Yamamoto Itoh achievable scheme for variable length feedback communication

Aditya Mahajan

Feedback communication refers to the idealized situation where the channel output is available at the encoder with one unit of delay. For the case of discrete memoryless channels, the availability of feedback does not increase capacity; it does, however, significantly boost the error exponents provided the transmission time is allowed to be random.

This result was proved by Burnashev [1], who also gave an exact characterization of the reliability function for variable length communication—a remarkable result given the fact that the reliability function of *fixed* length communication is still not completely characterized. Burnashev's proof proceeded in two steps. He first proved an upper bound on the reliability function and then proposed a variable length scheme that achieves that bound.

The Burnashev's achievable scheme operates for random number of epochs; each epoch consists of two phases: a message phase and a confirmation phase; each phase is of a random length. The performance analysis of the scheme is quite involved.

Later, Yamamoto and Itoh [2] presented a simpler achievable scheme, which structurally is similar to the one proposed by Burnashev—the only difference is that each phase is of fixed length, which makes the performance analysis quite simple. Unfortunately, the analysis in the original paper is incomplete: Yamamoto and Itoh only analyze one epoch of the scheme. They do not take into account the increase in transmission time due to retransmissions and its affect on the rate and error exponent. In this note, we give a complete proof of the Yamamoto-Itoh scheme.

The coding scheme

We present a family of coding schemes $\mathcal{S} = \{S_1, S_2, \dots\}$, which achieves the Burnashev exponent as the *index* of the coding scheme goes to infinity. This family asymptotically achieves a rate R, $R < C$, where C is the capacity of the channel.

Scheme S_n operates in multiple epochs; each epoch is of length n; the total number of epochs is a stopping time denoted by K_n. Each epoch consists of two phases: a *message phase* of length $\alpha_n n$ and a *confirmation phase*[1] of length $(1-\alpha_n)n$. The parameter α_n is chosen such that

$$\alpha_n > \frac{R}{C} \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = \frac{R}{C}.$$

The k-th epoch of the scheme operates as follows:

1. *Message mode*: The transmitter sends one of $M_n = 2^{nR}$ messages using a codebook of length $\alpha_n n$. Let $W_n(k)$ denote the transmitted message and $\hat{W}_n(k)$ denote the decoded message. Then, the probability of error is bounded by

$$\text{Prob}\left(\hat{W}_n(k) \neq W_n(k)\right) \le 2^{-\alpha_n n E_G(R/\alpha_n)}$$

where $E_G(R)$ is the random coding exponent [3] at rate R. Since $\alpha_n > R/C$, we have that $R/\alpha_n < C$ and hence

[1] Yamamoto and Itoh called the second phase as *control phase* but I think that *confirmation phase* is a better term

$$E_G(R/\alpha_n) > 0 \tag{$\star$}$$

2. *Confirmation mode*: The encoder transmits an ACCEPT message if $\hat{W}_n(k) = W_n(k)$, otherwise it transmits a REJECT message. ACCEPT consists of $(1-\alpha_n)n$ repetitions of x_A while REJECT consists of $(1-\alpha_n)n$ repetitions of x_R where

$$(x_A, x_R) = \arg\max_{(x_1,x_2)} D(Q(\cdot|x_1)\|Q(\cdot|x_2))$$

where $D(\cdot\|\cdot)$ is the KL divergence between two distributions and $Q(\cdot|x)$ is the output distribution when the input is x. Let

$$B = D(Q(\cdot|x_A)\|Q(\cdot|x_R)).$$

At the end of this phase, the receiver generates likelihood ratios for ACCEPT and REJECT, i.e.,

$$L_A(k) = \log\frac{p_A(k)}{1-p_A(k)} \quad \text{and} \quad L_R(k) = \log\frac{p_R(k)}{1-p_R(k)}$$

where

$$p_A(k) = \text{Prob}\left(\text{ACCEPT}\,\middle|\,\text{output of confirmation phase}\right)$$
$$p_R(k) = \text{Prob}\left(\text{REJECT}\,\middle|\,\text{output of confirmation phase}\right)$$

If

$$L_A(k) \geq T_n L_R(k)$$

the receiver declares ACCEPT otherwise it declares REJECT. Let $V_n(k)$ denote the transmitted symbol and $\hat{V}_n(k)$ denote the decoded symbol. As shown in [4] if the threshold T_n is chosen appropriately,

$$\text{Prob}\left(\hat{V}_n(k) \neq V_n(k)\,\middle|\,V_n(k) = \text{ACCEPT}\right) \leq 2^{-(1-\alpha_n)nH_n^A}$$
$$\text{Prob}\left(\hat{V}_n(k) \neq V_n(k)\,\middle|\,V_n(k) = \text{REJECT}\right) \leq 2^{-(1-\alpha_n)nH_n^R}$$

where

$$\lim_{n\to\infty} H_n^A = 0 \quad \text{and} \quad \lim_{n\to\infty} H_n^R = B \tag{$\star\star$}$$

Stopping Criterion *If $\hat{V}_n(k) =$ ACCEPT, communication stops and the receiver declares $\hat{W}_n(k)$ as the final decoded message; otherwise, the encoder and the decoder discard the output of the current epoch, and the encoder starts epoch $(k+1)$.*

Performance Analysis

A key feature of the above coding scheme is that asymptotically the number of retransmission go to zero.

Lemma on number of retransmissions *Let K_n denote the epoch when communication stops, i.e.,*

$$K_n = \{\inf k \in \mathbb{N} : \hat{V}_n(k) = \text{ACCEPT}\}$$

Then K_n is geometrically distributed with success probability going to one, i.e.,

$$\text{Prob}\,(K_n = k) = p_n(1-p_n)^{k-1}$$

where

$$\lim_{n\to\infty} p_n = 1$$

Consequently, for large values of n, only one transmission occurs, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\{K_n\} = 1 \tag{$\star\star\star$}$$

Proof of the Lemma For any $k \in \mathbb{N}$, we have

$$\begin{aligned}
\text{Prob}\,(K_n \geq k+1) &= \mathbb{E}\left\{\prod_{i=1}^{k}\{\hat{V}_n(i) = \text{REJECT}\}\right\} \\
&= \prod_{i=1}^{k} \text{Prob}\left(\hat{V}_n(i) = \text{REJECT}\right) \\
&= (1-p_n)^k
\end{aligned}$$

where

$$p_n = \text{Prob}\left(\hat{V}_n(1) = \text{ACCEPT}\right)$$

Hence, K_n has a geometric distribution.

Now, consider $1 - p_n = \text{Prob}\left(\hat{V}_n(1) = \text{REJECT}\right)$.

$$\begin{aligned}
0 \leq 1 - p_n &\leq \text{Prob}\left(\hat{W}_n(1) \neq W_n(1)\right) + \text{Prob}\left(\hat{V}_n(1) \neq \text{ACCEPT} \,\middle|\, \hat{W}_n(1) = W_n(1)\right) \\
&= \text{Prob}\left(\hat{W}_n(1) \neq W_n(1)\right) + \text{Prob}\left(\hat{V}_n(1) \neq V_n(1) \,\middle|\, V_n(1) = \text{ACCEPT}\right) \\
&\leq 2^{-\alpha_n n E_G(R/\alpha_n)} + 2^{-(1-\alpha_n) n H_n^A}
\end{aligned}$$

Taking limit $n \to \infty$ and using ($\star$) and ($\star\star$), we get that

$$\lim_{n\to\infty} 1 - p_n = 0$$

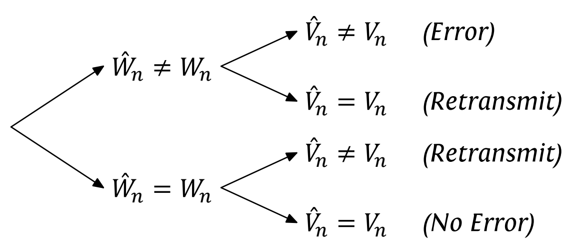
$\square$

An immediate consequence of the above Lemma is that the rate of the proposed coding scheme is R.

Rate of Coding Scheme

$$\text{Rate} = \lim_{n\to\infty} \frac{\log M_n}{\mathbb{E}\{K_n n\}} = \frac{R}{\lim_{n\to\infty} \mathbb{E}\{K_n\}} = R$$

Probability of error *The retransmission processes is a renewal process.*



Thus, the probability of error can be written as

$$P_e = \text{Prob}\left(\hat{W}_n \neq W_n\right) \text{Prob}\left(\hat{V}_n \neq V_n \,\middle|\, V_n = \text{REJECT}\right) + P_e\,\text{Prob}\left(\hat{V}_n = \text{REJECT}\right)$$

Rearranging, we get that

$$P_e = \frac{\text{Prob}\left(\hat{W}_n \neq W_n\right) \text{Prob}\left(\hat{V}_n \neq V_n \,\middle|\, V_n = \text{REJECT}\right)}{\text{Prob}\left(\hat{V}_n = \text{ACCEPT}\right)}$$

Error Exponents *The error exponent of the above scheme is*

$$E(R) = \lim_{n\to\infty} \frac{\log(1/P_e)}{\mathbb{E}\{K_n n\}} \geq \left(1 - \frac{R}{C}\right) B$$

Proof of Error Exponents We can simply the expression for error exponents as follows:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log(1/P_e)}{\mathbb{E}\{K_n n\}} &= \lim_{n\to\infty} \frac{\alpha_n n E_G(R/\alpha_n) + (1-\alpha_n) n H_n^R + \log p_n}{\mathbb{E}\{K_n n\}} \\
&= \lim_{n\to\infty} \frac{\alpha_n E_G(R/\alpha_n) + (1-\alpha_n) H_n^R}{\mathbb{E}\{K_n\}} \\
&\geq (1 - (R/C))B
\end{aligned}$$

where the first equality follows from the simplified expression for P_e, the second equality follows from the fact that $\lim_{n\to\infty} p_n = 1$, and the last inequality follows from ($\star$), ($\star\star$), and the choice of α_n. □

References

[1] M. V. Burnashev, "Data transmission over a discrete channel with feedback. Random transmission time," *Problemy peredachi informatsii*, vol. 12, no. 4, pp. 10-30, 1976.

[2] H. Yamamoto and K. Itoh, "Asymptotic performance of a modified Schalkwijk-Barron scheme for channels with noiseless feedback," *IEEE Trans. Inf. Theory*, vol. 25, no. 6, pp. 729–733, Nov. 1979.

[3] R. G. Gallager, *Information Theory and Reliable Communication*, Wiley, 1968.

[4] R. E. Blahut, "Hypothesis testing and information theory," *IEEE Trans. Inf. Theory*, Jul. 1974.