MAC with feedback : Structure of optimal block Markov superposition codes

Aditya Mahajan

ITA Workshop, February 5, 2010

Are **auxiliary random variables** in info theory related to **information states** in stochastic control?

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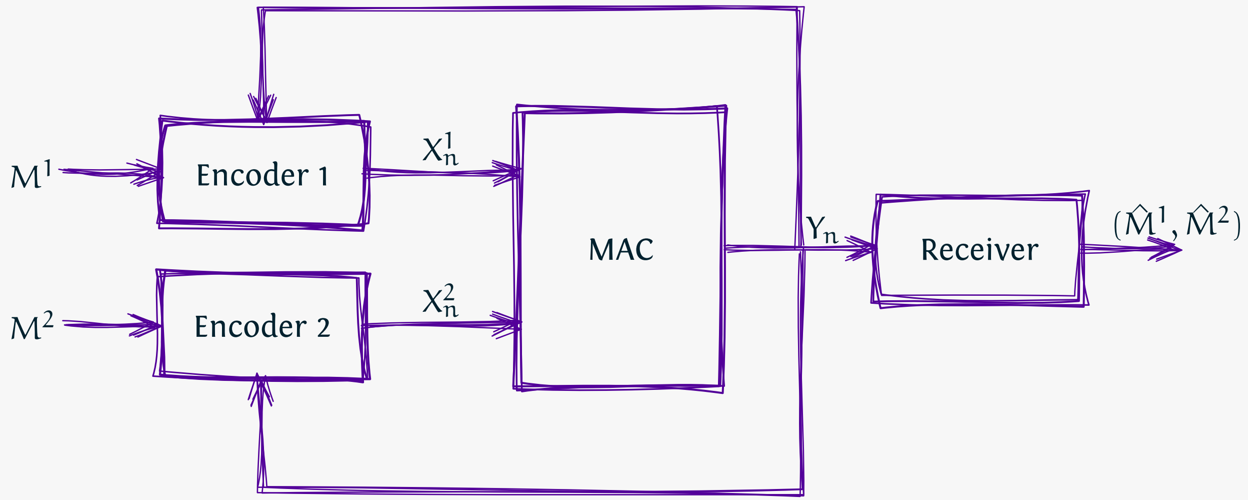
The Setup



- **i.i.d. messages**: M^1 and M^2
- **memoryless channel**: $P(Y_n \mid X^1_{[1:n]}, X^2_{[1:n]}) = P(Y_n \mid X^1_n, X^2_n)$
- **feedback**: $X^i_n = f^i_n(M^i, X^i_{[1:n-1]}, Y_{[1:n-1]}),\ i = 1, 2.$

Capacity of MAC with feedback

- Feedback can **increase** capacity. (Gaarder and Wolf, 1975).

The encoders can communicate
using feedback channel

- Capacity characterized by **multi-letter directed information** expression (Kramer, 1998)

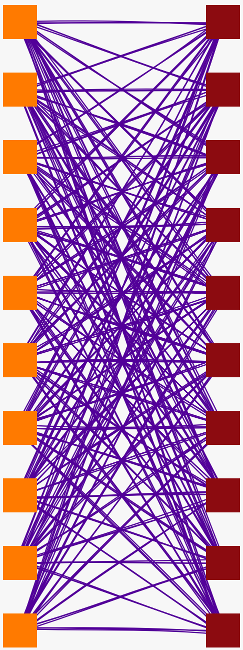
$$\bigcup \left\{ (R_1, R_2) : \begin{array}{r} R_1 \leqslant I(X^1 \to Y \| X^2) \\ R_2 \leqslant I(X^2 \to Y \| X^1) \\ R_1 + R_2 \leqslant I(X_1 X_2 \to Y) \end{array} \right\}$$

Achievable schemes

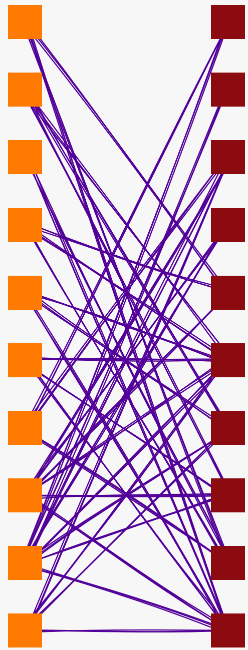
- Gaarder and Wolf, 1975: Code in two phases.
  - **Two-way communication** of independent messages
  - Point-to-point communication with **side information**
- Cover and Leung, 1981: **Superpose the two phases**.
- Bross and Lapidoth, 2005: **Add an explicit phase for two-way communication of correlated messages**
- Venkataramanan and Pradhan, 2009: **Superpose the three phases**
- The idea can be extended to more than three phases

Thinning of correlation graph

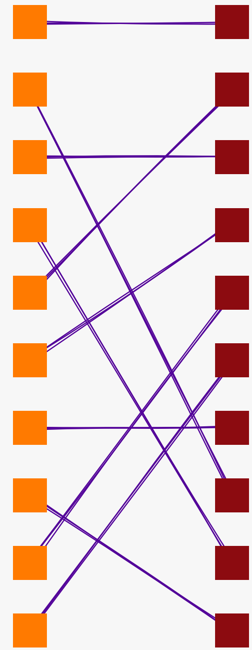
Venkataraman and Pradhan, 2009



Two way comm. of
independent messages



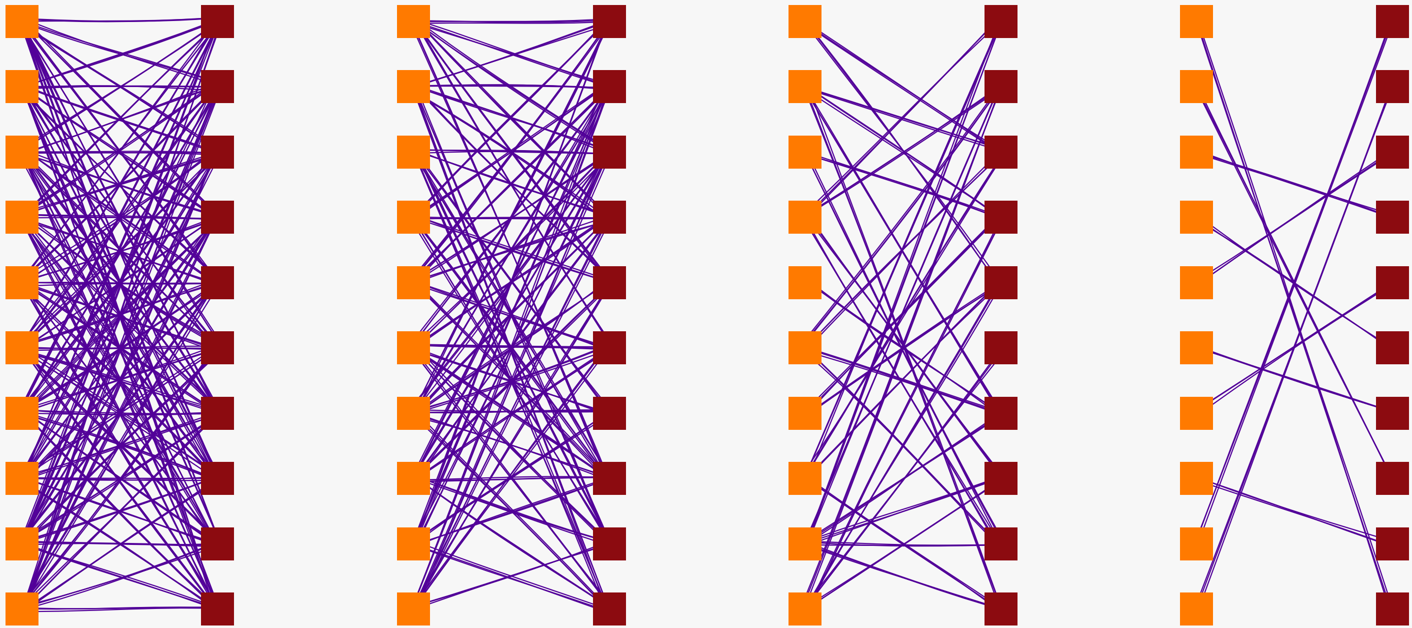
Two way comm. of
correlated messages



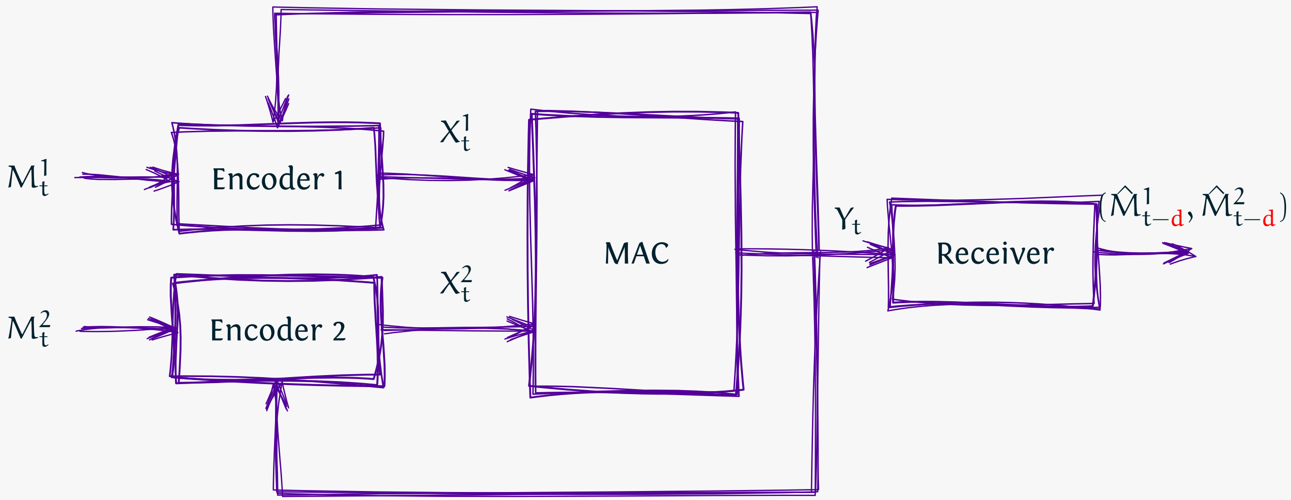
Point-to-point comm.
with **side information**

Can be extended to more than three phases

Venkataramanan and Pradhan, 2009



Block Markov superposition codes at the block level



- **i.i.d. source:** M_t^1 and M_t^2
- **memoryless channel:** $P(Y_t \mid X_{[1:t]}^1, X_{[1:t]}^2) = P(Y_t \mid X_t^1, X_t^2)$

Block Markov superposition codes at the block level

Let $Z_t = (M_t^1, M_t^2, X_t^1, X_t^2)$. Assume delayed sharing of information.

- **encoders**:

$$X_t^i = f_t^i(Z_{1:t-2}, Y_{1:t-1}, M_{t-1}^i, M_t^i, X_{t-1}^i), \quad i = 1, 2$$

- **decoders** with decoding delay d:

$$(\hat{M}_{t-d}^1, \hat{M}_{t-d}^2) = g_t(Z_{1:t-3}, Y_{1:t})$$

- **distortion** at time t:

$$\rho(M_{t-d}^1, \hat{M}_{t-d}^1) + \rho(M_{t-d}^2, \hat{M}_{t-d}^2)$$

Venkataramanan-Pradhan scheme

- **Encoder**:

$$X_t^i = f_t^i(M_t^i, U_t, W_t, A_t^i)$$

- **Three auxiliary random variables**:
  - W_t: Compression of the past

$$W_{t+1} = \psi_t(U_t, W_t, A_t^1, A_t^2, Y_t)$$

  - U_t: Common message

$$U_{t+1} = \hat{\psi}_t^1(A_t^1, M_t^1, Y_t) = \hat{\psi}_t^2(A_t^2, M_t^2, Y_t)$$

  - A_t^1 and A_t^2: Messages for two-way communication between encoders 1 and 2.

$$A_{t+1}^1 = \tilde{\psi}_t(X_t^1, U_t, W_t, A_t^1, A_t^2, Y_t)$$

Is the structure of
this scheme optimal?
**Will decentralized control
give the same structure?**

Information Structure

- **Encoder 1**: $(Z_{1:t-2}, Y_{1:t-1}, M^1_{t-1}, M^1_t, X^1_{t-1})$
- **Encoder 2**: $(Z_{1:t-2}, Y_{1:t-1}, M^2_{t-1}, M^2_t, X^2_{t-1})$
- **Decoder**: $(Z_{1:t-3}, Y_{1:t})$

Non-classical information structure

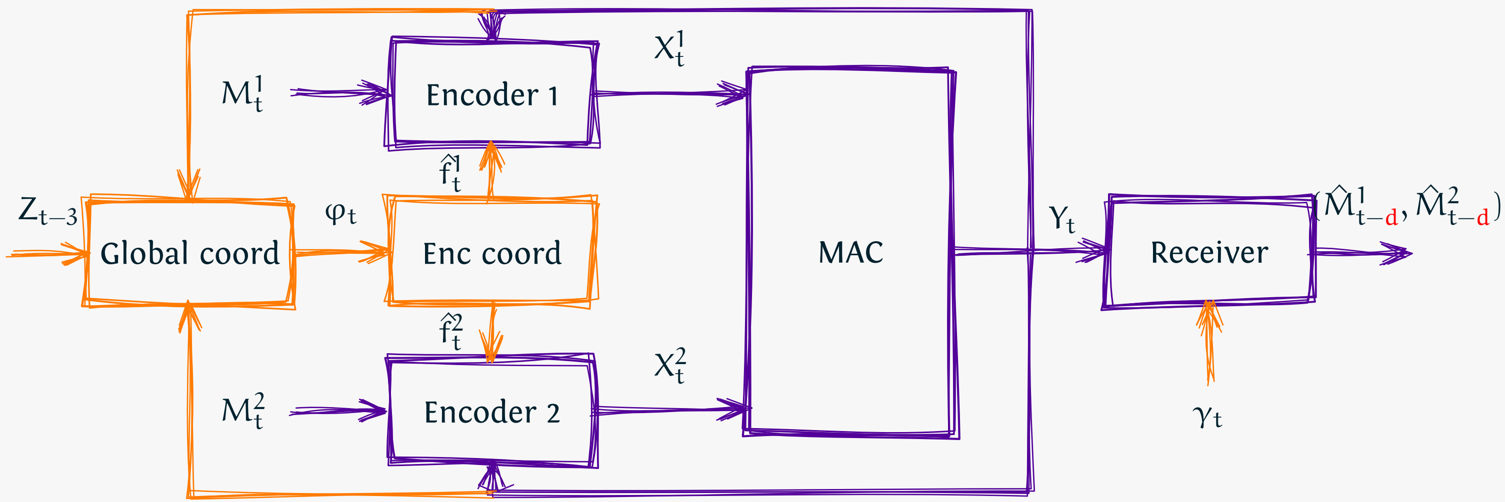
- Delayed sharing of information
  - Common info between encoders and decoder: $(Z_{1:t-3}, Y_{1:t-1})$
  - Common info between encoders: $(Z_{1:t-2}, Y_{1:t-1})$

Structural results in decentralized control

- Consider a **coordinator** that observes common information (but does not observe the private information).
- Formulate a **centralized optimization problem** from the point of view of the coordinator
- Show that the coordinator's problem is equivalent to the original problem
- Find **states** sufficient for **input-output mapping** for the **coordinator**
- Find **information states** (state sufficient for dynamic programming) for the coordinator

[Mahajan-Nayyar-Teneketzis 08]

Coordinator for the system



- Decisions made by the coordinators

$$(\varphi_t, \gamma_t) = \psi_t(Z_{1:t-3}, Y_{1:t-1})$$

$$(\hat{f}_t^1, \hat{f}_t^2) = \xi_t(\varphi_t, Z_{t-2})$$

- Encoders and decoders simply use these partial functions

$$X_t^i = \hat{f}_t^i(M_{t-1}^i, M_t^i, X_{t-1}^i), \qquad (\hat{M}_{t-2}^1, \hat{M}_{t-2}^2) = \gamma_t(Y_t)$$

Structural results

- **Common information**: $(Z_{1:t-3}, Y_{1:t-1})$
- **All private information**: $(Z_{t-2}, Z_{t-1}, M_t^1, M_t^2, Y^t)$

- Structural results

Replace **common information** with
P(state + all private info | common info)

[Mahajan-Nayyar-Teneketzis 08]

Structural results

- **Common information**: $(Z_{1:t-3}, Y_{1:t-1})$
- **All private information**: $(Z_{t-2}, Z_{t-1}, M_t^1, M_t^2, Y_t)$

- Structural results
  - Define $\pi_t = P(Z_{t-2}, Z_{t-1}, M_t^1, M_t^2, Y_t \mid Z_{1:t-3}, Y_{1:t-1})$
  - **Encoder**: $X_t^i = f_t^i(\pi_t, Z_{t-2}, M_{t-1}^i, M_t^i, X_{t-1}^i)$
  - **Decoder**: $(\hat{M}_{t-d}^1, \hat{M}_{t-d}^2) = g_t(\pi_t, Y_t)$

[Mahajan-Nayyar-Teneketzis 08, Anastasopoulos 09, Ardestanizadeh-Javidi-Kim-Wigger, 09]

Is this related to auxiliary random variables?

$$
\begin{aligned}
\pi_t &= P(Z_{t-2}, Z_{t-1}, M_t^1, M_t^2, Y^t \mid Z_{1:t-3}, Y_{1:t-1}) \\
&= P(Z_{t-2} \mid Z_{1:t-3}, Y_{1:t-1}) \\
&\quad \times P(Z_{t-1} \mid Z_{1:t-2}, Y_{1:t-1}) \\
&\quad \times P(M_t^1 \mid Z_{1:t-1}, Y_{1:t-1}) \\
&\quad \times P(M_t^2 \mid Z_{1:t-1}, Y_{1:t-1}) \\
&\quad \times P(Y_t \mid Z_{1:t-1}, Y_{1:t-1}, M_t^1, M_t^2)
\end{aligned}
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&\quad \times P(M^1_t \mid Z_{1:t-1}, Y_{1:t-1}) \\
&\quad \times P(M^2_t \mid Z_{1:t-1}, Y_{1:t-1}) \\
&\quad \times P(Y_t \mid Z_{1:t-1}, Y_{1:t-1}, M^1_t, M^2_t)
\end{aligned}
$$

These “chunks” of information state appear to be related to auxiliary random variables, but the relation is not convincing.

Reevaluate information states

- summary of past past information
- computable at all agents
- sufficient for performance evaluation
- time-invariant domain
- minimal (although the notion of minimality is not clear)

Conditional probability (also called belief states) traditionally capture these notions in **centralized stochastic control**

In **decentralized control**, can we derive information states in a different manner?

Salient features of decentralized control

- The "control action" of the coordinator is a partial function that tells each agent what to do with their private information.
- Reason for double exponential complexity in space

Salient features of decentralized control

- The "control action" of the coordinator is a **partial function** that tells each agent what to do with their private information.
- Reason for double exponential complexity in space

Can we **exploit** this "control action is a partial function feature" to **summarize the past** in a different way?

Affect of functions on future can be compressed by **partially evaluating the function**

Partially evaluating a function

Consider

$$X_{t+1} = f_t(X_t, Y_t)$$

How do we compress the affect (f_t, X_t) affect X_{t+1}?

Partially evaluating a function

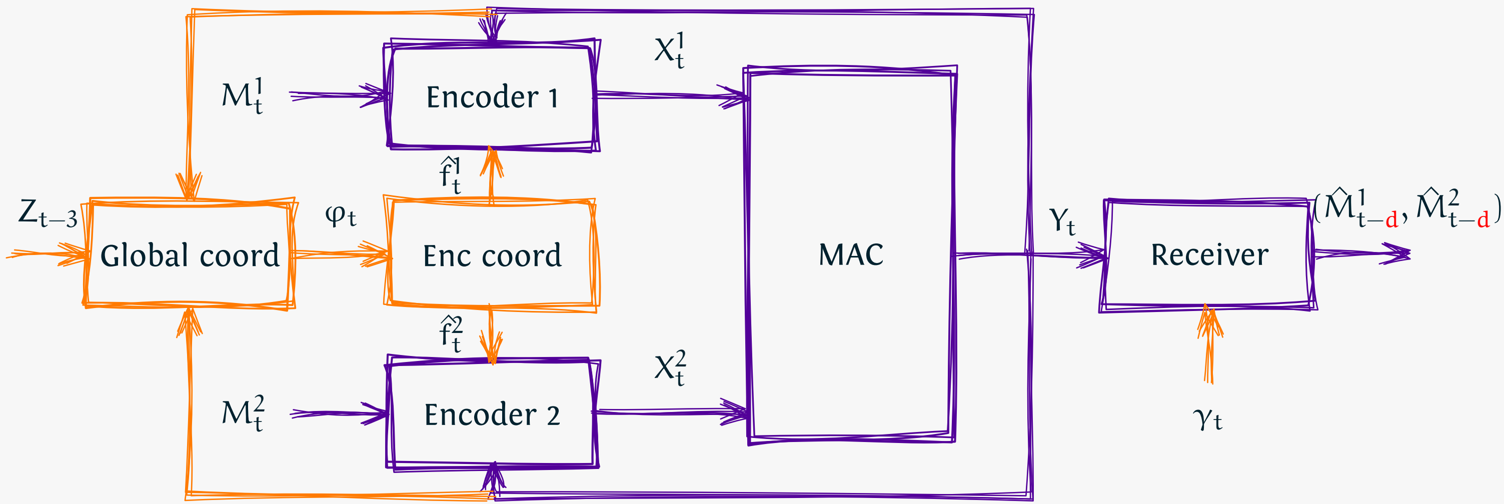
Consider

$$X_{t+1} = f_t(X_t, Y_t)$$

How do we compress the affect (f_t, X_t) affect X_{t+1}?

$$f_t(X_t, \cdot)$$

Coordinator for the system



- Decisions made by the coordinators

$$(\varphi_t, \gamma_t) = \psi_t(Z_{1:t-3}, Y_{1:t-1})$$

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- Encoders and decoders simply use these partial functions

$$X_t^i = \hat{f}_t^i(M_{t-1}^i, M_t^i, X_{t-1}^i), \qquad (\hat{M}_{t-2}^1, \hat{M}_{t-2}^2) = \gamma_t(Y_t)$$

The coordinator's point of view

- Information at coordinator:

$$(Z_{1:t-3}, Y_{1:t-1}, \varphi_{1:t-1}, \gamma_{1:t-1}, \hat{f}^1_{1:t-1}, \hat{f}^2_{1:t-1})$$

- **Information state**

$$(Z_{t-3}, Y_{t-2}, Y_{t-1}, \hat{f}^1_{t-1}, \hat{f}^2_{t-1}, \tilde{f}^1_{t-2}, \tilde{f}^2_{t-2})$$

where

$$\tilde{f}^i_{t-2}(\cdot) = \hat{f}^i_{t-2}(M^i_{t-3}, X^i_{t-3}, \cdot)$$

Information states

$$(Z_{t-3}, Y_{t-2}, Y_{t-1}, \hat{f}^1_{t-1}, \hat{f}^2_{t-1}, \tilde{f}^1_{t-2}, \tilde{f}^2_{t-2})$$

where

$$\tilde{f}^i_{t-2}(\cdot) = \hat{f}^i_{t-2}(M^i_{t-3}, X^i_{t-3}, \cdot)$$

Can show that

- the state is a controlled Markov process
- is sufficient to evaluate expected instantaneous cost

Structural Results

- Encoder

$$X_t^i = \hat{f}_t^i(M_t^i, M_{t-1}^i, X_{t-1}^i) = f_t^i(M_t^i, M_{t-1}^i, X_{t-1}^i, Z_{t-2}, \hat{f}_{t-1}^{1,2}, \tilde{f}_{t-2}^{1,2})$$

- components of **information state**
  - Information state $(Z_{t-3}, Y_{t-2}, Y_{t-1}, \hat{f}_{t-1}^1, \hat{f}_{t-1}^2, \tilde{f}_{t-2}^1, \tilde{f}_{t-2}^2)$
  - Partial function that determines the current message
    $\varphi_t = \psi_t(Z_{t-3}, Y_{t-2}, Y_{t-1}, \hat{f}_{t-1}^1, \hat{f}_{t-1}^2, \tilde{f}_{t-2}^1, \tilde{f}_{t-2}^2)$
  - Partial function that determines the one step old message
    $\hat{f}_{t-1}^i(\cdot) = \varphi_{t-1}(Z_{t-3}, \cdot)$
  - Partial function that determines the two step old message
    $\tilde{f}_{t-2}^i(\cdot) = \hat{f}_{t-2}^i(M_{t-3}^i, X_{t-3}^i, \cdot)$

Venkataramanan-Pradhan scheme

- **Encoder**:

$$X_t^i = f_t^i(M_t^i, U_t, W_t, A_t^i)$$

- **Three auxiliary random variables**:
  - W_t: Compression of the past

$$W_t \equiv (Z_{t-3}, Y_{t-1}, Y_{t-2})$$

  - U_t: Common message

$$U_t \equiv (Z_{t-2}, \tilde{f}_{t-1}^1, \tilde{f}_{t-1}^2)$$

  - A_t^1 and A_t^2: Messages for two-way communication between encoders 1 and 2.

$$A_t^i \equiv \hat{f}_{t-1}^i$$

Comparison of the two schemes

- The **dynamics** of auxiliary random variables and partial functions in the information state are **similar**
- The **semantics** are **different**
- The structure results derived using **stochastic control** are **simpler**
  - We do not need a “compression of the past”; rather a few recent common observations are sufficient.
- The achievable scheme of Venkataramanan-Pradhan can be **adapted** to use this simpler structure (or so it appears after a few back of the envelop calculations)

Conclusion
(for block Markov superposition coding)

- Horses for courses
  - Use **stochastic control** to determine the form of coding scheme **across blocks**
  - Use **information theory** to determine the form of functions **within a block**
- Future directions

Does this idea works for other channels
(broadcast, relay, interference, . . .)

Thoughts
(on relations to auxiliary random variables)

- Past can be summarized in two ways
  - conditional probability (belief)
  - partial function

Do auxiliary random variables
also exhibit this behavior?

- Can be useful for proving converses
- Can results that bound the cardinality of auxiliary random variables help in numerical computation of decentralized stochastic control problems?

Thank you