Learning to control networked-coupled subsystems with unknown dynamics

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Symcomore group meeting

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Networks are ubiquitous

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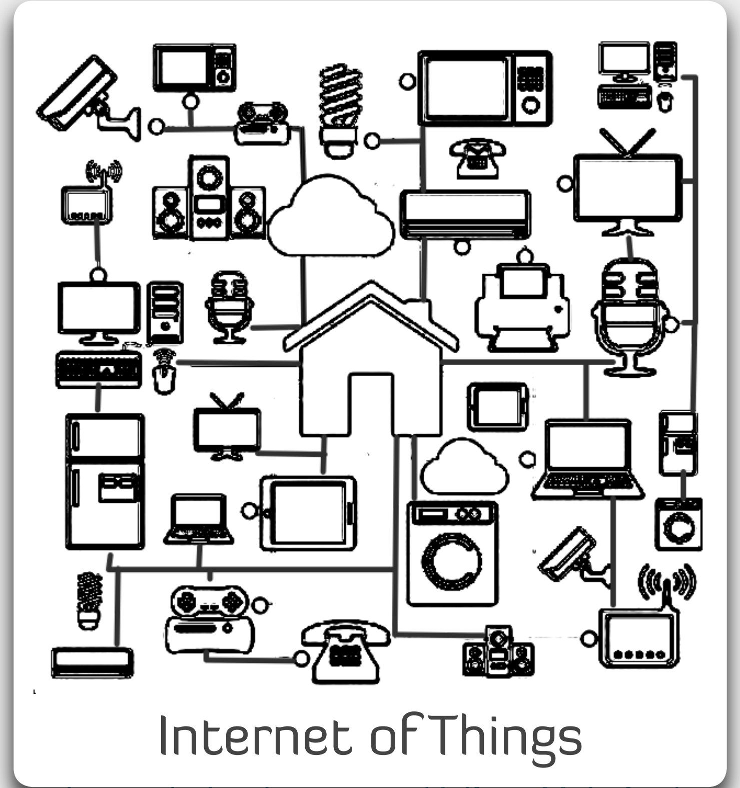
Transportation networks

Networks are ubiquitous



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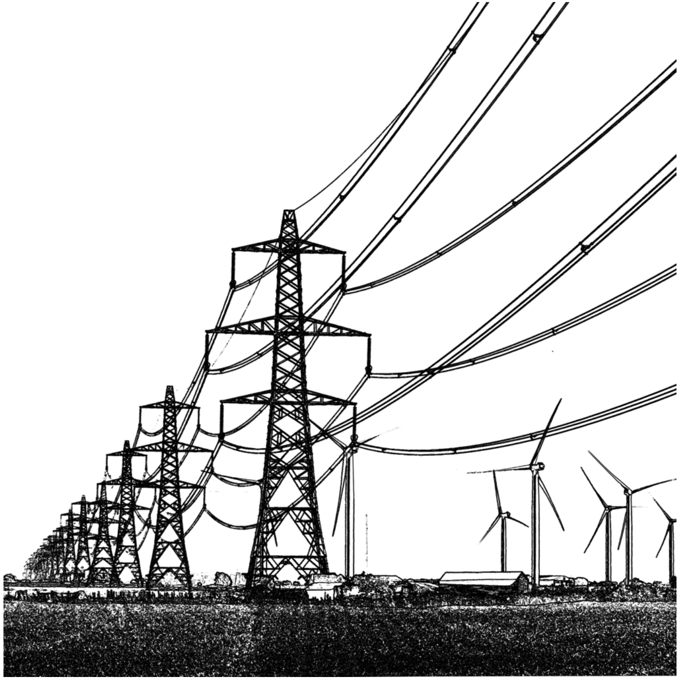

Energy network

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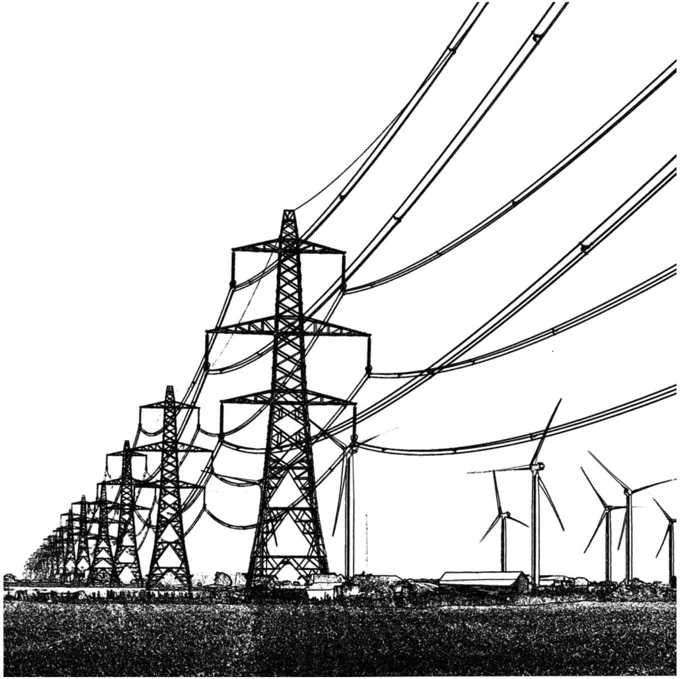

Energy network

Salient Features

- Large/growing size
- Nodes have local states
- Coupled dynamics and costs

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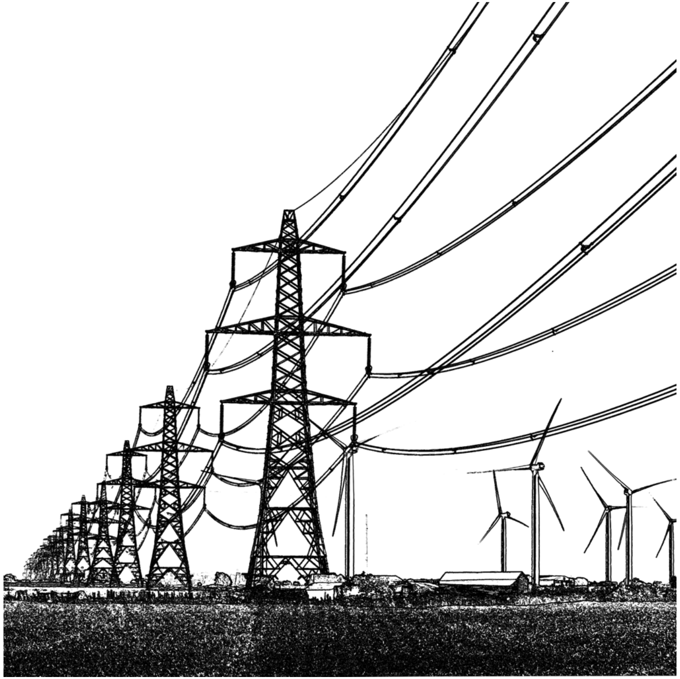

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Design challenges

- Scalability of the solution
- How to handle model uncertainty

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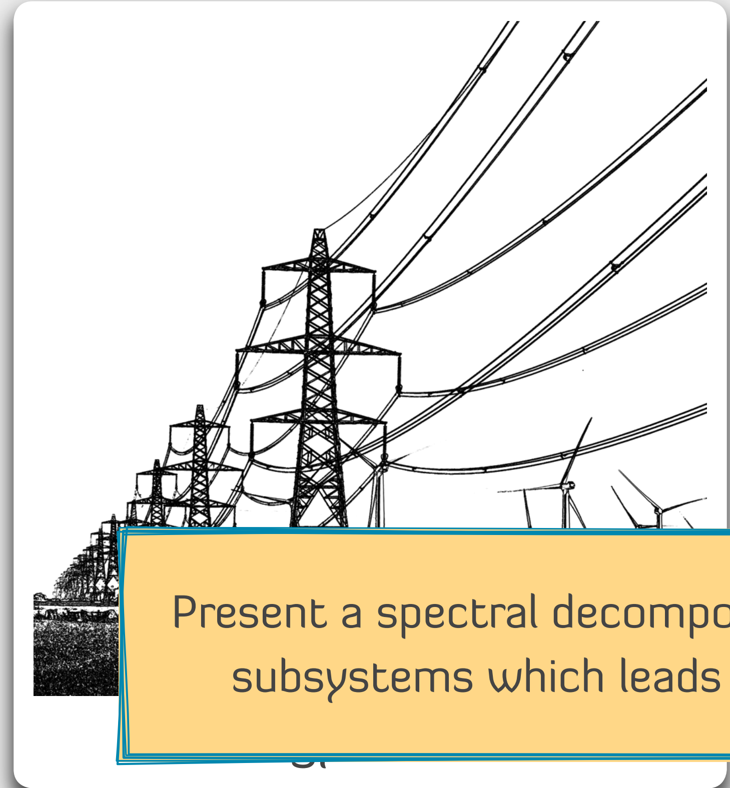

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Present a spectral decomposition method for network-coupled subsystems which leads to scalable planning and learning

Outline

System Model

- Network-coupled subsystems
  - Agents interacting over a graph
  - Coupled dynamics
  - Coupled costs

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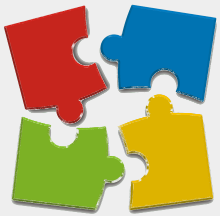

Planning solution

- Spectral factorization of dynamics and cost
- Decoupled Riccati equations

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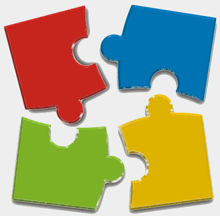

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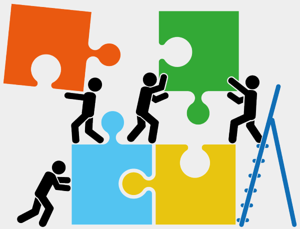

Learning solution

- Spectral factorization of learning
- Numerical experiments

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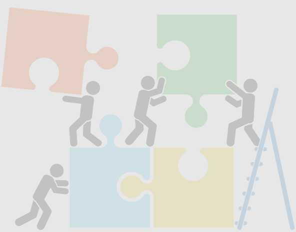

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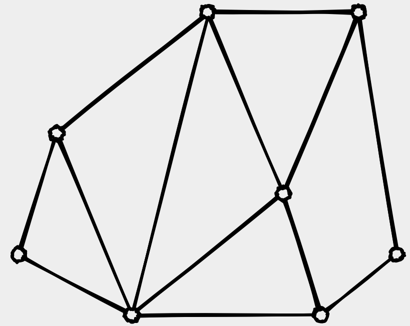
System Model

Weighted undirected graph $\mathcal{G}$

- Nodes $N = \{1, \dots, n\}$.
- Symmetric matrix $M = [m^{ij}]$ associated with $\mathcal{G}$ (e.g., weighted adjacency matrix, weighted Laplacian, etc.)



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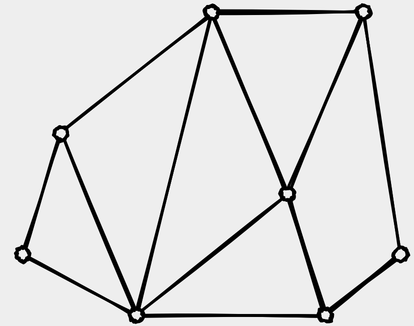

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System Dynamics

- A subsystem located at each node. State $x^i_t \in \mathbb{R}^{d_x}$. Control $u^i_t \in \mathbb{R}^{d_u}$.

$$x^i_{t+1} = Ax^i_t + Bu^i_t + D \sum_{j \in N} m^{ij} x^j_t + E \sum_{j \in N} m^{ij} u^j_t + w^i_t$$

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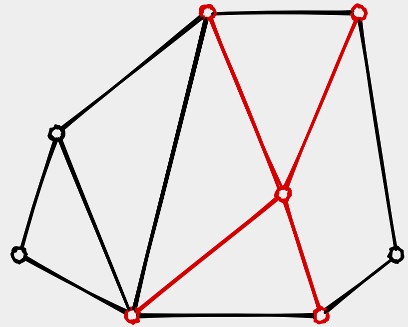

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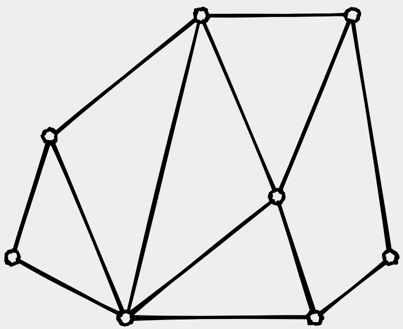
Network field of states $x^{\mathcal{G},i}_t$ **Network field** of control $u^{\mathcal{G},i}_t$

System Model (cont.)

Per-step cost

$$c(x_t, u_t) = \sum_{i,j \in N} \left[h_q^{ij} (x_t^i)^\intercal Q (x_t^j) + h_r^{ij} (u_t^i)^\intercal Q (u_t^j) \right]$$

where $H_q = [h_q^{ij}]$ and $H_r = [h_r^{ij}]$ are **symmetric matrices which have the same eigenvectors as M**.



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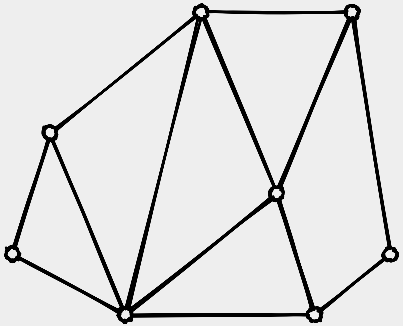

Remark

For two symmetric $n \times n$ matrices M_1 and M_2, the following statements are equivalent:

- M_1 and M_2 share the same eigenvectors.
- M_1 and M_2 communte (i.e., $M_1 M_2 = M_2 M_1$)
- M_1 and M_2 are simultaneously diagonalizable.

System Model (cont.)

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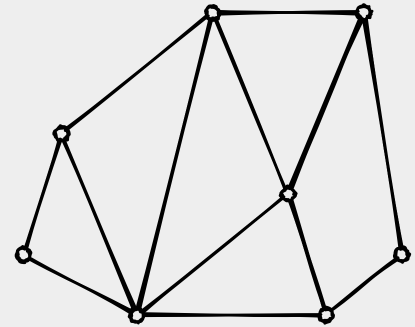
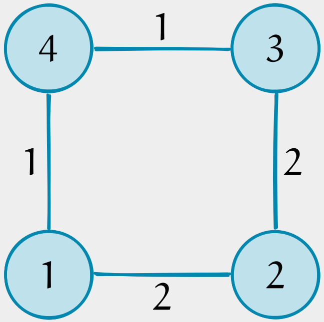


Important special case

- $H_q = \sum_{k=0}^{K_q} q_k M^k$ and $H_r = \sum_{k=0}^{K_r} r_k M^k$.
- Captures the intuition that the per-step cost respects the graph structure.
- Example: $H_q = q_0 I + q_1 M + q_2 M^2$ means that there is a cost coupling between the one- and two-hop neighbors.

An example to illustrate that nodes are not exchageable



A graph $\mathcal{G}$

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Dynamical coupling

- Nodes are not exchageable

$$x_t^{\mathcal{G},1} = 2\,x_t^2 + 1\,x_t^4, \qquad x_t^{\mathcal{G},2} = 2\,x_t^1 + 2\,x_t^3,$$

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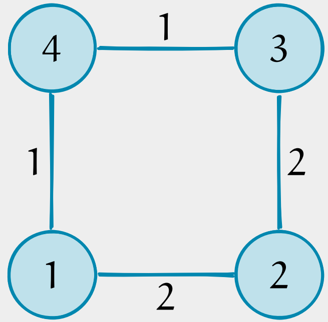

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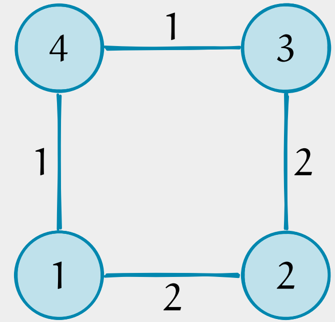

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Cost coupling

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Suppose $H_q = q_0 I + q_1 M + q_2 M^2$.

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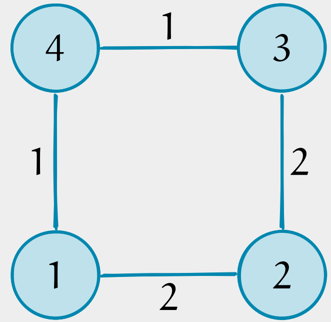

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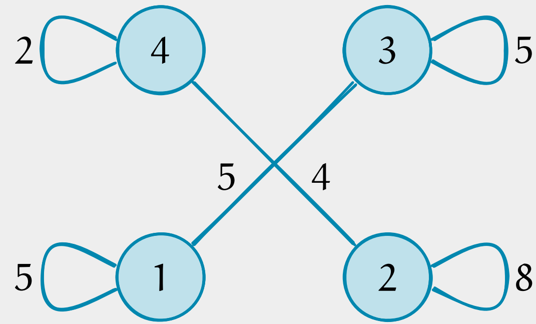

Two-hop neighborhood

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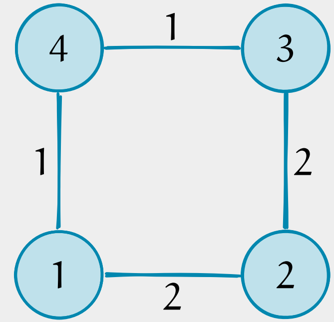

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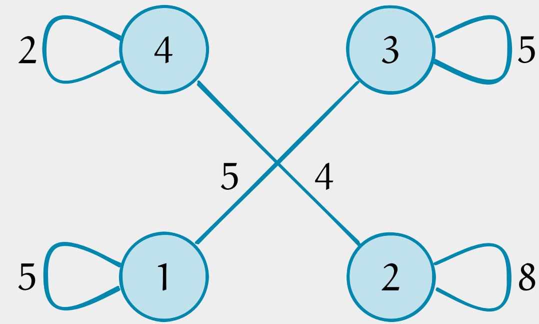
Suppose $H_q = q_0 I + q_1 M + q_2 M^2$. Then

$$H_q = \begin{bmatrix} q_0 + 5q_2 & 2q_1 & 5q_2 & q_1 \\ 2q_1 & q_0 + 8q_2 & 2q_1 & 4q_2 \\ 5q_2 & 2q_1 & q_0 + 5q_2 & q_1 \\ q_1 & 4q_2 & q_1 & q_0 + 2q_2 \end{bmatrix}$$

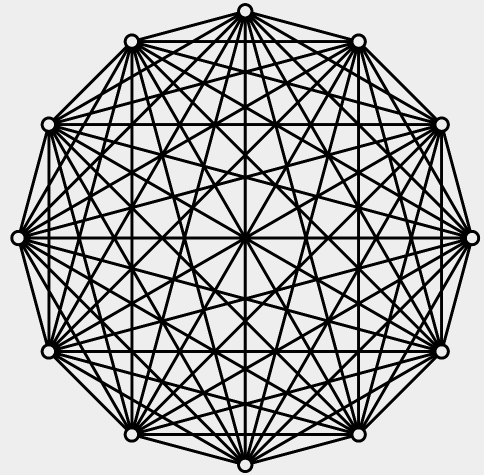


Two-hop neighborhood

Model generalizes mean-field control model

Special case

- Consider $M = \frac{1}{n}\mathbb{1}_{n \times n}$ and $H_q = H_r = \frac{1}{n}I + \frac{\kappa}{n}M$.
- Network-field $\frac{1}{n}\sum_{j \in N} x_t^j =: \bar{x}_t$ is the (empirical) mean-field.



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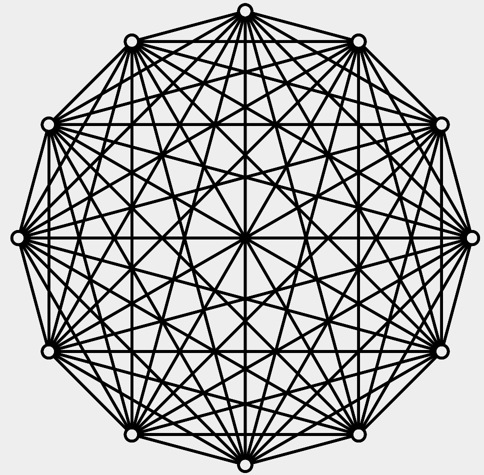


Dynamics

$$x_{t+1}^i = Ax_t^i + Bu_t^i + D\bar{x}_t + E\bar{u}_t + w_t^i.$$

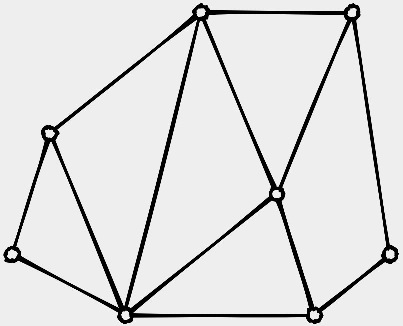
Per-step cost

$$\begin{aligned} c(x_t, u_t) &= (1+\kappa)\left[\bar{x}_t^\top Q\bar{x}_t + \bar{u}_t^\top R\bar{u}_t\right] \\ &\quad + \frac{1}{n}\sum_{i \in N}\left[(x_t^i - \bar{x}_t)^\top Q(x_t^i - \bar{x}_t) + (u_t^i - \bar{u}_t)^\top Q(u_t^i - \bar{u}_t)\right]. \end{aligned}$$

Problem formulation

Summary of the model

- Dynamics: $x^i_{t+1} = Ax^i_t + Bu^i_t + D\sum_{j\in N} m^{ij} x^j_t + E\sum_{j\in N} m^{ij} u^j_t + w^i_t$
- Per-step cost: $c(x_t, u_t) = \sum_{i,j\in N}\left[h^{ij}_q (x^i_t)^\top Q(x^j_t) + h^{ij}_r (u^i_t)^\top Q(u^j_t)\right]$
- Network structure: M, H_q, and H_r have the same eigenvectors.



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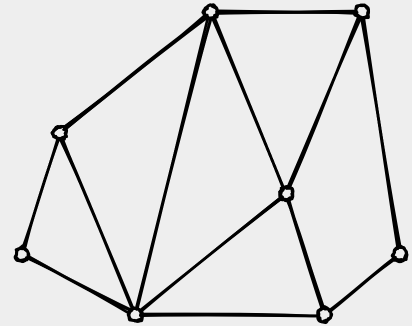

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Objective

Choose a policy π: $(x^1_t, \dots, x^n) \to (u^1_t, \dots, u^n_t)$ to minimize:

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^\pi \left[\sum_{t=1}^{T} c(x_t, u_t) \right]$$

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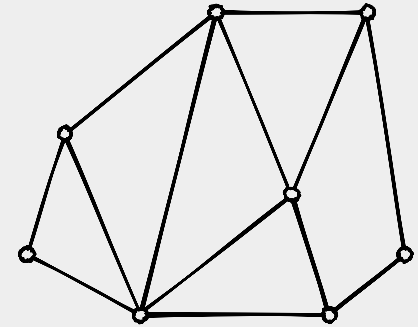

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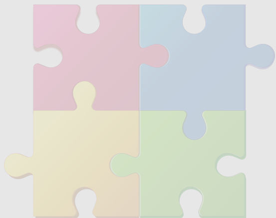
- Standard soln requires solving $nd_x \times nd_x$ Riccati Eq.
- Complexity scales $\mathcal{O}(n^3 d_x^3)$.

Outline



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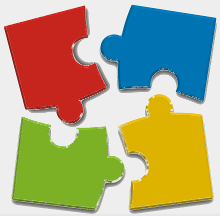

Planning solution

- Spectral factorization of dynamics and cost
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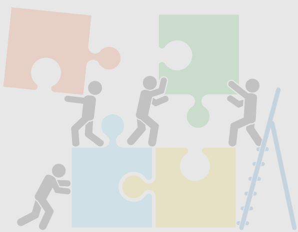

Learning solution

- Spectral factorization of learning
- Numerical experiments

Our result: Develop a decomposition which computes the optimal policy by solving at most n Riccati eqns of dimension $d_x \times d_x$.

- co-author: Shuang Gao
- paper: TCNS 2022

Spectral decomposition

Spectral decomposition of coupling matrices

$$M = \sum_{\ell=1}^{L} \lambda^{\ell} v^{\ell} (v^{\ell})^{\mathsf{T}},$$

Spectral decomposition

Spectral decomposition of coupling matrices

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Spectral decomposition of dynamics

At each node $i \in [n]$:

- For each $\ell \in [L]$, define **eigenstates**, **eigencontrols**, and **eigennoise** as

$$x_t^{\ell,i} = x_t^i v^\ell (v^\ell)^\top, \quad u_t^{\ell,i} = u_t^i v^\ell (v^\ell)^\top, \quad \text{and} \quad w_t^{\ell,i} = w_t^i v^\ell (v^\ell)^\top.$$

Spectral decomposition

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- Define **auxiliary state**, **auxiliary control**, **auxiliary noise** as

$$\breve{x}_t^i = x_t^i - \sum_{\ell=1}^{L} x_t^{\ell,i}, \quad \breve{u}_t^i = u_t^i - \sum_{\ell=1}^{L} u_t^{\ell,i}, \quad \text{and} \quad \breve{w}_t^i = w_t^i - \sum_{\ell=1}^{L} w_t^{\ell,i}.$$

Implication of Spectral Decomposition

Noise-coupled dynamics

$$x_{t+1}^{\ell, i} = (A + \lambda^{\ell} D)\, x_t^{\ell, i} + (B + \lambda^{\ell} E)\, u_t^{\ell, i} + w_t^{\ell, i}$$

and $\breve{x}_{t+1}^{i} = A \breve{x}_t^{i} + B \breve{u}_t^{i} + \breve{w}_t^{i}$

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Decoupled cost

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where $q^\ell = q_0 + q_1 \lambda_q^\ell$, $\quad r^\ell = r_0 + r_1 \lambda_r^\ell$, and

$$\breve{c}(\breve{x}_t^i, \breve{u}_t^i) = (\breve{x}_t^i)^\intercal Q \breve{x}_t^i + \frac{r_0}{q_0} (\breve{u}_t^i)^\intercal R \breve{u}_t^i$$

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Implication of Spectral Decomposition

Eigen-system (ℓ, i) with $\ell \in [L]$, $i \in [n]$

- State $x_t^{\ell,i}$. Control $u_t^{\ell,i}$.
- Dynamics: $x_{t+1}^{\ell,i} = (A + \lambda^\ell D)\, x_t^{\ell,i} + (B + \lambda^\ell E)\, u_t^{\ell,i} + w_t^{\ell,i}$
- Per-step cost: $c^\ell(x_t^{\ell,i}, u_t^{\ell,i})$.

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Certainty equivalence: Optimal policy of stochastic LQ system is same as that of deterministic LQ system.

The deterministic system has **decoupled dynamics and cost!**

Only coupled through the noise in the dynamics

Main result

Under standard assumptions, the optimal control action is given by

$$u_t^i = \breve{u}_t^i + \sum_{\ell=1}^{L} u_t^{\ell,i} = \breve{G}\breve{x}_t^i + \sum_{\ell=1}^{L} G^\ell x_t^{\ell,i}$$

where

$$\breve{G} = \mathsf{Gain}\left(A, B, Q, \frac{r_0}{q_0} R\right)$$

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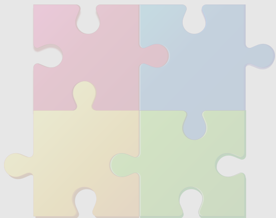
- The gains $\breve{G}$, $\{G^{\ell}\}_{\ell=1}^{L}$ are the same at all subsystems!
- Requires solving $(L+1)$ Riccati Eqn of dimension $d_x \times d_x$.
- Complexity scales $\mathcal{O}(L d_x^3)$ (cf. $\mathcal{O}(n^3 d_x^3)$ for naive solution).

Outline

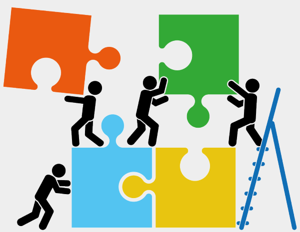


System Model

- Network-coupled subsystems
  - Agents interacting over a graph
  - Coupled dynamics
  - Coupled costs

Planning solution

- Spectral factorization of dynamics and cost
- Decoupled Riccati equations



Learning solution

- Spectral factorization of learning
- Numerical experiments

Review: LQ regulation with unknown/uncertain dynamics

Modeling uncertainty	Model $\theta_* = [A_*, B_*] \in \Theta$ (uncertain set)

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Comparing learning algorithms

$$\text{Regret}(T) = \sum_{t=1}^{T} \Big[\text{cost of learning algo}(t) - \text{cost of clairvoyant agent}(t) \Big]$$

- Large literature. Various classes of algos with different regret guarantees.

Review: Regret for learning in LQ regulation

Bounds on Regret

- **Lower bound**: No algorithm can do better than $\tilde{\Omega}(d_x^{0.5} d_u \sqrt{T})$.
- **Upper bound**: Various classes of algorithms achieve $\tilde{\mathcal{O}}(d_x^{0.5}(d_x + d_u)\sqrt{T})$.
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Challenge with learning in networks

- Effective dimensions are $n d_x$ and $n d_u$
- Directly using existing algos gives regret of $\tilde{\mathcal{O}}(n^{1.5} d_x^{0.5}(d_x + d_u)\sqrt{T})$.
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Regret per agent grows with size of the network!

Our result: Develop a learning algorithm which exploits the structure of the network and has a per agent regret of $\tilde{O}((1 + \frac{1}{n}) d_x^{0.5} (d_x + d_u) \sqrt{T})$.

- co-author: Sagar Sudhakara, Ashutosh Nayyar, Yi Ouyang
- paper: TCNS 2023

Learning model

Problem setting

- Known: Network (M, H_q, H_r). Cost (Q, R).
- Unknown: Dynamics (A, B, C, D).

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- $\breve{\theta}_* = [A_*, B_*] \in \breve{\Theta}$
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Implication of Spectral Decomposition

Recall: $c(x_t, u_t) = \sum_{i \in N} \Big[q_0 \breve{c}(\breve{x}_t^i, \breve{u}_t^i) + \sum_{\ell=1}^L q^\ell c^\ell(x_t^{\ell,i}, u_t^{\ell,i}) \Big]$

Thus, for any policy π,

$$J(\pi; \theta_*) = \sum_{i \in N} \Big[q_0 \breve{J}^i(\pi; \breve{\theta}_*) + \sum_{\ell=1}^L q^\ell J^{\ell,i}(\pi; \theta_*^\ell) \Big].$$

Key idea for learning

Separately learn $\{\theta^\ell\}_{\ell=1}^{L}$ and $\breve{\theta}$

- For learning θ^ℓ_*, select an agent $i^\ell_\circ$ such that $\nu^{\ell, i^\ell_\circ} \neq 0$.
- Learn $G^\ell(\theta^\ell_*)$ using $\{x_t^{\ell, i^\ell_\circ}, u_t^{\ell, i^\ell_\circ}\}_{t \geq 1}$.

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- Learn $\breve{G}(\breve{\theta}_*)$ using $\{\breve{x}_t^{j_t}, \breve{u}_t^{j_t}, \breve{x}_{t+1}^{j_t}\}_{t \geqslant 1}$.

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Since $J(\pi; \theta_*) = \sum_{i \in N} \Big[q_0 \breve{J}^i(\pi; \breve{\theta}_*) + \sum_{\ell=1}^{L} q^\ell J^{\ell, i}(\pi; \theta^\ell_*) \Big]$.

Thus, regret also decomposes as

$$R(T) = \sum_{i \in N} \Big[q_0 \breve{R}^i(T) + \sum_{\ell=1}^{L} q^\ell R^{\ell, i}(T) \Big].$$

Bounding regret

Bounding $R^{\ell,i}(T)$

- Since agent $i^{\ell}_{\circ}$ is learning in the standard manner, we have

$$R^{\ell, i^{\ell}_{\circ}}(T) = \tilde{\mathcal{O}}\big(W^{\ell, i^{\ell}_{\circ}} d_x^{0.5}(d_x + d_u)\sqrt{T}\big).$$

- We show that for other agents

$$R^{\ell, i}(T) = \left(\frac{v^{\ell, i}}{v^{\ell, i^{\ell}_{\circ}}}\right)^2 R^{\ell, i^{\ell}_{\circ}}(T) = \tilde{\mathcal{O}}\big(W^{\ell, i} d_x^{0.5}(d_x + d_u)\sqrt{T}\big).$$

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Bounding $\breve{R}^i(T)$

- Need to bound regret from first principles.
- Using the most informative observation allows us to bound the regret of auxiliary systems at all nodes.
- Show that $\breve{R}^i(T) = \tilde{\mathcal{O}}\big(\breve{W}^i d_x^{0.5}(d_x + d_u)\sqrt{T}\big)$.

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Overall Regret Bound

- Combining these, we have

$$R(T) = \tilde{\mathcal{O}}\big(\alpha^{\mathcal{G}} d_x^{0.5}(d_x + d_u)\sqrt{T}\big),$$

where $\alpha^{\mathcal{G}} = \sum_{\ell=1}^{L} q^{\ell} + q_0(n - L)$.

- Regret per agent is proportional to

$$\alpha^{\mathcal{G}}/n = \mathcal{O}\left(1 + \frac{L}{n}\right).$$

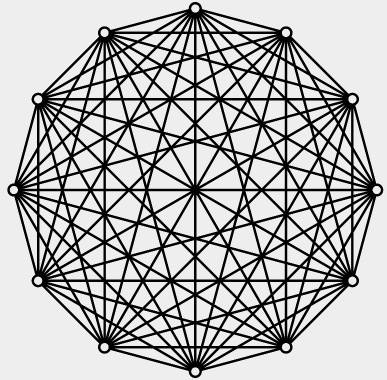
Thus, regret per agent reduces with the size of the network!

Some examples

Mean-field systems

Choice of parameters

- $M = \frac{1}{n}\mathbb{1}_{n \times n}$ and $H_q = H_r = \frac{1}{n}I + \frac{\kappa}{n}M$.



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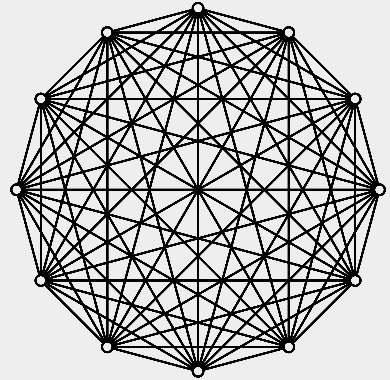

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Scaling of regret

- $q^1 = r^1 = (1+\kappa)/n$. Thus, (normalized) $\alpha^{\mathcal{G}} = \left(1 + \frac{\kappa}{n}\right)$.
- Regret per-agent goes down as the network becomes larger (mean-field effect).

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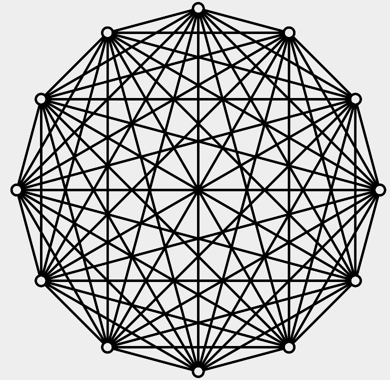

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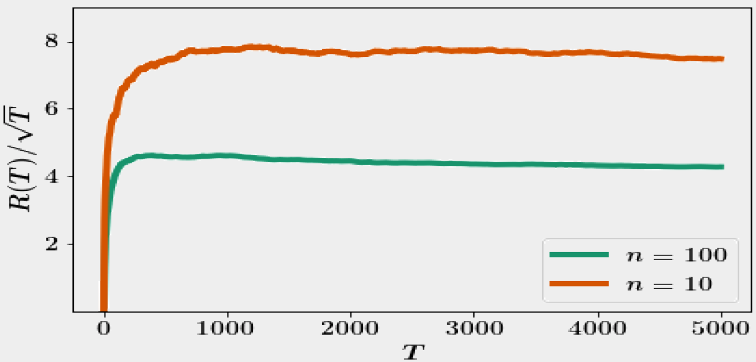
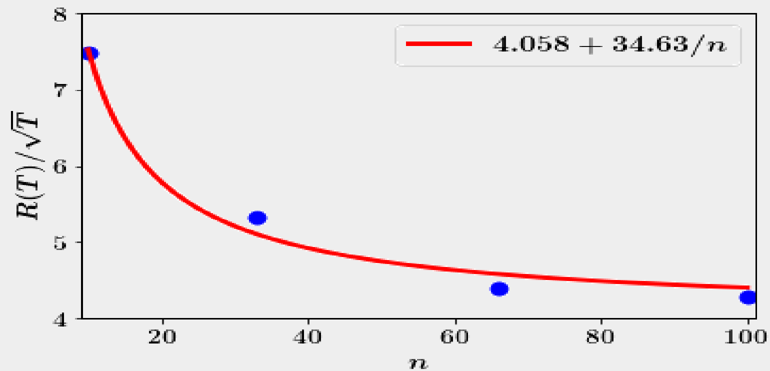


$R(T)/\sqrt{T}$ vs T

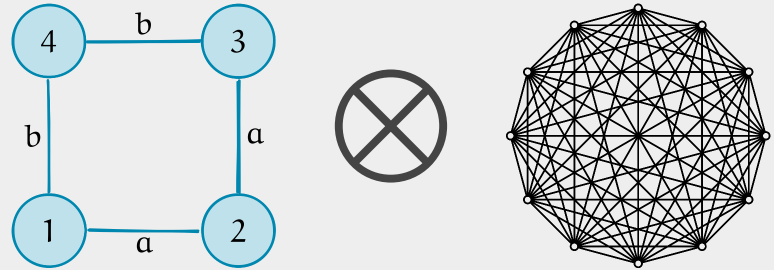


$R(T)/\sqrt{T}$ vs n

A general low-rank network

Choice of parameters

- $M = M^{\circ} \otimes \frac{1}{n}\mathbb{1}_{n \times n}$, $H_q = (I - M)^2$, and $H_r = I$.



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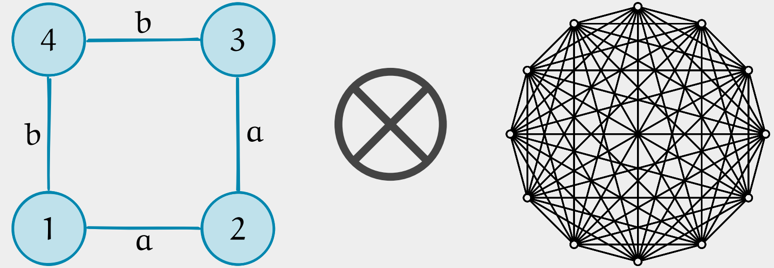

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- $\lambda^\ell = \pm\sqrt{2(a^2 + b^2)}$, $q^\ell = (1 - \lambda^\ell)^2$, $r^\ell = 1$. Thus, (unnormalized) $\alpha^{\mathcal{G}} = 4n + 4(a^2 + b^2)$.
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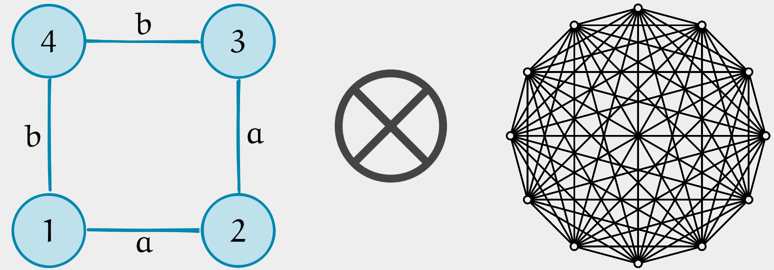
A general low-rank network



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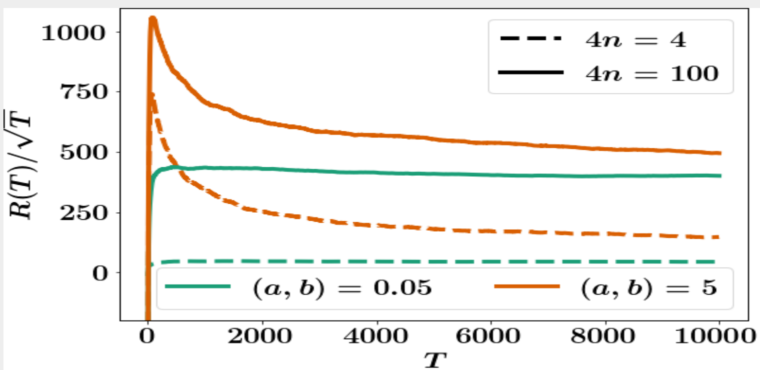
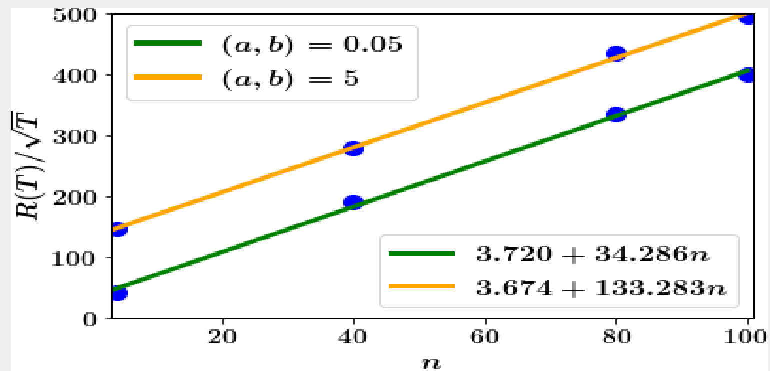


$R(T)/\sqrt{T}$ vs T



$R(T)/\sqrt{T}$ vs n

Conclusion

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- Regret per agent $\tilde{\mathcal{O}}((1+\frac{1}{n})\sqrt{T})$

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Future Directions

- Multiple types of agents, approximate symmetry, . . .
- Large networks, graphon limits?
- Other types of scalable network stuctures?

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Thank you

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References

- **planning**: TCNS 2022
- **learning**: TCNS 2023