Low-complexity optimal control of networked coupled subsystems

Aditya Mahajan
McGill University

Joint work with Shuang Gao (Polytechnique Montreal)

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Overview of my research

Decentralized stochastic control

- Multi-stage decision problems with multiple decision makers
- Each DM has different partial information about the global state
- Identify information structures for which a dynamic programming solution is possible.

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- POMDPs: Partially observed Markov decision processes
- Developed a principled framework to understand RL algorithms for POMDPs, and used theoretical insights to improve existing algorithms.

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"Structure-aware" planning and learning

- Leverage underlying structure of the model to develop efficient planning and learning algos.
- **Application domains:** Networked control systems, telecommunication systems, scheduling and resource allocation

This talk: “Structure-aware” planning in networked control systems

Motivating example: Demand response in smart grids

Problem setting

- Temperature control of large number of units

  [a multi-story building, a city block, . . .]
- $N = \{1, \dots, n\}$ is the set of users.
- Each user has a desired set point x_0^i, $i \in N$.
- Control objective: The average temperature should track a reference signal r_t.

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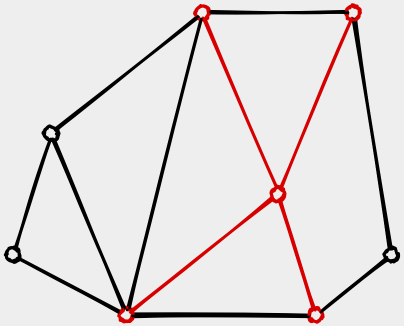

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System Dyanmics

- Model each user as a subsystem with state $x^i_t \in \mathbb{R}^{d_x}$ and control $u^i_t \in \mathbb{R}^{d_u}$.

$$x^i_{t+1} = Ax^i_t + Bu^i_t + D \sum_{j \in N} m^{ij} x^j_t + E \sum_{j \in N} m^{ij} u^j_t + w^i_t$$

Motivating example: Demand response in smart grids

Mean-field of states and control

- $\bar{x}_t := \dfrac{1}{n} \sum_{i \in N} x^i_t$ (emperical average of states)
- $\bar{u}_t := \dfrac{1}{n} \sum_{i \in N} u^i_t$ (emperical average of controls)

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Per-step cost

$$
\begin{aligned}
c(x_t, u_t) = \kappa\big[(\bar{x}_t - r_t)^\intercal Q(\bar{x}_t - r_t)\big] \\
+ \frac{1}{n} \sum_{i \in N} \big[(x^i_t - x^i_0)^\intercal Q(x^i_t - x^i_0) + (u^i_t)^\intercal Q u^i_t\big].
\end{aligned}
$$

Computing the optimal solution

System-model

- Define $\mathbf{x}_t = (x_t^1, \dots, x_t^n)$ and $\mathbf{u}_t = (u_t^1, \dots, u_t^n)$.
- Dynamics: $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t$
- Per-step cost

$$c(\mathbf{x}_t, \mathbf{u}_t) = (\mathbf{C}\mathbf{x}_t - \bar{\mathbf{C}}r_t)^\intercal \mathbf{Q}(\mathbf{C}\mathbf{x}_t - \bar{\mathbf{C}}r_t) + \mathbf{u}_t^\intercal \mathbf{R}\mathbf{u}_t.$$

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Standard solution approach

- This is a standard centralized reference tracking problem.
- Optimal solution is a state feedback of the form:

$$\mathbf{u}_t = \mathbf{G}_t \mathbf{x}_t$$

where the gains $\mathbf{G}_{1:T}$ are computed by solving a Riccati equation.

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Complexity of Riccati equation

- Dimension of system: $n d_x$.
- Complexity of solving Riccati equation $\mathcal{O}(n^3 d_x^3)$.
- **Does not scale to large networks!**

Our result: Develop a decomposition which computes the optimal policy by solving at most n Riccati eqns of dimension $d_x \times d_x$.

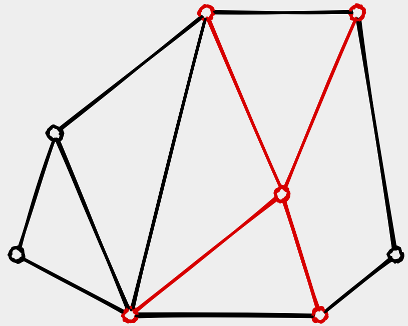
- co-author: Shuang Gao
- paper: TCNS 2022

Assumption

Structure of Per-step cost

$$c(x_t, u_t) = \sum_{i,j \in N} \left[h_q^{ij} (x_t^i)^\top Q (x_t^j) + h_r^{ij} (u_t^i)^\top Q (u_t^j) \right]$$

where $H_q = [h_q^{ij}]$ and $H_r = [h_r^{ij}]$ are **symmetric matrices which have the same eigenvectors as M**.



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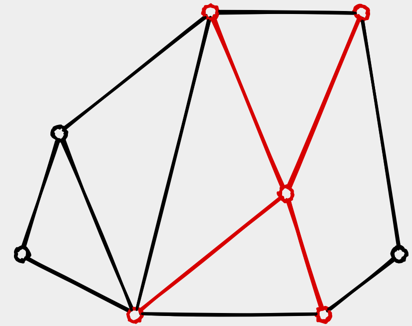

Remark

For two symmetric $n \times n$ matrices M_1 and M_2, the following statements are equivalent:

- M_1 and M_2 share the same eigenvectors.
- M_1 and M_2 communte (i.e., $M_1 M_2 = M_2 M_1$)
- M_1 and M_2 are simultaneously diagonalizable.

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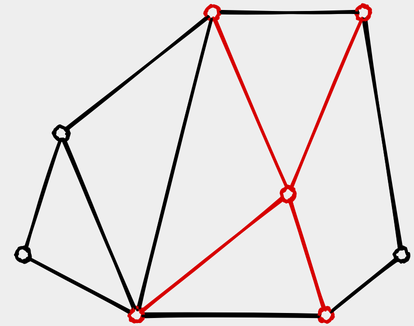
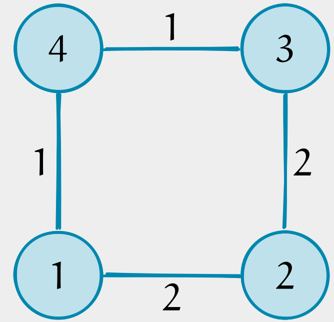


Important special case

- $H_q = \sum_{k=0}^{K_q} q_k M^k$ and $H_r = \sum_{k=0}^{K_r} r_k M^k$.
- Captures the intuition that the per-step cost respects the graph structure.
- Example: $H_q = q_0 I + q_1 M + q_2 M^2$ means that there is a cost coupling between the one- and two-hop neighbors.

An example to illustrate that nodes are not exchageable



A graph $\mathcal{G}$

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Dynamical coupling

- Nodes are not exchageable

$$x_t^{\mathcal{G},1} = 2\,x_t^2 + 1\,x_t^4, \qquad x_t^{\mathcal{G},2} = 2\,x_t^1 + 2\,x_t^3,$$

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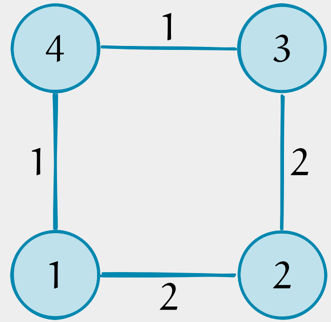

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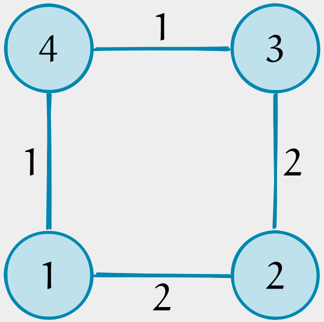

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Cost coupling

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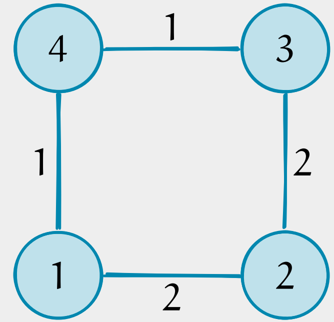

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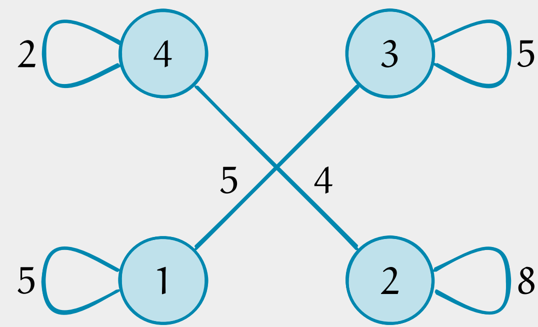

Two-hop neighborhood

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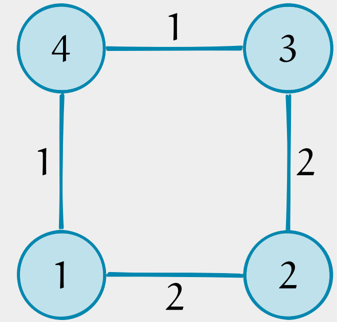

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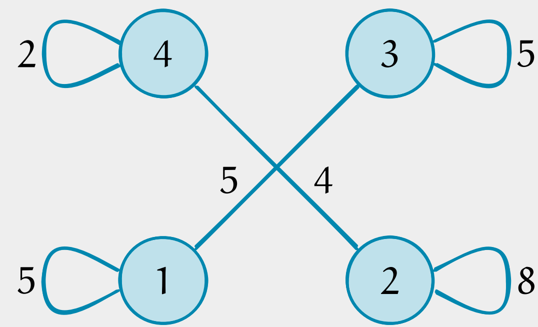
Suppose $H_q = q_0 I + q_1 M + q_2 M^2$. Then

$$H_q = \begin{bmatrix} q_0 + 5q_2 & 2q_1 & 5q_2 & q_1 \\ 2q_1 & q_0 + 8q_2 & 2q_1 & 4q_2 \\ 5q_2 & 2q_1 & q_0 + 5q_2 & q_1 \\ q_1 & 4q_2 & q_1 & q_0 + 2q_2 \end{bmatrix}$$



Two-hop neighborhood

Spectral factorization

Spectral decomposition of coupling matrices

$$M = \sum_{\ell=1}^{L} \lambda^{\ell} v^{\ell} (v^{\ell})^{\mathsf{T}},$$

Spectral factorization

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Spectral decomposition of dynamics

At each node $i \in [n]$:

- For each $\ell \in [L]$, define **eigenstates**, **eigencontrols**, and **eigennoise** as

$$x_t^{\ell,i} = x_t^i v^\ell (v^\ell)^\intercal, \quad u_t^{\ell,i} = u_t^i v^\ell (v^\ell)^\intercal, \quad \text{and} \quad w_t^{\ell,i} = w_t^i v^\ell (v^\ell)^\intercal.$$

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- Define **auxiliary state**, **auxiliary control**, **auxiliary noise** as

$$\breve{x}_t^i = x_t^i - \sum_{\ell=1}^{L} x_t^{\ell,i}, \quad \breve{u}_t^i = u_t^i - \sum_{\ell=1}^{L} u_t^{\ell,i}, \quad \text{and} \quad \breve{w}_t^i = w_t^i - \sum_{\ell=1}^{L} w_t^{\ell,i}.$$

Implication of Spectral factorization

Noise-coupled dynamics

$$x^{\ell,i}_{t+1} = (A + \lambda^{\ell} D) x^{\ell,i}_t + (B + \lambda^{\ell} E) u^{\ell,i}_t + w^{\ell,i}_t$$

and $\breve{x}^i_{t+1} = A \breve{x}^i_t + B \breve{u}^i_t + \breve{w}^i_t$

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Decoupled cost

$$c(x_t, u_t) = \sum_{i \in N} \left[q_0 \breve{c}(\breve{x}_t^{i}, \breve{u}_t^{i}) + \sum_{\ell=1}^{L} q^{\ell} c^{\ell}(x_t^{\ell,i}, u_t^{\ell,i}) \right]$$

where $q^{\ell} = q_0 + q_1 \lambda_q^{\ell}$, $\quad r^{\ell} = r_0 + r_1 \lambda_r^{\ell}$, and

$$\breve{c}(\breve{x}_t^{i}, \breve{u}_t^{i}) = (\breve{x}_t^{i})^{\intercal} Q \breve{x}_t^{i} + \frac{r_0}{q_0} (\breve{u}_t^{i})^{\intercal} R \breve{u}_t^{i}$$

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Eigen-system $(\ell, \mathbf{i})$ with $\ell \in [\mathbf{L}]$, $\mathbf{i} \in [\mathbf{n}]$

- State $x_t^{\ell,i}$. Control $u_t^{\ell,i}$.
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Auxiliary system $\mathbf{i}$ with $\mathbf{i} \in [\mathbf{n}]$

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Certainty equivalence: Optimal policy of stochastic LQ system is same as that of deterministic LQ system.

The deterministic system has **decoupled dynamics and cost!**

Only coupled through the noise in the dynamics

Main result

Under standard assumptions, the optimal control action is given by

$$u_t^i = \breve{u}_t^i + \sum_{\ell=1}^{L} u_t^{\ell,i} = \breve{G}\breve{x}_t^i + \sum_{\ell=1}^{L} G^\ell x_t^{\ell,i}$$

where

$$\breve{G} = \mathrm{Gain}\left(A, B, Q, \frac{r_0}{q_0} R\right)$$

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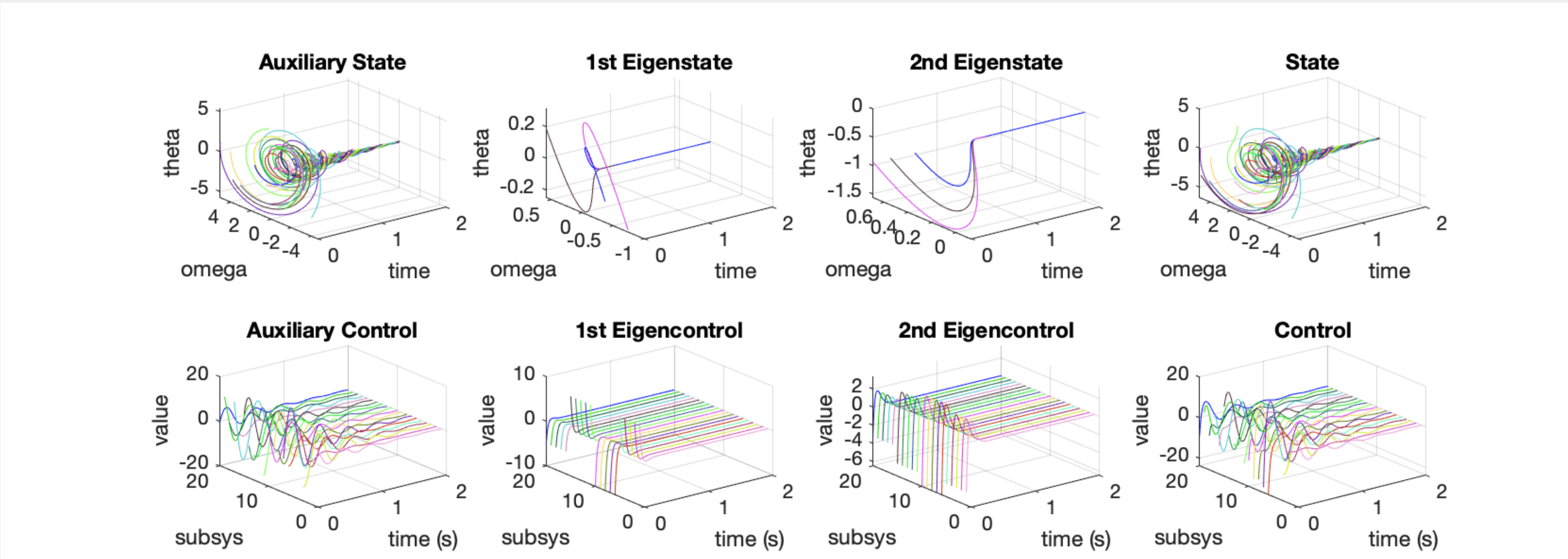
- The gains $\breve{G}$, $\{G^\ell\}_{\ell=1}^{L}$ are the same at all subsystems!
- Requires solving $(L+1)$ Riccati Eqn of dimension $d_x \times d_x$.
- Complexity scales $\mathcal{O}(L d_x^3)$ (cf. $\mathcal{O}(n^3 d_x^3)$ for naive solution).

Numerical Example

- 20 harmonic oscillators coupled via the adjacency matrix of a graph
- Solution obtained by solving three 2×2 Riccati equations



Conclusion

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Poissible generalizations/points of interest

- Multiple types of agents, approximate symmetry, ...
- Specific models for power management in microgrids with storage devices ...
- Adding constraints: Scalable MPC ...

- **email**: `aditya.mahajan@mcgill.ca`
- **web**: `http://cim.mcgill.ca/~adityam`

Thank you

Funding

- NSERC Discovery
- DND IDEaS Network

References

- **planning**: TCNS 2022
- **learning**: TCNS 2023