Mean-field games between teams

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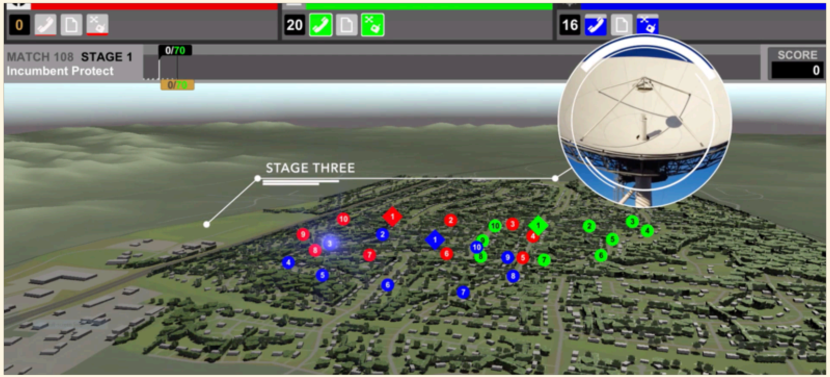
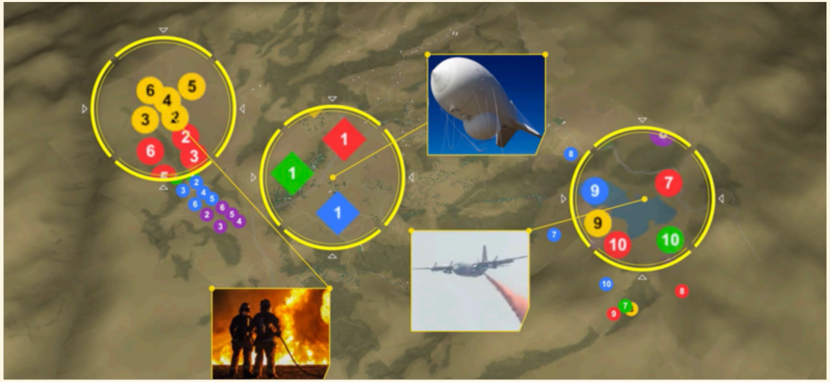
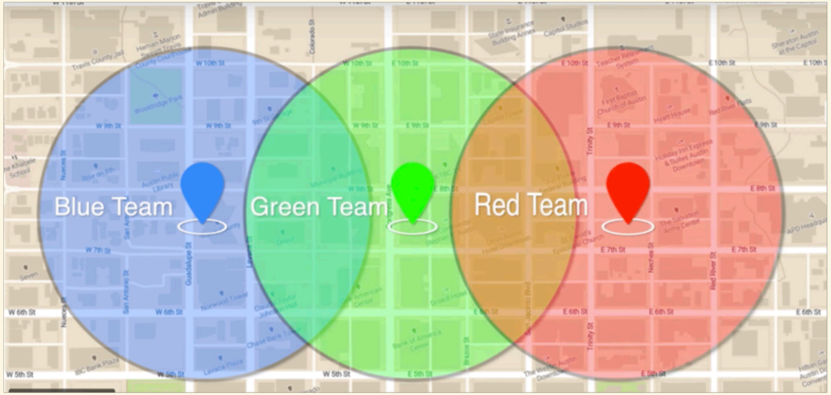
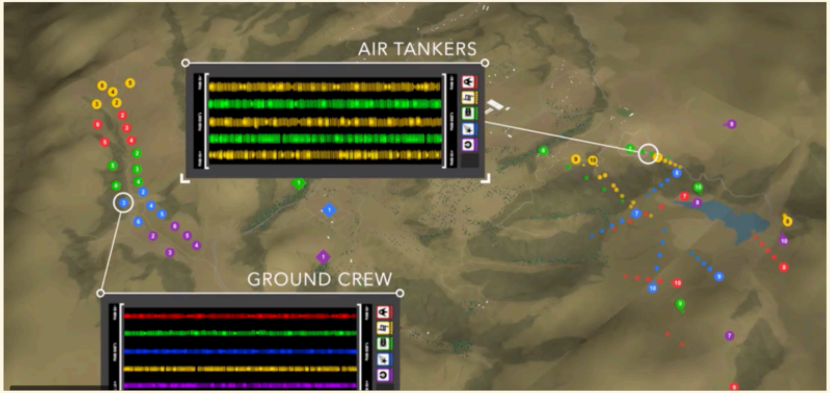
DARPA Spectrum Collaboration Challenge (SC2)





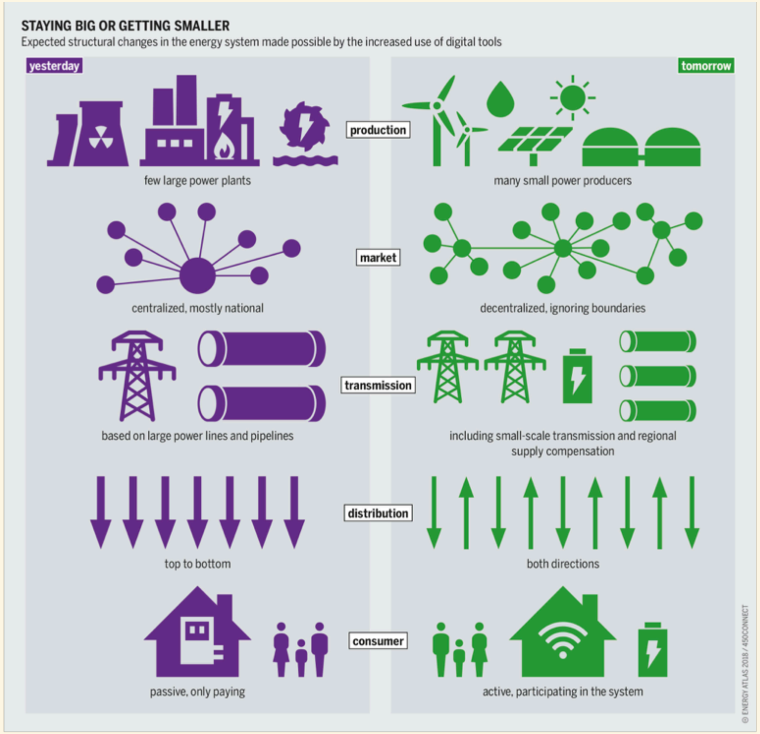




Source: https://www.spectrumcollaborationchallenge.com/media/

Mean-field games between teams–(Subramanian, Kumar, and Mahajan)

Multiple aggregators in energy markets



Source: https://en.wikipedia.org/wiki/Smart_grid#/media/File:Staying_big_or_getting_smaller.jpg

Salient Features

- Each "player" is a collection of multiple agents (i.e., a team).
- All agents in a team are exchangeable.
- Agents within a team only care about the utility of the team and don't have an individual utility.
- Teams are competing with one another.
- Information is decentralized and asymmetric.

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- Information is decentralized and asymmetric.

- What is the right solution concept for games between teams?
- How do find a solution in the dynamic case? Note that agents within a team as well as within the entire population have asymmetric information.

What are games between teams?

Preliminaries: Static Bayesian Games

- N players.
- Uncertainty lies in a probability space $(\Omega, \mathcal{F}, P)$.
- Player i receives a signal $t_i = t_i(\omega)$ and takes an action $a_i \in A_i$ using a STRATEGY $s_i \colon t_i \mapsto a_i$.
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A strategy $s = (s_1, \dots, s_n)$ is BNE if:

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- Note that in the definition of TGE, all agents in a team are allowed to deviate together!
- Therefore, game between teams are different than games in which subsets of agents have identical interests.

Exchangeable Markov processes and their mean-field projection

Notation: Mean-field of a sequence

Given a finite set $\mathcal{X}$ and a positive integer n, let Δ_n denote the set of probability distributions on $\mathcal{X}$ with denominator n.

Note that $|\Delta_n| \leqslant (n+1)^{|\mathcal{X}|}$.

EXAMPLE: Let $\mathcal{X} = \{0, 1\}$ and $n = 3$. Then

$$\Delta_n = \left\{ \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right\}.$$

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For a sequence $x \in \mathcal{X}^n$, let $\xi(x) \in \Delta_n$ denote the empericial distribution of x. We call $\xi(x)$ as the (empirical) MEAN-FIELD of a sequence.

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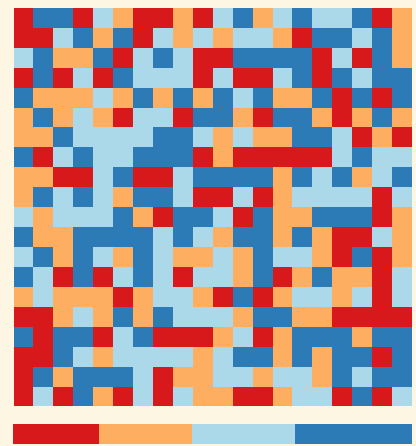

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EXAMPLE: Let $\mathcal{X} = \{0, 1\}$, $n = 3$, and p_{ijk} denotes $\mathbb{P}(X = (i, j, k))$. Then X is exchangeable if

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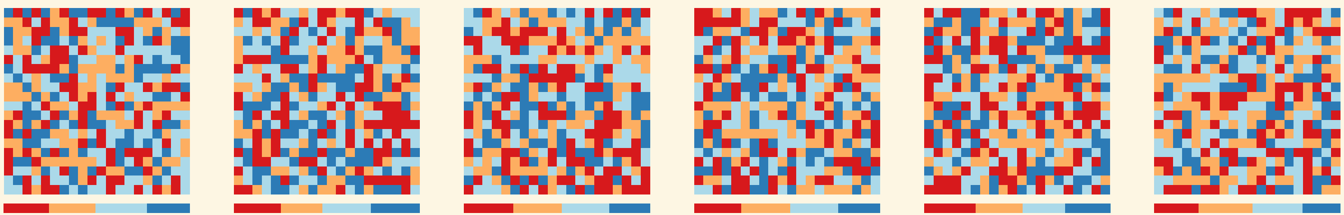
A Markov process $\{X_t\}_{t\geqslant 1}$, $X_t \in \mathcal{X}^n$ is called **exchangeable** if

- The initial state X_1 is exchageable
- The transition matrix is invariant under permutations, i.e., for any permultation σ,
  $$\mathbb{P}(X_{t+1} = \sigma y | X_t = \sigma x) = \mathbb{P}(X_{t+1} = y | X_t = x)$$

Note that if $\{X_t\}_{t\geqslant 1}$ is an exchangeable Markov process, then X_t is an exchangeable random vector.

Mean-field projection of exchageable Markov processes

Let $\{X_t\}_{t \geqslant 1}$, $X_t \in \mathcal{X}^n$ be an exchangeable Markov process. Its mean-field projection is the process $\{Z_t\}_{t \geqslant 1}$, where $Z_t = \xi(X_t)$.



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PROPOSITION:

- The mean-field projection is a Markov process, i.e.,
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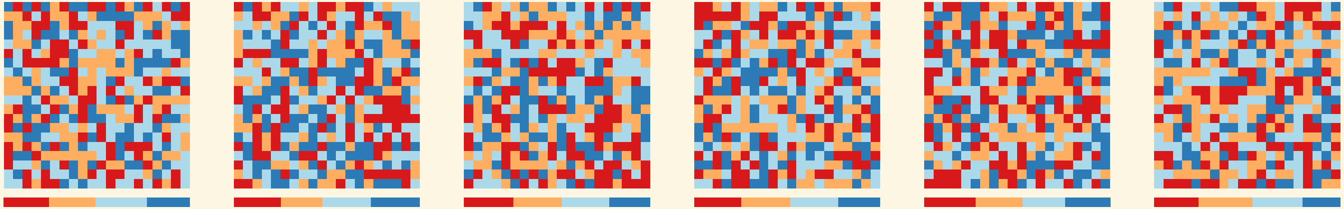

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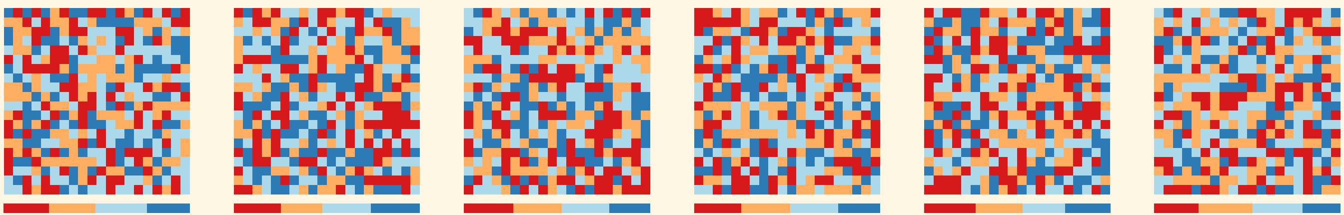
THEOREM: Conditioned on the mean-field, all feasible realizations are equally likely, i.e.,

$$\begin{aligned}\mathbb{P}(X_t \mid Z_{1:t}) &= \mathbb{P}(X_t \mid Z_t) \\ &= \mathbb{P}(\sigma X_t \mid Z_t) \\ &= \frac{\mathbb{1}\{\xi(X_t) = Z_t\}}{|\Xi(Z_t)|}.\end{aligned}$$



Mean-field games between teams

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- Similar interpretation holds of $U_t^{(k)}$ and U_t.

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- Similar interpretation holds of $U^{(k)}_t$ and U_t.
- $Z^{(k)}_t = \xi(X^{(k)}_t)$ is the (empirical) mean-field of team k.
- $Z_t = (Z^{(k)})_{k \in \mathcal{K}}$ is the mean-field profile of all teams.

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- $Z_t = (Z^{(k)})_{k \in \mathcal{K}}$ is the mean-field profile of all teams.

- Initial states are independent across all agents

$$\mathbb{P}(X_1 = (x^i_1)_{i \in \mathcal{N}}) = \prod_{k \in \mathcal{K}} \prod_{i \in \mathcal{N}^{(k)}} P^{(k)}_0(x^i_1)$$

System Model: States, actions, and dynamics

- K teams. Team k has $N^{(k)}$ exchangeable agents indexed by the set $\mathcal{N}^{(k)}$.
- $X^i_t \in \mathcal{X}^{(k)}$: State of agent i in team k.
- $U^i_t \in \mathcal{U}^{(k)}$: Action of agent i in team k.
- $X^{(k)}_t = (X^i_t)_{i \in \mathcal{N}^{(k)}}$ is the state profile of team k.
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- The population state evolves in a controlled Markov manner.
- Agents within a team are exchangeable and, therefore, are only coupled through the mean-field.
$$\mathbb{P}(X_{t+1} \mid X_{1:t}, U_{1:t}) = \prod_{k \in \mathcal{K}} \prod_{i \in \mathcal{N}^{(k)}} P^{(k)}(X^i_{t+1} \mid X^i_t, U^i_t, Z_t)$$

System Model: Information structure and cost

- Mean-field sharing information structure

$$I^i_t = \{X^i_t, Z_t\}.$$

Note that agents have asymmetric information (also called non-classical information structure in the team theory literature).

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$$U^i_t \sim g^{(k)}_t(X^i_t, Z_t)$$
where $g^{(k)} = \left(g^{(k)}_1, \dots, g^{(k)}_T\right)$ is called the strategy of team k.

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  $$C^{(k)} = \frac{1}{N^{(k)}} \sum_{i \in \mathcal{N}^{(k)}} c^{(k)}_t(X^i_t, U^i_t, Z_t).$$
- Cost incurred by team k:
  $$J^{(k)}(g^{(k)}, g^{(-k)}) = \mathbb{E}^{(g^{(k)}, g^{(-k)})}\Big[\sum_{t=1}^{T} C^{(k)}_t\Big]$$

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GAME 1: Identify a TGE $g = (g^{(k)})_{k \in \mathcal{K}}$ of the game between teams formulated above.

Such a (mixed-strategy) equilibrium always exists because each “player” has a finite number of strategies.

Conceptual difficulties

- The game formulated above is a dynamic game with asymmetric information. So, the TGE must satisfy **sequential rationality** and **consistency**. Such equilibrium are call Perfect Bayesian Equilibrium.
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- However, there is no general methodology to identify PBE in dynamic games.

- In recent years, there are some results that propose a common information based refinement of Nash Equilibrium for dynamic games with asymmetric information.
- These may be viewed as the extension of the common information approach [Nayyar, Mahajan, Teneketzis 2013] for teams with non-classical information to games with asymmetric information.

Common information based refinements of Nash equilibrium

- [Nayyar, Gupta, Langbort, Başar 2014] propose a common information based refinement of Markov perfect equilibrium for a subclass of dynamic games with asymmetric information.
- The key assumption is that the common information based beliefs are strategy independent. This may be viewed as games where there is **no signalling effect**.
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- We effectively show that mean-field games have no signalling effect.
- Following [NGLB 2014], we propose a common information based MPE for our model.

Preliminary results

For any strategy $g = (g^{(1)}, \dots, g^{(K)})$ and any realization $z_{1:T}$ of the mean-field $Z_{1:T}$, define the following partial functions, which we call prescriptions:

$$\gamma_t^{(k)} = g_t^{(k)}(\cdot, z_t), \quad \forall k \in \mathcal{K}.$$

- When the realization z_t of the mean-field is given, $\gamma_t^{(k)}$ is a function from $\mathcal{X}^{(k)}$ to $\mathcal{U}_t^{(k)}$.
- When the mean-field Z_t is a random variable, $\gamma_t^{(k)}$ is a random function from $\mathcal{X}^{(k)}$ to $\mathcal{U}^{(k)}$. We denote this by $\Gamma^{(k)}$.

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LEMMA: The mean-field process $\{Z_t\}_{t\geqslant 1}$ is a controlled Markov process that evolves conditionally independently across teams:

$$\mathbb{P}(Z_{t+1} \mid Z_{1:t}, \Gamma_{1:t}) = \prod_{k\in\mathcal{K}} Q^{(k)}(Z_{t+1}^{(k)} \mid Z_t, \Gamma_t^{(k)})$$

where $Q^{(k)}(z_{t+1}^{(k)} \mid z_t, \gamma_t^{(k)})$ can be computed by picking any $x_{t+1}^{(k)} \in \Xi(z_{t+1}^{(k)})$ and $x_t \in \Xi(z_t)$ and setting

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LEMMA: The mean-field is a sufficient statistic to predict the population state

$$\mathbb{P}(X_t \mid Z_{1:t}, \Gamma_{1:t}) = \prod_{k \in \mathcal{K}} \frac{\mathbb{1}\{\xi(X_t^{(k)}) = Z_t^{(k)}\}}{|\Xi^{(k)}(Z_t^{(k)})|}$$

Common information based Markov perfect equilibrium (CIB-MPE)

A virtual game between virtual players

- Consider a virtual Markov game between K virtual players with symmetric information.
- The state is $Z_t = (Z_t^{(k)})_{k \in \mathcal{K}}$.
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- Distribution of initial state
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Note that the cost does not depend on the action of other players!

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GAME 2: Identify a MPE $\psi = (\psi^{(k)})_{k \in \mathcal{K}}$ of the virtual game formulated above.

Such a MPE can be identified using dynamic programming.

Equivalence between Game 1 and Game 2

THEOREM (GAME 1 TO GAME 2)

Let $g = (g^{(k)})_{k \in \mathcal{K}}$ be a TGE of Game 1. Define a strategy $\psi = (\psi^{(k)})_{k \in \mathcal{K}}$ for Game 2 as follows:

$$\psi_t^{(k)}(z) = g_t^{(k)}(\cdot, z), \quad \forall z.$$

Then $\psi = (\psi^{(k)})_{k \in \mathcal{K}}$ is a MPE of Game 2.

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THEOREM (GAME 2 TO GAME 1)

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$$g_t^{(k)}(x, z) = \psi_t^{(k)}(z)(x), \quad \forall x in \mathcal{X}^{(k)}, \forall z.$$

Then $g = (g^{(k)})_{k \in \mathcal{K}}$ is a TGE of Game 1.

We call this the common information based MPE of Game 1.

Conclusion

SOLUTION IDEA

- Formulate a Markov game between virtual players.
- The virtual players represent the entire team and decide the prescription for all members of the team.
- Find an MPE of the virtual game using DP
- Any MPE of the virtual game is a TGE of the game between teams (called CIB-MPE).
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FUTURE WORK

- Zero-sum games
- LQG models
- Mean-field limits for large populations in each team ($N^{(k)} \to \infty$) and also for large number of teams ($K \to \infty$).
- ...