Optimal threshold strategies for remote state estimation with communication costs

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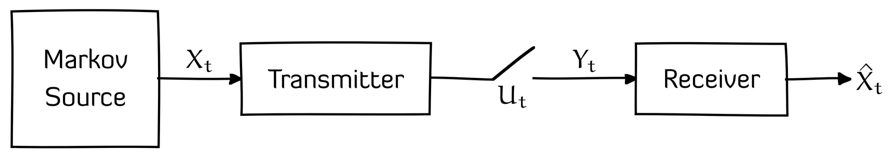
The communication system



Source
- $X_t \in \mathbb{Z}$
- Transition matrix P is Toeplitz, i.e., $P_{i,j} = p_{|i-j|}$, where $p_0 \geqslant p_1 \geqslant \cdots$.

Transmitter $U_t = f_t(X_{1:t}, U_{1:t-1})$ and $Y_t = \begin{cases} X_t & \text{if } U_t = 1 \\ \varepsilon & \text{if } U_t = 0 \end{cases}$

Receiver
- $\hat{X}_t = g_t(Y_{1:t})$
- Distortion: $d(X_t - \hat{X}_t)$ where $d(e) = d(-e) \leqslant d(e+1)$

Communication Strategies
- Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$.
- Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$.

Assumptions on the model

(A0) $X_t \in \mathbb{Z}$, and $X_0 = 0$.

(A1) The transition matrix is Toeplitz with decaying off-diagonal terms.

$$P = \begin{bmatrix} \ddots & p_0 & \ddots & & \\ \cdots & p_1 & p_0 & p_1 & \cdots \\ & \ddots & p_1 & p_0 & p_1 & \cdots \\ & & \ddots & \ddots & p_0 & \ddots \end{bmatrix} \quad \text{and} \quad p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$$

- Nayyar et al, assumed that the transistion matrix was banded, that is, $\exists b$ such that $p_k = 0$, for all $k \geqslant b$.

(A2) The distortion function is even and increasing on $\mathbb{Z}_{\geqslant 0}$.

$$\forall e \in \mathbb{Z}_{\geqslant 0}: \quad d(e) = d(-e) \quad \text{and} \quad d(e) \leqslant d(e+1).$$

The constrained optimization problem

$$\boxed{\min_{(f,g)} D_\beta(f,g) \quad \text{such that } N_\beta(f,g) \leqslant \alpha}$$

Minimize expected distortion such that expected # of transmissions is less than α

Discounted setup

$$D_\beta(f,g) = (1-\beta)\, \mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

$$N_\beta(f,g) = (1-\beta)\, \mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t \,\Big|\, X_0 = 0\Big]$$

Average cost setup

$$D_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

$$N_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\Big[\sum_{t=0}^{T-1} U_t \,\Big|\, X_0 = 0\Big]$$

Salient Features

Comparision to Information Theory
- As in information theory, the optimization problem may be viewed as minimizing average distortion under an average-power constraint.
- Unlike information theory, the source reconstruction must be done in real-time (or with zero delay).
- Therefore, classical information theory techniques do not work. Source-channel separation is not optimal.
- We use the decentralized control approach to real-time communication (following Witsenhausen, Walrand-Varaiya, Teneketzis, ...)

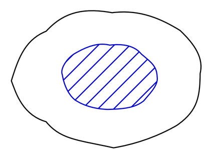
Comaprision to decentralized control
- Two decision makers—the transmitter and the receiver.
- (One-sided) nested information structure: the transmitter knows all the information available to the receiver.
- Constrained optimization problem, where the constraint does not depend on the "common information" (i.e., the information at the receiver).

Lagrange Relaxation

$$\boxed{\min_{(f,g)} D_\beta(f,g) \quad \text{such that } N_\beta(f,g) \leqslant \alpha}$$

Minimize expected distortion such that expected # of transmissions is less than α

Lagrange Relaxation $C^*_\beta(\lambda) := \inf_{(f,g)} C_\beta(f,g;\lambda)$ where $C_\beta(f,g;\lambda) = D_\beta(f,g) + \lambda N_\beta(f,g)$



Search space of strategies (f,g)

- Restrict the search space of strategies (f,g) by identifying structure of optimal tranmission and estimation strategies.
- Difficulty: Non-classical information structure

Structure of optimal estimator (Nayyar et al, 2013)

Transmitted Process Let Z_t denote the most recently transmitted value of the Markov source.

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} X_t & \text{if } U_t = 1; \\ Z_{t-1} & \text{if } U_t = 0. \end{cases}$$

The estimator can keep track of Z_t as follows:

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} Y_t & \text{if } Y_t \neq \varepsilon; \\ Z_{t-1} & \text{if } Y_t = \varepsilon. \end{cases}$$

Theorem 1 The process $\{Z_t\}_{t=0}^{\infty}$ is a sufficient statistic at the estimator and an optimal estimation strategy is given by

$$\hat{X}_t = g^*_t(Z_t) = Z_t \qquad (\star)$$

Remark
- The optimal estimation strategy is time-homogeneous and can be specified in closed form.

Structure of optimal transmitter (Nayyar et al, 2013)

Error process Let $E_t = X_t - Z_{t-1}$ denote the error process. $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process where

$$E_0 = 0 \quad \text{and} \quad \mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} P_{0n}, & \text{if } u = 1; \\ P_{en}, & \text{if } u = 0. \end{cases}$$

Theorem 2 When the estimation strategy is of the form $(\star)$, then $\{E_t\}_{t=0}^{\infty}$ is a sufficient statistic at the transmitter.

Furthermore, an optimal transmission strategy is characterized by a time-varying threshold $\{k_t\}_{t=0}^{\infty}$, i.e.,

$$U_t = f_t(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k_t; \\ 0 & \text{if } |E_t| < k_t. \end{cases}$$

Proof idea
- The proof of [Nayyar et al, 2013] was based on some majorization inequalities of [Hajek et al, 2009] for distributions with finite support.
- We extend these inequalities to distributions over integers using results of [Wang-Woo-Madiman, 2014].

Performance of a threshold based strategy

Threshold-based strategy We analyze the performace of $(f^{(k)}, g^*)$, where

$$f^{(k)}(e) := \begin{cases} 1, & \text{if } |e| \geqslant k; \\ 0, & \text{if } |e| < k. \end{cases}$$

Cost until first transmission Define $S^{(k)} = \{e \in \mathbb{Z} : |e| \leqslant k-1\}$ and let $\tau^{(k)}$ be the stopping time when the Markov process starting at state 0 at time $t = 0$ escapes the set $S^{(k)}$.

$$\text{Define} \quad L^{(k)}_\beta := \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = 0\Big]$$

$$M^{(k)}_\beta := \frac{1 - \mathbb{E}[\beta^{\tau^{(k)}} \mid E_0 = 0]}{1-\beta}$$

and

$$L^{(k)}_1 := \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} d(E_t) \Big| E_0 = 0\Big]$$

$$M^{(k)}_1 := \mathbb{E}[\tau^{(k)} - 1 \mid E_0 = 0]$$

Performance of a threshold based strategy (cont.)

Renewal relationships

$$D^{(k)}_\beta := D_\beta(f^{(k)}, g^*) = \frac{L^{(k)}_\beta}{M^{(k)}_\beta}$$

$$N^{(k)}_\beta := N_\beta(f^{(k)}, g^*) = \frac{1}{M^{(k)}_\beta} - (1-\beta)$$

Vanishing discount relationships

$$L^{(k)}_1 = \lim_{\beta\uparrow 1} L^{(k)}_\beta, \quad M^{(k)}_1 = \lim_{\beta\uparrow 1} M^{(k)}_\beta.$$

and

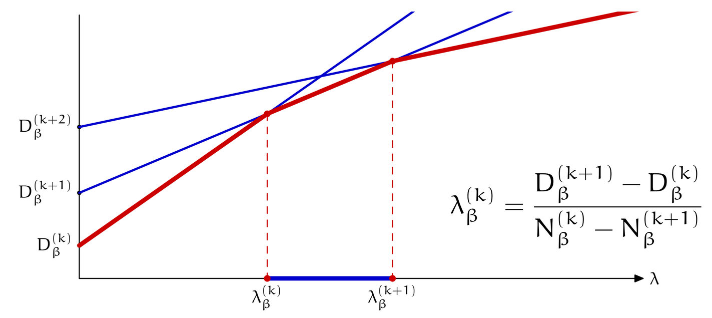
$$D^{(k)}_1 = \lim_{\beta\uparrow 1} D^{(k)}_\beta = \frac{L^{(k)}_1}{M^{(k)}_1}$$

$$N^{(k)}_1 = \lim_{\beta\uparrow 1} N^{(k)}_\beta = \frac{1}{M^{(k)}_1}$$

Optimal stategy for the Lagrange relaxation

Some inequalities $L^{(k)}_\beta < L^{(k+1)}_\beta, \quad M^{(k)}_\beta < M^{(k+1)}_\beta, \quad D^{(k)}_\beta < D^{(k+1)}_\beta.$

Lagrangian cost $C^{(k)}_\beta(\lambda) := C(f^{(k)}, g^*; \lambda) = D^{(k)}_\beta + \lambda N^{(k)}_\beta$



$$\lambda^{(k)}_\beta = \frac{D^{(k+1)}_\beta - D^{(k)}_\beta}{N^{(k)}_\beta - N^{(k+1)}_\beta}$$

Optimal performance
- For all $\lambda \in (\lambda^{(k)}_\beta, \lambda^{(k+1)}_\beta]$ the threshold strategy $f^{(k+1)}$ is optimal.
- $C^*_\beta(\lambda) = \min_{k\in\mathbb{Z}} C^{(k)}_\beta$ is piecewise linear, continuous, concave, and increasing function of λ.

Back to the constrained optimization problem

Bernoulli randomized strategy Let $\theta \in [0,1]$ and f_1 and f_2 be two stationary strategies. The Bernoulli randomized strategy (f_1, f_2, θ) randomizes between f_1 and f_2 at each stage, choosing f_1 with probability θ and f_2 with probability $(1-\theta)$.

Simple rand. strategy A Bernoulli randomized strategy (f_1, f_2, θ) is simple if the actions prescribed by f_1 and f_2 differ only at one state.

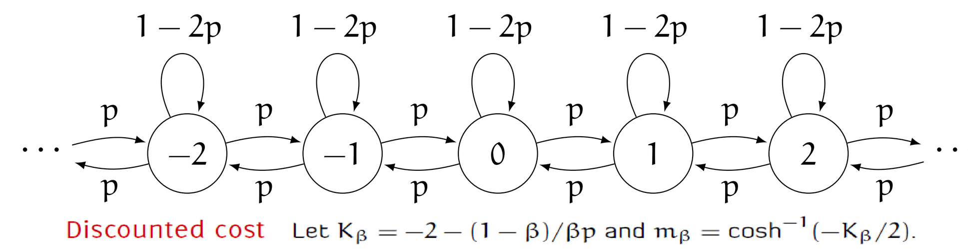
Main result Define $k^*_\beta = \sup\{k \in \mathbb{Z}_{\geqslant 0} : N^{(k)}_\beta \geqslant \alpha\}$ and let θ be such that

$$\theta N^{(k^*_\beta)}_\beta + (1-\theta) N^{(k^*_\beta+1)}_\beta = \alpha$$

Then, the Bernoulli simple randomized strategy $(f^{(k^*_\beta)}, f^{(k^*_\beta+1)}, \theta)$ is optimal for the constrained optimization problem for $\beta \in (0,1]$.

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$



Discounted cost Let $K_\beta = -2 - (1-\beta)/\beta p$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$.

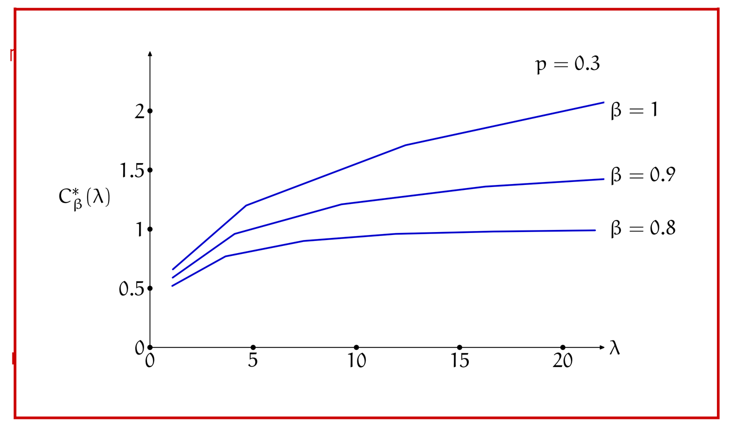
$$D^{(k)}_\beta = \frac{\sinh(k m_\beta) - k \sinh(m_\beta)}{2\sinh^2(k m_\beta/2)\sinh(m_\beta)}$$

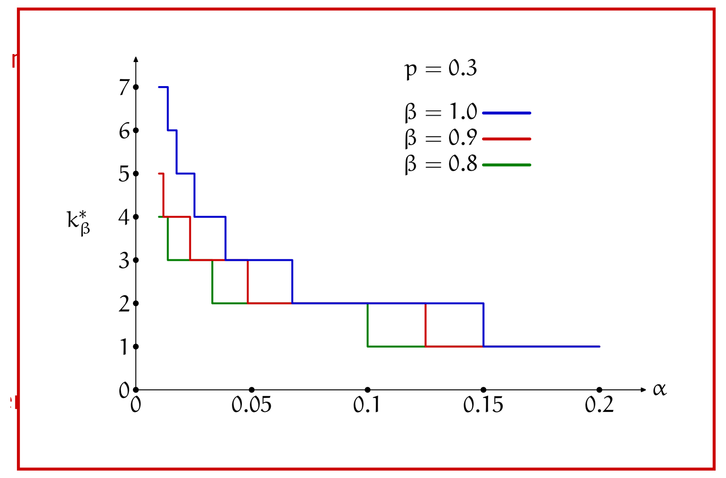
$$N^{(k)}_\beta = \frac{2\beta p \sinh^2(m_\beta/2)\cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} - (1-\beta)$$

$\boxed{\lambda^{(k)}_\beta \text{ can be computed in terms of } D^{(k)}_\beta \text{ and } N^{(k)}_\beta.}$

Average cost $D^{(k)}_1 = \dfrac{k^2-1}{3k}$ and $N^{(k)}_1 = \dfrac{2p}{k^2}$

$$\boxed{\lambda^{(k)}_1 = \frac{k(k+1)(k^2+k+1)}{6p(2k+1)}} \qquad \boxed{k^*_1 = \left\lfloor \sqrt{\frac{2p}{\alpha}} \right\rfloor}$$





Summary and Conclusion

Problem formulation
- Real-time transmission of a Markov source under constraints on the number of transmissions.
- Investigated both discounted and average cost infinite horizon setups.
- Modeled as a decentralized stochastic control problem with two decision maker.
- As long as the transmitter uses a symmetric threshold based strategy, the estimation strategy does not depend on the transmission strategy.
- The problem of find the "best response" transmitter is a centralized stochastic control problem.

Main results
- Simple Bernoulli randomized strategies $(f^{(k^*)}, f^{(k^*+1)}, \theta)$ are optimal.
- k^* and θ can be computed easily.