Estimation with active sensing

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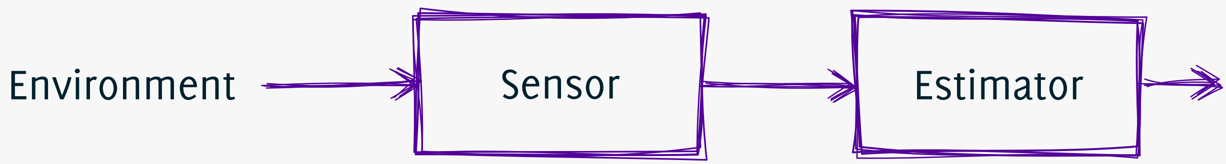
System Model



Sensor transmits packet to the estimator

- Each transmission incurs a constant cost
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- Trade-off estimation quality with transmission energy
- The sensor decides when and how to transmit
  (as opposed to the estimator scheduling the transmissions
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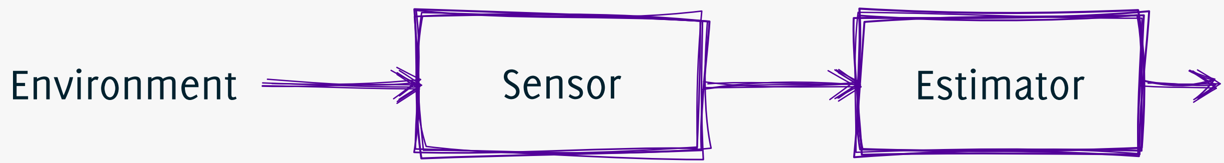

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When to communicate? And how to communicate?

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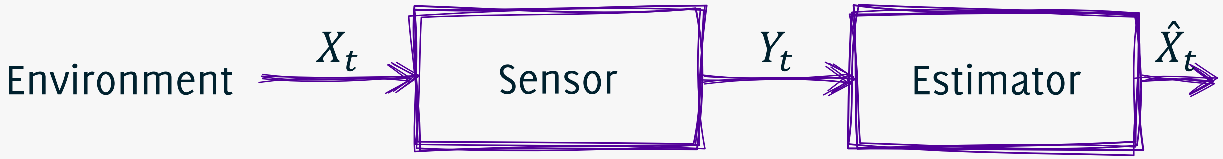

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  First order Markov process $\{X_t, t = 1, 2, \ldots\}$, $X_t \in \mathbb{X}$

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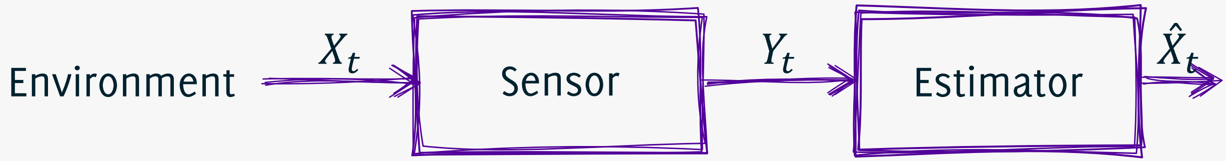

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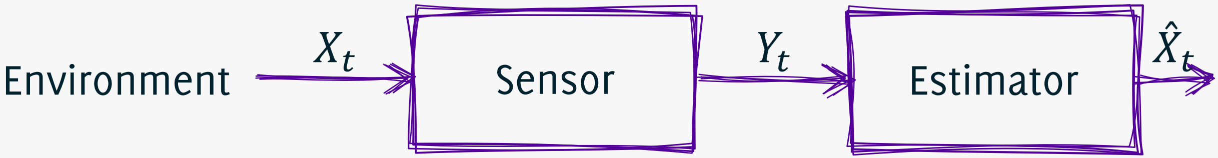

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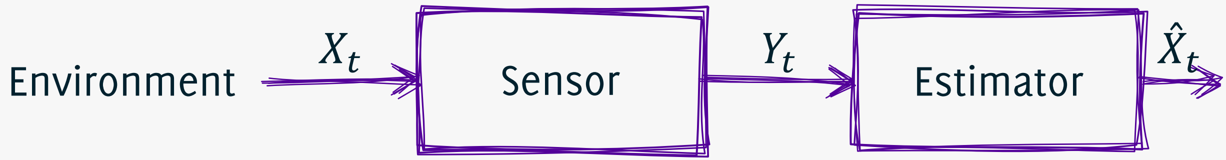

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- Cost $c(Y_t) + d(X_t, \hat{X}_t)$

  - $c(y) = \begin{cases} 0, & y = \mathbb{b}, \\ c^*, & y \in \mathbb{X} \end{cases}$
  - d is a metric on $\mathbb{X}$

Problem Formulation

- Transmission policy and estimation policies

$$\boldsymbol{f} = (f_1, f_2, \dots, f_T) \qquad \boldsymbol{g} = (g_1, g_2, \dots, g_T)$$

- Performance of a policy

$$\mathcal{J}(\boldsymbol{f}, \boldsymbol{g}) \coloneqq \mathbb{E}^{\boldsymbol{f},\boldsymbol{g}}\Big[\sum_{t=1}^{T} c(Y_t) + d(X_t, \hat{X}_t)\Big]$$

- Causal Policies $(\mathcal{F}, \mathcal{G})$

$$\mathcal{F} = \prod_{t=1}^{T} F_t, \qquad F_t = \big(\mathbb{X}^t \times \mathbb{Y}^{t-1} \mapsto \mathbb{Y}\big)$$

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P1: Choose $(\boldsymbol{f}, \boldsymbol{g}) \in (\mathcal{F}, \mathcal{G})$ to minimize $\mathcal{J}(\boldsymbol{f}, \boldsymbol{g})$

Salient Features

- Sequential dynamic team

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- Causal real-time communication

  Equivalent to a real-time communication system with input cost

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- Implication: Optimal policy is easy to implement . . .

  . . . but the intuition relies of the Markov process being Gaussian.

For non-Gaussian processes, is the optimal policy easy to implement?

Solution approach

- Identify irrelevant information at sensor

  Fix estimator. Set up the problem at the sensor as a MDP

- Find structure of optimal transmission and estimation policy: Coding does not improve performance

  A sequence of interchange arguments

- Identify a dynamic programming decomposition

  Dynamic team problem. Use the coordinator approach proposed by [Mahajan, Nayyar, Teneketzis, 2008]

- Simplify DP using structural result

- Implementation issues

Removing irrelevant information

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or equivalently if $f_t \in F'_t$ then $y_t = f_t(x_t, y_{1:t-1})$

- Proposition

There is no loss of optimality in restricting attention to simple causal policies.

Proof Outline

Fix an arbitrary estimation policy. Look at the problem of choosing the best transmission policy.

Define $Z_t = (X_t, Y_{1:t-1})$. We can show that

- $\mathbb{P}(Z_{t+1} | Z_{1:t}, Y_{1:t}) = \mathbb{P}(Z_{t+1} | Z_t, Y_t)$
- $\mathbb{E}[c(Y_t) + d(X_t, \hat{X}_t) | Z_{1:t}, Y_{1:t}] = \mathbb{E}[c(Y_t) + d(X_t, \hat{X}_t) | Z_t, Y_t]$

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Hence, $\{Z_t, t = 1, 2, \dots\}$ is a controlled Markov process. Thus, there is no loss of optimality in restricting attention to

$$Y_t = f_t(Z_t) = f_t(X_t, Y_{1:t-1})$$

Structure of optimal policies

- Structured Policies $(\mathcal{F}^*, \mathcal{G}^*)$

$$\mathcal{F}^\star = \prod_{t=1}^{T} F_t^\star,$$
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Theorem: There is no loss of optimality in restricting attention to structured policies.

Implication: Real-time coding does not improve performance

Proof Outline

- Define $(\mathcal{F}^{(s)}, \mathcal{G}^{(s)})$, $s = 0, 1, \ldots, T$ as follows

$$(F_t^{(s)}, G_t^{(s)}) = \begin{cases} (F_t', G_t) & t < s \\ (F_t^\star, G_t^\star) & t \geq s \end{cases}$$

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- Lemma

For any $(\boldsymbol{f}^{(s+1)}, \boldsymbol{g}^{(s+1)}) \in (\mathcal{F}^{(s+1)}, \mathcal{G}^{(s+1)})$, there is $(\boldsymbol{f}^{(s)}, \boldsymbol{g}^{(s)}) \in (\mathcal{F}^{(s)}, \mathcal{G}^{(s)})$, such that

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- Lemma $\implies$ Theorem.

$$(\mathcal{F}^{(T+1)}, \mathcal{G}^{(T+1)}) = (\mathcal{F}', \mathcal{G}), \qquad (\mathcal{F}^{(0)}, \mathcal{G}^{(0)}) = (\mathcal{F}^\star, \mathcal{G}^\star)$$

So how do we find an optimal policy?

Dynamic programming decomposition

- Non-classical information structure

$$Y_t = f_t(X_t, Y_{1:t-1}) \qquad \hat{X}_t = g_t(Y_{1:t-1}, Y_t)$$

Sensor | Estimator

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Sensor — Estimator

- Belongs to the class of tractable non-classical information structures identified in [Mahajan, Nayyar, Teneketzis, 2008]
- Look at the problem from the p.o.v. of a coordinator that observes the common data between the sensor and the estimator.
- The local data does not increase with time.

The coordinated system

$$f_t \; \boxed{X_t, Y_{1:t-1}} \; Y_t \qquad\qquad g_t \; \boxed{Y_{1:t-1}, Y_t} \; \hat{X}_t$$

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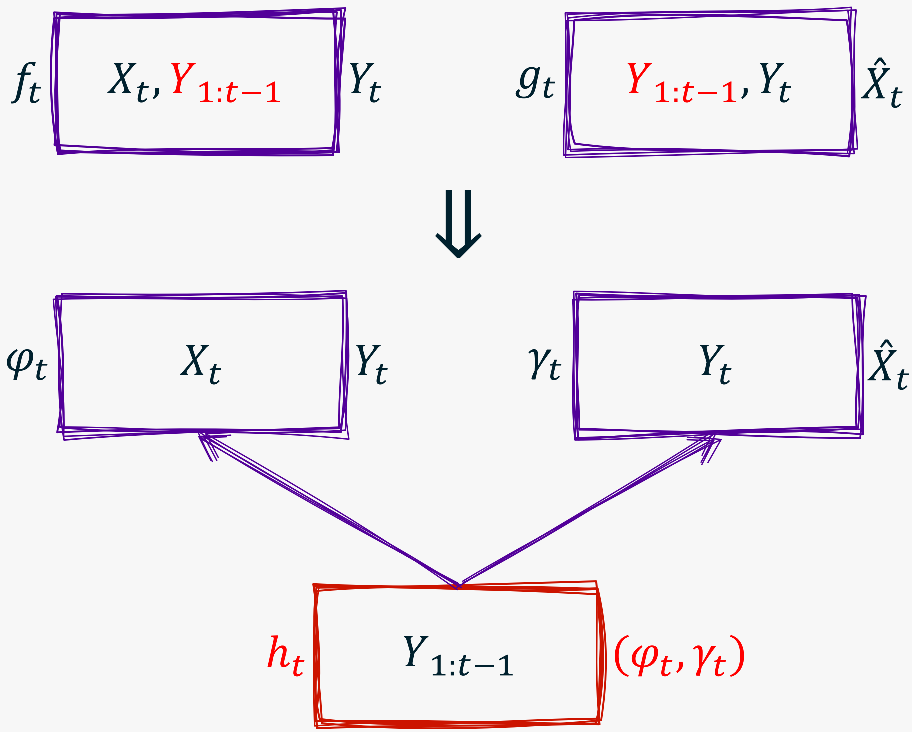

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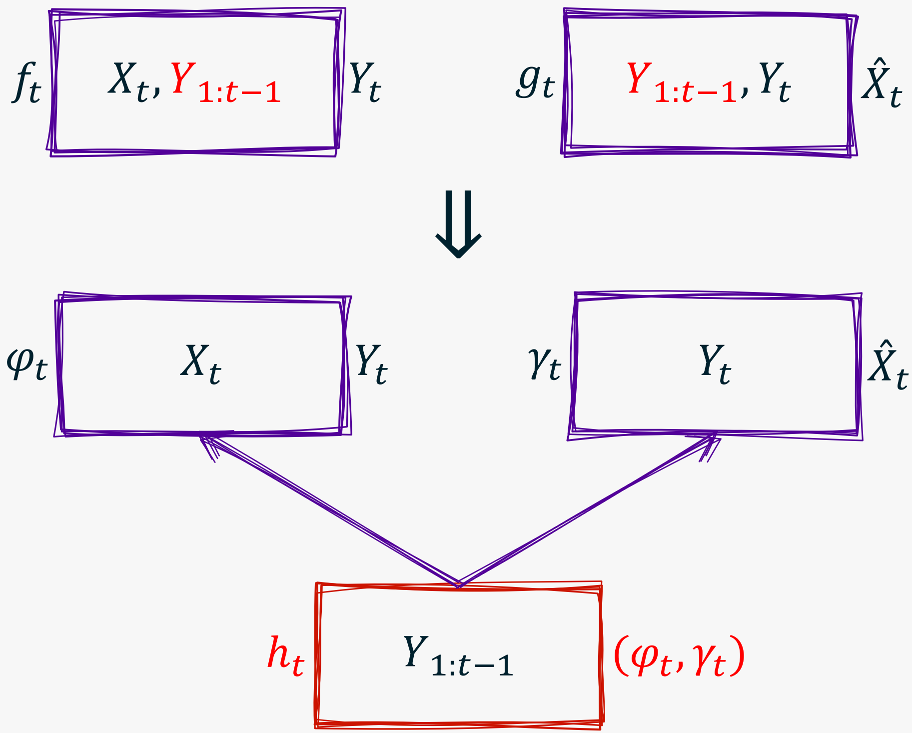

Both systems are equivalent

The dynamic programming decomposition

The coordinated system is a centralized (single agent) POMDP

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Structure Result

Let $\pi_t = \mathbb{P}(X_t | Y_{1:t-1}, \varphi_{1:t-1}, \gamma_{1:t-1})$.

$$(\varphi_t, \gamma_t) = h_t(\pi_t)$$

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$$V_t(\pi_t) = \min_{\varphi_t, \gamma_t} \left\{ \mathbb{E}\Big[c(X_t) + d(X_t, \hat{X}_t) + V_{t+1}(\text{update}(\pi_t, Y_t)) \,\Big|\, \pi_t, \varphi_t, \gamma_t \Big] \right\}$$

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- Each step is a functional optimization problem

Refinement of DP

Restrict attention to structured policies

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- $\forall f_t \in F_t^\star, \quad f_t(x_t, y_{1:t-1}) \in \{x_t, \mathbb{b}\}.$

  Thus, $\varphi_t(\cdot) \equiv \left\{x \in \mathbb{X} : \varphi_t(x_t) = x_t\right\} =: C_t$ (communication set)

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- Can be converted to countable state MDP.

Implementation issues

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- Sensor: $Y_t = f_t(X_t, U_t, N_t)$
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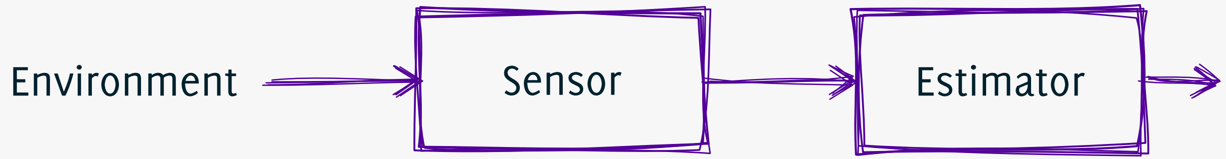
Conclusion



- Use ideas from dynamic team theory and real-time communications
  - remove irrelevant data at sensor
  - coding does not improve performance
  - DP decomposition by investigating the coordinated system
  - Simplify DP using structure of optimal policy
- Transformed to denumerable MDP with set valued actions

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  Consider "symmetric" Markov chains and a fixed decoding rule. Finding the optimal transmission policy is tangentially related to [Katehakis and Veinott, 1987] characterization of the Gittin's index as a restart in i problem.

  Could we reduce the problem to a 1-1/2 armed bandit problem?

Thank you.

Questions?