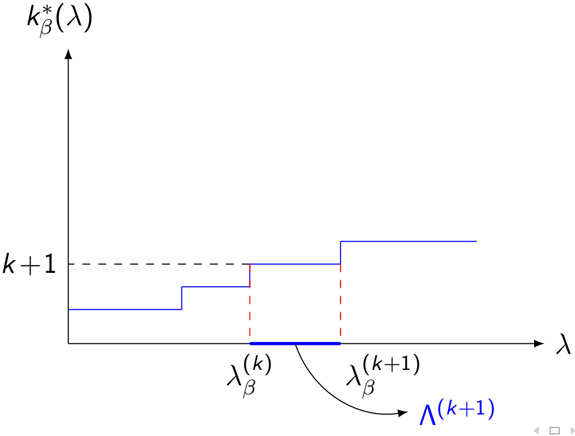

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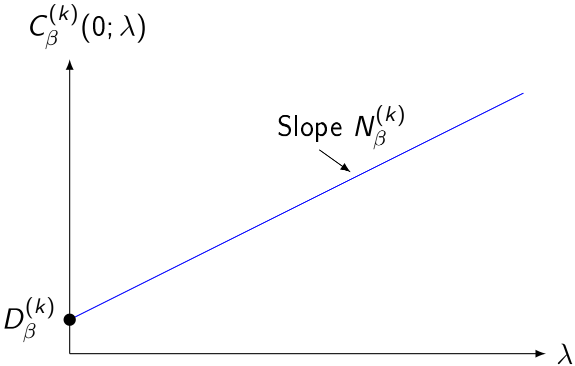

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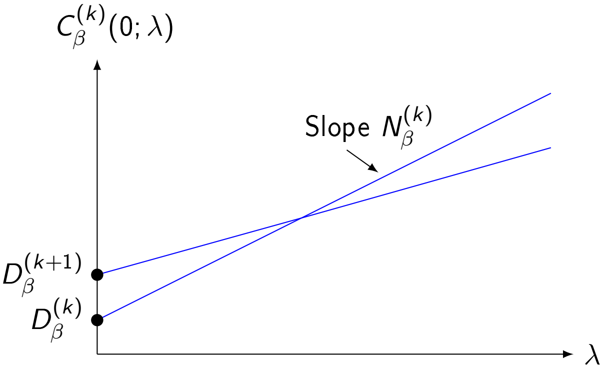

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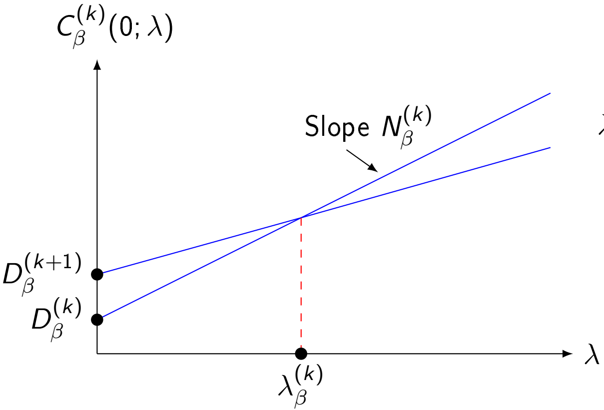

$$\lambda_\beta^{(k)} := \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

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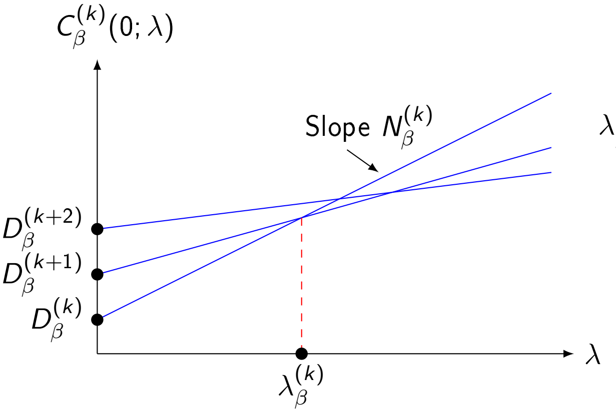

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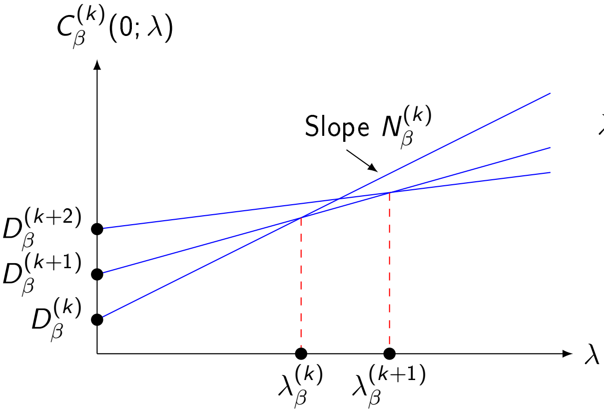

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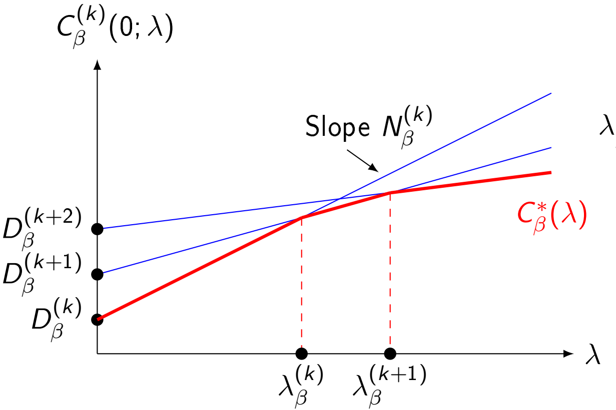

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$C_\beta^*(\lambda)$ is piecewise linear, increasing, concave function of λ.

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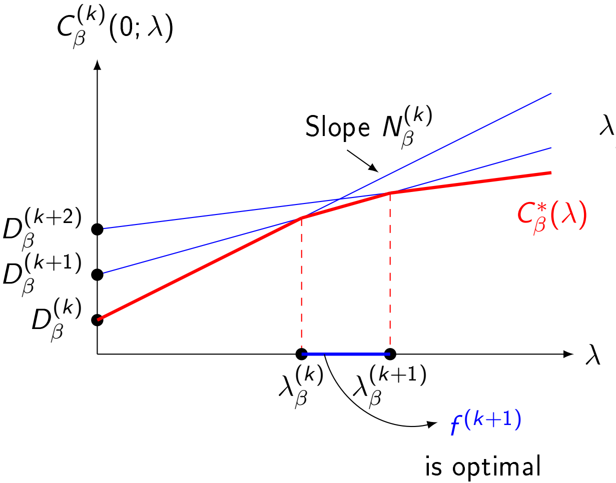

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Summary

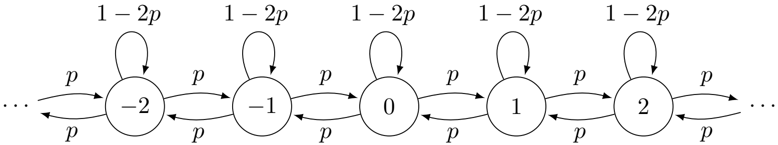
- For a remote state estimation problem, identify the suffcient staistic and the dynamic program
- For a particular type of source process (AR), the following are optimal
  Threshold based transmission strategy: $f^* = f^{(k^*_\beta(\lambda))}$
  Time homogeneous Kalman-like estimation strategy:

$$\hat{X}_t = \begin{cases} x, & \text{if } Y_t = x \\ a\hat{X}_{t-1}, & \text{if } Y_t = \mathfrak{E} \end{cases}$$

- Given a λ, find $\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}$ such that $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$
  Optimal threshold $k^*_\beta(\lambda) = k + 1$
- Optimal costly communication - piecewise linear, concave, increasing

Numerical results: $\beta \in \{0.9, 0.95, 1.0\}$, $a = 1$, $p = 0.3$

Symmetric birth-death Markov chain



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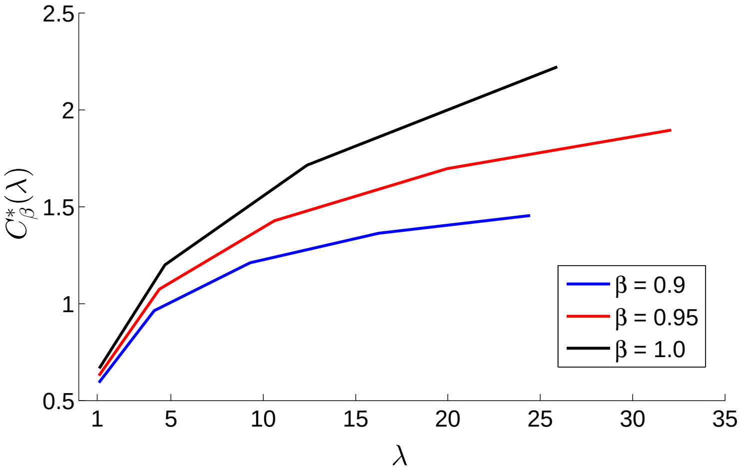

Extensions

Other results

- Gauss-Markov setup
- Constrained communication
- Packet-drops in communication channel

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Chakravorty J. and Mahajan A., "Fundamental limits of remote estimation of Markov processes under communication constraints", submitted in IEEE TAC, 2015.

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Future directions

- Communication in wireless network
- Multiple transmitters sharing a channel
- Signaling in decentralized control

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Issues of delay, quantization.

Conclusion

- Two-agent system with asymmetric information: DP decomposition - hard !
- Team-theory literature - common information approach
- For dynamic team with a particular type of source: complete characterization of optimal strategies

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- Use of common information approach in dynamic games: Markov pefect equilibrium - Gupta et al 2014; perfect Bayesian equilibrium - Ouyang et al, 2015

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Thank you!

Some parameters

Let $\tau^{(k)}$ be the stopping time of first transmission (starting from $E_0 = 0$), under $f^{(k)}$. Then

- $L_\beta^{(k)}(e) = (1-\beta)\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \;\Big|\; E_0 = 0\Big]$.
- $M_\beta^{(k)}(e) = (1-\beta)\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \;\Big|\; E_0 = 0\Big]$.

Regenerative process: The process $\{X_t\}_{t=0}^\infty$, if there exist $0 \le T_0 < T_1 < T_2 < \cdots$ such that $\{X_t\}_{t=T_k+s}^\infty$, $s \ge 0$,

- has the same distribution as $\{X_t\}_{t=T_0+s}^\infty$,
- is independent of $\{X_t\}_{t=0}^{T_k}$.

Technical assumption: extension from finite horizon to infinite horizon

For every $a > 0$ and $\lambda \geq 0$, there exists a function $\rho : \mathbb{Z} \to \mathbb{R}$ and positive and finite constants μ_1 and μ_2 such that for all $e \in \mathbb{Z}$, we have that

$$\max\{\lambda, d(e)\} \leq \mu_1 \rho(e),$$

$$\max\left\{\sum_{n=-\infty}^{\infty} p_{n-ae}\rho(n), \sum_{n=-\infty}^{\infty} p_n \rho(n)\right\} \leq \mu_2 \rho(ae).$$

Step 1: Main idea

For Model B: Proof technique followed after Lerma, Lasserre - Discrete-time Markov control processes: basic optimality criteria, Springer

- The model satisfies certain assumptions (4.2.1, 4.2.2)
- Hence, the structural results extend to the infinite horizon discounted cost setup (Theorem 4.2.3)
- The discounted model satisfies some more assumptions (4.2.1, 5.4.1)
- Hence, structural results extend to long-term average setup (Theorem 5.4.3)

- Assumption 4.2.1 - The one-stage cost is l.s.c, non-negative and inf-compact on the set of feasible state-action pairs. The stochastic kernel ϕ is strongly continuous.
- Assumption 4.2.2 - There exists a strategy π such that the value function $V(\pi, x) < \infty$ for each state $x \in X$.
- Theorem 4.2.3 - Suppose Assumptions 4.2.1 and 4.2.2 hold. Then, in the discounted setup, there exists a selector which attains the minimum V_β^* and the optimal strategy, if it exists, is deterministic stationary.
- Assumption 5.4.1 - There exixts a state $z \in X$ and scalars $\alpha \in (0, 1)$ and $M \geq 0$ such that
  1. $(1 - \beta)V_\beta^*(z) \leq M$, $\forall \beta \in [\alpha, 1)$.
  2. Let $h_\beta(x) := V_\beta(x) - V_\beta(z)$. There exists $N \geq 0$ and a non-negative (not necessarily measurable) function $b(\cdot)$ on X such that $-N \leq h_\beta(x) \leq b(x)$, $\forall x \in X$ and $\beta \in [\alpha, 1)$.

- Theorem 5.4.3 - Suppose that Assumption 4.2.1 holds. Then the optimal stategy for average cost setup is deterministic stationary and is obtained by taking limit $\beta \uparrow 1$. The vanishing discount method is applicable and is employed to compute the optimal performance.

Step 1: Optimal threshold-type transmitter strategy for long-term average setup

The DP satisfies some suitable conditions so that, the vanishing discount approach is applicable.

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The DP satisfies some suitable conditions so that, the vanishing discount approach is applicable.

- For discounted setup, $\beta \in (0, 1]$, optimal transmitting strategy $f^*_\beta(\cdot; \lambda)$ is deterministic, threshold-type.
- Let $f^*(\cdot; \lambda)$ be any limit point of $f^*_\beta(\cdot; \lambda)$ as $\beta \uparrow 1$.
  Then the time-homogeneous transmission strategy $f^*(\cdot; \lambda)$ is optimal for $\beta = 1$ (the long-term average setup).
- Performance of optimal strategy:
  $C^*(\lambda) := C(f^*, g^*; \lambda) := \inf_{(f,g)} C(f, g; \lambda) = \lim_{\beta \uparrow 1} C^*_\beta(\lambda)$

Step 1: The SEN conditions

For any $\lambda \geq 0$, the value function $V_\beta(\cdot;\lambda)$, as given by a suitable DP, satisfies the following SEN conditions of [Sennot; Lerma, Lasserre]:

SEN conditions

(S1) There exists a reference state $e_0 \in \mathbb{Z}$ for Model A and $e_0 \in \mathbb{R}$ for Model B and a non-negative scalar M_λ such that $V_\beta(e_0, \lambda) < M_\lambda$ for all $\beta \in (0, 1)$.

(S2) Define $h_\beta(e;\lambda) = (1-\beta)^{-1}[V_\beta(e;\lambda) - V_\beta(e_0;\lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e;\lambda) \leq K_\lambda(e)$ for all $e \in \mathbb{Z}$ for Model A and for all $e \in \mathbb{R}$ for Model B and $\beta \in (0, 1)$.

(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leq h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ for Model A and for all $e \in \mathbb{R}$ for Model B and $\beta \in (0, 1)$.

Step 2: Performance of threshold based strategies

Cost until first transmission: solution of FIE

Let $\tau^{(k)}$ be the stopping time when the Gauss-Markov process starting at state 0 at time $t = 0$ enters the set $\{e \in \mathbb{R} : |e| \geq k\}$. Expected distortion incurred until stopping and expected stopping time under $f^{(k)}$ are solutions of Fredholm integral equations of second kind.

$$L^{(k)}(e) = e^2 + \int_{-k}^{k} \phi(w-e) L^{(k)}(w) dw;$$
$$M^{(k)}(e) = 1 + \int_{-k}^{k} \phi(w-e) M^{(k)}(w) dw.$$

Note that we have dropped the subscript 1 for ease of notation.

Step 2: Performance of threshold based strategies

Solutions to FIE

- Let $\mathcal{C}^{(k)}$ denote the space of bounded functions from $[-k, k]$ to $\mathbb{R}$. Define the operator $\mathcal{B}^{(k)} : \mathcal{C}^{(k)} \to \mathcal{C}^{(k)}$ as follows. For any $v \in \mathcal{C}^{(k)}$,

$$[\mathcal{B}^{(k)} v](e) = \int_{-k}^{k} \phi(w - e) v(w) dw.$$

- The operator $\mathcal{B}^{(k)}$ is a contraction
- Hence, FIE has a unique bounded solution $L^{(k)}$ and $M^{(k)}$.

Step 2: Performance of threshold based strategies

Renewal relationship

$$D^{(k)}(0) = \frac{L^{(k)}(0)}{M^{(k)}(0)}, \quad N^{(k)}(0) = \frac{1}{M^{(k)}(0)}$$

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$$D^{(k)}(0) = \frac{L^{(k)}(0)}{M^{(k)}(0)}, \quad N^{(k)}(0) = \frac{1}{M^{(k)}(0)}$$

Properties

- $L^{(k)}$ and $M^{(k)}$ are continuous, differentiable and monotonically increasing in k.
- $D^{(k)}(0)$ and $N^{(k)}(0)$ are continuous and differentiable in k. Furthermore, $N^{(k)}(0)$ is strictly decreasing in k.
- $D^{(k)}(0)$ is increasing in k.

Step 3: Identify critical Lagrange multipliers

Critical Lagrange multipliers

$$\lambda = \frac{D_\beta^{(k+1)}(0) - D_\beta^{(k)}(0)}{N_\beta^{(k)}(0) - N_\beta^{(k+1)}(0)} \text{(Model A)}; \ \lambda = -\frac{\partial_k D_\beta^{(k)}(0)}{\partial_k N_\beta^{(k)}(0)} \text{(Model B)} \tag{1}$$

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Optimal transmission startegy: Lagrange relaxation

$(f^{(k)}, g^*)$ is $\lambda^{(k)}$-optimal for Lagrange relaxation. Furthermore, for any $k > 0$, there exists a $\lambda = \lambda^{(k)} \geq 0$ that satisfies (1).

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Proof

- For Model A, for any $\lambda \in (\lambda^{(k-1)}, \lambda^{(k)}]$, $f^{(k)}$ is optimal.
- For Model B, the choice of λ implies that $\partial_k C^{(k)}(0; \lambda) = 0$. Hence strategy $(f^{(k)}, g^*)$ is λ-optimal.
- $\lambda^{(k)} \geq 0$, by the properties of $D^{(k)}(0)$ and $N^{(k)}(0)$.

Step 4: The constrained setup

A strategy (f°, g°) is optimal for a constrained optimization problem, if

Sufficient conditions for optimality [Sennott, 1999]

(C1) $N(f^\circ, g^\circ) = \alpha$,

(C2) There exists a Lagrange multiplier $\lambda^\circ \geq 0$ such that (f°, g°) is optimal for $C(f, g; \lambda^\circ)$.

Step 4: The constrained setup

- For Model A, $\alpha \in (0,1)$, let $k^*(\alpha), \theta^*(\alpha)$ be such that $\theta^*(\alpha)N^{(k^*(\alpha))} + (1-\theta^*(\alpha))N^{(k^*(\alpha)+1)} = \alpha$ (for Model B, $N^{(k^*(\alpha))} = \alpha$). Find $k^*(\alpha)$, $\theta^*(\alpha)$ for a given α;

Optimal deterministic strategy: f^* (Model A); $f^{(k^*(\alpha))}$ (Model B).

Step 4: The constrained setup

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Step 4: The constrained setup

Proof

- (C1) is satisfied by $f^\circ = f^*$ for Model A ($f^\circ = f^{(k^*(\alpha))}$ for Model B) and $g^\circ = g^*$ (by definition of $k^*(\alpha)$ and $\theta^*(\alpha)$)
- For $k^*(\alpha)$, we can find a λ satisfying (1). Hence we have that for Model A, (f^*, g^*) $((f^{(k^*(\alpha))}, g^*)$ for Model B) is optimal for $C(f, g; \lambda)$.
- Thus, for Model A, (f^*, g^*) $((f^{(k^*(\alpha))}, g^*)$ for Model B) satisfies (C2), since for $\lambda = \lambda_\beta^{(k^*)}$, both $f^{(k^*)}$ and $f^{(k^*+1)}$ are optimal for $C_\beta(f, g; \lambda)$. Hence, any strategy randomizing between them, in particular f^*, is also optimal for $C_\beta(f, g; \lambda)$.
- For Model A, $D^*(\alpha) := D(f^*, g^*) = \theta^*(\alpha) D^{(k^*(\alpha))}(0) + (1 - \theta^*(\alpha)) D^{(k^*(\alpha)+1)}(0)$ (for Model B, $D^*(\alpha) := D(f^{(k^*(\alpha))}, g^*) = D^{(k^*(\alpha))}(0)$)

Algorithm

Algorithm 1: Computation of $D^*_\beta(\alpha)$ for Model B

input : $\alpha \in (0,1)$, $\beta \in (0,1]$, $\varepsilon \in \mathbb{R}_{>0}$

output: $D_\beta^{(k^\circ)}(\alpha)$, where $|N_\beta^{(k^\circ)}(0) - \alpha| < \varepsilon$

Pick $\underline{k}$ and $\bar{k}$ such that $N_\beta^{(\underline{k})}(0) < \alpha < N_\beta^{(\bar{k})}(0)$

$k^\circ = (\underline{k} + \bar{k})/2$

while $|N_\beta^{(k^\circ)}(0) - \alpha| > \varepsilon$ **do**

  if $N_\beta^{(k^\circ)}(0) < \alpha$ **then**

    $\underline{k} = k^\circ$

  else

    $\bar{k} = k^\circ$

  $k^\circ = (\underline{k} + \bar{k})/2$

return $D_\beta^{(k^\circ)}(\alpha)$