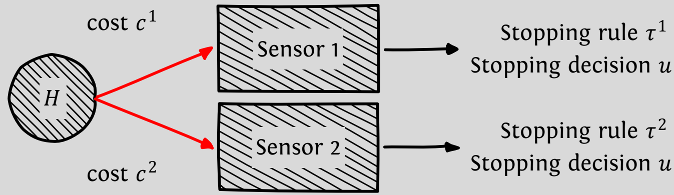
Decentralized sequential hypothesis-testing with common observations

Aditya Mahajan

McGill University

Allerton Conference
Oct 2, 2013

Decentralized sequential hypothesis testing



Setup

- Two (or more) sensors with correlated observations
- Have coupled stopping cost when sequentially testing a hypothesis.

Objective

$$\min_{(\tau^1,\tau^2,u^1,u^2)} \mathbb{E}[\tau^1 c^1 + \tau^2 c^2 + \ell(u^1, u^2, H)]$$

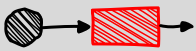
Well posed

- $\ell(u^1, u^2, h) \neq \ell_1(u^1, h) + \ell_2(u^2, h)$
- $\ell(n, n, n) \leqslant \left\{ \begin{matrix} \ell(n, m, n) \\ \ell(m, n, n) \end{matrix} \right\} \leqslant \ell(m, m, n)$

Motivation

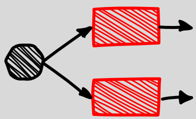
- Sensor networks
- Cognitive radio

(Selected) variations of sequential hypothesis testing



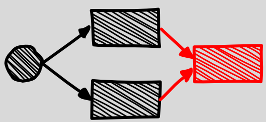
Sequential hypothesis testing

- Wald (1945), Arrow, Blackwell, Girshick (1949)
- Two threshold rule (equivalent to SLRT) is optimal



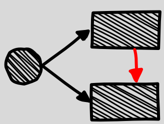
Multiple sensors making individual decisions

- Teneketzis, Ho (1987); LaVigna, Makowski, Baras (1986)
- Coupled cost function $\ell(u_1, u_2, h)$.
- Two threshold stopping rules (equiv. SLRT) are optimal.



Multiple sensors communicating to fusion center

- Veeravalli, Başar, Poor (1993)
- Feedback from the fusion center to the sensors
- Two threshold stopping rule (equiv. SLRT) is optimal at fusion ctr



Multiple sensors communicating to one another

- Nayyar, Teneketzis (2011)
- Two threshold stopping rule (equiv. SLRT) is optimal at 2nd sensor
- **. . . but not optimal at 1st sensor**

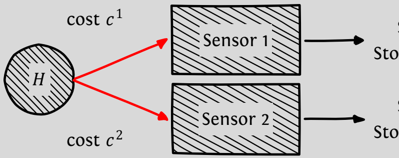
Problem formulation

Observation model

Sensor i gets local observation Y^i_t and common observation Z_t.

Under h_0: $Y^i_t \sim f^i_0$ and $Z_t \sim f^0_0$

Under h_1: $Y^i_t \sim f^i_1$ and $Z_t \sim f^0_1$



Decision set

At each time, sensor i has two alternatives

- Take another measurement at a cost c^i.
- Decide to stop and declare a value in $\{h_0, h_1\}$.

After both sensors stop, a cost $\ell(u^1, u^2, H)$ is incurred.

Objective

Choose decision strategies (g^1, g^2) to minimize

$$J(g^1, g^2) = \min_{(\tau^1,\tau^2,u^1,u^2)} \mathbb{E}[\tau^1 c^1 + \tau^2 c^2 + \ell(u^1, u^2, H)]$$

Features of the setup

- Non-classical information structure.
- Correlation modeled by common and local observations.
- Actions coupled by cost and not dynamics.

Main results (1/2)

Definition $P^i_t(h) = \mathbb{P}(H = h \mid Y^i_{1:t}, Z_{1:t})$. Let $P^i_{0,t} = P^i_t(h_0)$.

Theorem 1 There is no loss of optimality in restricting attention to control strategies of the form

$$U^i_t = \bar{g}^i_t(P^i_{0,t}, Z_{1:t}), \qquad i \in \{1, 2\}.$$

where $\bar{g}^i_t$ is a threshold strategy given by

$$\bar{g}^i_t(p^i_t, z_{1:t}) = \begin{cases} h_1 & \text{if } p^i_t \leqslant \alpha^i_{1,t}(z_{1:t}) \\ \mathsf{C} & \text{if } \alpha^i_{1,t}(z_{1:t}) < p^i_t < \alpha^i_{0,t}(z_{1:t}) \\ h_2 & \text{if } \alpha^i_{0,t}(z_{1:t}) \leqslant p^i_t \end{cases}$$

and the person-by-person optimal threshold functions $\alpha^i_{1,t}(z_{1:t})$ and $\alpha^i_{0,t}(z_{1:t})$ are obtained by solving two coupled dynamic programs.

Main results (2/2)

Definitions

- $S^i_t = \mathbb{1}\{\exists t' < t : u^i_{t'} = C\}$
- $Q^i_t(h) = \mathbb{P}(H = h \mid Y^i_{1:t})$; and $Q^i_{0,t} = Q^i_t(h_0)$.
- $D_t(h, s^1, s^2) = \mathbb{P}(H = h, S^1 = s^1, S^2 = s^2 \mid Z_{1:t})$

Theorem 2 There is no loss of optimality in restricting attention to control strategies of the form

$$U^i_t = \hat{g}^i_t(Q^i_{0,t}, D_t), \qquad i \in \{1, 2\}.$$

where $\hat{g}^i_t$ is a threshold strategy given by

$$\hat{g}^i_t(q^i_t, d_t) = \begin{cases} h_1 & \text{if } p^i_t \leqslant \beta^i_{1,t}(d_t) \\ C & \text{if } \beta^i_{1,t}(d_t) < p^i_t < \beta^i_{0,t}(d_t) \\ h_2 & \text{if } \beta^i_{0,t}(d_t) \leqslant p^i_t \end{cases}$$

and the threshold functions $\beta^i_{1,t}(d_t)$ and $\beta^i_{0,t}(d_t)$ are obtained by solving a dynamic program.

Salient features of the result

Threshold rules (or SLRTs) are optimal

For the finite horizon setup, the optimal strategy is a threshold rule in q^i_t, where the threshold curves $\beta^i_{1,t}$ and $\beta^i_{0,t}$ depend on d_t.

Equivalent to a sequential likelihood ration test (SLRT) where the value of the thresholds depend on d_t.

Time-invariant thresholds

For the infinite horizon set, the threshold curves $\beta^i_{1,t}$ and $\beta^i_{0,t}$ are time-invariant.

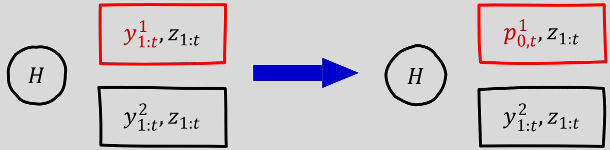
Therefore, the optimal strategy is easy to implement.

DP to compute thresholds

We present a dynamic program to compute the globally optimal thresholds.

This is in contrast to previous work on independent observations [Teneketzis, Ho (1987), LaVigna, Makowski, Baras (1986)] where coupled dynamic programs to compute person-by-person optimal thresholds were presented.

Step 1 ▸ Identify information state for local observations



where $P^i_t(h) = \mathbb{P}(H = h \mid Y^i_{1:t}, Z_{1:t})$; and $P^i_{0,t} = P^i_t(h_0)$.

Proof ▸ Arbitrarily fix strategy of j; consider best response strategy of i

▸ Define: $F^i_t(h^j \mid h, z_{1:t}; g^j) = \mathbb{P}(U^j_{\tau^j} = h^j \mid H = h, Z_{1:t} = z_{1:t}; g^j)$.

Lemma 1 $\mathbb{P}(H = h, U^j_{\tau^j} = h^j \mid Y^i_{1:t}, Z_{1:t}; g^j) = F^i(h^j \mid h, z_{1:t}; g^j) P^i_t(h)$

Lemma 2 $\mathbb{E}[\ell(u^i, U^j, H) \mid Y^i_{1:t}, Z_{1:t}; g^j] = \langle L^i_t(Z_{1:t}, u^i),\ P^i_t \rangle$

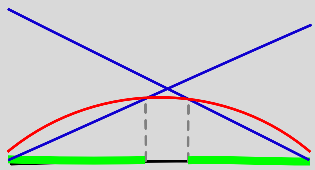
Dynamic Program

$$V^i_t(p^i_t, z_{1:t}; g^j) = \min \Big\{ \langle L^i_t(z_{1:t}, h_0),\ p^i_t \rangle,\quad \langle L^i_t(z_{1:t}, h_1),\ p^i_t \rangle,\\ c + \mathbb{E}[V^i_{t+1}(P^i_{t+1}, Z_{1:t+1}; g^j) \mid P^i_t = p^i_t, Z_{1:t} = z_{1:t}] \Big\}$$

Proof of Step 1 (cont.)

Dynamic Program

$$V^i_t(p^i_t, z_{1:t}; g^j) = \min \Big\{ \big\langle L^i_t(z_{1:t}, h_0),\ p^i_t \big\rangle, \quad \big\langle L^i_t(z_{1:t}, h_1),\ p^i_t \big\rangle, \\ c + \mathbb{E}[V^i_{t+1}(P^i_{t+1}, Z_{1:t+1}; g^j) \mid P^i_t = p^i_t, Z_{1:t} = z_{1:t}] \Big\}$$



Theorem 1 There is no loss of optimality in restricting attention to control strategies of the form

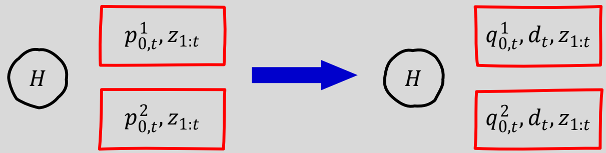
$$U^i_t = \bar{g}^i_t(P^i_{0,t}, Z_{1:t}), \qquad i \in \{1, 2\}.$$

where $\bar{g}^i_t$ is a threshold strategy given by

$$\bar{g}^i_t(p^i_t, z_{1:t}) = \begin{cases} h_1 & \text{if } p^i_t \leqslant \alpha^i_{1,t}(z_{1:t}) \\ C & \text{if } \alpha^i_{1,t}(z_{1:t}) < p^i_t < \alpha^i_{0,t}(z_{1:t}) \\ h_2 & \text{if } \alpha^i_{0,t}(z_{1:t}) \leqslant p^i_t \end{cases}$$

and the person-by-person optimal threshold functions $\alpha^i_{1,t}(z_{1:t})$ and $\alpha^i_{0,t}(z_{1:t})$ are obtained by solving two coupled dynamic programs.

Step 2 ▸ Alternative description of information state



Definitions

- $S^i_t = \mathbb{1}\{\exists t' < t : u^i_{t'} = C\}$
- $Q^i_t(h) = \mathbb{P}(H = h \mid Y^i_{1:t})$; and $Q^i_{0,t} = Q^i_t(h_0)$.
- $D_t(h, s^1, s^2) = \mathbb{P}(H = h, S^1 = s^1, S^2 = s^2 \mid Z_{1:t})$

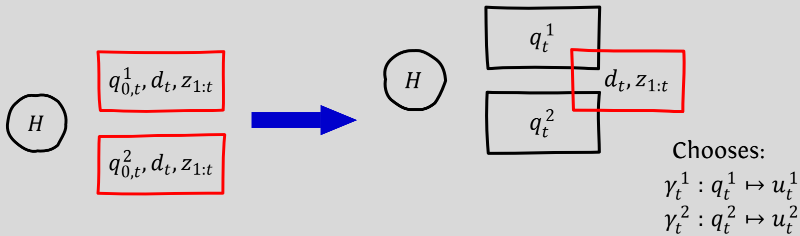
Lemma 3 P^i_t is a function of D_t and Q^i_t

- Proof: Let $D_t(h)$ denote the marginal of $D_t(h, s^1, s^2)$. Then,

$$\frac{P^i_t(h_0)}{P^i_t(h_1)} = \left[\frac{D_t(h_0)(1-p)}{D_t(h_1)p}\right] \frac{Q^i_t(h_0)}{Q^i_t(h_1)}$$

Lemma 4 $P^i_t \lesseqgtr \alpha(Z_{1:t}) \equiv Q^i_t \lesseqgtr \hat{\alpha}(D_t, Z_{1:t})$

Step 3 ▸ Information state for common observation



Proof

- Use the common information approach of Nayyar, M, Teneketzis ('13).
- Equivalent centralized coordinated system.
- Coordinator observes common information $(D_t, Z_{1:t})$.
- . . . and chooses prescriptions (Γ^1_t, Γ^2_t), where $\Gamma^i_t : Q^i_t \mapsto U^i_t$.
- . . . sensors passively use this prescription: $U^i_t = \Gamma^i_t(Q^i_t)$.

Information state

$$\Pi_t(h, s^1, s^2, q^1, q^2) = \mathbb{P}(H = h, S^1 = s^1, S^2 = s^2, Q^1_{0,t} = q^1, Q^2_{0,t} = q^2 \mid D_t, Z_{1:t})$$

Proof of Step 3 (cont.)

Structural results

In the coordinated system, without loss of optimality use

$$(\Gamma^1_t, \Gamma^2_t) = \psi(\Pi_t)$$

Therefore, in the original system, without loss of optimality use

$$U^i_t = \hat{g}^i_t(Q^i_t, \Pi_t)$$

Also obtain a corresponding dynamic program.

Lemma 5 Π_t is a function of D_t.

Theorem 2 There is no loss of optimality in restricting attention to control strategies of the form

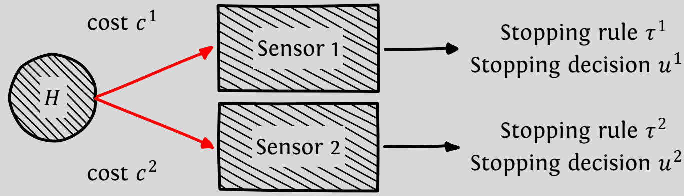
$$U^i_t = \hat{g}^i_t(Q^i_{0,t}, D_t), \qquad i \in \{1, 2\}.$$

where $\hat{g}^i_t$ is a threshold strategy given by

$$\hat{g}^i_t(q^i_t, d_t) = \begin{cases} h_1 & \text{if } p^i_t \leqslant \beta^i_{1,t}(d_t) \\ C & \text{if } \beta^i_{1,t}(d_t) < p^i_t < \beta^i_{0,t}(d_t) \\ h_2 & \text{if } \beta^i_{0,t}(d_t) \leqslant p^i_t \end{cases}$$

and the threshold functions $\beta^i_{1,t}(d_t)$ and $\beta^i_{0,t}(d_t)$ are obtained by solving a dynamic program.

Summary



Theorem 2 There is no loss of optimality in restricting attention to control strategies of the form

$$U^i_t = \hat{g}^i_t(Q^i_{0,t}, D_t), \qquad i \in \{1, 2\}.$$

where $\hat{g}^i_t$ is a threshold strategy given by

$$\hat{g}^i_t(q^i_t, d_t) = \begin{cases} h_1 & \text{if } p^i_t \leqslant \beta^i_{1,t}(d_t) \\ C & \text{if } \beta^i_{1,t}(d_t) < p^i_t < \beta^i_{0,t}(d_t) \\ h_2 & \text{if } \beta^i_{0,t}(d_t) \leqslant p^i_t \end{cases}$$

and the threshold functions $\beta^i_{1,t}(d_t)$ and $\beta^i_{0,t}(d_t)$ are obtained by solving a dynamic program.

Summary

Key ideas of the proof

- Step 1: Find information state for local information.
- Step 2: Obtain an alternative description of information state.
- Step 3: Find information state for common information.
- Step 4: Simplify the information state.

Features of the result

- Threshold based rules are optimal.
- The threshold curves depends on D_t.
- Obtain a dynamic program to compute globally optimal threshold curves.

Future directions

- Computation of the threshold curves (cf. Smallwood and Sondik)
- Approximation of the threshold curves based on type-I and type-II error probabilities (cf. Wald, Shiryaev)
- Asymptotically optimal threshold curves (cf. Chernoff)