Chapter 2

Coupled Pendula

Objectives

1. Find the spring constant of different springs

2. To find the normal mode frequencies

3. To establish beats and compare the beat frequency with the difference between the normal mode frequencies

4. To set-up arbitrary oscillations of the system and do a Fourier transform of the data and compare the frequencies with the normal mode frequencies

Introduction/Theory/Background

Consider the following system

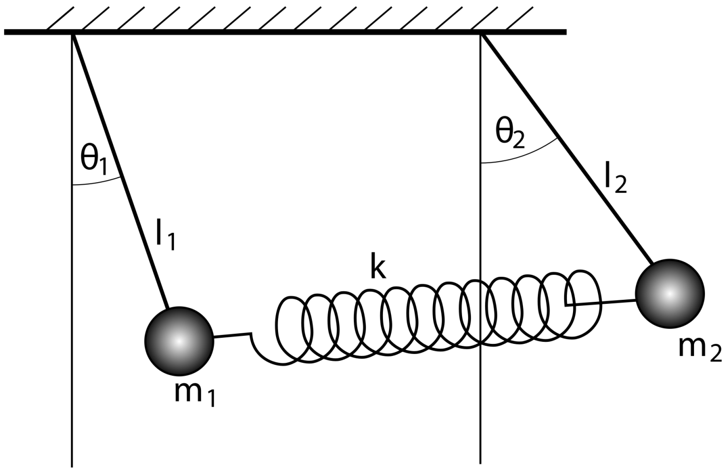


Figure 2.1: Coupled pendulum set-up

We use Newton's second law to set-up the equations of motion of the two bobs. We consider the angular displacement to be small, and look at the horizontal(x) displacement. It is clear that there are two forces on each bob- one due to gravity, and the other due to the stretching of the spring. Whereas the force due to gravity on the bob depends only on its displacement, the force due to stretching(or compression) of the spring depends on the difference between the amounts by which both the bobs are displaced. (To understand that, imagine what happens if both the bobs are displaced by the same amount in the same direction, the spring is un-stretched and hence the spring force is zero) (here, $m_1 = m_2$)

$$\frac{d^2x_1}{dt^2} = -g\frac{x_1}{l} + \frac{k}{m}(x_2 - x_1)$$

$$\frac{d^2x_2}{dt^2} = -g\frac{x_2}{l} - \frac{k}{m}(x_2 - x_1)$$

Notice that the two differential equations, while ordinary, are coupled. This implies that, in this form, the solution to each equation would require the solution of the other. This leads to what appears to be a mathematical impasse.
To get out of this impasse, we need to remember that if we are looking for the values of two variables but are given two independent linear combinations of them, we can solve for the variables. (This is what you called solving simultaneous equations in school.) So we need to find two independent linear combinations of x_1 and x_2. For an arbitrary pair of couple differential equations, this will require a specialised procedure, but for the pair corresponding to this system, we find the linear combinations by inspection.

Because the coupling term $k(x_2 - x_1)$ has opposite signs in the two equations, adding the equations eliminates it, leaving us an equation in the single variable $x_1 + x_2$. Now we need to find another linear combination of x_1 and x_2. If we stare at the equations a little, we realise that if we subtract one equation from the other, we get another equation in a single variable $x_1 - x_2$. We define $X = x_1 + x_2$, and $Y = x_1 - x_2$.

In terms of X and Y, the equations of motion of the system are

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = -\omega_1^2 X \tag{2.1}$$

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} = -\omega_2^2 Y \tag{2.2}$$

These are just simple harmonic oscillator equations, with $\omega_1 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$, and can easily be solved. Now all we need to do is invert the linear combinations that lead to X and Y in terms of x_1 and x_2, and we will get the solutions we are looking for. As we know these are two differential equations of second order, so the solutions will have four arbitrary constants to be fixed by four initial conditions $(x_1(0), x_2(0), \dot{x_1}(0), \dot{x_2}(0))$ or equivalently $(X(0), Y(0), \dot{X}(0), \dot{Y}(0))$.

The solutions to the eqs. 2.1 and 2.2 are

$$X = a_1 \sin \omega_1 t + a_2 \cos \omega_1 t \tag{2.3}$$

$$Y = b_1 \sin \omega_2 t + b_2 \cos \omega_2 t \tag{2.4}$$

and inverting using $x_1 = \frac{X+Y}{2}$ and $x_2 = \frac{X-Y}{2}$ and performing some algebra, we get

$$x_1 = c_1 \sin \omega t + c_2 \cos \omega t \tag{2.5}$$

$$x_2 = d_1 \sin \omega t + d_2 \cos \omega t \tag{2.6}$$

where c_1, c_2, d_1, d_2 can be expressed in terms of a_1, a_2, b_1, b_2 , but as they are arbitrary constants to be fixed by initial conditions we can leave them as it.

Notice that when the coupled pendulums are in an arbitrary state of motion, both normal-model frequencies are needed to describe the motion. However, there can be special states of motion in which one only frequency appears.
For example, if $x_1 = x_2$ at the beginning and $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} = 0$ at the beginning, then $Y = 0$ at all times. Thus, in this state of motion the motions of both bobs and thus of the whole system are described by just the single frequency ω_1.
Similarly, if $x_1 = -x_2$ at the beginning and $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} = 0$ at the beginning, then $X = 0$ at all times, and the other frequency. ω_2, is the only one needed to describe the motion of the system.
These special states, in each of which only a single frequency appears in the motion of all moving parts of the system, are called normal modes.

General solution to the equations that describe the motion of the coupled pendulum can be written as a superposition of the normal mode solutions.

An interesting case is when the system shows beats. The initial conditions that correspond to this is

$$x_1 = A, x_2 = 0$$

$$\dot{x_1} = 0, \dot{x_2} = 0$$

$$x_1 = A\ \cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\ \cos\left(\frac{\omega_1 - \omega_2}{2}t\right)$$

$$x_2 = A\ \sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\ \sin\left(\frac{\omega_1 - \omega_2}{2}t\right)$$

Question: What can you infer from the above solutions, and how does this correspond to what you see in the experiment?

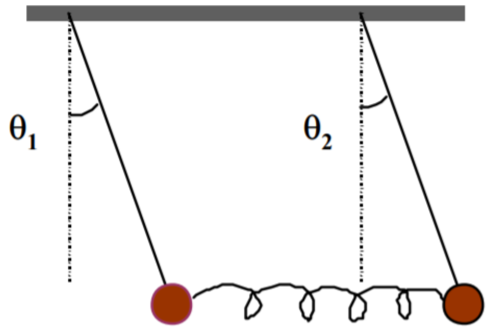


Figure 2.2: In-phase mode

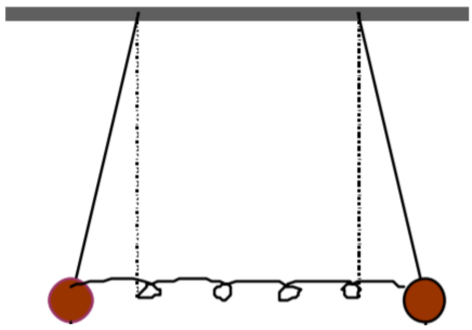


Figure 2.3: Out of Phase mode

The above system we have solved is two simple pendula coupled using a spring. What you have as apparatus are two compound pendula. For the above equations to hold, figure out the conditions for when a compound pendulum can be approximated by a simple pendulum. Some properties of a simple pendulum are

1. The string is massless
2. All the mass (of the bob) is concentrated at the end of the string
3. The motion is in one plane

Experimental Setup

Apparatus

1. Two pendulum rods and masses
2. Springs of different spring constants
3. Stop-watch
4. Weights
5. Weighing machine
6. Camera and tripod
7. Tracker software

Description

Warnings

- Level both the pendulums
- Ensure that when you put the spring, both the masses are vertical and at equilibrium position
- Put a white background behind the setup for tracking

Procedure

In this experiment, you will study the motion of the coupled pendulum with various initial conditions to better understand normal modes of a system.

2.1 Part I

- Take 3-4 springs of various stiffness and hook them vertically onto a clamp stand to measure their spring constants.
- Hung different weights from the spring and note its oscillation with the help of the stopwatch.
- Also measure the extension of each of the springs with the help of a scale.

2.2 Part II

- Attach the spring to both the masses to make the system coupled.
- Try to carefully excite the in-phase and out of phase modes for each of the springs and take the video with the help of a camera (which should be placed parallel to the setup).
- Find the frequencies by doing the Fourier transform of the x vs t data sets for each of the bobs and measure the error analysis.

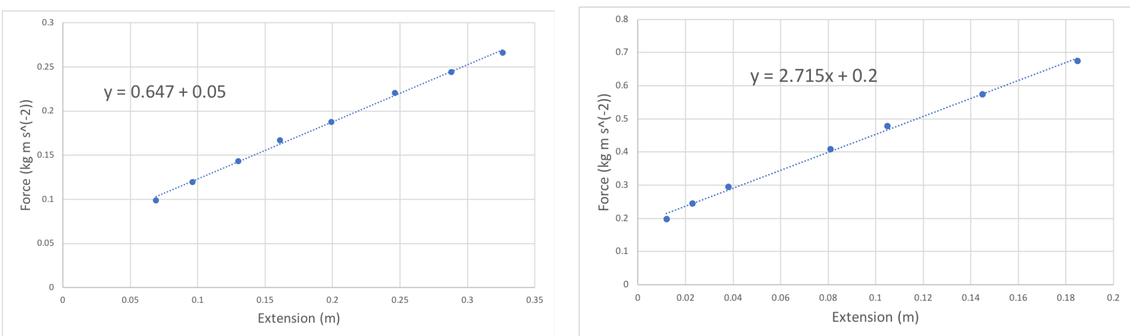
2.3 Part III

- Slightly displace one of the bobs ($\theta_i = \theta$) and observe the motion
- If the beats have been setup, one of the bobs will oscillate at a time and after it stops the other bob will start oscillating.

2.4 Part IV

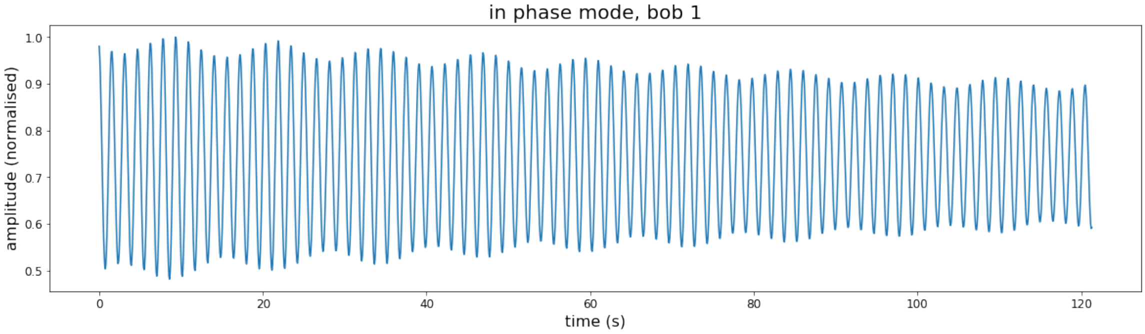
- Choose the spring with appropriate spring constant (i.e. large), so that the normal modes are well separated
- Displace both the bobs by different amounts and record the motion using a camera
- Perform a Fourier transform for various arbitrary motions (different initial conditions) and you will observe different contributions of different modes

Expected Outcomes



(a) Spring 1: low k (b) Spring 2: high k

Figure 2.4: The spring constants: the slope of each of these graphs gives k.



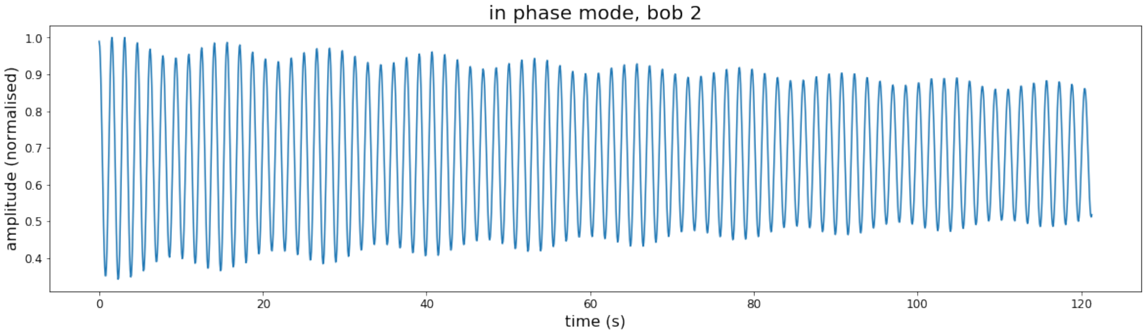
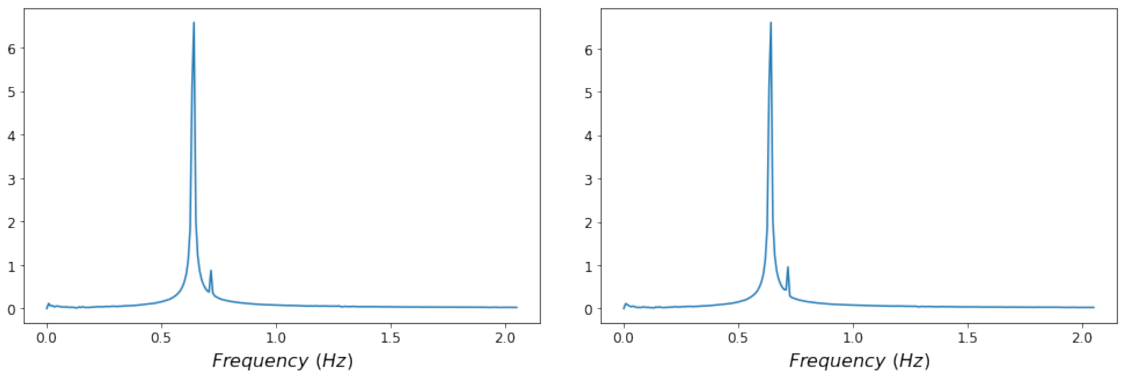


Figure 2.5: The position vs. time graphs of both the bobs



(a) FT of Bob 1

(b) FT of Bob 2

Figure 2.6: The FTs of the bobs for the in-phase normal mode.

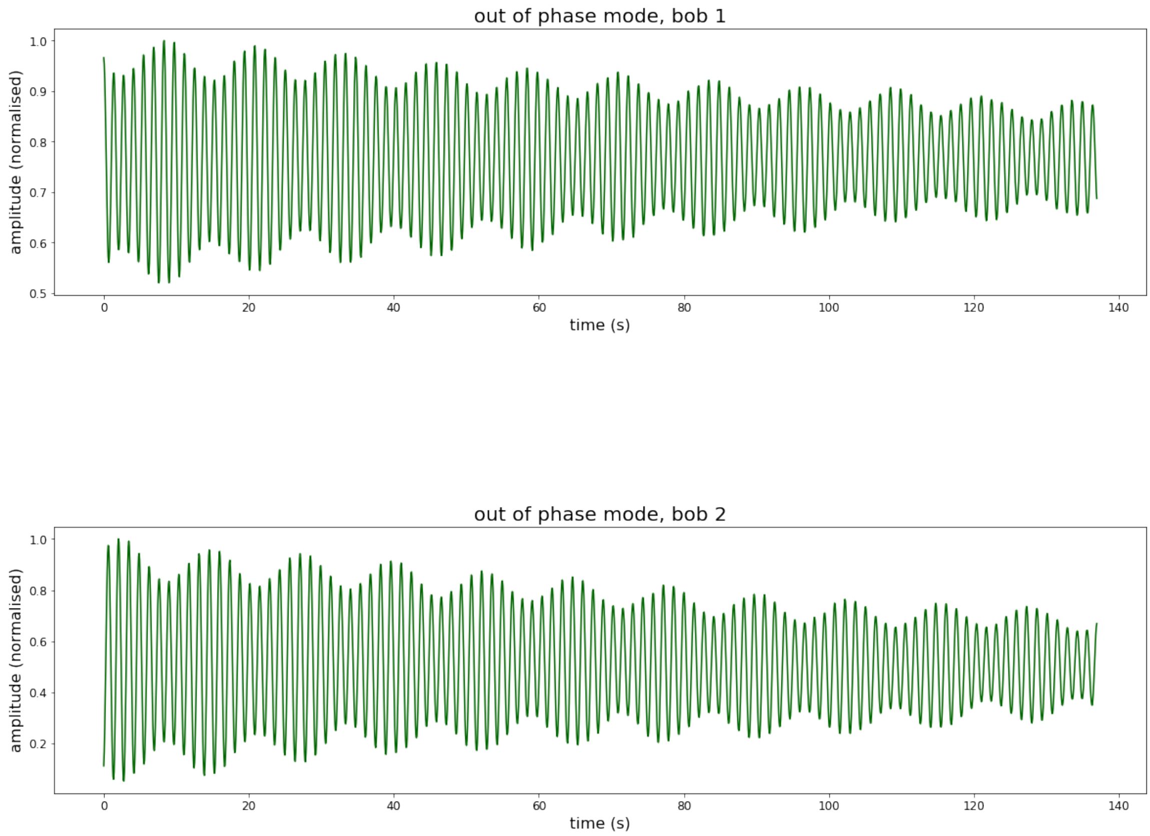
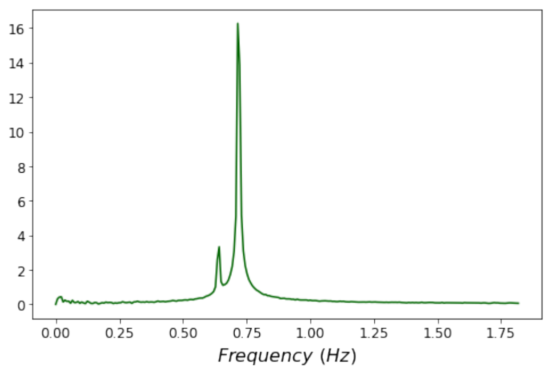
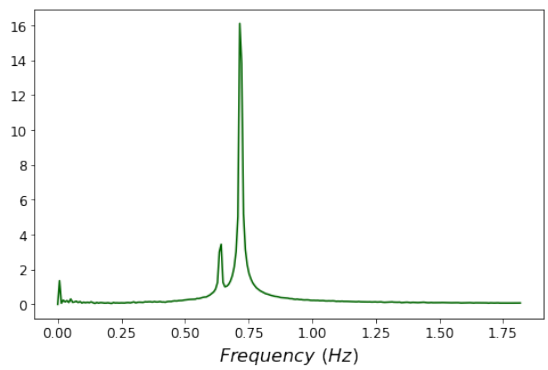


Figure 2.7: The position vs. time graphs of both the bobs

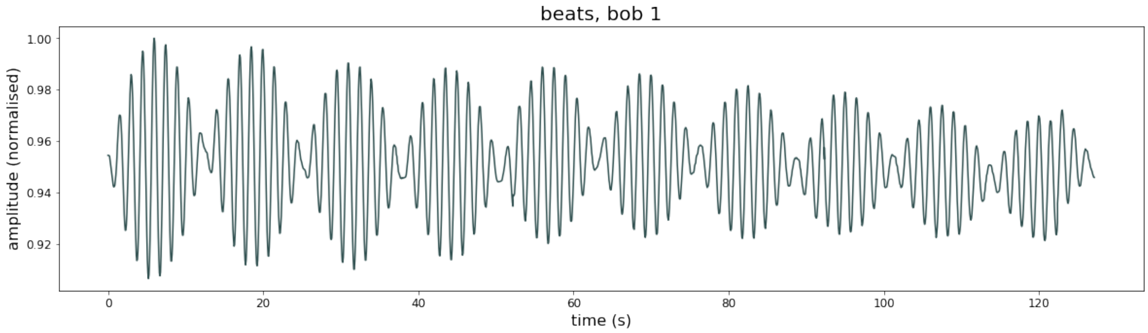


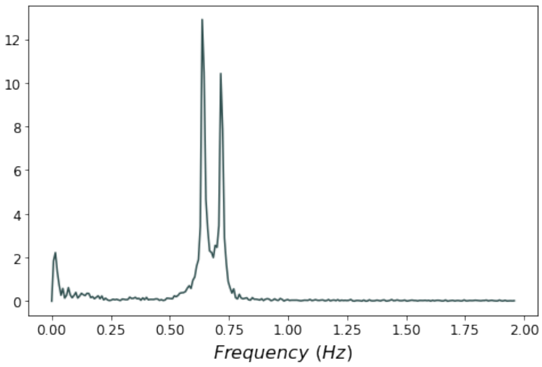
(a) FT of Bob 1



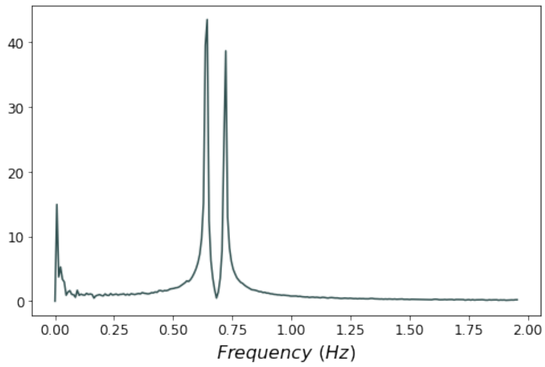
(b) FT of Bob 2

Figure 2.8: The FTs of the bobs for the in-phase normal mode.





(a) FT of Bob 1



(b) FT of Bob 2

Figure 2.9: The FTs of the bobs for the in-phase normal mode.