Optimal decentralized stochastic control: The designer and common information approaches

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Joint work: Ashutosh Nayyar (UIUC/UC Berkeley) and Demosthenis Teneketzis (Univ of Michigan)

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Common theme:

multi-stage multi-agent decision making under uncertainty

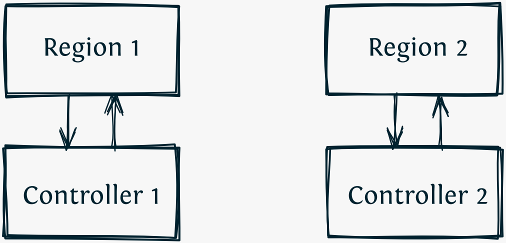
Interconnected Power Systems

Interconnected Power Systems

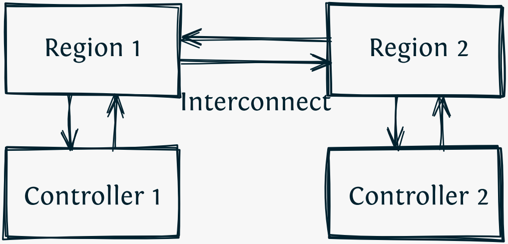
Region 1

Region 2

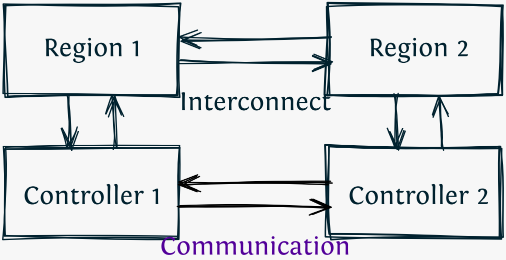
Interconnected Power Systems



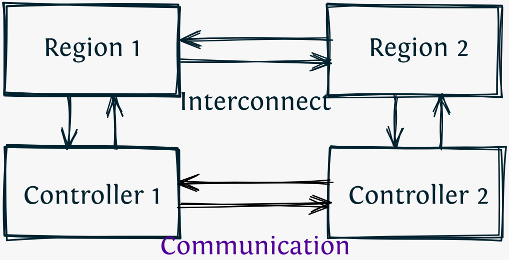
Interconnected Power Systems



Interconnected Power Systems



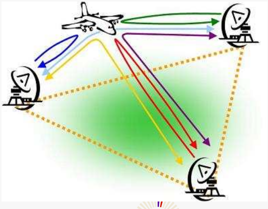
Interconnected Power Systems



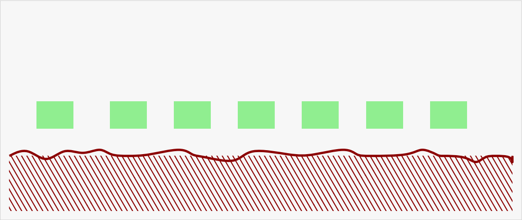
Challenges

- How to **coordinate**?
- When, what, and how to **communicate**?

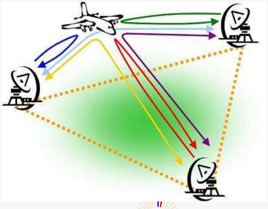
Sensor and Surveillance Networks



Sensor and Surveillance Networks



Limited resources



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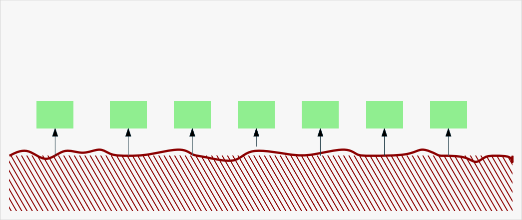
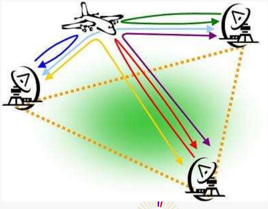
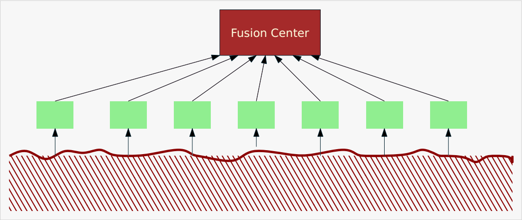
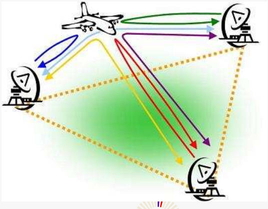


Limited resources   Noisy observations



Sensor and Surveillance Networks

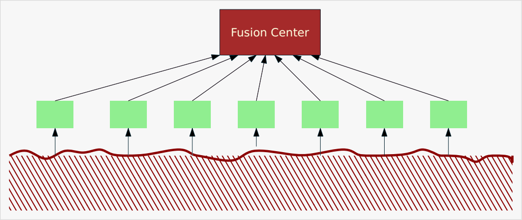


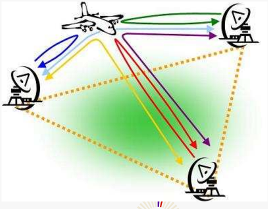


Limited resources Noisy observations

Communication

Sensor and Surveillance Networks



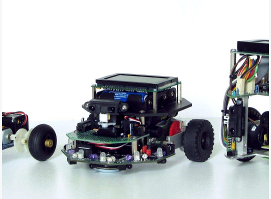


Limited resources Noisy observations
Communication

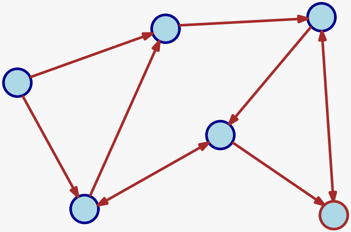
Challenges

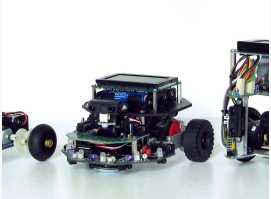
- Real-time communication
- Scheduling measurements and communication
- Detect node failures

Networked Control Systems



Networked Control Systems





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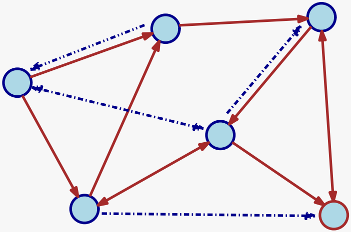
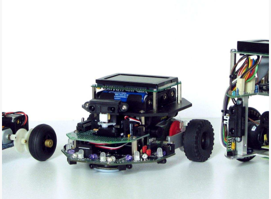
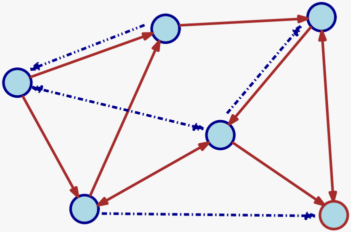
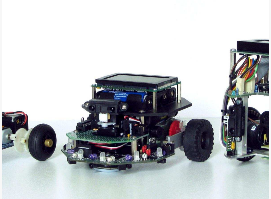




Networked Control Systems

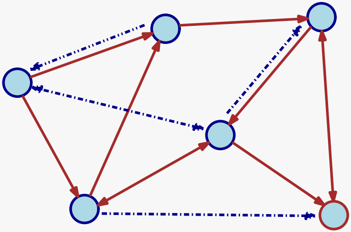


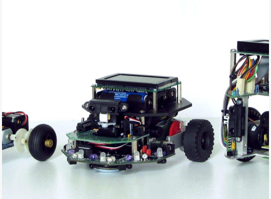


Challenges

- Control and communication over networks
  (internet $\Rightarrow$ delay, wireless $\Longrightarrow$ losses)

Networked Control Systems

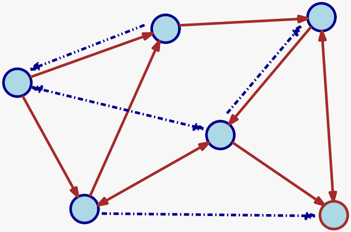


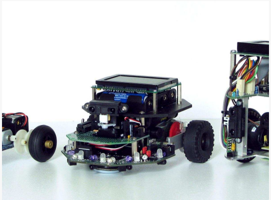


Challenges

- Control and communication over networks (internet $\Rightarrow$ delay, wireless $\Longrightarrow$ losses)
- Distributed estimation

Networked Control Systems





Challenges

- Control and communication over networks
  (internet $\Rightarrow$ delay, wireless $\Longrightarrow$ losses)
- Distributed estimation
- Distributed learning

Salient features in decentralized decision making

Multiple decision makers

Decisions made by multiple controllers in a stochastic environment

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Decisions made by multiple controllers in a stochastic environment

Coordination issues

All controllers must coordinate to achieve a system-wide objective

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Controllers can communicate either directly or indirectly

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All controllers must coordinate to achieve a system-wide objective

Communication issues

Controllers can communicate either directly or indirectly

Robustness

System model may not be completely known

Classification of decentralized systems

Controllers/agents are coupled in two ways:

1. Coupling due to cost/utility
2. Coupling due to dynamics

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Decentralized systems may be classified according to:

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Decentralized systems may be classified according to:

1. Objective

   Team vs Games

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Decentralized systems may be classified according to:

1. Objective
   Team vs Games
2. Dynamics
   Static vs Dynamic

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This talk will focus on Dynamic Teams

Classification of decentralized systems

Controllers/agents are coupled in two ways:

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Decentralized systems may be classified according to:

1. Objective

   Team vs Games

2. Dynamics

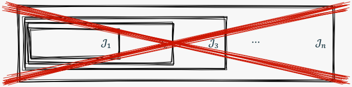
   Static vs Dynamic

This talk will focus on Dynamic Teams

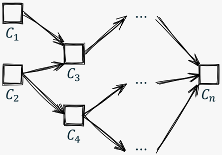
- Studied in economics and systems and control since the mid 50s.
- Unlike games, agents have no incentive to cheat.
- Instead of equilibrium, we seek globally optimal strategies.

Key features of decentralized teams

- Non-Classical information structure



- Fixed **partial** order in which agents act



Why is decentralized stochastic control difficult?

An example of centralized static optimization

$$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$$

ω_1	ω_2	ω_3	ω_4

An example of centralized static optimization

$$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2

An example of centralized static optimization

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$$u = g(x) \in \{1, 2, 3\}$$

An example of centralized static optimization

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	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2

$$u = g(x) \in \{1, 2, 3\}$$

$c(\omega, u)$

	ω_1	ω_2	ω_3	ω_4
$u = 1$	•	•	•	•
$u = 2$	•	•	•	•
$u = 3$	•	•	•	•

$$\boxed{J(g) = \mathbb{E}^g[c(\omega, u)]}$$

An example of centralized static optimization

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$u = 3$	•	•	•	•

$$J(g) = \mathbb{E}^g[c(\omega, u)]$$

Brute force search $\quad \min_g J(g), \quad |g| = |\mathcal{U}|^{|\mathcal{X}|} = 9$ possibilities.

An example of centralized static optimization

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Brute force search $\min_g J(g), \quad |g| = |\mathcal{U}|^{|\mathcal{X}|} = 9$ possibilities.

Systematic search $3 + 3 = 6$ possibilities

$$u_1 = g(1) \qquad\qquad u_2 = g(2)$$

$$\min_{u_1} \mathbb{E}[c(\omega, u_1) \mid x = 1] \qquad\qquad \min_{u_2} \mathbb{E}[c(\omega, u_2) \mid x = 2]$$

An example of centralized static optimization

$P = [\ \bullet\ \ \bullet\ \ \bullet\ \ \bullet\ ]$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2

$c(\omega, u)$

	ω_1	ω_2	ω_3	ω_4
$u = 1$	•	•	•	•
$u = 2$	•	•	•	•
$u = 3$	•	•	•	•

$u = g(x) \in \{1, 2, 3\}$

$J(g) = \mathbb{E}^g[c(\omega, u)]$

(functional opt.)

Brute force search $\min_g J(g), \quad |g| = |\mathcal{U}|^{|\mathcal{X}|} = 9$ possibilities.

Systematic search $3 + 3 = 6$ possibilities **(parametric opt.)**

$u_1 = g(1)$ $\qquad$ $u_2 = g(2)$

$\min_{u_1} \mathbb{E}[c(\omega, u_1) \mid x = 1]$ $\qquad$ $\min_{u_2} \mathbb{E}[c(\omega, u_2) \mid x = 2]$

An example of decentralized static optimization

$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$

ω_1	ω_2	ω_3	ω_4

An example of decentralized static optimization

$$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
$y =$	2	1	1	2

An example of decentralized static optimization

$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
$y =$	2	1	1	2

$u = g(x) \in \{1, 2, 3\} \quad v = h(y) \in \{1, 2\}$

An example of decentralized static optimization

$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
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$u = g(x) \in \{1,2,3\} \quad v = h(y) \in \{1,2\}$

$c(\omega, u, v)$

	ω_1		ω_2		ω_3		ω_4	
$u = 1$	•	•	•	•	•	•	•	•
$u = 2$	•	•	•	•	•	•	•	•
$u = 3$	•	•	•	•	•	•	•	•
$v =$	1	2	1	2	1	2	1	2

$$\boxed{J(g,h) = \mathbb{E}^{g,h}[c(\omega,u,v)]}$$

An example of decentralized static optimization

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	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
$y =$	2	1	1	2

$u = g(x) \in \{1, 2, 3\} \quad v = h(y) \in \{1, 2\}$

$c(\omega, u, v)$

$$\begin{array}{c|cc|cc|cc|cc|}
 & \omega_1 & & \omega_2 & & \omega_3 & & \omega_4 & \\
u = 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
u = 2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
u = 3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
v = & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2
\end{array}$$

$J(g, h) = \mathbb{E}^{g,h}[c(\omega, u, v)]$

Brute force search $\quad \min_{g,h} J(g, h), \quad |g| = |\mathcal{U}|^{|\mathcal{X}|}, |h| = |\mathcal{V}|^{|\mathcal{Y}|},$

$9 \times 4 = 36$ possibilities.

An example of decentralized static optimization

$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
$y =$	2	1	1	2

$c(\omega, u, v)$

	ω_1		ω_2		ω_3		ω_4	
$u = 1$	•	•	•	•	•	•	•	•
$u = 2$	•	•	•	•	•	•	•	•
$u = 3$	•	•	•	•	•	•	•	•
$v =$	1	2	1	2	1	2	1	2

$u = g(x) \in \{1, 2, 3\} \quad v = h(y) \in \{1, 2\}$

$$J(g, h) = \mathbb{E}^{g,h}[c(\omega, u, v)]$$

Brute force search $\quad \min_{g,h} J(g), \quad |g| = |\mathcal{U}|^{|\mathcal{X}|},\ |h| = |\mathcal{V}|^{|\mathcal{Y}|},$

$9 \times 4 = 36$ possibilities.

> For one controller/agent to choose an optimal action, it must second guess the other controller's/agent's policy

An example of decentralized static optimization

$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
$y =$	2	1	1	2

$c(\omega, u, v)$

	ω_1		ω_2		ω_3		ω_4	
$u = 1$	•	•	•	•	•	•	•	•
$u = 2$	•	•	•	•	•	•	•	•
$u = 3$	•	•	•	•	•	•	•	•
$v =$	1	2	1	2	1	2	1	2

$u = g(x) \in \{1, 2, 3\} \quad v = h(y) \in \{1, 2\}$

$$J(g, h) = \mathbb{E}^{g,h}[c(\omega, u, v)]$$

Orthogonal search

1. Suppose h is fixed: $\min\limits_{u_i} \mathbb{E}^{h}[c(\omega, u_i, v) \mid x = i], \quad i = 1, 2, 3.$
2. Suppose g is fixed: $\min\limits_{v_j} \mathbb{E}^{g}[c(\omega, u, v_j) \mid y = j], \quad j = 1, 2.$

An example of decentralized static optimization

$P = [\ \bullet \quad \bullet \quad \bullet \quad \bullet\ ]$

	ω_1	ω_2	ω_3	ω_4
$x =$	1	1	2	2
$y =$	2	1	1	2

$c(\omega, u, v)$

$$\begin{array}{c|cc|cc|cc|cc|}
 & \multicolumn{2}{c}{\omega_1} & \multicolumn{2}{c}{\omega_2} & \multicolumn{2}{c}{\omega_3} & \multicolumn{2}{c}{\omega_4} \\
u = 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
u = 2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
u = 3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
v = & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2
\end{array}$$

$u = g(x) \in \{1, 2, 3\} \quad v = h(y) \in \{1, 2\}$

$J(g, h) = \mathbb{E}^{g,h}[c(\omega, u, v)]$

Orthogonal search yields person-by-person opt strategy

1. Suppose h is fixed: $\min_{u_i} \mathbb{E}^{h}[c(\omega, u_i, v) \mid x = i], \quad i = 1, 2, 3.$
2. Suppose g is fixed: $\min_{v_j} \mathbb{E}^{g}[c(\omega, u, v_j) \mid y = j], \quad j = 1, 2.$

To find globally optimal strategies, in general, we cannot do better than brute force search

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4
ω_5	ω_6	ω_7	ω_8

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

$u_1 = g_1(y_1) \in \{1, 2\}$

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

$u_1 = g_1(y_1) \in \{1, 2\}$

$u_1 = 1 \implies y_2 =$	1	1	2	2
$u_1 = 1 \implies y_2 =$	1	1	2	2

$u_1 = 2 \implies y_2 =$	1	2	2	1
$u_1 = 2 \implies y_2 =$	1	2	2	1

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

$u_1 = g_1(y_1) \in \{1, 2\}$

$u_1{=}1 \implies y_2{=}$	1	1	2	2
$u_1{=}1 \implies y_2{=}$	1	1	2	2
$u_1{=}2 \implies y_2{=}$	1	2	2	1
$u_1{=}2 \implies y_2{=}$	1	2	2	1

$u_2 = g_2(y_1, y_2, u_1) \in \{1, 2\}$

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

$u_1 = g_1(y_1) \in \{1, 2\}$

$u_1{=}1 \implies y_2{=}$	1	1	2	2
$u_1{=}1 \implies y_2{=}$	1	1	2	2
$u_1{=}2 \implies y_2{=}$	1	2	2	1
$u_1{=}2 \implies y_2{=}$	1	2	2	1

$u_2 = g_2(y_1, y_2, u_1) \in \{1, 2\}$

$c_1(\omega, u_1) + c_2(\omega, u_2)$

$$J(g_1, g_2) = \mathbb{E}^{g_1, g_2}[c_1(\omega, u_1) + c_2(\omega, u_2)]$$

An example of centralized multi-stage optimization

	ω_1	ω_2	ω_3	ω_4	$y_1 = 1$	$u_1 = g_1(y_1) \in \{1, 2\}$
	ω_5	ω_6	ω_7	ω_8	$y_1 = 2$	$\boxed{d_1 = \{y_1\}}$
$u_1{=}1 \implies y_2{=}$	1	1	2	2		$u_2 = g_2(y_1, y_2, u_1) \in \{1, 2\}$
$u_1{=}1 \implies y_2{=}$	1	1	2	2		$\boxed{d_2 = \{y_1, y_2, u_1\}}$
$u_1{=}2 \implies y_2{=}$	1	2	2	1		
$u_1{=}2 \implies y_2{=}$	1	2	2	1		$c_1(\omega, u_1) + c_2(\omega, u_2)$

$$J(g_1, g_2) = \mathbb{E}^{g_1, g_2}[c_1(\omega, u_1) + c_2(\omega, u_2)]$$

An example of centralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

$u_1{=}1 \implies y_2{=}$	1	1	2	2
$u_1{=}1 \implies y_2{=}$	1	1	2	2

$u_1{=}2 \implies y_2{=}$	1	2	2	1
$u_1{=}2 \implies y_2{=}$	1	2	2	1

$u_1 = g_1(y_1) \in \{1, 2\}$

$\boxed{d_1 = \{y_1\}}$

$u_2 = g_2(y_1, y_2, u_1) \in \{1, 2\}$

$\boxed{d_2 = \{y_1, y_2, u_1\}}$

$c_1(\omega, u_1) + c_2(\omega, u_2)$

$$J(g_1, g_2) = \mathbb{E}^{g_1, g_2}[c_1(\omega, u_1) + c_2(\omega, u_2)]$$

Critical Assumption: Centralized information $\boxed{d_1 \subseteq d_2}$

Solution approach for centralized multi-stage optimization

Brute force search $\displaystyle \min_{g_1, g_2} J(g_1, g_2).$

$$|g_1| = |\mathcal{U}_1|^{|\mathcal{Y}_1|}, \quad |g_2| = |\mathcal{U}_2|^{|\mathcal{Y}_1| \times |\mathcal{Y}_2| \times |\mathcal{U}_1|}. \quad 2^2 \times 2^8 = 1024 \text{ possiblities.}$$

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Dynamic programming decomposition

$$V_2(d_2) = \min_{u_2} \mathbb{E}[c_2(\omega, u_2) \mid d_2, u_2]$$

$$V_1(d_1) = \min_{u_1} \mathbb{E}[c_1(\omega, u_1) + V_2(d_2) \mid d_1, u_1]$$

Solution approach for centralized multi-stage optimization

Brute force search $\min_{g_1, g_2} J(g_1, g_2).$ **(functional opt.)**

$$|g_1| = |\mathcal{U}_1|^{|\mathcal{Y}_1|}, \quad |g_2| = |\mathcal{U}_2|^{|\mathcal{Y}_1| \times |\mathcal{Y}_2| \times |\mathcal{U}_1|}. \quad 2^2 \times 2^8 = 1024 \text{ possiblities.}$$

Dynamic programming decomposition **(parametric opt.)**

$$V_2(d_2) = \min_{u_2} \mathbb{E}[c_2(\omega, u_2) \mid d_2, u_2]$$

$$V_1(d_1) = \min_{u_1} \mathbb{E}[c_1(\omega, u_1) + V_2(d_2) \mid d_1, u_1]$$

Solution approach for centralized multi-stage optimization

Brute force search $\min_{g_1, g_2} J(g_1, g_2).$ **(functional opt.)**

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- Step 1 works because $\mathbb{P}(\omega \mid d_2)$ **does not** depend on g_1.
- Step 2 works because $\mathbb{P}(d_2 \mid d_1, u_1)$ **does not** depend on g_1.

Solution approach for centralized multi-stage optimization

Brute force search $\min_{g_1, g_2} J(g_1, g_2).$ **(functional opt.)**

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Dynamic programming decomposition **(parametric opt.)**

$$V_2(d_2) = \min_{u_2} \mathbb{E}[c_2(\omega, u_2) \mid d_2, u_2]$$

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- Step 1 works because $\mathbb{P}(\omega \mid d_2)$ **does not** depend on g_1.
- Step 2 works because $\mathbb{P}(d_2 \mid d_1, u_1)$ **does not** depend on g_1.
- $\boxed{\text{Both steps work because } d_1 \subseteq d_2}$

An example of decentralized multi-stage optimization

ω_1	ω_2	ω_3	ω_4	$y_1 = 1$
ω_5	ω_6	ω_7	ω_8	$y_1 = 2$

$$u_1 = g_1(y_1) \in \{1, 2\}$$

$$\boxed{d_1 = \{y_1\}}$$

$u_1{=}1 \implies y_2{=}$	1	1	2	2
$u_1{=}1 \implies y_2{=}$	1	1	2	2

$$u_2 = g_2(y_2) \in \{1, 2\}$$

$$\boxed{d_2 = \{y_2\}}$$

$u_1{=}2 \implies y_2{=}$	1	2	2	1
$u_1{=}2 \implies y_2{=}$	1	2	2	1

$$c_1(\omega, u_1) + c_2(\omega, u_2)$$

$$J(g_1, g_2) = \mathbb{E}^{g_1, g_2}[c_1(\omega, u_1) + c_2(\omega, u_2)]$$

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ω_1	ω_2	ω_3	ω_4	$y_1 = 1$	$u_1 = g_1(y_1) \in \{1, 2\}$
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$u_1{=}1 \implies y_2{=}$	1	1	2	2	$\boxed{d_2 = \{y_2\}}$
$u_1{=}2 \implies y_2{=}$	1	2	2	1	
$u_1{=}2 \implies y_2{=}$	1	2	2	1	$c_1(\omega, u_1) + c_2(\omega, u_2)$

$$J(g_1, g_2) = \mathbb{E}^{g_1, g_2}[c_1(\omega, u_1) + c_2(\omega, u_2)]$$

Critical Assumption: Decentralized information $\boxed{d_1 \not\subseteq d_2}$

Can we do better than brute force search?

Usual Dynamic programming does not work?

$$V_2(d_2) \stackrel{?}{=} \min_{u_2} \mathbb{E}^{g_1}[c_2(\omega, u_2) \mid d_2, u_2]$$

$$V_1(d_1) \stackrel{?}{=} \min_{u_1} \mathbb{E}^{g_1}[c_1(\omega, u_1) + V_2(d_2) \mid d_1, u_1]$$

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$$V_2(d_2) \stackrel{?}{=} \min_{u_2} \mathbb{E}^{g_1}[c_2(\omega, u_2) \mid d_2, u_2]$$

$$V_1(d_1) \stackrel{?}{=} \min_{u_1} \mathbb{E}^{g_1}[c_1(\omega, u_1) + V_2(d_2) \mid d_1, u_1]$$

A sequential decomposition is possible (Witsenhausen, 1973)

Define $\pi_t = \mathbb{P}(\omega \mid g_{1:t-1})$.

$$V_t(\pi_t) = \min_{g_t} \; \mathbb{E}^{g_t}[c_t(\omega, u_t) + V_{t+1}(\pi_{t+1}) \mid \pi_t]$$

But, the worst case complexity remains the same.

Finding optimal strategies

Can we obtain a systematic approach to find optimal strategies that does better than brute force search?

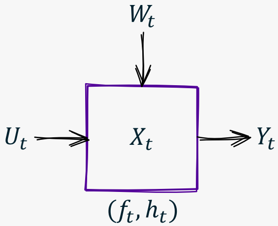
Finding optimal strategies

Can we obtain a systematic approach to find optimal strategies that does better than brute force search?

- Designer approach
- Common information approach

Designer approach

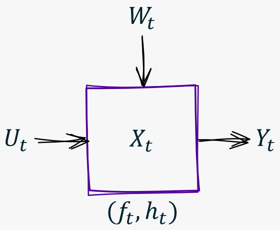
Block notation



$$Y_t = h_t(X_t, U_t, W_t)$$

$$X_{t+1} = f_t(X_t, U_t, W_t)$$

Block notation



$$Y_t = h_t(X_t, U_t, W_t)$$
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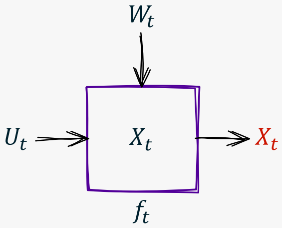
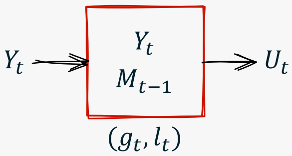


$$X_{t+1} = f_t(X_t, U_t, W_t)$$

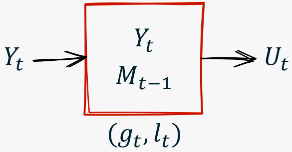
Block notation



$$U_t = g_t(Y_t, M_{t-1})$$

$$M_t = l_t(Y_t, M_{t-1})$$

Block notation



$$U_t = g_t(Y_t, M_{t-1})$$
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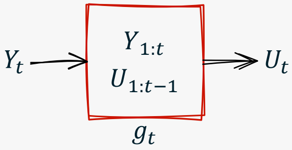
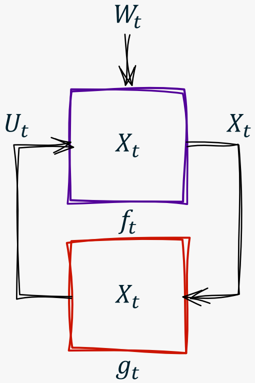
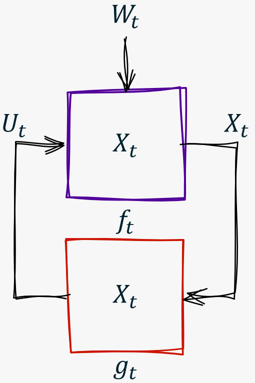


$$U_t = g_t(Y_{1:t}, U_{1:t-1})$$

Another look at dynamic consistency



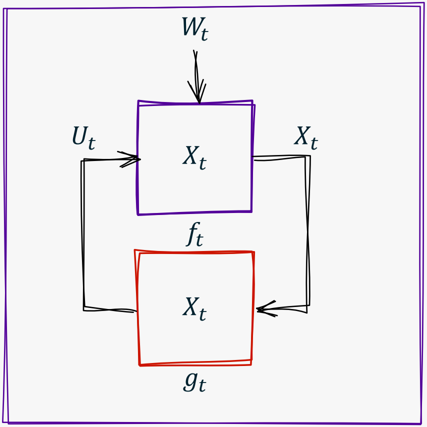
Another look at dynamic consistency



Define: $\pi_t = \mathbb{P}(X_t | g_{1:t-1})$. Then,

$$V_t(\pi_t) = \min_{g_t} \mathbb{E}[c(X_t, U_t) + V_{t+1}(L^g \pi_t)]$$

An alternative derivation



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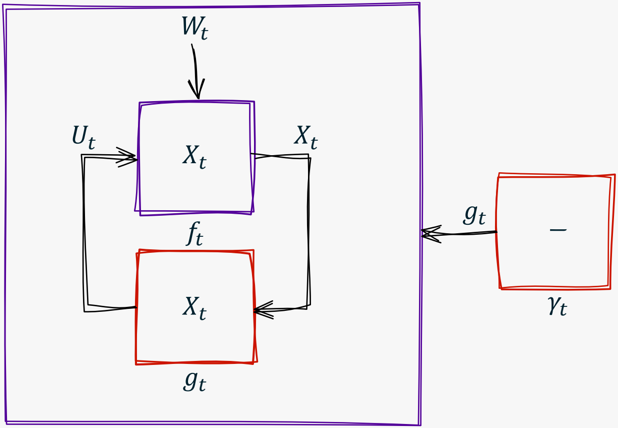
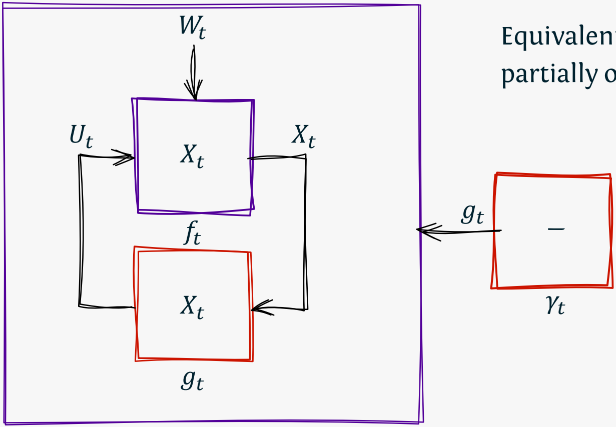
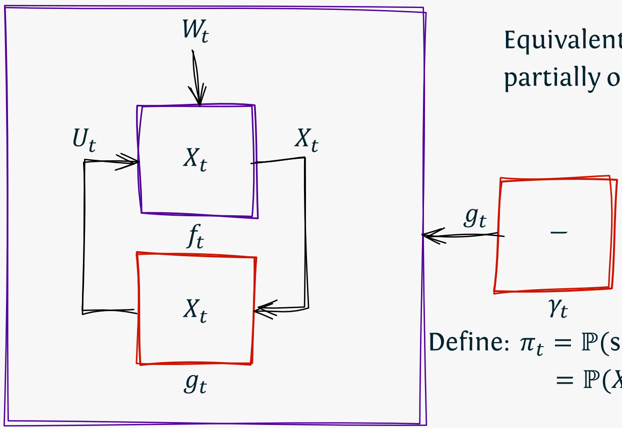


An alternative derivation



Equivalent to a **centralized** partially observed system

An alternative derivation

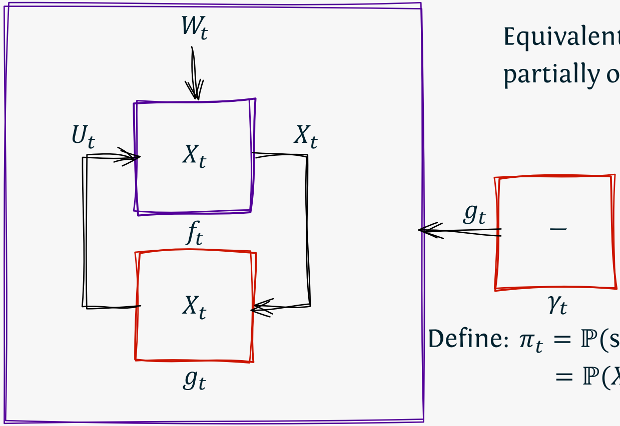


Equivalent to a **centralized** partially observed system

Define: $\pi_t = \mathbb{P}(\text{state} \mid \text{history of data})$

$= \mathbb{P}(X_t \mid g_{1:t-1})$

An alternative derivation



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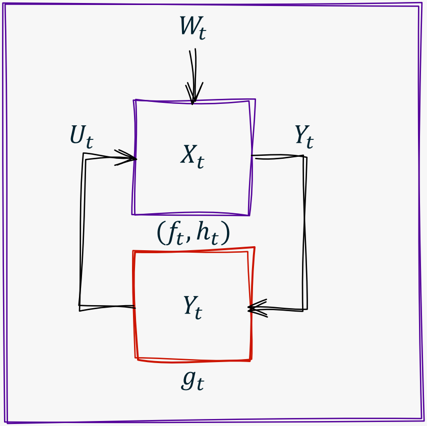
$$V_t(\pi_t) = \min_{g_t} \mathbb{E}[c(X_t, U_t) + V_{t+1}(L^g \pi_t)]$$

Generalization to decentralized system: finite memory controller



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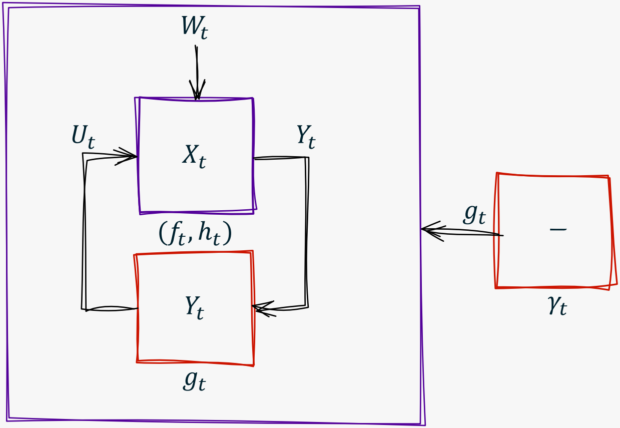
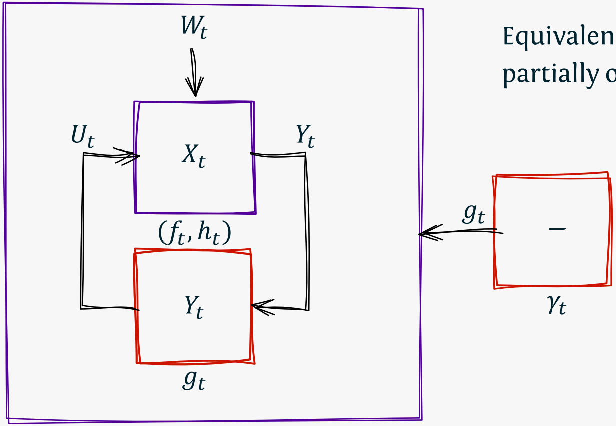


Generalization to decentralized system: finite memory controller



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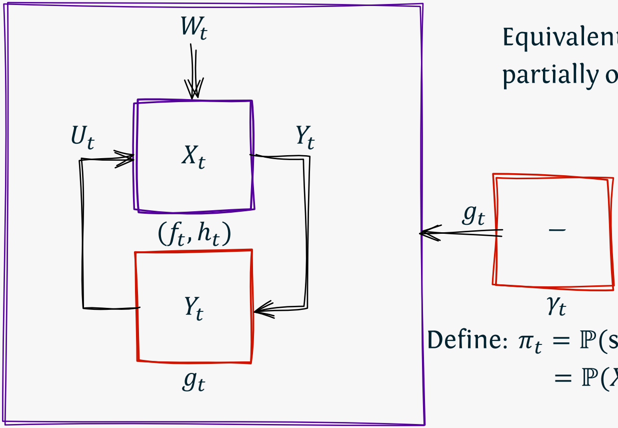

Equivalent to a **centralized** partially observed system

Define: $\pi_t = \mathbb{P}(\text{state} \mid \text{history of data})$
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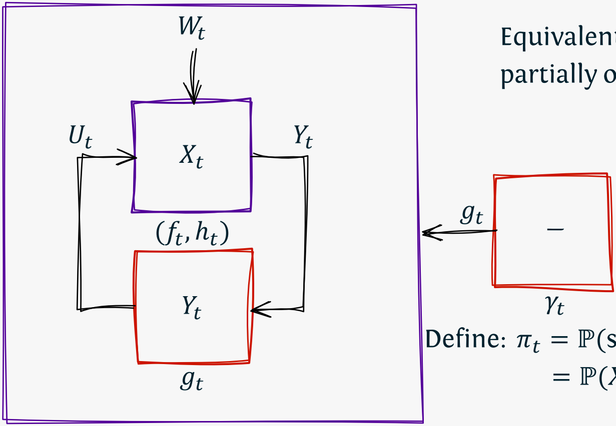
Generalization to decentralized system: finite memory controller



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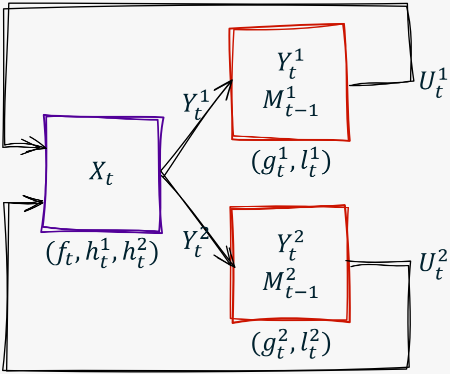
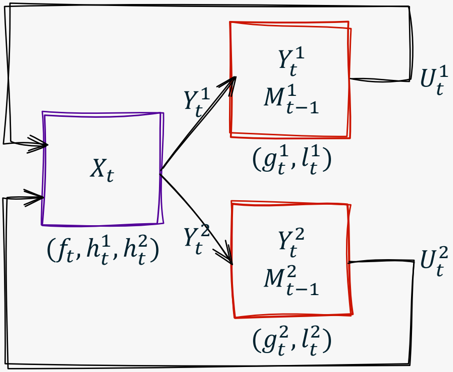
Same idea works for arbitrary systems with finite memory controllers



$$\min_{g^{1,2}_{1:T}, l^{1,2}_{1:T}} \mathbb{E}\Big[\sum_{t=1}^{T} c(X_t, U^1_t, U^2_t)\Big]$$

Same idea works for arbitrary systems with finite memory controllers



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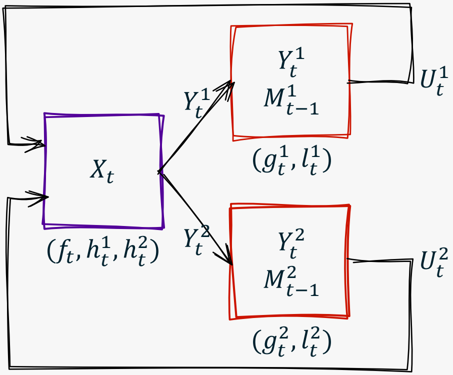
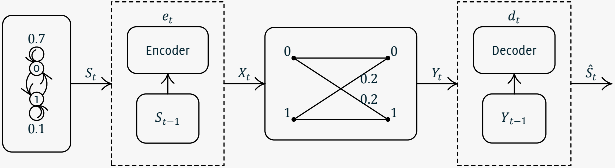


$$\min_{g_{1:T}^{1,2}, l_{1:T}^{1,2}} \mathbb{E}\Big[\sum_{t=1}^{T} c(X_t, U_t^1, U_t^2)\Big]$$

$$\pi_t = \mathbb{P}(X_t, Y_t^{1,2}, M_{t-1}^{1,2} \mid g_{1:t-1}^{1,2}, l_{1:t-1}^{1,2})$$

$$V_t(\pi_t) = \min_{g_t^{1,2},\ l_t^{1,2}} \mathbb{E}[c(X_t, U_t^1, U_t^2) + V_{t+1}(L^{g_t^{1,2},\ l_t^{1,2}} \pi_t)]$$

Example: Real-time communication



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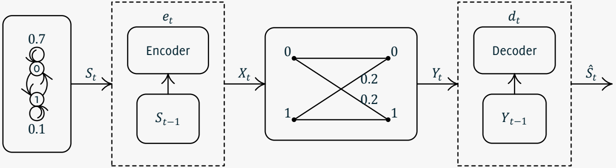


Finite Memory

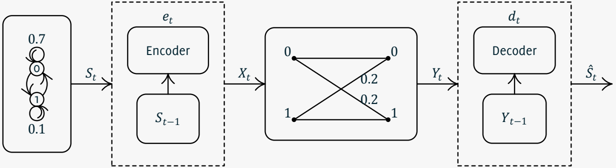
Encoder

$x_t = e_t(s_t, s_{t-1})$

Decoder

$\hat{s}_t = d_t(y_t, y_{t-1})$

Example: Real-time communication



Finite Memory

Encoder

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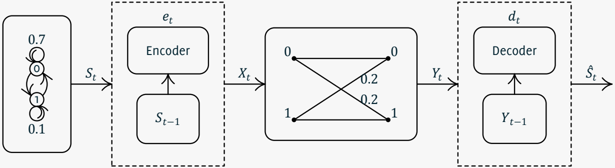
Decoder

$$\hat{s}_t = d_t(y_t, y_{t-1})$$

- Communication Strategy

$$E = (e_1, e_2, \ldots, e_T), \quad D = (d_1, d_2, \ldots, d_T)$$

Example: Real-time communication



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Encoder

$$x_t = e_t(s_t, s_{t-1})$$

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- Communication Strategy

$$E = (e_1, e_2, \dots, e_T), \quad D = (d_1, d_2, \dots, d_T)$$

- Performance

$$\mathcal{J}(E, D) = \lim_{T \to \infty} \mathbb{E}\left\{ \sum_{t=2}^{T} \beta^{t-1}\, \mathbb{P}(\hat{s}_t \neq s_{t-1}) \right\}$$

Gaarder and Slepian's (1982) approach

Brute force search of an optimal policy

- Pick a **time invariant** strategy $E = (e, e, ..., e)$, $D = (d, d, ..., d)$.
- Find the steady-state distribution of the MC $\{S_{t-1}, S_t, Y_{t-1}\}$
- Find the steady-state probability of error

$$\lim_{t\to\infty} \mathbb{E}\left\{\mathbb{P}(\hat{s}_t \neq \hat{s}_{t-1})\right\}$$

- **Repeat** for all time invariant strategies.

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$$\lim_{t \to \infty} \mathbb{E}\left\{ \mathbb{P}(\hat{s}_t \neq \hat{s}_{t-1}) \right\}$$

- **Repeat** for all time invariant strategies.

Difficulty with the approach

- Steady-state distribution of a Markov chain is discontinuous in its transition matrix
- For some (E, D), the Markov chain may not have a unique steady-state distribution

Designer approach based solution

Define $\pi_t = \mathbb{P}(s_{t-1}, s_t, y_{t-1} | e_{1:t-1}, d_{1:t-1})$.

Designer approach based solution

Define $\pi_t = \mathbb{P}(s_{t-1}, s_t, y_{t-1} | e_{1:t-1}, d_{1:t-1})$.

Finite horizon: An optimal communication strategy can be determined by the solution of the following nested optimality equations

$$V_T(\pi_T) = \min_{e_T, d_T} \mathbb{E}\Big[\, \mathbb{P}(\hat{s}_T \neq s_{T-1}) \Big| \pi_T, e_T, d_T \Big]$$

$$V_t(\pi_t) = \min_{e_t, d_t} \mathbb{E}\Big[\, \mathbb{P}(\hat{s}_t \neq s_{t-1}) + V_{t+1}(\pi_{t+1}) \Big| \pi_t, e_t, d_t \Big]$$

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Infinite horizon: . . . fixed point equation

$$V(\pi) = \min_{e,d} \mathbb{E}\Big[\, \mathbb{P}(\hat{s}_t \neq s_{t-1}) + \beta V(\pi_+) \Big| \pi, e, d \Big]$$

Designer approach based solution

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Finite horizon: An optimal communication strategy can be determined by the solution of the following nested optimality equations

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$$V_t(\pi_t) = \min_{e_t, d_t} \mathbb{E}\Big[\, \mathbb{P}(\hat{s}_t \neq s_{t-1}) + V_{t+1}(\pi_{t+1}) \Big| \pi_t, e_t, d_t \Big]$$

Infinite horizon: . . . fixed point equation

$$V(\pi) = \min_{e,d} \mathbb{E}\Big[\, \mathbb{P}(\hat{s}_t \neq s_{t-1}) + \beta V(\pi_+) \Big| \pi, e, d \Big]$$

The designer strategy $\gamma_t : \pi_t \to (e_t, d_t)$ is time-invariant. The choice of (e_t, d_t) is **not time invariant**.

Optimal communication scheme

- Example with $\beta = 0.9$

$$(e_t, d_t) = \begin{pmatrix} s_t & , & 0 \\ s_{t-1} \oplus s_t & , & y_{t-1} \oplus y_t \\ s_{t-1} & , & y_{t-1} \oplus y_t \end{pmatrix}_{t \pmod 3}$$

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Source	s_1	s_2	s_3	s_4	s_5	s_6	s_7
Encoder							
Decoder							
Estimate	–	s_1	s_2	s_3	s_4	s_5	s_6

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$$(e_t, d_t) = \begin{pmatrix} s_t & , & \color{red}{0} \\ s_{t-1} \oplus s_t & , & y_{t-1} \oplus y_t \\ s_{t-1} & , & y_{t-1} \oplus y_t \end{pmatrix}_{t \pmod 3}$$

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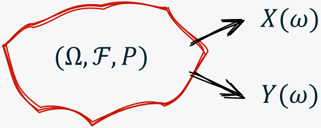
Coordinator approach

Common Knowledge (Aumann, 1976)



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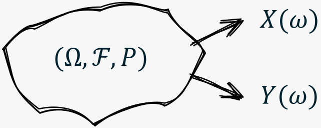

$\sigma(X) \cap \sigma(Y)$

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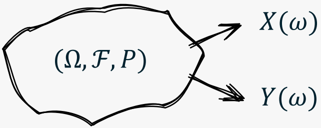
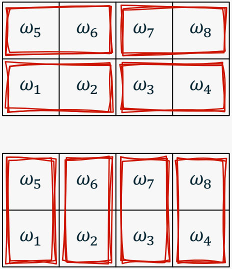


$\sigma(X) \cap \sigma(Y)$

ω_5	ω_6	ω_7	ω_8
ω_1	ω_2	ω_3	ω_4



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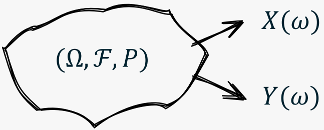

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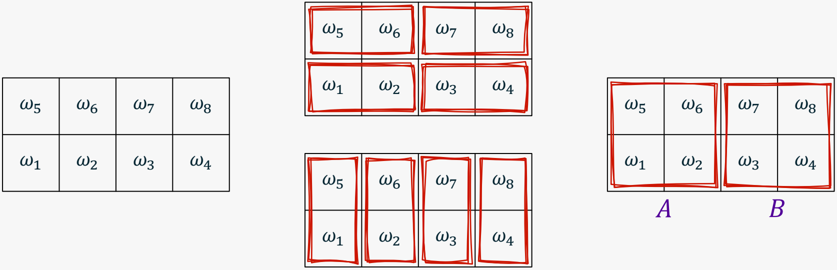
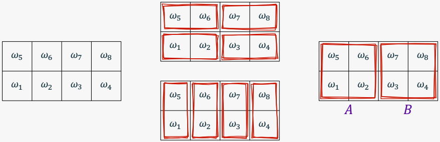


Exploiting common knowledge to simplify decentralized static optimization



$$u = g(x), \quad v = h(y)$$

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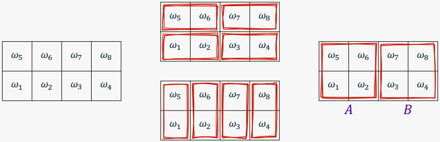

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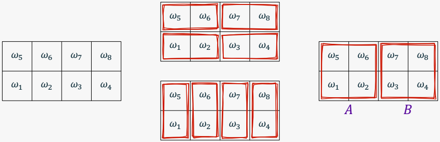

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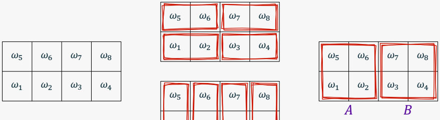

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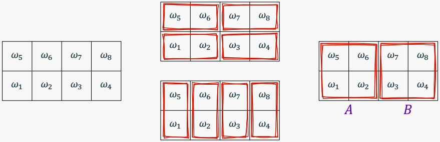

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A common knowledge based solution

$$\min_{\gamma, \eta} \mathbb{E}^{\gamma,\eta}[c(\omega, u, v) | k]$$

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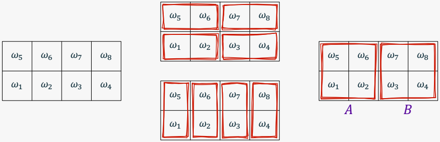

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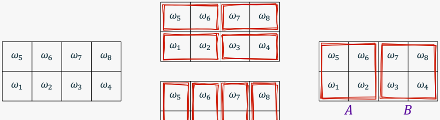

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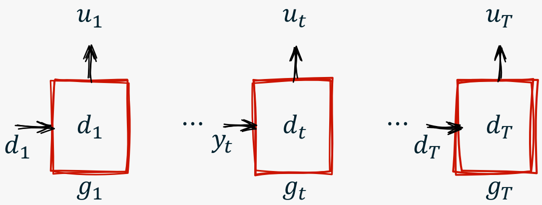
Brute force: $2^4 \times 2^4$ possiblities. CI-based soln: $2 \cdot (2^2 \times 2^2)$ possibilities.

Main idea: Extend CI-based approach to decentralized multi-stage systems.

A common information based approach for decentralized multi-stage systems

(Nayyar, 2010; Nayyar, Mahajan, Teneketzis, 2011)



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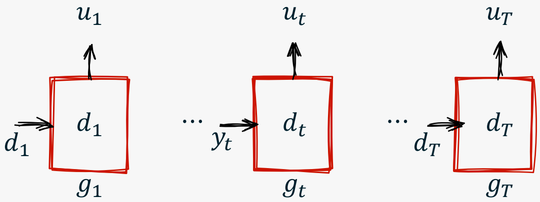

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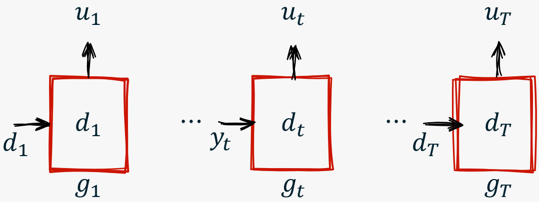

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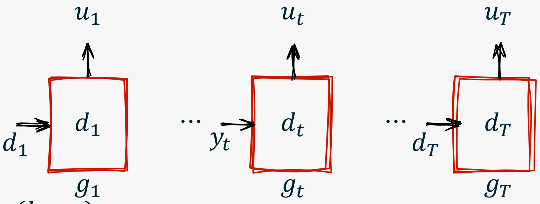

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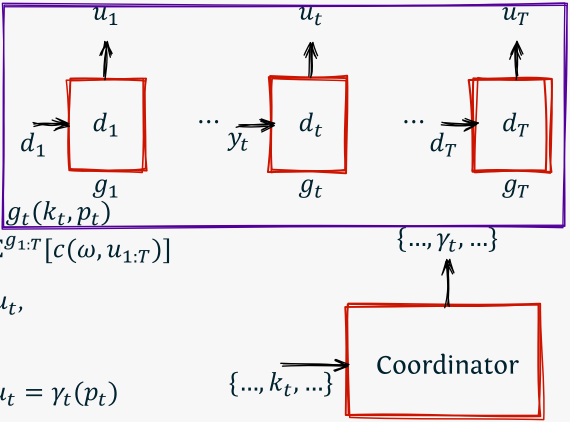
- **Prescription**: $\gamma_t : p_t \mapsto u_t$, chosen according to
  $$\gamma_t = \psi_t(k_t, \gamma_{1:t-1}), \qquad u_t = \gamma_t(p_t)$$

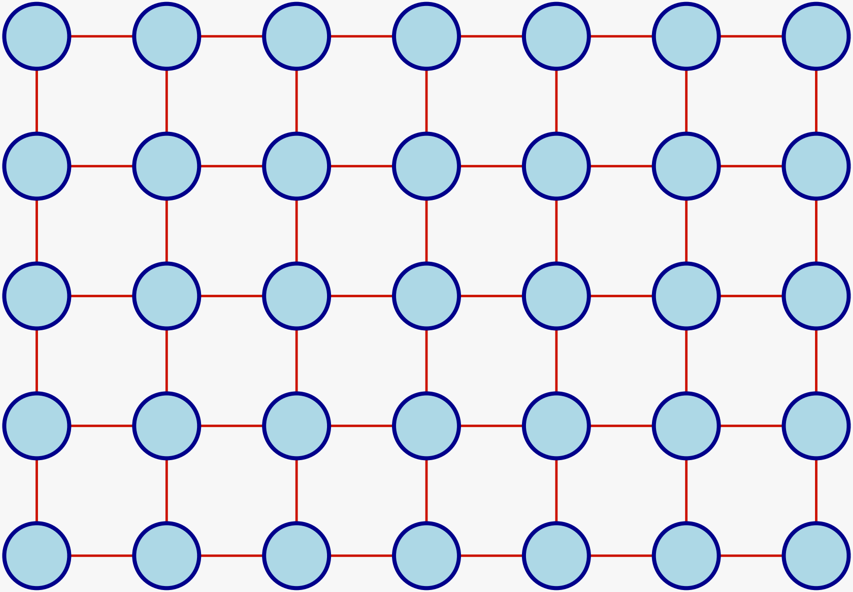


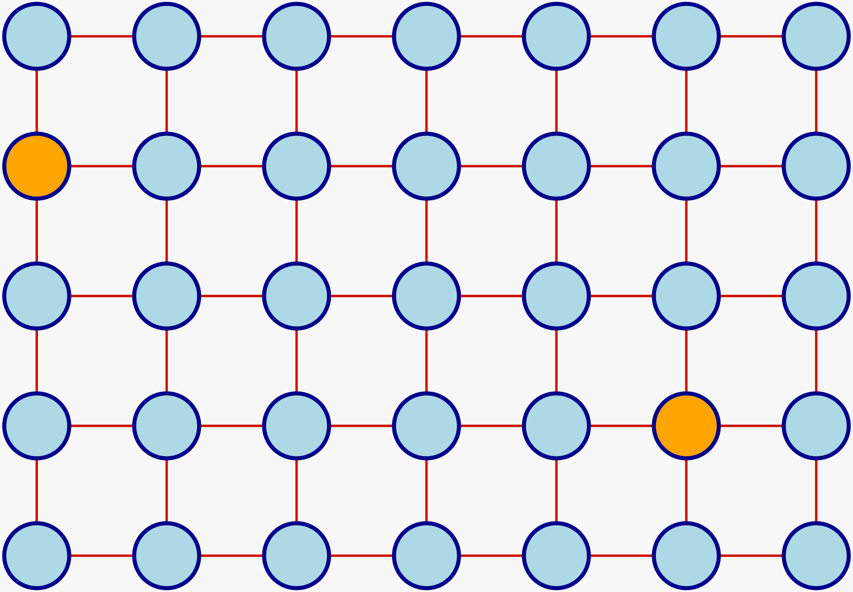
A common information based approach for decentralized multi-stage systems

Solution approach

1. Construct a **coordinated system** (that has classical info-struct.)
2. Show that coordinated system $\equiv$ original system.
3. Find a solution to coordinated system using centralized stoc. control.
4. Translate the result back to original system

An example: delayed sharing information structure





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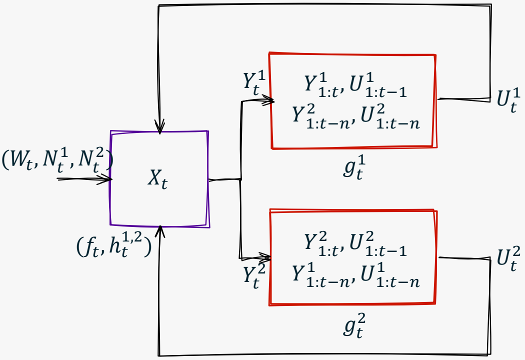

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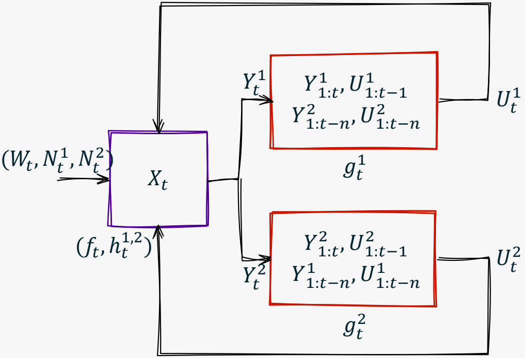

$$J(g^{1,2}_{1:T}) = \mathbb{E}^{g^{1,2}_{1:T}}[c(X_t, U^{1,2}_t)]$$

Literature Overview

How to compress data into a sufficient statistic?

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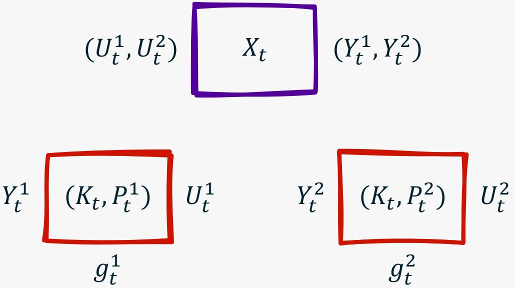
A good model for many applications The result of one-step delayed sharing used in various applications in queuing theory, communication networks, stochastic games, and economics. (Kuri Kumar, 1995; Altman *et al*, 2009; Grizzle *et al* 1982; Papavassilopoulos, 1982; Chang and Cruz, 1983; Li and Whu, 1991).

The coordinated system: state for I/O mapping



Common information $K_t = (Y^{1,2}_{1:t-n}, U^{1,2}_{1:t-n}).$

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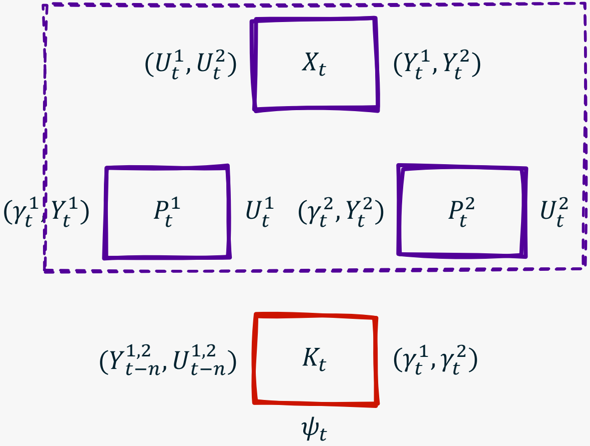

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Structural Result There is no loss of optimality in restricting prescriptions of the form

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Setting $g^i_t(\pi_t, P^i_t) = \psi^i_t(\pi_t)(P^i_t)$ gives optimal control strategy.

An easy solution to long standing open problem

Connections

(Nayyar, Mahajan, Teneketzis, 2011)

Many existing results on decentralized control are special cases

- Delayed state sharing (Aicardi *et al*, 1987)
- Periodic sharing information structures (Ooi *et al*, 1997)
- Control sharing (Bismut, 1972; Sandell and Athans, 1974; Mahajan 2011)
- Finite state memory controllers (Sandell, 1974, Mahajan, 2008)

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(Nayyar, Mahajan, Teneketzis, 2011)

Many existing results on decentralized control are special cases

- Delayed state sharing (Aicardi *et al*, 1987)
- Periodic sharing information structures (Ooi *et al*, 1997)
- Control sharing (Bismut, 1972; Sandell and Athans, 1974; Mahajan 2011)
- Finite state memory controllers (Sandell, 1974, Mahajan, 2008)

Generalization to other models

- **Infinite horizon** (discounted and average cost) models using standard results for POMDPs
- **Computation algorithms** based on algorithms for POMDPs
- Extend results to systems with **unknown models** based on Q-learning and adaptive control algorithms

Summary of the main idea

- Find **common information** at the controllers
- Look from the point of view of a **coordinator** that observes common information and chooses prescriptions to the controllers
- Find **information state** for the coordinated system and use it to set up a dynamic program
- When common information is nil, the approach reduces to **designer's approach**

Future Directions

Identify other tractable information structures

The common information approach (almost) all known results for Markov chain setup. Are there other structures that are tractable?

Computational algorithms

Develop computation algorithms that are tuned to the type of DP equations that arise in decentralized control.

Connections with stochastic optimization

Can techniques from stochastic optimization used to solve decentralized stochastic control problems?

Connections with sequential games

Does the common information approach help in identifying sequential equilibrium in sequential games?

Thank you

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